GROWTH IN ILLYRIA: THE ROLE OF MERITOCRACY IN THE ACCUMULATION OF HUMAN CAPITAL*

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Abstract

In this paper we present a dynamic model of cooperative production with human capital accumulation. We assume CES preferences on consumption and leisure in each period. When agents do not care about future generations, sustained growth occurs iff the elasticity of substitution between consumption and leisure is larger or equal than one. Meritocracy always has a positive effect on output, but when the elasticity of substitution is less than one, is only a level effect. When agents care about future generations, under Cobb-Douglas preferences in each period and some extra conditions, there is constant growth at a rate that is larger than the one when future generations do not count. For any discount rate between generations, there is a unique level of meritocracy for which efficiency is achieved.

Keywords: Cooperative Production, Growth, Meritocracy.

JEL classification: D2, D9, O4.

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1. Introduction

The theory of cooperative production studies an economy in which workers offer their labor to a production center (the coop). The forerunners of this theory were Ward (1958) who imagined an ideal ("Illyrian") world, Vanek (1970) who focussed on connecting the theory with the cooperative movement and Sen (1966), who modeled the coop as a sharing rule to distribute the output.[1] Later on, Moulin (1987), Romer and Silvestre (1993)) provided foundations for different sharing rules. In this paper, we follow Sen's approach. The (impossible) goal of the theory of cooperative production is to mimic the theory of markets with profit maximizing firms by offering a detailed study of all the possible outcomes. In this (grand) research picture the theory of cooperative production lags in dynamic models that illustrate the engines and the consequences of growth. Our paper is addressed to explore these questions.

Our model is very simple. Agents are identical in all respects: preferences, labor endowments, and the human capital (capital in the sequel) inherited in the first period from their ancestors.[2] They care about consumption and leisure. Each individual produces her own capital using the inherited capital and labor. Output is produced by capital and it is distributed according to a sharing rule. Finally, individuals consume and die, leaving to their successors the human/social capital they have accumulated.

The coop is composed of a large number of individuals, capital is the product of the inherited capital and labor, the utility function is CES, the production function has constant elasticity and the sharing rule is a weighted average of the proportional and the egalitarian sharing rules. There are 6 parameters in our model:

1. Initial conditions: capital in the initial period.
2. Productivity: the returns to scale in the production of consumption.
3. Degree of meritocracy: the weight of the proportional sharing rule in the sharing rule.
4. Taste for consumption: the weight of consumption versus leisure.
5. Substitutability between consumption and leisure: the elasticity of substitution.
6. Labor endowment.

[1] Illyrian has been often used in the literature to represent an ideal world: "Shakespeare's Illyria is a fantasy land of make-believe and illusion" (Shakespeare, p. 169, 1993) or even a scientific Utopia (Beckett, 2003).

[2] Another interpretation of capital is that refers to social capital, see Dasgupta (2002).

Our main task will be to identify the role of these variables in the dynamic equilibrium of our model.

We focus on the case in which, along the equilibrium path, agents make a positive effort. This is always the case if the elasticity of substitution is less or equal to one. If the elasticity is greater than one this requires a certain bound on how increasing are the returns to scale.

We start our analysis by considering the limit case in which agents do not care about future generations. Under Cobb-Douglas preferences (i.e. when the elasticity of substitution equals to one) we show that capital grows at a constant rate. This rate depends positively on the taste of consumption and the degree of meritocracy because both encourage effort which in turn produces more capital.

For any other value of the elasticity of substitution, there is a steady state value of capital. When the elasticity of substitution is smaller than one (i.e. consumption and leisure are complements), the steady state is stable, i.e. growth eventually disappears. This is explained by the fact that consumption is a poor substitute of leisure. So when consumption grows, in order to take full advantage of this, leisure must grow as well which dampens growth. The value of capital at steady state depends positively on the degree of meritocracy and the taste of consumption.

If the elasticity of substitution is larger than one (i.e. consumption and leisure are substitutes), capital at the steady state depends negatively on meritocracy and the taste of consumption. This may sound paradoxical, but it is explained by the fact that the steady state is now unstable. Thus, given the initial stock of capital, more meritocracy/taste for consumption may boost the economy from negative to positive growth. In this case, when human capital grows, leisure is advantageously replaced by consumption, so unbounded growth is now possible.

We then consider a more general set up in which individuals care about future generations and they maximize a discounted sum of utilities.[3] Due to the technical difficulties we focus in the case in which preferences are Cobb-Douglas. We show that when future generations do not count much, there is a solution to the intertemporal maximization problem in which the growth rate is constant. Such a rate does not exist for certain values of the parameters, but when it does, is larger than the corresponding rate in the zero discounting case. This is due to the fact that when future counts, there are more incentives to invest in capital which in turn stimulates growth. As before,

[3] Thus the appoach developed above is a special case in which the discount factor is zero.

meritocracy encourages work which produces human capital, which makes the economy grow.

Finally, we study efficiency. Too much meritocracy encourages too much work because effort not only increases the agent's capital but increases her share. On the other hand, too little meritocracy encourages free riding. This suggests that there is an optimal degree of meritocracy and Sen (1966) proved that this is indeed the case: the optimal degree of meritocracy is achieved when the weight of the proportional sharing rule equals the elasticity of output with respect to capital. We prove that Sen's result, which corresponds to a static economy without capital accumulation, holds in our framework.

The role of human capital in models of growth with profit maximizing firms has been stressed from different angles. Uzawa (1965) presented the first model, to the best of our knowledge, in which human capital played a central role in growth. But it is fair to say that this topic caught fire only after the influential paper by Lucas (1988). The modern literature is enormous, see Acemoglu (2008), chapter 10 for a survey. From the empirical point of view, the interest in human capital stems from the highly influential paper by Barro (1991) which showed that growth is correlated with the initial value of human capital. See Lucas (2015) for a new model of the role of human capital in growth. These models share with our's the assumption that human capital can be accumulated without bound and without diminishing returns in the production of capital. But as far as we can tell, there is no correlate in this literature to our results linking preferences and growth because in all the above models, leisure does not play any role. But constant growth due to human capital is possible as in our model, see Lucas (1988), pp. 21-25. In the latter, socially optimal and equilibrium paths diverge, as they do in our model. But there is no "exogenous" variable (the meritocratic parameter in our model) that can used to minimize welfare losses.

2. The Model and Preliminary Results

Time is countable infinite. There are n families (dynasties) with a member alive in each period. An individual alive in period t from the family i receives from her predecessor a capital of H_i^{t-1}, the capital accumulated by the previous generation. We assume that in the first period, all agents inherit the same amount of capital, that is, $H_i^0 = H_j^0$ for all i, j. All individuals have the same endowment of labor time in every period, ω. We assume that $\omega > 1$. At time t, each individual produces her own capital from the inherited capital, H_i^{t-1}, and her labor, $l_i^t \in [0, \omega]$, in the following way

$$H_i^t = H_i^{t-1} l_i^t. \tag{2.1}$$

Let $H^t = \sum_{j=1}^n H_j^t$ be the aggregate capital at period t. Note that given (2.1), $\omega > 1$ is essential here to allow for capital growth.

The assumption that human capital is produced by inputs that enter in a multiplicative form has been used in many models of endogenous growth since the pioneering work of Uzawa (1965), see e.g. Lucas (1988, p. 18). This multiplicative form has the strong implication that if one of its terms is zero, it is impossible to produce human capital. A form like $H_i^t = (H_i^{t-1} + a)(l_i^t + b)$, $a, b > 0$ would be preferable. In this case capital does not fully depreciate even when the effort is zero. Unfortunately, this formulae does not yield a tractable model. We may think of (2.1) as an approximation of this more realistic production function when a and b are small.

The consumption good is produced by means of capital represented by the production function $(H^t)^\gamma$. We have decreasing/constant/increasing returns to scale as long as γ is less, equal or larger than one.

The consumption of each individual, c_i^t, is determined by a sharing rule which is written as follows

$$c_i^t = (H^t)^\gamma \left(\frac{1-\rho}{n} + \rho \frac{H_i^t}{H^t} \right), \ \rho \in [0,1]. \tag{2.2}$$

When $\rho = 1$ the sharing rule allocates consumption proportional to relative capital and when $\rho = 0$ the sharing rule allocates consumption in a totally egalitarian manner. The parameter ρ is the weight attached to the relative contribution of a particular agent to the aggregate capital and we refer to it as the degree of meritocracy. Kang (1988) showed that (2.2) for $n > 2$, is the unique differentiable sharing rule which is symmetric and homogeneous of degree zero (i.e. independent of the units in which capital is measured). Moulin (1987) axiomatized this sharing rule by means of two properties: additivity (if two agents merge without changing their efforts the merged agent receives the sum of the shares of the two agents) and No-Advantageous Reallocation (agents cannot increase the sum of their shares by reallocating money inside any coalition). Both properties have a strong strategic flavor.

An individual alive in period t from the family i derives utility from consumption, c_i^t, and leisure, $\omega - l_i^t$, represented by a CES utility function

$$U_i(c_i^t, \omega - l_i^t) = \left(\beta (c_i^t)^{\frac{s-1}{s}} + (\omega - l_i^t)^{\frac{s-1}{s}}\right)^{\frac{s}{s-1}}, \ \beta > 0, \ s \in (0, \infty). \tag{2.3}$$

where s is the elasticity of substitution between consumption and labor. When $s \to 1$, the utility function is Cobb-Douglas. When $s \to \infty$ utility is linear in consumption and leisure. When $s \to 0$, the utility function tends to be Leontieff. We refer to β as the taste for consumption.

The life of an individual, say i, is simple. She inherits H_i^{t-1} capital and she chooses consumption and leisure in order to maximize her utility function subject to (2.1), and (2.2). Throughout the paper, we will assume that when an individual maximizes her utility she takes as given total capital, H^t. This can be justified when there is a very large number of individuals in the society so when an individual decides about her investment she disregards these terms. Under this assumption and (2.1), the restriction (2.2) defines the consumption as a linear function of labor. Thus, the maximization problem of individual i is simple and becomes

$$\begin{aligned} & Max_{(c_i^t, l_i^t)} \ U_i(c_i^t, \omega - l_i^t) \\ & s.t. \ c_i^t = (H^t)^\gamma \left(\tfrac{1-\rho}{n} + \rho \tfrac{H_i^{t-1} l_i^t}{H^t}\right) \\ & l_i^t \in [0, \omega_i^t] \end{aligned} \tag{2.4}$$

We now define our equilibrium notion.

Definition 1. *A short run equilibrium (SRE) in period t is a list $\{(\tilde{c}_i^t, \tilde{l}_i^t)\}_{i=1}^n$, such that for any agent i, $(\tilde{c}_i^t, \tilde{l}_i^t)$ is a solution to the maximization problem (2.4) taking $\tilde{H}^t$ as given and $\tilde{H}^t = \sum_{j=1}^n H_j^{t-1} \tilde{l}_j^t$.*

We close this section showing that at each period t the short run equilibrium exists and is symmetric, and under certain conditions on the parameters of the model, the equilibrium is interior. We offer the formal proofs of these preliminary results in the Appendix.

Proposition 1. *At every t, a short run equilibrium exists, is unique and symmetric.*

That a short run equilibrium exists, is a direct consequence of the continuity of the utility as a function of l_i^t given that, $c_i^t = (H^t)^\gamma \left(\frac{1-\rho}{n} + \rho \frac{H_i^{t-1} l_i^t}{H^t}\right)$ and the domain is a compact set ($l_i^t \in [0, \omega]$). At period 1, all agents inherit the same capita, H_i^0, and all have the same endowment of labor time. Since they take H^1 as given, all of them face the same maximization problem which has a unique solution and, therefore, they choose the same amount of labor time. Consequently, the inherited capital in period 2 is also the same for all agents, and so on. In each period, all agents inherit the same capital.

Proposition 2. *If $\gamma(\frac{s-1}{s}) < 1$ and $\rho > 0$, the short run equilibrium is interior, i.e. $\tilde{l}_i^t \in (0, \omega)$ for all i.*

This condition holds under constant returns to scale ($\gamma = 1$), or if the utility function is Cobb-Douglas, or if $s < 1$. But if $s > 1$, returns to scale must be sufficiently decreasing. Note that if $\rho = 0$ agents expect their income to come entirely from the aggregate capital which, by assumption, does not change with the efforts of a single player. Thus they make zero effort.

3. Equilibrium paths

Along this section we assume that $\gamma(\frac{s-1}{s}) < 1$ and $\rho > 0$ which guarantees that the equilibrium is interior.

In order to analyze the equilibrium it is useful to consider the Lagrange function associated to (2.4),

$$U_i(c_i^t, \omega - l_i^t) - \lambda_i^t \left(c_i^t - (H^t)^\gamma \left(\frac{1-\rho}{n} + \rho \frac{H_i^{t-1} l_i^t}{H^t} \right) \right)$$

where λ_i^t is the Lagrange multiplier. Since we assume that the parameters of the model are such that the equilibrium is interior, the first order conditions give us the maximum.

$$\begin{aligned} \frac{\partial U_i(c_i^t, \omega - l_i^t)}{\partial c_i^t} &= \lambda_i^t. \\ \frac{\partial U_i(c_i^t, \omega - l_i^t)}{\partial l_i^t} &= -\lambda_i^t \rho \frac{(H^t)^\gamma H_i^{t-1}}{H^t}. \end{aligned}$$

Denoting the marginal rate of substitution between consumption and leisure as $M_i(c_i^t, \omega - l_i^t)$ we have that

$$M_i(c_i^t, \omega - l_i^t) = \frac{H^t}{\rho H_i^{t-1} (H^t)^\gamma}. \tag{3.1}$$

For a CES utility function, we have that

$$M_i(c_i^t, \omega - l_i^t) = \beta \left(\frac{\omega - l_i^t}{c_i^t} \right)^{\frac{1}{s}}. \tag{3.2}$$

Combining (3.1) and (3.2),

$$l_i^t = \omega - \left(\frac{(H^t)^{1-\gamma}}{\beta\rho H_i^{t-1}}\right)^s c_i^t. \tag{3.3}$$

Since all individuals are identical, in an interior symmetric equilibrium, $H_i^{t-1} = H_j^{t-1}$; $H_i^t = H_j^t$; $H^{t-1} = nH_i^{t-1}$; $H^t = nH_i^t$. Thus, $l_i^t = H_i^t/H_i^{t-1} = H^t/H^{t-1}$, and $c_i^t = (H^t)^\gamma/n$. From (3.3),

$$\frac{H^t}{H^{t-1}} = \omega - n^{s-1}\left(\frac{1}{\beta\rho H^{t-1}}\right)^s (H^t)^{s(1-\gamma)+\gamma}. \tag{3.4}$$

We will refer to (3.4) as the transition function, that defines an equilibrium path of the human capital.

We start analyzing the case of a linear transition function.

3.1. Equilibrium paths with a linear transition function

The transition function is linear iff the utility function is Cobb-Douglas. In this case, the transition (3.4) can be written as:

$$H^t = \frac{\omega\beta\rho}{1+\beta\rho}H^{t-1}.$$

The growth rate is given by

$$g^t = \frac{H^t - H^{t-1}}{H^{t-1}} = \frac{\omega\beta\rho}{1+\beta\rho} - 1. \tag{3.5}$$

The growth rate is constant, and it is positive iff $\beta\rho(\omega-1) > 1$. It increases with labor endowments (so countries rich in endowments tend to grow faster than countries poor in endowments); in the taste of consumption (so consumerism is good for growth because incentivizes hard work) and the degree of meritocracy (meritocracy encourages hard work). Returns to scale (γ) do not play any role. As noted before, full egalitarianism ($\rho = 0$) destroys the whole economy in a single period!. Since $H^t/H^{t-1} = l_i^t$, $g^t = l_i^t - 1$ and from (3.5) we see that each individual makes an effort proportional ($\beta\rho/(1+\beta\rho)$) to her endowment of labor time. This proportion increases with the taste of consumption and the degree of meritocracy.

3.2. Equilibrium paths with a non linear transition function

When $s \neq 1$, the transition function has a unique fixed point, $\hat{H}$, different from zero given by

$$\hat{H} = \left(\frac{\omega - 1}{n^{s-1}(\frac{1}{\beta\rho})^s}\right)^{\frac{1}{\gamma(1-s)}}. \tag{3.6}$$

Note that the assumption that $\omega > 1$ implies that the fixed point is positive.

The following propositions states the properties of the transition function around $\hat{H}$. The formal proofs can be found in the Appendix.

Proposition 3. *The unique fixed point, $\hat{H}$, is globally stable iff $s < 1$. Furthermore, if $s < 1$, $\hat{H}$ is increasing with ω, β, and ρ.*

When $s < 1$ growth finally ends and we converge to a stationary society in which the human capital $\hat{H}$ is increasing in ω, β and in the level of meritocracy. This is because when $s < 1$ consumption is a poor substitute of leisure. So when consumption grows, in order to take full advantage of this, leisure must grow as well which eventually exhausts growth.

Proposition 4. *If $s > 1$ the unique fixed point, $\hat{H}$, is unstable. Furthermore, if $s > 1$, $\hat{H}$ is decreasing with ω, β, and ρ. If $H^0 < \hat{H}$, capital always decreases. Otherwise capital always increases, and when $t \to \infty$, $g^t \to \omega - 1$.*

The case $s > 1$ in the long run is somehow similar to the Cobb-Douglas case, in particular, in the limit, the growth rate is independent of the returns to scale. But there are striking similarities. Neither the taste of consumption nor the degree of egalitarianism play any role whatsoever in the determination of the long run growth rate, even though they play a role in the determination of $\hat{H}$ and therefore in the fact that the economy grows positively or negatively. The possibility of a positive growth depends on $\hat{H}$ to be small! This explains the, apparently paradoxical comparative statics results, that when $s > 1$, $\hat{H}$ is decreasing in ω, β and ρ. This is because high initial endowments (or taste for consumption or meritocracy) make $\hat{H}$ small and thus it is more likely that growth is positive! Notice that in the limit $g^t \simeq \omega - 1$ and, given that $g^t = l_i^t - 1$, leisure disappears in the limit. This is due to the fact that consumption and leisure are very substitutable, i.e. $s > 1$.

Finally, we note that the same assumptions that yield intuitive comparative static results yield stability. This is a familiar situation since Samuelson advanced the Correspondence Principle (1941), namely that most assumptions that are sufficient to insure stability of equilibrium turn out to be very useful when doing comparative statics (and uniqueness).

4. Dynasties

The previous approach does not take into account that individuals may care about future generations. Thus, when choosing labor they disregard the effect of today's labor on tomorrow's capital. In this section we consider a model in which each agent maximizes a utility function which is the infinite sum of discounted utility (at rate δ) namely

$$W_i = \sum_{t=1}^{t=\infty} \delta^{t-1} U(c_i^t, \omega - l_i^t). \tag{4.1}$$

Now, when an individual decides on today's effort takes into account today's and tomorrow's impact of this effort.

Equilibrium arises from the maximization of (4.1) with the following set of constraints

$$\begin{aligned} c_i^t &= (H^t)^\gamma \left(\frac{1-\rho}{n} + \rho \frac{H_i^t}{H^t} \right), \ t \in \{1, 2, ...\} \\ H_i^t &= H_i^{t-1} l_i^t, \ t \in \{1, 2, ...\}, \ H_i^0 > 0. \end{aligned}$$

Or equivalently,

$$c_i^t = (H^t)^\gamma \left(\frac{1-\rho}{n} + \rho \frac{H_i^0 \Pi_{T=1}^t l_i^T}{H^t} \right), \ t \in \{1, 2....\}; \ H_i^0 > 0.$$

Let λ_i^t be the Lagrange multiplier associated with the constraint at time t. First order conditions of maximization for c_i^t and l_i^1 yield

$$\begin{aligned} \frac{\partial U(c_i^t, \omega - l_i^t)}{\partial c_i^t} \delta^{t-1} &= \lambda_i^t, \ t \in \{1, 2....\} \\ -\frac{\partial U(c_i^1, \omega - l_i^1)}{\partial l_i^1} &= \rho \left(\lambda_i^1 \frac{(H^t)^\gamma H_i^0}{H^1} + \sum_{t=2}^{\infty} \lambda_i^t \frac{(H^t)^\gamma H_i^0 \Pi_{T=2}^t l_i^T}{H^t} \right), \\ \lim_{T\to\infty} \delta^{T-1} \frac{\partial U(c_i^T, \omega - l_i^T)}{\partial l_i^t} H_i^T &= 0. \end{aligned} \tag{4.2}$$

Where (4.2) is the transversality condition. Combining the first two equations,

$$\begin{aligned} -\frac{1}{\rho} \frac{\partial U(c_i^1, \omega - l_i^1)}{\partial l_i^1} &= \frac{\partial U(c_i^1, \omega - l_i^1)}{\partial c_i^1} (H^1)^{\gamma-1} H_i^0 + \\ &+ \sum_{t=2}^{\infty} \delta^{t-1} \frac{\partial U(c_i^t, \omega - l_i^t)}{\partial c_i^t} (H^t)^{\gamma-1} H_i^0 \Pi_{T=2}^t l_i^T. \end{aligned} \tag{4.3}$$

Given the complexity of the difference equations that characterize the optimum, we will assume that the utility function is Cobb-Douglas, namely $U_i(c_i^t, \omega - l_i^t) = (c_i^t)^{\frac{\beta}{1+\beta}} (\omega_i^t - l_i^t)^{\frac{1}{1+\beta}}$, $\beta > 0$. Thus,

$$\frac{\frac{\partial U(c_i^t, \omega - l_i^t)}{\partial c_i^t}}{-\frac{\partial U(c_i^1, \omega - l_i^1)}{\partial l_i^1}} = \frac{\beta (c_i^t)^{\frac{-1}{1+\beta}} (\omega - l_i^t)^{\frac{1}{1+\beta}}}{(c_i^1)^{\frac{\beta}{1+\beta}} (\omega - l_i^1)^{\frac{-\beta}{1+\beta}}}. \tag{4.4}$$

Dividing (4.3) by $-\partial U(c_i^1, \omega - l_i^1)/\partial l_i^1$ and using (4.4) we obtain

$$\begin{aligned}\frac{1}{\rho} &= \frac{\beta(\omega - l_i^1)}{(c_i^1)}(H^1)^{\gamma-1}H_i^0 + \\ &+ \sum_{t=2}^{\infty}\delta^{t-1}\frac{\beta(c_i^t)^{\frac{-1}{1+\beta}}(\omega - l_i^t)^{\frac{1}{1+\beta}}}{(c_i^1)^{\frac{\beta}{1+\beta}}(\omega - l_i^1)^{\frac{-\beta}{1+\beta}}}(H^t)^{\gamma-1}H_i^0\Pi_{T=2}^t l_i^T. \qquad (4.5)\end{aligned}$$

In an interior symmetric equilibrium, $l_i^t = H^t/H^{t-1}$, and $c_i^t = (H^t)^\gamma/n$. Thus, substituting in (4.5) and simplifying we get:

$$\begin{aligned}\frac{1}{\rho} &= \beta(\omega - \frac{H^1}{H^0})\frac{H^0}{H^1} + \\ &+ \sum_{t=2}^{\infty}\delta^{t-1}\frac{\beta(H^t)^{-\frac{\gamma}{1+\beta}}(\omega - \frac{H^t}{H^{t-1}})^{\frac{1}{1+\beta}}}{(H^1)^{\frac{\gamma\beta}{1+\beta}}(\omega - \frac{H^1}{H^0})^{\frac{-\beta}{1+\beta}}}(H^t)^{\gamma-1}H^0\Pi_{T=2}^t\frac{H^T}{H^{T-1}}. \qquad (4.6)\end{aligned}$$

Inspired by the case in which $\delta = 0$, we conjecture that when $\delta > 0$ and the utility function is Cobb-Douglas, human capital would grow at a constant rate, at least if δ is small. Our method of proof here is the familiar "propose and check", i.e., we propose a certain solution and we show that this solution fulfills the necessary and sufficient conditions of an equilibrium. For a constant growth rate to be a solution, $G = H^t/H^{t-1}$ for all t should satisfy (4.6) and the transversality condition. This implies that G should be a solution of

$$G(1+\beta\rho) - \delta G^{\frac{\gamma\beta}{1+\beta}+1} = \omega\beta\rho. \qquad (4.7)$$

The following proposition gives the conditions under which this equation has a solution. It happens that, when (4.7) has a solution, it has one or two of them, the constant growth rate solution to our problem is the smallest one when there are two. Formal details of the proofs are given in the Appendix.

Proposition 5. *If the utility function is Cobb-Douglas:*
i) There is constant growth rate iff

$$\left(\frac{1+\beta\rho}{\frac{\gamma\beta}{1+\beta}+1}\right)^{\frac{1+\beta}{\gamma\beta}+1} \geq \delta^{\frac{1+\beta}{\gamma\beta}}\frac{\omega\rho(1+\beta)}{\gamma}.$$

ii) The growth rate is given by the smallest solution to the equation

$$G(1+\beta\rho) - \delta G^{\frac{\gamma\beta}{1+\beta}+1} = \omega\beta\rho.$$

The necessary and sufficient condition holds for δ sufficiently small. In particular, it always holds when $\delta = 0$, and the growth rate coincides with the growth rate obtained with the linear transition function in Section 3, namely $\omega\beta\rho/(1+\beta\rho)$. Finally, we show in the following proposition how the constant growth rate varies with the parameters of the model.

Proposition 6. *The growth rate is increasing in ω, δ and the level of meritocracy ρ. If $G > 1$, the growth rate is also increasing in γ.*

Summing up, the introduction of the future in the plans of agents has several important consequences. First, the dynamics become extremely complex. Second, in the Cobb-Douglas case, for a sufficiently small δ, there is a solution to the intertemporal maximization problem in which the growth rate is constant. Such a rate does not exist for certain values of the parameters, but when it does, is larger than the corresponding rate in the zero discounting case. This is due to the fact that when future counts, there are more incentives to invest in capital which in turn stimulates growth. The role of meritocracy is identical to the case $\delta = 0$. Meritocracy encourages work which produces human capital, which makes the economy grow. For $\delta > 0$ returns to scale does play a role. The larger they are, the larger is the growth rate.

5. Efficiency

We turn our attention to the relationship between efficiency and equilibrium. Sen (1966) proved in a static model without capital that equilibrium and efficiency can be reconciled by setting $\rho = \gamma$. The intuition is that, on the one hand an egalitarian sharing rule gives incentives to work very little. On the other hand, meritocracy gives incentives to work more than it is socially efficient because an increase in work, not only means more production, it also means that the share of the labour of the person doing this extra effort increases. Thus, by balancing egalitarianism and meritocracy, efficiency is achieved. We now investigate if Sen's result holds in our model. In this section we will assume that $\gamma \leq 1$.[4]

We say that an intertemporal allocation $\{(\tilde{c}_i^t, \tilde{l}_i^t, \tilde{H}_i^t)\}_{t=1}^{\infty}$ is feasible if for all t, $\sum_{i=1ur}^{n} \tilde{c}_i^t = (\tilde{H}^t)^{\gamma}$, and for all i, $\tilde{l}_i^t \in [0, \omega]$ and $\tilde{H}_i^t = \tilde{H}_i^{t-1}\tilde{l}_i^t$.

[4] Since our method of proof is looking at first order conditions, we need $\gamma \leq 1$ to make sure that they characterize the efficient allocation.

Definition 2. *A feasible intertemporal allocation* $\{(\tilde{c}_i^t, \tilde{l}_i^t, \tilde{H}_i^t)\}_{t=1}^{\infty}$ *is Pareto efficient if there is no other feasible intertemporal allocation in which all agents are better off.*

It is well known that if utility functions and the production function are concave, a Pareto Efficient allocation maximizes the weighted sum of utilities (for some weights) under the feasible set. With this fact in hand, we state the following result whose proof is in the Appendix.

Proposition 7. *For all* $\delta \geq 0$, *the equilibrium is Pareto efficient iff* $\gamma = \rho$.

Thus, Sen's result is robust in the sense that it holds in a dynamic world with capital. Unfortunately, his original result is heavily dependent on the assumption that utility functions are identical. Working independently, Beviá and Corchón (2009) and Moulin (2010) proved that when individuals have different tastes, a completely different sharing rule has to be used. Furthermore, the production function must be polynomial.

6. Conclusion

In this paper, we presented a model of accumulation of human/social capital in a cooperative. The model has interesting features like the existence of a constant growth rate in the case of Cobb-Douglas preferences (dynastic or not) and the possibility of accelerated growth when the elasticity of substitution is larger than one and agents do not care about future generations. We show that by equating the weight of meritocracy in the sharing rule to the elasticity of production with respect to human capital, the intertemporal allocation of resources is efficient.

We hope that our paper, which, to the best of our knowledge is the first to deal with growth of cooperatives, stimulates more research in this area.

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7. APPENDIX

Proof of Proposition 1. Given that $l_i^t \in [0, \omega]$, and $c_i^t = (H^t)^\gamma \left(\frac{1-\rho}{n} + \rho \frac{H_i^{t-1} l_i^t}{H^t}\right)$, $U_i(c_i^t, \omega - l_i^t)$ is a continuous function of l_i^t, and therefore a maximum exists. In fact, this maximum is unique because the second order condition of utility maximization is

$$\frac{\partial^2 U_i}{\partial (c_i^t)^2} \frac{\partial c_i^t}{\partial l_i^t} + \frac{\partial U_i}{\partial c_i^t} \frac{\partial^2 c_i^t}{\partial (l_i^t)^2} + \frac{\partial^2 U_i}{\partial (l_i^t)^2} \tag{7.1}$$

where

$$c_i^t = (H^t)^\gamma \left(\frac{1-\rho}{n} + \rho \frac{H_i^{t-1} l_i^t}{H^t}\right). \tag{7.2}$$

Given that

$$\frac{\partial c_i^t}{\partial l_i^t} = \rho H_i^{t-1} (H^t)^{\gamma - 1} \tag{7.3}$$

and that we assume that when individuals maximize they take aggregate capital as given, the second term in (7.1) is zero. Given that the utility function is CES, the first term and the third term are negative. Thus, the maximum is unique. Finally, note that at period 1 all agents inherit the same human capital, $H_i^0 = H_j^0$, and since agents maximize taking aggregate capital as given, the maximization at time 1 is the same for all agents. Given that the maximization has a unique solution, equilibrium at $t = 1$ is symmetric, which implies that all agents in period 2 inherit the same human capital. Therefore, the equilibrium is also symmetric in period 2 and in any other subsequent period. ■

Proof of Proposition 2.

Let us see first that $\tilde{l}_i^t < \omega$. Given that $c_i^t = (H^t)^\gamma \left(\frac{1-\rho}{n} + \rho \frac{H_i^{t-1} l_i^t}{H^t}\right)$, and agents take H^t as given, it is sufficient to show that

$$\frac{\partial U_i}{\partial c_i^t} \frac{\partial c_i^t}{\partial l_i^t} + \frac{\partial U_i}{\partial l_i^t} < 0 \text{ when } l_i^t \to \omega, \tag{7.4}$$

that is,

$$\left(\beta(c_i^t)^{\frac{s-1}{s}}+(\omega-l_i^t)^{\frac{s-1}{s}}\right)^{\frac{1}{s-1}}\left(\beta(c_i^t)^{-\frac{1}{s}}\frac{\partial c_i^t}{\partial l_i^t}-(\omega-l_i^t)^{-\frac{1}{s}}\right)<0 \text{ when } l_i^t\rightarrow\omega. \tag{7.5}$$

When $l_i^t \rightarrow \omega$, both c_i^t and $\frac{\partial c_i^t}{\partial l_i^t}$ are strictly positive.

If $s>1$,to require (7.5) is equivalent to require that

$$\lim_{l_i^t\rightarrow\omega}\left(\beta(c_i^t)^{-\frac{1}{s}}\frac{\partial c_i^t}{\partial l_i^t}-(\omega-l_i^t)^{-\frac{1}{s}}\right)<0. \tag{7.6}$$

Since $\frac{1}{s}>0$, (7.6) holds for i independently of what other individuals do.

If $s<1$, we can write (7.5) as $A+B$ where

$$A = \left(\beta(c_i^t)^{\frac{s-1}{s}}+(\omega-l_i^t)^{\frac{s-1}{s}}\right)^{\frac{1}{s-1}}\left(\beta(c_i^t)^{-\frac{1}{s}}\frac{\partial c_i^t}{\partial l_i^t}\right); \text{ and} \tag{7.7}$$

$$B = -\left(\beta(c_i^t)^{\frac{s-1}{s}}+(\omega-l_i^t)^{\frac{s-1}{s}}\right)^{\frac{1}{s-1}}(\omega-l_i^t)^{-\frac{1}{s}}. \tag{7.8}$$

Since both, c_i^t and $\frac{\partial c_i^t}{\partial l_i^t}$, are strictly positive when $l_i^t\rightarrow\omega$, and $\lim_{l_i^t\rightarrow\omega}\left(\beta(c_i^t)^{\frac{s-1}{s}}+(\omega-l_i^t)^{\frac{s-1}{s}}\right)^{\frac{1}{s-1}}=0$, $\lim_{l_i^t\rightarrow\omega}A=0$. Furthermore, B can be written as

$$\begin{aligned} B &= -\left(\beta\left(\frac{c_i^t}{\omega-l_i^t}\right)^{\frac{s-1}{s}}+1\right)^{\frac{1}{s-1}} = \\ &= -\left(\frac{1}{\beta(\frac{\omega-l_i^t}{c_i^t})^{\frac{1-s}{s}}+1}\right)^{\frac{1}{1-s}} \end{aligned}$$

For $s<1$, $\lim_{l_i^t\rightarrow\omega}B=-1$ and therefore, (7.4) holds.

Finally, when $s\rightarrow 1$, the utility function is Cobb-Douglas, that is

$$U_i(c_i^t,\omega-l_i^t)=(c_i^t)^{\frac{\beta}{1+\beta}}(\omega_i^t-l_i^t)^{\frac{1}{1+\beta}},\ \beta>0. \tag{7.9}$$

Let us see that, also in this case, (7.4) holds because it can be written as

$$\frac{\beta}{1+\beta}(c_i^t)^{\frac{-1}{1+\beta}}(\omega_i^t-l_i^t)^{\frac{1}{1+\beta}}\frac{\partial c_i^t}{\partial l_i^t}-\frac{1}{1+\beta}(c_i^t)^{\frac{\beta}{1+\beta}}(\omega_i^t-l_i^t)^{\frac{-\beta}{1+\beta}}. \tag{7.10}$$

When $l_i^t\rightarrow\omega$, the first term converges to zero and the second term converges to $-\infty$. Consequently, (7.4) holds.

Thus, for any $s\in(0,\infty)$, in equilibrium $\tilde{l}_i^t<\omega$.

Secondly, let us show under which conditions $l_i^t = 0$ for all i is not an equilibrium. To show that is enough to prove that when $l_j^t = 0$ for all $j \neq i$,

$$\frac{\partial U_i}{\partial c_i^t}\frac{\partial c_i^t}{\partial l_i^t} + \frac{\partial U_i}{\partial l_i^t} > 0 \text{ when } l_i^t \to 0, \tag{7.11}$$

that is,

$$\left(\beta (c_i^t)^{\frac{s-1}{s}} + (\omega - l_i^t)^{\frac{s-1}{s}}\right)^{\frac{1}{s-1}} \left(\beta (c_i^t)^{-\frac{1}{s}} \frac{\partial c_i^t}{\partial l_i^t} - (\omega - l_i^t)^{-\frac{1}{s}}\right) > 0 \text{ when } l_i^t \ \to 0. \tag{7.12}$$

If $s > 1$,to require (7.12) is equivalent to require that

$$\beta (c_i^t)^{-\frac{1}{s}} \frac{\partial c_i^t}{\partial l_i^t} > \omega^{-\frac{1}{s}} \text{ when } l_i^t \ \to 0. \tag{7.13}$$

By plugging (7.2) and (7.3) stated in the proof of Proposition 1 into (7.13), we get,

$$\beta \left(\frac{1-\rho}{n}\right)^{-\frac{1}{s}} \rho H_i^{t-1} (H^t)^{\gamma(\frac{s-1}{s})-1} > \omega^{-\frac{1}{s}} \tag{7.14}$$

Thus if $\gamma(\frac{s-1}{s}) < 1$ and $\rho > 0$, when $l_i^t \to 0$, $H_i^t \to 0$ and (7.11) holds.

If $s < 1$,we can write (7.12) as $A + B$ as in (7.7) and (7.8). Note that $\lim_{l_i^t \to 0} B = 0$, and A can be written as

$$\begin{aligned} A &= \left(\beta + (\frac{\omega - l_i^t}{c_i^t})^{\frac{s-1}{s}}\right)^{\frac{1}{s-1}} \left(\beta \frac{\partial c_i^t}{\partial l_i^t}\right) = \\ &= \left(\frac{1}{\beta + (\frac{c_i^t}{\omega - l_i^t})^{\frac{1-s}{s}}}\right)^{\frac{1}{1-s}} \left(\beta \frac{\partial c_i^t}{\partial l_i^t}\right). \end{aligned}$$

The first term converges to $\left(\frac{1}{\beta}\right)^{\frac{1}{1-s}}$ when $l_i^t \to 0$ because and the second term converges to zero if $\gamma > 1$, converges to $+\infty$ if $\gamma < 1$, and to ρH_i^{t-1} if $\gamma = 1$. In any case, $\lim_{l_i^t \to 0} A \geq 0$ and (7.11) holds.

Finally, when $s \to 1$, the utility function is Cobb-Douglas as in (7.9). Let us see that, also in this case, (7.11) holds because it can be written as in (7.10) where the second term converges to zero when $l_i^t \to 0$, and the first term converges to zero if $\gamma(\frac{\beta}{1+\beta}) - 1 > 0$, and converges to $+\infty$ if $\gamma(\frac{\beta}{1+\beta}) - 1 < 0$. In any case, (7.11) holds.

Summarizing, if $\gamma(\frac{s-1}{s}) < 1$ and $\rho > 0$, the short run equilibrium is interior ■

Proof of Proposition 3. The transition function (3.4) can be written implicitly as

$$\phi(H^{t-1}, H^t) = \frac{H^t}{H^{t-1}} - \omega + n^{s-1} \left(\frac{1}{\beta\rho}\right)^s (H^{t-1})^{-s} (H^t)^{s(1-\gamma)+\gamma} = 0.$$

Clearly, at $(\hat{H}, \hat{H})$ we have that $\phi(\hat{H}, \hat{H}) = 0$. Furthermore, $\partial\phi(\hat{H}, \hat{H})/\partial H^t > 0$ because,

$$\frac{\partial\phi(H^{t-1}, H^t)}{\partial H^t} = \frac{1}{H^{t-1}} + (s(1-\gamma) + \gamma)n^{s-1}\left(\frac{1}{\beta\rho}\right)^s (H^{t-1})^{-s}(H^t)^{s(1-\gamma)+\gamma-1},$$

and $s(1-\gamma) + \gamma > 0$ given that $\gamma(\frac{s-1}{s}) < 1$. Thus, $\partial\phi(H^{t-1}, H^t)/\partial H^t > 0$ for any (H^{t-1}, H^t) in a rectangular domain $\Omega \subset \mathbb{R}^2_+$. Therefore,

$$\frac{\partial H^t}{\partial H^{t-1}} = \frac{\frac{H^t}{(H^{t-1})^2} + sn^{s-1}\left(\frac{1}{\beta\rho}\right)^s (H^{t-1})^{-s-1}(H^t)^{s(1-\gamma)+\gamma}}{\frac{1}{H^{t-1}} + (s(1-\gamma)+\gamma)n^{s-1}\left(\frac{1}{\beta\rho}\right)^s (H^{t-1})^{-s}(H^t)^{s(1-\gamma)+\gamma-1}},$$

which implies that H^t is an increasing function of H^{t-1} in any rectangular domain.

Finally, let us see that $\left|\frac{\partial H^t}{\partial H^{t-1}}\right|_{\hat{H}} < 1$ if and only if $s < 1$. Note that $\left|\frac{\partial H^t}{\partial H^{t-1}}\right|_{\hat{H}} < 1$ if and only if

$$\frac{1}{\hat{H}} + sn^{s-1}\left(\frac{1}{\beta\rho}\right)^s (\hat{H})^{\gamma(1-s)-1} < \frac{1}{\hat{H}} + (s(1-\gamma)+\gamma)n^{s-1}\left(\frac{1}{\beta\rho}\right)^s (\hat{H})^{\gamma(1-s)-1},$$

that is, if and only if, $s < 1$. Furthermore, since H^t is an increasing function of H^{t-1}, with a unique fixed point, the fixed point is also globally stable.

That $\hat{H}$ is increasing with ω, β, and ρ when $s < 1$ follows directly from (3.6). ■

Proof of Proposition 4. Instability when $s > 1$ follows from Proposition 3. That $\hat{H}$ is decreasing with ω, β, and ρ when $s > 1$ follows directly from (3.6). In order to show the last part of the proposition, note that the transition function can be written as

$$\frac{H^t}{H^{t-1}} + n^{s-1}\left(\frac{1}{\beta\rho}\right)^s \left(\frac{H^t}{H^{t-1}}\right)^s (H^t)^{\gamma(1-s)} = \omega.$$

Which in terms of the growth rate, $1 + g^t = H^t/H^{t-1}$, can be written as follows

$$1 + g^t + n^{s-1}\left(\frac{1}{\beta\rho}\right)^s (1+g^t)^s (H^t)^{\gamma(1-s)} = \omega. \tag{7.15}$$

The left hand side of (7.15) is increasing in g^t but decreasing in H^t because $s > 1$. Therefore, if $H^t > H^{t-1}$ then $g^t > g^{t-1}$. Thus, if $H^0 > \hat{H}$, when $t \to \infty$, $H^t \to \infty$, and

$$\lim_{t\to\infty}(1 + g^t) \simeq \omega,$$

which implies that g^t tends to $\omega - 1$. ■

In the proof of Proposition **5** we use Berge maximum theorem (Berge 1963). For completeness, we recall the version of the theorem that we are using here.

Berge maximum theorem (Berge 1963). Let E and Y be topological spaces. If $u : E \times Y \to \mathbb{R}$ is a continuous real-valued function and Y is a compact set, Then the correspondence $M : E \to 2^Y$ defined for each $e \in E$ as

$$M(e) = \{y \in Y : u(e, y) \geq u(e, x), \forall x \in Y\}$$

is upper hemi-continuous.

Proof of Proposition 5.

(i) If there is a constant growth rate, $1 + g = H^t/H^{t-1}$ for all t. Let $G = 1 + g$. Thus, $H^t = H^0(G)^t$. Note that $(G)^t$ means G at the power of t, abusing notation and whenever no confusion arrives, we denote G at the power of t as G^t. Using (4.6), G should be a solution of the following equation:

$$\begin{aligned} \frac{1}{\rho} &= \beta(\omega - G)\frac{1}{G} + \\ &+ \sum_{t=2}^{\infty} \delta^{t-1} \frac{\beta (G^t)^{-\frac{\gamma}{1+\beta}}(\omega - G)}{(H^0)^\gamma (G)^{\frac{\gamma\beta}{1+\beta}}} (G^t)^{\gamma-1}(H^0)^\gamma G^{t-1}, \end{aligned}$$

or equivalently,

$$\frac{1}{\rho} = \beta(\omega - G)\frac{1}{G}\left(1 + \frac{1}{\delta(G)^{\frac{\gamma\beta}{1+\beta}}} \sum_{t=2}^{\infty} \left(\delta G^{\frac{\gamma\beta}{1+\beta}}\right)^t\right). \tag{7.16}$$

We need to make sure that the sum in the above equation is finite. If the growth rate is constant, the transversality condition in the optimum implies that,

$$\begin{aligned} \lim_{T\to\infty} -\delta^{T-1} \left(\frac{\left(H_0 G^T\right)^\gamma}{n}\right)^{\frac{\beta}{1+\beta}} \frac{H_0 G^T}{n} (\omega - G)^{\frac{-\beta}{1+\beta}} &= 0, \\ \lim_{T\to\infty} -\frac{H_0^{\frac{\gamma\beta}{\beta+1}+1}}{\delta n^{\frac{\beta}{\beta+1}+1}} (\omega - G)^{\frac{-\beta}{1+\beta}} (\delta G^{\frac{\gamma\beta}{\beta+1}+1})^T &= 0. \end{aligned}$$

Thus, the transversality condition implies that $\delta G^{\frac{\gamma\beta}{\beta+1}+1} < 1$, which also implies that $\delta G^{\frac{\gamma\beta}{\beta+1}} < 1$ (because if $\delta G^{\frac{\gamma\beta}{\beta+1}} \geq 1$ will imply $G \geq 1$ and therefore $\delta G^{\frac{\gamma\beta}{\beta+1}+1} \geq 1$) . Then, (7.16) becomes

$$\frac{1}{\rho} = \beta(\omega - G)\frac{1}{G}(\frac{1}{1 - \delta G^{\frac{\gamma\beta}{1+\beta}}}).$$

Arranging terms,

$$G(1 + \beta\rho) - \delta G^{\frac{\gamma\beta}{1+\beta}+1} = \omega\beta\rho. \tag{7.17}$$

Any constant growth rate must be a solution of (7.17).

Note that when $\delta = 0$, the growth rate coincides with the growth rate obtained with the linear transition function in Section 3, namely $\omega\beta\rho/(1+\beta\rho)$.

For $\delta > 0$, let us see under which conditions a solution to (7.17) exists.

The derivative of the left hand side of (7.17) with respect to G is $1+\beta\rho-(\frac{\gamma\beta}{1+\beta}+1)\delta G^{\frac{\gamma\beta}{1+\beta}}$, which has the following properties: 1) It is zero for a unique value of G, namely

$$\hat{G} = \left(\frac{1+\beta\rho}{\left(\frac{\gamma\beta}{1+\beta}+1\right)\delta}\right)^{\frac{1+\beta}{\gamma\beta}}.$$

2) For values of G less (resp. more) than this value the left hand side of (7.17) is increasing (resp. decreasing) with respect to G. Therefore, the left hand side of (7.17) is strictly concave with respect to G. Note that (7.17) has a solution if and only if the left hand side of (7.17) evaluated at $\hat{G}$ is greater or equal than $\omega\beta\rho$, that is,

$$\left(\frac{1+\beta\rho}{\frac{\gamma\beta}{1+\beta}+1}\right)^{\frac{1+\beta}{\gamma\beta}+1} \geq \delta^{\frac{1+\beta}{\gamma\beta}}\frac{\omega\rho(1+\beta)}{\gamma}. \tag{7.18}$$

(ii) Note that the left hand side of (7.17) is strictly concave on G because the second derivative is negative. Thus, generically, there are two values of G which solve this equation. We will end this proof by showing that only the smallest root is a solution to our problem.

Note that the one period utility function can be written as

$$U_i((H^t)^\gamma(\frac{1-\rho}{n}+\rho\frac{H_i^t}{H^t}),\omega-\frac{H_i^t}{H_i^{t-1}}).$$

Let us write the previous equation as $V(H_i^{t-1}, H_i^t, H^t)$. Given that we are assuming that the growth rate of human capital is constant, the intertemporal utility function can be written as

$$W_i(\delta, G) = \sum_{t=1}^{\infty}\delta^{t-1}V(H_i^{t-1}, H_i^t, H^t) = F(\delta, H_i^0, H_i^0 G, H_i^0 G^2, \ldots\ldots, H^1, H^2, \ldots.)$$

in which agents choose (a constant) G. Applying Berge maximum theorem where $Y = [0,\omega]$, $E = [0,1]$, $G \in [0,\omega]$, and $\delta \in [0,1]$, we now that the correspondence

$$G(\delta) = \{G \in [0,\omega]/W_i(\delta, G) \geq W_i(\delta, G'), \forall G' \in [0,\omega]\}$$

has to be upper hemi-continuous.

We know that at $\delta = 0$, the solution is such that $G < \hat{G}$ (this is the case where the equation (7.17) only has one solution). If for some δs the maximum root of (7.17) were a solution, then it can be proved that the correspondence $G(\delta)$ will fail to be upper hemi-continuous, contradicting Berge theorem. ■

Proof of Proposition 6. Recall that, since $l_i^t < \omega$, $G < \omega$. Let us write (7.17) as $H(G, \omega, \rho, \delta, \gamma) = 0$. Then,

$$\begin{aligned}
\frac{\partial G}{\partial \omega} &= -\frac{\frac{\partial H}{\partial \omega}}{\frac{\partial H}{\partial G}} = \frac{\beta\rho}{\frac{\partial H}{\partial G}}; \\
\frac{\partial G}{\partial \delta} &= -\frac{\frac{\partial H}{\partial \delta}}{\frac{\partial H}{\partial G}} = \frac{G^{\frac{\gamma\beta}{1+\beta}+1}}{\frac{\partial H}{\partial G}}; \\
\frac{\partial G}{\partial \rho} &= -\frac{\frac{\partial H}{\partial \rho}}{\frac{\partial H}{\partial G}} = \frac{\beta(\omega - G)}{\frac{\partial H}{\partial G}}; \\
\frac{\partial G}{\partial \gamma} &= -\frac{\frac{\partial H}{\partial \gamma}}{\frac{\partial H}{\partial G}} = \frac{\delta(\frac{\gamma\beta}{1+\beta}+1)G^{\frac{\gamma\beta}{1+\beta}}\frac{\beta}{1+\beta}\ln G}{\frac{\partial H}{\partial G}}.
\end{aligned}$$

In the proof of Proposition 5 we have shown that for all $G < \hat{G}$, $\frac{\partial H}{\partial G} > 0$. Thus, the proposition follows. ■

Proof of Proposition 7. A useful characterization of efficiency in this case can be stated as follows: A feasible intertemporal allocation $\{(\tilde{c}_i^t, \tilde{l}_i^t, \tilde{H}_i^t)\}_{t=1}^{\infty}$ is Pareto efficient if there exist a list $(\alpha_1, \alpha_2, ...\alpha_n)$ with $\alpha_i \geq 0$ for all i, such that,

$$\begin{aligned}
&\{(\tilde{c}_i^t, \tilde{l}_i^t, \tilde{H}_i^t)\}_{t=1}^{\infty} \in \arg\max \textstyle\sum_{i=1}^n \alpha_i \sum_{t=1}^{\infty} \delta^{t-1} U(c_i^t, \omega - l_i^t). \\
&s.t. \textstyle\sum_{i=1}^n c_i^t = (H^t)^{\gamma},\ t \in \{1, 2, ...\} \\
&H_i^t = H_i^0 \Pi_{T=1}^t l_i^T,\ t \in \{1, 2, ...\},\ i \in \{1, .., n\} \\
&H_i^0 > 0\,,\ i \in \{1, .., n\}
\end{aligned}$$

Now Lagrange multipliers have a time super index and the lagrange function is given by

$$\sum_{i=1}^n \alpha_i \sum_{t=1}^{\infty} \delta^{t-1} U(c_i^t, \omega - l_i^t) + \sum_{t=1}^{\infty} \lambda^t (\sum_{i=1}^n c_i^t - (H^t)^{\gamma}) + \sum_{i=1}^n \sum_{t=1}^{\infty} \mu_i^t (H_i^t - H_i^0 \Pi_{T=1}^t l_i^T).$$

FOC are

$$\alpha_i \delta^{t-1} \frac{\partial U(c_i^t, \omega - l_i^t)}{\partial c_i^t} + \lambda^t = 0; \quad (7.19)$$

$$\alpha_i \frac{\partial U(c_i^1, \omega - l_i^1)}{\partial l_i^1} - \mu_i^1 H_i^0 - \sum_{t=1}^{\infty} \mu_i^t H_i^0 \Pi_{T=2}^t l_i^T = 0; \quad (7.20)$$

$$-\lambda^t \gamma (H^t)^{\gamma-1} + \mu_i^t = 0. \quad (7.21)$$

From (7.21) $\mu_i^t = \lambda^t \gamma (H^t)^{\gamma-1}$, and from (7.19), $\mu_i^t = -\alpha_i \delta^{t-1} \frac{\partial U(c_i^t, \omega - l_i^t)}{\partial c_i^t} \gamma (H^t)^{\gamma-1}$. Thus, (7.21) can be written as:

$$\alpha_i \frac{\partial U(c_i^1, \omega - l_i^1)}{\partial l_i^1} + \alpha_i \frac{\partial U(c_i^1, \omega - l_i^1)}{\partial c_i^1} \gamma (H^1)^{\gamma-1} H_i^0 + \sum_{t=2}^{\infty} \alpha_i \delta^{t-1} \frac{\partial U(c_i^t, \omega - l_i^t)}{\partial c_i^t} \gamma (H^t)^{\gamma-1} H_i^0 \Pi_{T=2}^t l_i^T = 0,$$

or equivalently,

$$-\frac{1}{\gamma} \frac{\partial U(c_i^1, \omega - l_i^1)}{\partial l_i^1} = \frac{\partial U(c_i^1, \omega - l_i^1)}{\partial c_i^1} (H^1)^{\gamma-1} H_i^0 + \sum_{t=2}^{\infty} \delta^{t-1} \frac{\partial U(c_i^t, \omega - l_i^t)}{\partial c_i^t} (H^t)^{\gamma-1} H_i^0 \Pi_{T=2}^t l_i^T \quad (7.22)$$

Comparing (7.22) with (4.3), namely

$$-\frac{1}{\rho} \frac{\partial U(c_i^1, \omega - l_i^1)}{\partial l_i^1} = \frac{\partial U(c_i^1, \omega - l_i^1)}{\partial c_i^1} (H^1)^{\gamma-1} H_i^0 + \sum_{t=2}^{\infty} \delta^{t-1} \frac{\partial U(c_i^t, \omega - l_i^t)}{\partial c_i^t} (H^t)^{\gamma-1} H_i^0 \Pi_{T=2}^t l_i^T,$$

we see that they are identical when $\gamma = \rho$ and it is independent on δ. ■