Abstract

We extend the concept of competitive search to an environment where, as is often common, sellers cannot observe the willingness to pay of their clients. This theoretical contribution is applied to model retail trade. We find that in equilibrium the ratio of buyers over sellers exceeds that of the first best allocation. Furthermore, upon meeting a seller, buyers purchase lower quantities than in the first best. Nevertheless, a planner who faces the same informational constraints as the market and whose only policy instrument is to regulate market prices cannot improve upon the equilibrium allocation when the production technology is linear. In contrast, if the planner can use lump-sum taxes to subsidize sales, the first best allocation can be attained by introducing a linear sales subsidy. The model we construct is highly tractable, and when inserted in a Neoclassical growth framework it is sufficiently rich to be taken to the data for estimation. The search parameters can be identified with data on commercial margins and on the households’ allocation of time.

Keywords: Competitive search, private information, commerce, retail trade.
1 Introduction

Commercial margins constitute a very large fraction of the cost of purchasing a typical good. On average, for every dollar spent before tax in a retail store in the United States, only 57 cents reach the producers of the merchandise. The remaining 43 cents cover the margins of the merchant wholesaler (15 cents) and the retailer (28 cents).\textsuperscript{1} Models with Walrasian markets abstract from commercial activities. This abstraction would be of little importance if these activities were limited to transporting commodities from factories to consumers, for then we could think of commerce as a stage of production with a conventional technology. However, a large fraction of commercial costs are incurred in the process of contacting and dealing with buyers in bilateral meetings. Moreover, the size of these costs is independent of the quantities that buyers end up acquiring and cannot be recovered if the buyer chooses to buy nothing. This fact can only be properly modeled with non-Walrasian markets. To develop an appropriate framework to analyze this, we advance a search model of commerce. This model incorporates the competitive pressures observed in retail trade by adopting the concept of competitive search equilibrium in Moen (1997) and Shimer (1996). In addition, our model incorporates the fact that sellers cannot directly observe how much their clients are willing to pay for a product. Consequently, the model we advance extends the concept of competitive search equilibrium to an environment with private information.

Our model has the following five main features: First, commodities are traded in decentralized markets where buyers and sellers meet randomly in bilateral pairs according to a matching technology as in the two-sided search model of Mortensen (1982) and Pissarides (1990). Second, as in Shi (1997), individuals belong to large households and are endogenously assigned to one of the following tasks: production, purchase, and sale of goods.\textsuperscript{2} Third, buyers are heterogeneous in the goods they like most, and this information is private. In contrast, the costs of the sellers are homogeneous and are assumed common knowledge. Also, when they meet a seller, buyers observe the good that the seller carries. Therefore, the bilateral interaction between buyers and sellers takes place in an environment with one-sided incomplete information. Fourth, the revelation principle

\textsuperscript{1}This break-up assumes that the good goes through one merchant wholesaler and one retailer. In our calculations, we use the gross commercial margins over sales ratios calculated by the Bureau of the Census, Service Sector Statistics Division for 2000 (0.278 for retail and 0.2054 for wholesale). This data was downloaded in July 1, 2002 from http://www.census.gov/svsd/www/artstbl.html and http://www.census.gov/svsd/www/atspur.txt.

\textsuperscript{2}Thanks to the large household assumption each household faces no uncertainty, even though its buyers and sellers are uncertain about the number and the type of trading partners they will find while searching in the market place.
allows us to summarize the possible terms of exchange that result from these bilateral meetings by a price schedule that maps quantities purchased onto payments. For a given price schedule, buyers choose how much of the good to purchase. They make no payment if they decide not to buy anything. Fifth, consistent with our perception of the retail sector, sellers can attract clients to visit their outlets by announcing and committing to low price schedules prior to the search process.

To combine price competition and search, we adopt the concept of competitive search equilibrium advanced in Moen (1997) and Shimer (1996). This concept assumes that the market for a given good is subdivided into potential submarkets where buyers and sellers search for trading opportunities and meet randomly in bilateral pairs. The matching technology is identical in all submarkets. Each potential submarket is characterized by a price schedule and a degree of congestion (ratio of buyers over sellers). Before the search process begins, traders optimally choose, based on the prices and on their expectations of degrees of congestion in each submarket, which submarket to enter. This formulation allows buyers and sellers to direct their search efforts while maintaining some degree of randomness in the search process. In equilibrium, the congestion in each submarket is determined by the joint decisions of the traders and must be consistent with the traders’ expectations (i.e. expectations are rational), only those submarkets which attract positive measures of buyers and sellers are active, and it is not possible to create a new submarket that would attract positive masses of both buyers and sellers. Also, competition among potential submarkets implies that the equilibrium price schedules must respect individual rationality and must be pairwise efficient given the informational constraints. As Moen (1997) argues, this equilibrium construct is a reduced form of the equilibrium of an extended game in which sellers publicly announce their prices prior to the search process, and buyers choose the group of sellers in which they are going to restrict their search.

We derive efficiency results for a competitive search equilibrium which are in the same spirit of those of Moen (1997) and Shimer (1996) but more restricted due to the private information of buyers’ preferences. Moen (1997) and Shimer (1996) show that, with full information, competitive search equilibria are efficient. Our main welfare result is that a planner who faces the same informational constraints as the market and whose only policy instrument is to regulate market prices cannot improve upon the equilibrium allocation when the production technology is linear.

3 There are many related contributions that also assume price posting to attract clients. See, for example, Peters (1991) and Montgomery (1991) for earlier work, and Mortensen and Wright (2000), Burdett, Shi and Wright (2001), Shi (2001b), and Shimer (2001) for recent work.
Thus, if the production technology is linear, competitive search markets are second best efficient. However, if the planner can use lump-sum taxes to subsidize sales, the first best allocation, that is the optimal allocation with full information, can be attained by introducing a linear sales subsidy. The first best also coincides with a competitive equilibrium if sellers can hide the characteristics of their products and charge the buyers a flat fee to reveal such information.

The second best efficiency of competitive search markets in an environment with a linear production technology does not preclude important welfare costs due to the private information of buyers’ preferences. Intuitively, to attain an efficient allocation of resources, prices must play two conflicting roles in our model. First, for consumption goods to be optimally allocated among the households, prices ought to signal to buyers the opportunity cost of the goods they intend to purchase. To perform this role efficiently, the per unit price of each good should be equal to the marginal cost of production, and commercial margins should be zero. Second, for labor to be optimally allocated within a household, prices must provide incentives for the household to choose the relative amounts of time devoted to buying and selling commodities optimally. To perform this role efficiently, prices must include sizable commercial margins to remunerate retail costs. The presence of private information precludes price schedules that perfectly combine these two roles. For example, it precludes situations where buyers pay a flat fee to interact with a seller and then purchase commodities at marginal cost. Due to the tension between the two roles of prices, in equilibrium commercial margins are positive, but they are not large enough to generate the first best ratio of buyers over sellers. As a result, markets are too congested of buyers. Moreover, because they must pay a commercial margin, buyers purchase too little when they meet a seller.

Despite the combination of search, competition, and private information, that we believe is essential for a good model of commerce, our model is highly tractable. For instance, the model can

---

4 Without private information, the restriction to linear technologies is not necessary. Numerical simulations show that for reasonable non-linear technologies the competitive search equilibrium attains almost the same welfare as the planner’s allocation.

5 Households should also choose the time devoted to production optimally relative to other activities. With a linear technology, however, the return to each unit of labor in the production sector is fixed. Thus, the allocation of labor in the household will be optimal as long as buyers and sellers are given the right incentives in the commercial sector.

6 Consistent with the empirical results in Betancourt and Gautschi (1993), our model implies that commercial margins are not due to monopolistic power, but they are necessary to cover the cost of retail services. In our model, for simplicity, the labor cost of sellers is the only retail cost. Retail services are captured in the contribution of sellers to the matching technology, which reduces the search costs of buyers. The pricing of these services is explicitly modeled in the competition among potential submarkets.
be easily incorporated in a Neoclassical growth framework with two sectors. One of these sectors produces goods combining capital and labor as is typical in the Neoclassical framework, while the other exchanges goods in retail markets. The combined model can be estimated using standard empirical data. In particular, the parameters of the retail sector can be identified and estimated using commercial margins and the average time allocation of households.\textsuperscript{7} The tractability of the model makes it suitable for further extensions. For example, one can use this framework to introduce money along the lines of the search theoretical approach of Kiyotaki and Wright (1989 and 1993). Faig (2001) constitutes a first attempt in this direction in a simpler version of the present model. In this simpler version, sellers are restricted to make offers to the buyers they are paired with which consist of a single quantity-payment pair. Our main improvement with respect to Faig (2001) is the relaxation of this constraint by allowing sellers to make offers that consist of a price schedule that maps the quantity chosen by a buyer into the payment to be made to the seller. This price schedule serves as a mechanism to reveal the buyers’ private information about their preferences. Also, in Faig (2001) search is undirected and sellers make take-or-leave it offers to buyers.

Two recent papers, Soller-Curtis and Wright (2000) and Camera and Delacroix (2001) also study search-theoretic models where the buyers’ willingness to pay for a good is private information. In both of these papers, goods are indivisible and search is undirected. Soller-Curtis and Wright assume that sellers post prices and they focus on the coexistence of two prices for the same good in equilibrium. Camera and Delacroix focus on the endogeneous determination of the trade mechanism - sellers can choose if they want to commit to a posted price or to bargain once they meet a buyer. When buyers have identical preferences, pre-committing to a price is preferable from the sellers point of view because it allows them to extract the whole trading surplus. However, when the preferences of the buyers are heterogeneous, the bargaining process allows the seller to infer information about the buyers’ preferences and hence to price discriminate. In our paper, goods are divisible, so sellers can commit to a non-linear price schedule which allows a restricted form of price discrimination even without bargaining. Also, we incorporate the equilibrium concept of competitive search which endogeneizes the market power of sellers in a reasonable fashion.

The paper is organized as follows. Section 2 sets up the model and characterizes the optimal

\textsuperscript{7} Shi (1999 and 2001a) also incorporates a search model in a Neoclassical growth framework. The model in Shi does not differentiate between producers and sellers as we do. Also, it does not have neither private information nor competitive search.
behavior of the households for a given price schedule. Section 3 endogeneizes this price schedule by analyzing the interaction between buyers and sellers in the market place. Section 4 combines the analyses of the previous two sections into a general equilibrium model where both the behavior of households and the price schedule are endogenous. Section 5 studies the welfare properties of a competitive search equilibrium. Section 6 incorporates our model in a Neoclassical growth framework and discusses how to identify the parameters using standard data. Section 7 briefly discusses some of the issues one must confront when extending the present model to a monetary framework and concludes. The proofs are gathered in the Appendix.

2 The Households

The economy consists of a continuum of infinitely lived households with measure one who produce and consume differentiated goods in discrete time. Households do not consume the goods they produce so they need to trade. Each household is composed of a large countable number of individuals. We measure subsets of individuals by the fraction of their size over the size of the household, so the measure of individuals in a household is normalized to one. The production and exchange of goods is performed by the individuals of the household, who are endogenously assigned to one of three of occupations: produce, buy, and sell goods. All members equally share the consumption of goods which is their only source of utility, so there is no conflict of interests among members of the same household.

There are $H$ types of divisible goods, indexed by $h$, each of which comes in a continuum of varieties distributed around a unit circle and indexed by $\varepsilon \in (0, 1)$. Each household produces one variety of a given good and is labeled by the good and the variety that it produces. Household $h\varepsilon$ produces good $h$ variety $\varepsilon$ and may consume all varieties of good $h + 1$ modulus $H$. The household, however, does not like all varieties the same. Household $h\varepsilon$ likes variety $\varepsilon$ of good $h + 1$ best, and the utility of consuming the other varieties declines the further apart they are from $\varepsilon$. Households are uniformly distributed over the set of goods and varieties they produce.

Each of the $H$ goods has a different market place where individuals trade bilaterally. The market for good $h$ attracts the buyers and sellers of this good in all its varieties. When a buyer and a seller meet in one of these markets, the buyer observes the variety offered by the seller. The seller, however, does not know the most preferred variety of the buyer. Therefore, buyers and sellers interact in an environment with one-sided incomplete information. This interaction is modeled in
the next section. As we will show there, the terms of exchange that will prevail in the market in period $t$ can be summarized by a price schedule $Z_t(q_t)$ that maps the quantities purchased $q_t$ by the buyer onto the payments received by the seller $z_t$. In this section, we assume that the price schedule is exogenous and known by all market participants. We also assume that $Z_t(\cdot)$ is continuously differentiable, and concave with $Z_t(0) = 0$. The next section endogeneizes $Z_t(\cdot)$ and validates these assumptions.

All payments are denominated in an abstract numeraire. In the version of the model analysed in this paper, buyers do not need to carry money with them. Instead, all traders have access to a central clearing-house that records the credits (payments received by sellers) and debits (payments made by buyers) of all households. In principle, we could allow the clearing-house to pay or charge interest on the balance of credits and debits imposing only an intertemporal budget constraint on the households. However, since all households behave identically, all balances must be zero in equilibrium. Therefore, we can simplify the exposition by assuming that the balance of credits and debits for each household must be zero in each period.

2.1 Households’ Decisions

In this subsection, we describe the problem of a household for a given price schedule. Since the model is symmetric, without loss of generality, we describe the actions of household $h1$ and adopt the following notation. Lower-case letters denote the decision variables of household $h1$. Upper-case letters denote the decisions of the other households and hence aggregate quantities, which are taken as given by household $h1$. In a symmetric equilibrium, lower-case letters are equal to the corresponding upper-case letters.

Each period, the individuals of the household are assigned to one of three different tasks: production (producers), purchase (buyers), and sale of commodities (sellers). Each individual is endowed with one unit of labor and can perform at most one task every period. A typical day in the life of household $h1$ proceeds as follows. In the morning of day $t$, the members of the household are divided into producers, buyers, and sellers. The measure of individuals assigned to each activity is denoted by $n_t$, $b_t$, and $s_t$, respectively. Producers use their labor $n_t$ to generate output of good $h$ variety $1$, which immediately becomes available for sale. Goods are perishable so all output must be sold in the same period in which it is produced. Sellers go to the market for good $h$ to sell the output that the producers are generating during the period. When they meet a buyer who purchases a quantity $Q_t$ from them, they collect a payment $Z_t = Z_t(Q_t)$, which is immediately
transferred to the clearing-house. Buyers go to the market of good $h+1$. Upon meeting a seller, they observe the variety for sale, choose the quantity $q_t$ they want to acquire, and pay $z_t = Z_t(q_t)$. This payment is immediately debited from the household’s account in the clearing-house. In the evening, all the individuals of the household get together and equally share the consumption of the goods purchased.

In all markets, buyers and sellers are randomly matched according to an exogenous matching function $M(B_t, S_t)$, where $B_t$ and $S_t$ denote the measures of buyers and sellers in the market.\(^8\) This matching function is continuously differentiable, increasing in both arguments, concave, and homogeneous of degree one. Let

$$\theta_t = \frac{B_t}{S_t},$$

be the degree of congestion in the market, or the market tightness. The measure of buyers located by a seller in period $t$ depends positively on $\theta_t$ and is given by

$$m(\theta_t) = \frac{M(B_t, S_t)}{S_t} = M(\theta_t, 1).$$

Similarly, the measure of sellers located by a buyer depends negatively on $\theta_t$ and is given by

$$\frac{m(\theta_t)}{\theta_t} = \frac{M(B_t, S_t)}{B_t} = M(1, \theta_t^{-1}).$$

It follows that the measure of sellers of good $h+1$ contacted by the buyers of household $h1$ is

$$B(b_t, \theta_t) = b_t m(\theta_t) \theta_t^{-1}. $$

Since the model is symmetric, each one of these sellers carries with equal probability any one of the varieties of good $h+1$. Analogously, the measure of buyers of good $h$ contacted by the sellers of household $h1$ is

$$S(s_t, \theta_t) = s_t m(\theta_t).$$

Each one of these buyers belongs with equal probability to any one of the households demanding good $h$. We can view $S(s_t, \theta_t)$ as the production function in the retail sector, where the output is the number of potential buyers contacted and the input is the labor $s_t$ of the sellers. The productivity in this sector depends positively on the market congestion $\theta_t$.\(^9\)

\(^8\)As we shall see in Section 3, the matching process is slightly more involved. Potentially, each market is subdivided into submarkets where traders can choose to operate and where buyers and sellers meet randomly according to the matching function. In equilibrium, however, all traders operate in a single submarket. This result allows us to simplify the description of the household’s problem.

\(^9\)In more elaborate versions of the model, we could include other inputs such as capital and different types of labor in this production function.
The objective of the household is to maximize the utility from consumption. The household’s utility function is

$$\sum_{t=0}^{\infty} \beta^t U(c_t)$$

(6)

where \( \beta \in (0, 1) \) is the household’s discount factor. The function \( U : \mathbb{R}_+ \to \mathbb{R} \) is continuously differentiable, strictly increasing, and concave. The variable \( c_t \) is a hedonic measure of consumption. This measure depends on the quantity \( q_{jt} \) and the variety \( \varepsilon_{jt} \) acquired in each match \( j \) between a buyer of the household and a seller of good \( h + 1 \), where \( j = 1, 2, ... \)

$$c_t = \left( \lim_{J^0 \to \infty} \frac{1}{J} \sum_{j=1}^{J} \varepsilon_{jt} q_{jt}^{1-\sigma} \right)^{\frac{1}{1-\sigma}}.$$

(7)

Here, \( J \) and \( J^0 \) denote the number of matches and the number of household members respectively, and \( \sigma \in (0, 1) \) measures the preference for diversity. This consumption aggregator combines a preference for diversity with different valuations of the varieties acquired. The preference for diversity increases with \( \sigma \) and vanishes when \( \sigma \to 0 \). The varieties acquired have different valuations for the household. For a given quantity purchased \( q_{jt} \), the household gets maximum utility when acquiring its most preferred variety \( \varepsilon_{jt} = 1 \), and this utility declines linearly to 0 as \( \varepsilon_{jt} \) decreases to 0. Because there is a countable number of potential purchases and a continuum of varieties for sale, with probability one an additional purchase brings a new variety to the set of goods consumed by the household.

Since the number of household members is large, the measure of matches between the buyers of the household and a potential seller is \( \lim_{J^0 \to \infty} J/J^0 = B(b_t, \theta_t) \). Hence,

$$c_t = \left( B(b_t, \theta_t) \lim_{J^0 \to \infty} \frac{1}{J} \sum_{j=1}^{J} \varepsilon_{jt} q_{jt}^{1-\sigma} \right)^{\frac{1}{1-\sigma}}.$$

(8)

Moreover, conditional on a meeting taking place, the variety encountered by the buyer is the realization of an independent random variable, to be denoted \( \varepsilon \), uniformly distributed on the interval (0, 1]. Let \( q_t(\varepsilon) \) denote the quantity purchased by a buyer who is matched with a seller of variety \( \varepsilon \) in period \( t \). The Law of Large Numbers implies the following formula, which we employ from now on:

$$c_t = \left\{ B(b_t, \theta_t) \int_{0}^{1} \varepsilon \left[ q_t(\varepsilon) \right]^{1-\sigma} d\varepsilon \right\}^{\frac{1}{1-\sigma}}.$$

(9)

Production \( y_t \) in period \( t \) depends on the amount of labor \( n_t \) employed:

$$y_t = f(n_t).$$

(10)
The production function \( f : \mathbb{R}_+ \rightarrow \mathbb{R}_+ \) is assumed continuously differentiable, strictly increasing, and concave.

The sales of household \( h1 \) in period \( t \) are equal to the measure of buyers contacted by the sellers of the household times the average quantity traded in a match. Let \( Q_t(\varepsilon) \) denote the quantity sold by a seller who is matched with a buyer of household \((h-1)\varepsilon\) in period \( t \). The average quantity sold by each seller is \( \overline{Q}_t = \int_0^1 Q_t(\varepsilon) \, d\varepsilon \). Since goods are perishable and there is no aggregate uncertainty in a large household, the amount of output both produced and sold in period \( t \) is identical:

\[
y_t = S(s_t, \theta_t) \overline{Q}_t. \tag{11}
\]

The household must satisfy the budget constraint in each period, that is, total expenditure cannot exceed total sales revenue:

\[
S(s_t, \theta_t) \overline{Z}_t - B(b_t, \theta_t) \overline{z}_t \geq 0. \tag{12}
\]

Here, \( \overline{Z}_t = \int_0^1 Z_t [Q_t(\varepsilon)] \, d\varepsilon \) is the average revenue of a seller, and \( \overline{z}_t = \int_0^1 Z_t [q_t(\varepsilon)] \, d\varepsilon \) is the average expenditure of a buyer.

The measure of individuals assigned to the three different activities in the household must add up to one:

\[
b_t + s_t + n_t = 1. \tag{13}
\]

The problem of household \( h1 \) is to choose \( \{q_t(\varepsilon)\}_{\varepsilon \in [0,1]} \), \( b_t \), \( s_t \) and \( n_t \) to maximize (6) subject to (9) to (13), and non-negativity constraints for all variables. Condition (11) can be substituted into (10) to form a single resource constraint. Also, (9) can be substituted into the objective function (6). Using Lagrange multipliers \( \beta^t \mu_t \), \( \beta^t \lambda_t \), and \( \beta^t \vartheta_t \) for the resource constraint, the budget constraint, and the labor allocation constraint respectively, the first order conditions for an interior maximum are:

\[
U'(c_t) c_t^\sigma [g_t(\varepsilon)]^{-\sigma} = \lambda_t \overline{Z}_t[I][g_t(\varepsilon)] \text{ for } \varepsilon \in [0,1], \tag{14}
\]

\[
\frac{m(\theta_t)}{\theta_t} \left\{ \frac{U'(c_t) c_t^\sigma}{1 - \sigma} \int_0^1 \varepsilon [g_t(\varepsilon)]^{1-\sigma} \, d\varepsilon - \lambda_t \overline{Z}_t \right\} = \vartheta_t, \tag{15}
\]

\[
m(\theta_t) \left( \lambda_t \overline{Z}_t - \mu_t \overline{Q}_t \right) = \vartheta_t, \tag{16}
\]

\[
\mu_t f'(n_t) = \vartheta_t. \tag{17}
\]

Condition (14) states that a buyer who meets a seller of variety \( \varepsilon \) must equate the marginal utility of purchasing an extra unit of this variety with the marginal value of the payment required in
return. Also, the value of the marginal product of labor in all three occupations must be the same. This common value is $\vartheta_t$. Condition (15) equates $\vartheta_t$ to the expected consumer surplus obtained by a buyer. Condition (16) equates $\vartheta_t$ to the expected surplus generated by a seller. Finally, condition (17) equates $\vartheta_t$ to the value of the marginal product of labor of a producer.

### 3 The Commercial Equilibrium

In this section, we model the interaction between buyers and sellers in the marketplace when sellers do not know how much buyers are willing to pay for their merchandise. Given the symmetry of our framework, this interaction is identical across markets. A bilateral trade is described by a pair $(q, z) \in \mathbb{R}^2_+$ specifying the quantity $q$ supplied by the seller and the payment $z$ given in return by the buyer. In the previous section, the two elements of this vector were linked by an exogenous price schedule $z = Z(q)$. The purpose of this section is to endogenize $Z$. Since all variables in this section refer to the same period, time subscripts are omitted.

We extend the competitive search equilibrium concept in Moen (1997) and Shimer (1996) to our private information environment. To do this, we assume that each market is subdivided into potential submarkets where traders can choose to operate. In all submarkets, buyer and seller pairs are randomly matched according to the matching function $M$. Each submarket is characterized by a price schedule $Z$ and a degree of congestion $\theta$ that buyers and sellers expect to find. Since all households behave identically, in a given submarket the proportion of buyers (sellers) belonging to each of the households demanding (producing) the good is the same, and the distribution of traded varieties is uniform on $(0, 1]$. Prior to the search process, buyers and sellers optimally choose, based on the prices and on their expectations of degrees of congestion in each submarket, which submarket to enter. In equilibrium, the actual degree of congestion in each submarket is determined by the joint decisions of the traders and is consistent with the traders’ expectations (i.e., expectations are rational), only those submarkets which attract positive measures of buyers and sellers are active, and it is not possible to create a new submarket that would attract positive masses of buyers and sellers. Competition among submarkets to attract traders directly implies that, in equilibrium, price schedules must respect individual rationality and must be pairwise efficient given the informational constraints.

We begin our construction of $Z$ by characterizing the pairwise incentive efficient trading assignments - that is, the bilateral trading assignments that are optimal from the set of incentive
compatible and individually rational assignments. As one would expect, there is continuum of elements in this set. One particular element is the monopolistic allocation where sellers maximize their profit subject to the incentive compatibility and individual rationality constraints of the buyers as in Maskin and Riley (1984) and Mussa and Rosen (1978). We show that all the trading assignments that are pairwise incentive efficient can be implemented by varying a parameter $\gamma$ of a simple price schedule. A potential submarket is thus fully characterized by a value of $\gamma$ and a degree of congestion $\theta$. Given our assumed (ex ante) symmetry about buyers and sellers, in equilibrium, all traders ought to get the same utility from participating in any active submarket $(\gamma, \theta)$. Furthermore, there must be no gains from forming a new submarket -there is no potential submarket $(\gamma', \theta')$ that, if opened, would attract positive masses of buyers and sellers. As we shall see, these two conditions imply that a single submarket $(\gamma, \theta)$ will be active in a competitive search equilibrium.

It should be emphasized that the uniqueness of $(\gamma, \theta)$ in equilibrium depends, among other things, on our simplifying assumption that the opportunity cost of time is identical across buyers. When this cost is heterogeneous, many submarkets coexist in equilibrium. Some of these submarkets have low congestion and high prices to attract buyers with a high opportunity cost of their time, while other submarkets have high congestion and low prices to attract the buyers with a low opportunity cost of their time.

### 3.1 Pairwise Incentive Efficient Trading Assignments

Without loss of generality, we focus on matches that involve sellers of variety 1, so our notation is consistent with the one in the previous section. Because trading meetings are random, sellers are uncertain about their clients’ preferences. Specifically, for sellers, the most preferred variety $\varepsilon$ of any buyer they meet is a random variable uniformly distributed on the interval $[0,1]$.\footnote{Strictly speaking the interval is $(0,1]$. However, we close this interval in this subsection to simplify the exposition.} We shall refer to the realization of $\varepsilon$ as the buyer’s type. Buyers’ types are private information. The matching technology and the underlying preference structure are common knowledge.

All traders maximize the objectives of the households they belong to. Specifically, the incremental utility of a buyer’s purchase is the consumer surplus:

$$U^b(q,z;\varepsilon) = \frac{\varepsilon \psi q^{1-\sigma}}{1-\sigma} - \lambda z,$$

where

$$\psi \equiv U'(c)e^\sigma.$$
The incremental utility of a seller’s sale is the gross commercial margin:

\[ U^s(q, z) = \lambda z - \mu q. \]  

(20)

By symmetry, the variables \( \psi, \lambda, \) and \( \mu \) are identical across households. These variables are taken as given by both buyers and sellers because each trader is small in his/her household. We shall refer to \( U^b \) and \( U^s \) as the utility functions of buyers and sellers respectively.

We characterize trading assignments by a schedule of type-contingent trades \( \{q_\varepsilon, z_\varepsilon\}_{\varepsilon \in [0,1]} \).\(^{11} \) A trading assignment is said to be pairwise incentive efficient if it maximizes a weighted sum of the (ex ante) expected utilities of buyers and sellers:\(^{12} \)

\[ \{q_\varepsilon, z_\varepsilon\}_{\varepsilon \in [0,1]} = \arg \max \left[ (1 - \omega) \int_0^1 U^b(q_\varepsilon, z_\varepsilon; \varepsilon) d\varepsilon + \omega \int_0^1 U^s(q_\varepsilon, z_\varepsilon) d\varepsilon \right] \]  

(21)

where \( \omega \in [0,1] \), subject to the following constraints:

1. **Incentive Compatibility:** By the revelation principle, buyers must have no incentive to lie about their type:

\[ \varepsilon' \in \arg \max_{\varepsilon \in [0,1]} U^b(q_\varepsilon, z_\varepsilon; \varepsilon') \text{, for all } \varepsilon' \in [0,1]. \]  

(22)

2. **Individual Rationality:** Buyers and sellers must receive non-negative utility in all meetings:\(^{13} \)

\[ U^b(q_\varepsilon, z_\varepsilon; \varepsilon) \geq 0 \text{, for all } \varepsilon \in [0,1], \text{ and} \]  

\[ U^s(q_\varepsilon, z_\varepsilon) \geq 0 \text{, for all } \varepsilon \in [0,1]. \]  

(23)

(24)

Let the indirect utility of a type-\( \varepsilon \) buyer be defined as

\[ v_\varepsilon \equiv U^b(q_\varepsilon, z_\varepsilon; \varepsilon). \]  

(25)

Using \( v_\varepsilon \), the incentive compatibility constraint (22) can be restated with the help of the following standard proposition (see Mas-Colell, Winston and Green, 1995, Proposition 23.D.2):

**Proposition 1** A trading assignment satisfies the incentive compatibility constraint (22) if and only if \( q_\varepsilon \) is non-decreasing in \( \varepsilon \) and the indirect utility function satisfies

\[ v_\varepsilon = \int_0^\varepsilon \frac{\partial}{\partial x} U^b(q_x, z_x; x) dx = \int_0^\varepsilon \psi q_x^{1-\sigma} dx, \text{ for all } \varepsilon \in [0,1]. \]  

(26)

\(^{11}\) For notational ease we use subscripts to denote that \( q \) and \( z \) are functions of \( \varepsilon \).

\(^{12}\) To define the buyers’ expected utility note that \( (1 - \varepsilon) \) represents the distance between the buyers most preferred variety and the variety found in search. Ex ante, this distance is a random variable uniformly distributed on \([0,1]\).

\(^{13}\) This property follows from the combination of two facts: the buyer observes the variety offered by the seller as soon as they meet, and both buyers and sellers have the option of not trading. If sellers could hide their variety for sale, then they could demand a payment for buyers to reveal this information. In this case, buyers could end up with negative utility in some meetings.
Using Proposition 1 and constraints (23) and (24) together with the definitions (18), (20), and (25), the pairwise incentive efficient trading assignments can be characterized as the solution to the following program:

$$\max_{\{q_{\varepsilon}, v_{\varepsilon}\}_{\varepsilon \in [0,1]}} \int_0^1 \left[ (1 - \omega) v_{\varepsilon} + \omega \left( \psi_{\varepsilon} q_{\varepsilon}^{1-\sigma} - \mu q_{\varepsilon} - v_{\varepsilon} \right) \right] d\varepsilon$$

subject to\(^{14}\)

$$\dot{v}_{\varepsilon} = \frac{\psi_{\varepsilon} q_{\varepsilon}^{1-\sigma}}{1 - \sigma},$$

$$q_{\varepsilon} \geq 0,$$

$$v_{\varepsilon} \geq 0,$$

$$\pi_{\varepsilon} \equiv \frac{\varepsilon \psi_{\varepsilon} q_{\varepsilon}^{1-\sigma}}{1 - \sigma} - v_{\varepsilon} - \mu q_{\varepsilon} \geq 0,$$

$$q_{\varepsilon} \text{ is non-decreasing},$$

$$q_0 = 0, \text{ and } v_0 = 0.$$ \hspace{1cm} (34)

Here, $$\pi_{\varepsilon}$$ denotes the seller’s surplus upon meeting a buyer of type $$\varepsilon$$.

This program can be solved with a standard application of the Pontryagin’s Maximum Principle (see Appendix). The solution is summarized in the following proposition:

**Proposition 2** The pairwise incentive efficient trading assignments, which solve program (27) subject to (28) to (33), are the following:

For the case $$\omega \in (0.5,1],$$

$$q_{\varepsilon} = \begin{cases} 0 & \text{for } 0 \leq \varepsilon < \widehat{\varepsilon} \\ \left(\frac{\psi_{\varepsilon} - \varepsilon}{\mu (1 - \varepsilon)}\right)^{\frac{1}{\sigma}} & \text{for } \widehat{\varepsilon} \leq \varepsilon \leq 1 \end{cases}$$

and

$$v_{\varepsilon} = \begin{cases} 0 & \text{for } 0 \leq \varepsilon < \widehat{\varepsilon} \\ \frac{\sigma}{1 - \sigma} \psi_{\varepsilon}^{\frac{1}{\sigma}} \left((1 - \varepsilon) \mu^{\frac{1}{\sigma}} (\varepsilon - \widehat{\varepsilon})^{\frac{1}{\sigma}} \right) & \text{for } \widehat{\varepsilon} \leq \varepsilon \leq 1 \end{cases}$$

where

$$\widehat{\varepsilon} = \frac{2\omega - 1}{3\omega - 1}. \hspace{1cm} (36)$$

For the case $$\omega \in [0,0.5],$$

$$q_{\varepsilon} = \left(\frac{\psi_{\varepsilon}}{\mu}\right)^{\frac{1}{\sigma}}$$

and

$$v_{\varepsilon} = \frac{\sigma}{1 - \sigma} (\psi_{\varepsilon})^{\frac{1}{\sigma}} \mu^{\frac{\sigma - 1}{\sigma}}. \hspace{1cm} (38)$$

\(^{14}\) $$\dot{v}_{\varepsilon}$$ denotes the derivative of $$v$$ with respect to $$\varepsilon$$ evaluated at $$\varepsilon$$.\hspace{1cm}
The following proposition shows that the trading assignments that are pairwise incentive efficient can be implemented with a simple price schedule:

**Proposition 3** The pairwise incentive efficient trading assignments are implemented with a mechanism in which buyers choose \( q \) facing the following price schedule:

\[
Z(q) = \frac{1}{\lambda} \left[ \gamma \mu q + (1 - \gamma) \frac{\psi q^{1-\sigma}}{1-\sigma} \right],
\]

(39)

where \( \gamma = 1 \) if \( \omega \in [0, 0.5] \) and \( \gamma = \frac{\omega}{0.5 - \omega} \) if \( \omega \in (0.5, 1] \).

We will refer to a price schedule with parameter \( \gamma \) as the price schedule \( \gamma \). Given (18), a buyer facing a price schedule \( \gamma \) chooses the following quantities:

\[
q_\varepsilon = \begin{cases} 
0 & \text{for } 0 \leq \varepsilon < 1 - \gamma \\
\psi \left( \frac{\varepsilon - (1 - \gamma)}{\gamma} \right)^{\frac{1}{\sigma}} & \text{for } 1 - \gamma \leq \varepsilon \leq 1 
\end{cases}
\]

(40)

and has indirect utility:

\[
v_\varepsilon = \begin{cases} 
0 & \text{for } 0 \leq \varepsilon < 1 - \gamma \\
\sigma - \psi \left( \gamma \mu \right)^{\frac{1}{\sigma}} \left( \varepsilon - (1 - \gamma) \right)^{\frac{1}{\sigma}} & \text{for } 1 - \gamma \leq \varepsilon \leq 1 
\end{cases}
\]

(41)

Direct comparison of (40) and (41) with (34) to (38) proves Proposition 3.

Commercial margins are positive if and only if \( \gamma < 1 \). In this case, the pricing schedule \( Z(q) \) is strictly concave, which implies that the per unit price of goods declines with \( q \), or equivalently buyers obtain quantity discounts. This is not unrealistic. In retail trade, we observe not only quantity discounts that are explicit, but also some that are implicit. For example, it is common practice to reduce the per unit price of an item when it is presented in large packages.

When buyers have full market power, that is when \( \omega \in [0, 0.5] \) and \( \gamma = 1 \), prices cover only the cost of production. In this case, buyers capture the whole trading surplus. In contrast, even when sellers have full market power, that is when \( \omega = 1 \) and \( \gamma = 0.5 \), they are not able to extract the whole trading surplus because they do not know their clients’ type. The latter case is an interesting benchmark, which we denote monopolistic search. With monopolistic search, sellers post the prices that maximize their expected profits subject to the incentive compatibility and individual rationality constraints of the buyers in an environment where their price schedule has no effect on the number of clients visiting their outlets. This might be the relevant equilibrium concept for some tourist areas where buyers have little knowledge about where to shop. However, in most commercial areas sellers are aware that by posting low prices they can attract clients to their outlets. To capture these competitive pressures on prices we adopt the concept of competitive search in the next subsection.
3.2 The Competitive Search Equilibrium

Let $V_b(\theta, \gamma)$ and $V_s(\theta, \gamma)$ respectively be the expected utility of buyers and sellers when they operate in a submarket with congestion $\theta$ and price schedule $\gamma$:

$$V_b(\theta, \gamma) = \frac{m(\theta)}{\theta} \int_0^1 \left( \frac{\varepsilon \psi q^{1-\sigma} \mu \varepsilon_\theta - \lambda z_\varepsilon}{1 - \sigma} \right) d\varepsilon,$$

and

$$V_s(\theta, \gamma) = m(\theta) \int_0^1 (\lambda z_\varepsilon - \mu q_\varepsilon) d\varepsilon. \tag{43}$$

Substituting (39) and (40) into (42) and (43) and integrating, we obtain

$$V_b(\theta, \gamma) = \frac{m(\theta)}{\theta} \int_0^1 \left( \frac{\varepsilon \psi q^{1-\sigma} \mu \varepsilon_\theta - \lambda z_\varepsilon}{1 - \sigma} \right) d\varepsilon, \tag{44}$$

and

$$V_s(\theta, \gamma) = m(\theta) \int_0^1 (\lambda z_\varepsilon - \mu q_\varepsilon) d\varepsilon. \tag{45}$$

In a competitive search equilibrium, all buyers (sellers) get the same expected utility. Thus, any active submarket $(\gamma, \theta)$ lies in the intersection of an indifference curve for the buyer and an indifference curve for the seller for some fixed values of the functions (44) and (45). Since potential submarkets compete in combinations of congestion and prices, there must be no gains from creating a new submarket. Hence, buyers and sellers must have a common marginal rate of substitution between $\theta$ and $\gamma$ (see Moen 1997). That is,

$$\frac{\partial V_b(\theta, \gamma)}{\partial \theta} \frac{\partial V_s(\theta, \gamma)}{\partial \gamma} = \frac{\partial V_s(\theta, \gamma)}{\partial \gamma}. \tag{46}$$

Since marginal rates of substitution are strictly monotone, a single submarket will be active in a competitive search equilibrium. Differentiating (44) and (45) and substituting into (46), we obtain

$$\gamma = \frac{1 + \eta(\theta)}{2}, \tag{47}$$

where $\eta(\theta)$ is the elasticity of the function $m$:

$$\eta(\theta) = \frac{\theta m'(\theta)}{m(\theta)}. \tag{48}$$

If the matching function is Cobb-Douglas, that is $M(B_t, S_t) = A_0 B^\eta_t S^{1-\eta}_t$, then $\eta(\theta)$ is constant and equal to $\eta$. In this case, as the contribution of buyers in the matching technology (measured by $\eta$) increases, buyers capture a larger fraction of the trading surplus ($\gamma$ rises) in a competitive search equilibrium.
4 General Equilibrium

In this section, we characterize a competitive search equilibrium by combining the optimal behavior of households in Section 2 with the commercial equilibrium modeled in Section 3.

Definition: The set \( \{ \theta_t, n_t, b_t, s_t, c_t, y_t, q_t(\varepsilon), z_t(\varepsilon), \psi_t, \mu_t, \lambda_t, Z_t(q) \}_{t=0}^{\infty} \) is a symmetric search equilibrium for a given set of pricing parameters \( \{ \gamma_t \}_{t=0}^{\infty} \) if

1. All households choose \( \{ n_t, b_t, s_t, c_t, y_t, q_t(\varepsilon) \}_{t=0}^{\infty} \) for a given set of payment schedules \( \{ Z_t(q) \}_{t=0}^{\infty} \) and market congestion ratios \( \{ \theta_t \}_{t=0}^{\infty} \), and the implied set of payments is \( \{ z_t(\varepsilon) \}_{t=0}^{\infty} = \{ Z_t(q_t(\varepsilon)) \}_{t=0}^{\infty} \).

2. For all \( t \), the payment schedule \( Z_t(q_t) \) satisfies (39) when \( \psi_t \) satisfies (19) and \( \mu_t \) and \( \lambda_t \) are the current value Lagrange multipliers associated with the resource constraint and the budget constraint in the household’s optimization program.

3. Congestion ratios are consistent with individual behavior: \( \theta_t = b_t/s_t \).

Definition: A search equilibrium is competitive if \( \gamma_t \) satisfies (47) for all \( t \).

Definition: A search equilibrium is monopolistic if \( \gamma_t = 0.5 \) for all \( t \).

Since in a symmetric search equilibrium all households behave identically, lower-case and upper-case letters coincide. Also, in the absence of capital and with a constant price parameter \( \gamma_t \) all periods are identical, so the equilibrium allocation is time invariant. Indeed, in both a competitive and a monopolistic search equilibrium, \( \gamma_t \) is constant (see footnote 17), so we omit time subscripts until Section 6 where capital is introduced.

In equilibrium, a household must obtain the same utility from allocating an individual to any of the three activities, as it is implied by the first order conditions (15) to (17), that is \( V^b(\theta, \gamma) \), \( V^s(\theta, \gamma) \), and \( \mu f'(n) \) must be equal. Using (44) and (45), this equality implies

\[
\theta = \frac{\gamma}{2(1-\gamma)}, \quad \text{and} \quad f'(n) = \frac{m(\theta) \sigma^2}{\theta} \left( \frac{\psi}{\mu} \right)^\frac{1}{2} \gamma^2.
\]

15 A monopsonistic search equilibrium would be one in which \( \gamma_t = 1 \). However, a monopsonistic search equilibrium with a positive \( s_t \) does not exist, because then the commercial margin is zero, so sellers cannot cover the opportunity cost of their time.
According to (49), the ratio of buyers to sellers depends positively on $\gamma$. As $\gamma$ increases buyers capture a larger fraction of the trade surplus (see Proposition 3), so households respond by sending more buyers and fewer sellers to the marketplace.\footnote{Note that for all values of $\gamma$ the ratio is greater than or equal to 0.5, so there are more buyers than sellers.}

Using (40) to integrate for the average sales $\bar{q}$ in a trading meeting with a price schedule $\gamma$, the total amount of output sold by a household is:

$$S(s, \theta)\bar{q} = sm(\theta) \frac{\sigma}{1 + \sigma} \left( \frac{\psi}{\mu} \right)^{1/\sigma} \gamma. \quad (51)$$

Combining (10), (11), (50), and (51) gives

$$\frac{f'(n)}{f(n)} = \frac{1}{s\theta} \frac{\sigma}{1 - \sigma} \gamma. \quad (52)$$

Using the definition of $\theta$ in (1), and the labor allocation constraint (13), we have

$$s = \frac{1 - n}{1 + \theta}. \quad (53)$$

Finally, combining (49), (52), and (53), we obtain:

$$\frac{1 - n}{n} \alpha(n) = \frac{\sigma}{1 - \sigma} (2 - \gamma), \quad (54)$$

where $\alpha(n)$ is the elasticity of $f$, that is $\alpha(n) = f'(n)n/f(n)$. Assuming that $\alpha(n)$ is non-increasing and bounded away from 0, the left-hand side of (54) is a downward slopping function that goes from $\infty$ to 0 when $n$ goes from zero to one. Therefore, condition (54) uniquely determines the equilibrium value of $n$ for a given $\gamma$. The equilibrium value of $\gamma$ depends on the degree of competition in the commercial market. With competitive search, $\gamma$ is given by (47). In the latter case, $\theta$ and $\gamma$ are simultaneously determined by combining (47) and (49).\footnote{Note that if we had not omitted time subscripts, this system of equations, and thus the equilibrium values of $\theta_t$ and $\gamma_t$, would be identical across periods. This proves that the equilibrium is stationary.} Therefore, congestion $\theta$ is the solution to

$$\theta = \frac{1 + \eta(\theta)}{2[1 - \eta(\theta)]}. \quad (55)$$

The existence of an equilibrium depends on (55) having a non-negative solution. Assuming that $\eta(\theta)$ is non-increasing and bounded away from zero and one, a solution exists and is unique. Therefore, there is a unique competitive search equilibrium.

The following proposition summarizes the conditions for existence and the characterization of a competitive search equilibrium:
Proposition 4 If the elasticities $\eta(\theta)$ and $\alpha(n)$ are non-increasing and bounded away from zero, and $\eta(\theta)$ is bounded away from one, a competitive search equilibrium exists and is unique. Recursively, $\theta, \gamma, n, s, b, \psi/\mu, q_\varepsilon, c, y, \psi$, and $\mu$ are determined by equations (55), (47), (54), (53), (1), (50), (40), (9), (4), (10), and (19). The utility value of the payment schedule $\lambda Z(q)$ is determined by (39), but the precise values of $\lambda$ and $Z(q)$ are indeterminate because they depend on the units in which payments are measured.

To calculate the average commercial margin, we use (40) to obtain the following relationship:

$$\psi \int_0^1 \varepsilon q_\varepsilon^{1-\sigma} d\varepsilon = (1 + \sigma - \gamma \sigma) \mu \bar{q}. \tag{56}$$

Substituting (56) and (49) into (15) and (16), we get

$$\frac{2(1 - \gamma)}{\gamma} \left( \frac{1 + \sigma - \gamma \sigma}{1 - \sigma} \mu \bar{q} - \lambda \bar{q} \right) = \lambda \bar{q} - \mu \bar{q}. \tag{57}$$

Therefore, the average commercial margin is

$$\frac{\lambda \bar{q} - \mu \bar{q}}{\mu \bar{q}} = \frac{2\sigma(1 - \gamma)}{1 - \sigma}. \tag{58}$$

The commercial margin increases with the preference for diversity $\sigma$ and decreases with the price schedule parameter $\gamma$. In a competitive search equilibrium, $\gamma$ is given by (47), so the commercial margin is positive and increasing with the weight of the sellers in the matching technology.

5 Welfare

This section studies the welfare properties of a competitive search equilibrium. To this end, it characterizes the optimal allocation that a benevolent central planner would choose in order to maximize the utility of a representative household. Following standard practice, the central planner is not only bound by the resources available in the economy, but also by the bilateral matching among traders. In addition, it is sensible that we restrict the central planner to information that is publicly available. However, imposing this restriction is much more subtle than it first appears. The incentive compatibility constraint (22) arising from the fact that the buyers’ types are private information is only binding when selling costs must be financed with the revenue from sales (see below). Moreover, the central planner can easily affect this revenue with policy tools that, in principle, are respectful to private information, for example, a sales tax or a sales subsidy. Due to
the subtleties of restricting the central planner to public information, we start by characterizing
the first best allocation, in which the central planner is not bound by the incentive compatibility
constraints imposed by private information. This allocation is an interesting benchmark in itself
and is useful to evaluate the welfare costs of private information. Later, we introduce private
information with specific assumptions about the policy tools at the disposal of the central planner.

5.1 The Optimum with Complete Information

The first best allocation is one in which the central planner maximizes the utility of a representative
household subject to the resources available in the economy and the bilateral matching among
traders. Since resources cannot be transferred across periods, the objective of the central planner
must be to maximize utility at each period. Therefore, the planner must solve the following program:

$$\max_{b,s,n,q} U(c), \text{ where } c = \left[ \mathcal{M}(b,s) \int_0^1 \varepsilon q_x^{1-\sigma} d\varepsilon \right]^{\frac{1}{1-\sigma}},$$

subject to (13) and

$$\mathcal{M}(b,s) \int_0^1 q_x d\varepsilon = f(n).$$

The first order conditions of this problem are:

$$\varepsilon \psi q_x^{-\sigma} = \mu,$$

$$\mathcal{M}_b(b,s) \int_0^1 \frac{\varepsilon \psi q_x^{1-\sigma}}{1-\sigma} d\varepsilon - \mu \mathcal{M}_b(b,s) \int_0^1 q_x d\varepsilon = \mu f'(n),$$

$$\mathcal{M}_s(b,s) \int_0^1 \frac{\varepsilon \psi q_x^{1-\sigma}}{1-\sigma} d\varepsilon - \mu \mathcal{M}_s(b,s) \int_0^1 q_x d\varepsilon = \mu f'(n),$$

where $\mu$ is the Lagrange multiplier of (60) and $\psi = U'(c)c^\sigma$. In the first best, the marginal utility
of consuming each variety of merchandise must be equal to the marginal production cost. Also,
the marginal social benefit of employment in all three activities must be the same. Comparing (62)
and (63), we obtain

$$\mathcal{M}_b(b,s) = \mathcal{M}_s(b,s).$$

Equality (64) implies that for a given number of traders the meetings between a buyer and a seller
are maximized. Equations (13) and (60) to (63) can be easily manipulated to obtain an almost explicit solution of the first best allocation (see the Appendix for details):
Proposition 5  The first best allocation, in which the central planner has both complete information and control over all variables, is characterized by the following equations:

\[ \theta = \frac{\eta(\theta)}{1 - \eta(\theta)}, \]  
\[ \frac{1 - n}{n} \alpha(n) = \frac{\sigma}{1 - \sigma}, \]  
\[ s = [1 - \eta(\theta)](1 - n), \]  
\[ b = \theta s, \text{ and} \]  
\[ q_{e} = \frac{1 + \sigma f(n)}{\sigma M(b, s)^{\frac{1}{2}}} \varepsilon. \]

The following proposition compares the allocations in the first best and in the competitive search equilibrium. To facilitate this comparison, it specializes the production and matching technologies to standard functional forms:

Proposition 6  Let an asterisk denote first best and no asterisk denote competitive search equilibrium. When \( f \) is isoelastic and \( M \) is Cobb-Douglas, the following relations hold:

\[ n < n^*, \]  
\[ \frac{b}{s} > \frac{b^*}{s^*}, \]  
\[ s = s^*, \]  
\[ \frac{n}{n^*} = \frac{s}{s^*}, \]  
\[ q_{e} < q_{e}^*. \]

In a competitive search equilibrium, selling costs are financed with commercial margins that create a wedge between the marginal production cost of merchandises and the price paid by consumers. This wedge induces buyers to purchase smaller quantities in each transaction relative to the first best allocation. However, competition among potential submarkets narrows commercial margins so the congestion of buyers to sellers in equilibrium is higher than in the first best. As a result, the number of sellers and producers is lower in equilibrium than in the first best.

Intuitively, the equilibrium price schedule must play two conflicting allocational roles: it must signal buyers the opportunity cost of the goods they are considering purchasing, and it must finance retail costs. The equilibrium price schedule settles on a compromise between these two roles. Prices are higher than the social opportunity cost of goods but not high enough to finance the efficient number of sellers.
As in Moen (1997) and Shimer (1996), the first best allocation can be implemented as a competitive search equilibrium when buyers and sellers have complete information. In this case, all pairwise efficient allocations must maximize the joint surplus of buyers and sellers. Moreover, competition among potential submarkets leads to sharing this surplus according to Hosios’s (1990) rule, that is buyers get a fraction $\eta(\theta)$ of the surplus and sellers get $1 - \eta(\theta)$, and $\theta$ is the first best level of congestion. With private information about buyers’ types, this equilibrium breaks down because the incentive compatibility constraint (22) is violated.

The first best allocation can also be decentralized as a competitive search equilibrium were buyers’ types are private information if sellers can charge a flat fee in order for them to reveal the variety type they carry. In this case, the fee covers selling costs without having to add a positive commercial margin on the price of merchandises. In our model, such a fee is prevented by the assumption that buyers can immediately observe the sellers’ variety type. More generally, one could realistically assume that the buyers’ satisfaction from a commercial transaction depends on the service effort provided by the seller. In this more complicated model, the flat unconditional fee is also discouraged by the moral hazard problem it generates on the effort exercised by sellers. In reality, we find flat fees in warehouse clubs. However, sales in warehouse clubs are a small fraction of the economy wide retail sales, and even these clubs charge fees that cover a small fraction of their commercial costs.

The following proposition summarizes these two interesting ways of decentralizing the first best allocation:

**Proposition 7** The first best allocation where the planner has complete information can be implemented as a competitive search equilibrium if buyers and sellers have complete information or if sellers can charge a lump-sum fee before revealing the variety they carry to the buyers they meet.

### 5.2 The Optimum with Private Information

This subsection characterizes second best allocations when the central planner has limited information. Given that buyers’ types are private information it is natural to assume that the central planner cannot directly observe them. However, if the central planner can monitor the allocation of labor in each household, then the first best allocation can be easily implemented by dictating to households the allocation of labor, charging buyers the marginal cost of producing merchandise, and transferring the proceeds of these payments to sellers. With this mechanism, buyers truthfully
reveal their types without any efficiency loss. To make our analysis more realistic and more interesting than this simple result, we assume that the central planner cannot directly observe how households allocate their labor. This assumption is in line with the unobservability of leisure (and so the allocation of time when there are two activities) in the standard theory of taxation.

Specifically, we assume that the central planner can only observe market transactions and only has control of the price schedules faced by buyers and sellers. Given these price schedules, buyers choose the quantities they want to purchase, which sellers supply as long as prices are above the marginal costs of production. Households decide the allocation of labor taking into account the expected returns from each activity.

In principle, buyers and sellers could face different price schedules. If so, the gap between the two schedules is as a sales tax (or a sales subsidy) to be collected (or distributed) by a government who balances its budget with a lump-sum subsidy (or a lump-sum tax) on households. In the three propositions that follow, we make alternative assumptions on the central planner’s ability to control price schedules and to impose lump-sum taxes on households.

In the absence of any restriction on these policy instruments, the central planner can implement the first best allocation despite the presence of private information:

**Proposition 8** If the central planner can resort to lump-sum taxes on households to finance a linear sales subsidy, then the first best allocation characterized in Proposition 5 can be decentralized with the following price schedule faced by buyers:

\[ Z(q) = \frac{\eta(\theta)\mu q}{\lambda}, \quad (74) \]

and a linear sales subsidy at the gross rate:

\[ T = \frac{1 - \sigma \eta(\theta)}{(1 - \sigma) \eta(\theta)}, \quad (75) \]

where \( \theta \) is the solution to (65). (The price schedule faced by sellers is \( T Z(q) \)).

This proposition implies that the planner could not do better by introducing additional control instruments. Intuitively, the proposition holds because the government picks the tab for the selling costs with the sales subsidy. Hence, the prices faced by the buyers can be equated to the marginal social cost of production of merchandises.

The implementation of the first best allocation in Proposition 8 depends on the existence of lump-sum taxes. In the model, this is not problematic because all households are identical. How-
ever, this homogeneity is unrealistic and has been assumed only for simplicity. The next proposition assumes the absence of lump-sum taxes.

**Proposition 9** In the absence of a lump-sum tax on households, the price schedule in a competitive search equilibrium is optimal if the production function is linear.

In the important case where the production function is linear, which with a single input corresponds to constant returns to scale, a central planner without recourse to lump-sum taxes cannot improve upon the competitive search equilibrium. When the production function is not linear, regulation of the price schedule can be welfare enhancing because it has an indirect effect on the marginal product of labor which differs from its first best value.

Finally, we consider the converse to the assumptions in Proposition 9 - that is, we assume that the price schedule is beyond the direct control of the central planner but that one can resort to a lump-sum tax on households:

**Proposition 10** Assume that the production function is linear and the matching function is Cobb-Douglas, that is \( f(n) = wn \) and \( m(\theta) = \theta^\eta \). If the price schedule is determined in a competitive search equilibrium, but a benevolent government can subsidize retail sales by equilibrating its budget with a lump-sum tax on households, the optimal sales subsidy is:

\[
T = \frac{1 + \eta}{2\eta}.
\]  

(76)

With this subsidy, the equilibrium price schedule is:

\[
Z(q) = \frac{1}{\lambda} \left[ \eta \mu q + \frac{1 - \eta}{2} \psi \frac{q^{1-\sigma}}{1-\sigma} \right].
\]  

(77)

The resulting allocation of labor is the same as in the first best, but quantities purchased by buyers are

\[
g_\varepsilon = \begin{cases} 
\frac{1-\sigma^2}{1-\sigma^2} \frac{w}{\eta \rho (1-\eta)^{1-\eta}} \left( \frac{2}{1+\eta} \right)^{1-\frac{1}{\sigma}} \left( \varepsilon - \frac{1-\eta}{2} \right)^{\frac{1}{\sigma}} & \text{if } \varepsilon \geq \frac{1-\eta}{2}, \\
0, & \text{otherwise.}
\end{cases}
\]  

(78)

6 Commerce in a Neoclassical Growth Framework

This section sketches how to embed the model developed in the previous sections in a Neoclassical growth framework. In the resulting synthesis, the economy has two sectors: one produces goods combining capital and labor as in the Neoclassical growth model, the other exchanges goods in
competitive search markets where buyers’ types are private information. To make our model as close as possible to the basic Neoclassical growth model, we assume that investment does not require installation or commercial costs. However, future work could incorporate these features. For brevity, we omit all proofs, which are either standard in the Neoclassical growth theory or parallel to the arguments in the previous sections.

In the extended model, the production of goods requires not only labor but also capital:

$$y_t = F(k_t, n_t).$$

The production function $F : \mathbb{R}_+^2 \rightarrow \mathbb{R}_+$ maps capital and labor into output. This function is continuously differentiable, increasing in both arguments, concave, and homogeneous of degree one. Also, the Inada conditions for an interior solution are assumed to apply.

Goods can be used for both consumption and investment. When goods are used for consumption, they are exchanged in the same type of markets as those described in the previous sections. When goods are used for investment, they are perfect substitutes for one another. To save commercial costs, households use part of their own output to increase their capital stock. Therefore, households allocate a fraction of their output to be sold and another fraction to investment:

$$y_t = k_{t+1} - k_t (1 - \delta) + S(s_t, \theta_t)\pi_t,$$

where $\delta \in (0,1)$ is the depreciation rate. The fraction destined for sale is consumed by the purchasing households.

The optimal behavior of a household is characterized by the same first order conditions as those in Section 2, that is (14) to (17), with the obvious modification that output and the marginal product of labor now depend on the capital stock. Moreover, the following two conditions must hold:

$$\mu_t = \mu_{t+1} \left[ F_k(k_{t+1}, n_{t+1}) + 1 - \delta \right] \beta,$$

and

$$\lim_{t \rightarrow \infty} \beta^{-t} \mu_t k_t = 0.$$

Equation (81) states that the value of one unit of output today is equal to the present discounted value of the gross marginal product of capital, while equation (82) is a standard transversality condition.

The definition of a competitive search equilibrium is analogous to the one in Section 4. This equilibrium is now described by a system of difference equations for the variables $k_t$ and $\mu_t$: (80)
and (81) together with (1), (5), (13), (19), (40), (47), (49), (50), (53), and (79). This system together with the initial condition \(k_0\) and the terminal condition (82) determines the equilibrium path. For all capital stocks, equations (47) and (49) still determine the pricing parameter \(\gamma_t\) and the congestion \(\theta_t\) of buyers over sellers in the market for consumption goods. Therefore, these two variables are constant along an equilibrium path.

Qualitatively, the dynamics of capital accumulation are identical to those of the Neoclassical growth model. Capital converges monotonically to a steady state where \(k_t\) and \(\mu_t\) are constant. For low capital stocks, both the marginal product of capital and the utility price of capital \(\mu_t\) are high relative to the steady state. High levels of \(\mu_t\) induce low consumption and high supply of labor into production, and as a result high saving. As capital is accumulated, households not only increase the fraction of output allocated for consumption, but also the fraction of labor allocated to the exchange of commodities.

In the steady state, equation (81) implies that the net marginal product of capital is equal to the subjective discount rate:

\[
F_k \left( \frac{k}{n}, 1 \right) - \delta = \frac{1}{\beta_0} - 1. \tag{83}
\]

When net investment is zero, analogous steps to the derivation of (54) yield:

\[
\frac{F_n \left( \frac{k}{n}, 1 \right)}{F \left( \frac{k}{n}, 1 \right) - \delta \frac{k}{n}} \frac{1 - n}{n} = \frac{\sigma (2 - \gamma)}{1 - \sigma}. \tag{84}
\]

Equations (83) and (84) determine the steady state capital stock and labor allocated into production. The steady state values of the remaining variables \((b, s, q, \varepsilon, \text{and } \mu)\) are obtained from equations analogous to those in Section 4.

As in the version of the model without capital, a central planner who faces the same informational constraints as the market and whose only policy instrument is to regulate market prices cannot improve upon the allocation in a competitive search equilibrium when the production technology is linear. With the existence of capital, linearity of the production function is stronger than constant returns to scale. However, with constant returns to scale the difference in welfare between a competitive search equilibrium and the allocation that would be achieved by a central planner regulating the price schedule are negligible for reasonable parameters (see the numerical example in the following subsection).

\(^{18}\) Taking into account that the marginal product of labor depends on both capital and labor.
As in the Neoclassical growth model, output and capital converge to a steady growth path if the utility function $U$ is isoelastic and the efficiency of labor in the production of goods grows at a constant rate. Also, the utility function can be easily extended to include leisure or home services. In this case, the restrictions on $U$ for convergence to a steady growth path are the standard ones.

6.1 Numerical Calibration

In the context of the Neoclassical growth framework, our model of commerce can be estimated using standard economic data. In this subsection, we discuss how to identify the parameters of the model, and we provide a numerical calibration of the model. For this purpose, we assume logarithmic preferences: $U(c_t) = \ln c_t$, and Cobb-Douglas matching and production technologies:

$$\mathcal{M}(b_t, s_t) = A_0 b_t^{1-\eta}$$

and

$$\mathcal{F}(k_t, e_t n_t) = A_1 k_t^\alpha (e_t n_t)^{1-\alpha}.19$$

We also assume that the efficiency of labor in the production of goods ($e_t$) grows at a constant rate $g$.

As is standard in the Neoclassical growth model, the parameters $\alpha$, $\beta$, $\delta$, and $g$ can be estimated using capital and labor income shares in the sector producing goods, the real return on capital, the durability of capital, and the average growth rate of output. In our numerical example in Table 2, we pick standard estimates for these parameters. The two remaining parameters of the model to estimate are the preference for diversity ($\sigma$) and the elasticity of matches with respect to the number of buyers ($\eta$).20 These two parameters can be identified with empirical estimates of the average commercial margin and the congestion in the retail sector. In a competitive search equilibrium, equations (47), (55), and (58) imply:

$$\text{Average commercial margin \left( \frac{\lambda_t \eta_t - \mu_t \eta_t}{\mu_t \eta_t} \right) = \frac{\sigma (1 - \eta)}{1 - \sigma}, \quad \text{and} \quad (85)}$$

$$\text{Congestion \left( \frac{b_t}{s_t} \right) = \frac{1 + \eta}{2(1 - \eta)}. \quad (86)}$$

Therefore, with estimates of the average commercial margin and market congestion we can solve for $\sigma$ and $\eta$.

The Bureau of the Census of the United States reports that the average commercial margin for retail trade has been around 0.28 during the last decade.21 The Bureau of Labor Statistics

---

19 When preferences are logarithmic, the system of difference equations characterizing a competitive search equilibrium is greatly simplified because then $S(s_t, \theta_t)_{t+1} = [1 + \sigma(1 - \gamma)]^{-1} \mu_t^{-1}$.

20 The values of the technological constants $A_0$ and $A_1$ are irrelevant for the calculations in Table 1, and affect only the units in which we measure output and hedonic consumption.

21 See http://www.census.gov/svsd/www/artstbl.html for data since 1992. Although the average commercial margin varies little over time, it varies widely by the type of business of the retailers. For example, in 2000, at the lower
measures the number of production employees in retail trade and the average weekly hours that these employees work. The product of these two measures is the empirical counterpart of $s_t$ (for example, 465 million hours/week in 1986).\textsuperscript{22} To calculate $b_t$, we multiply the average time spent shopping by an adult (3.4 hours/week in 1986) reported in Robinson, Andreyenkov, and Patrushev (1989, p. 84)\textsuperscript{23} by the number of shoppers in the economy (United States population 16 and over). This product measures the empirical counterpart of $b_t$ (630 million hours/week in 1986). This implies that congestion in 1986 was 1.35 ($\approx 630/465$). Applying these estimates to the system (85) and (86), we obtain $\eta = 0.46$ and $\sigma = 0.42$.

Using the estimated parameters, Table 1 compares the competitive search equilibrium and the first best allocation.\textsuperscript{24} In equilibrium, households spend less time producing and selling goods and more time shopping than in the first best. Also, in equilibrium buyers leave empty handed from 27 percent of the trading meetings while they always acquire a positive amount of goods in the first best. Welfare in the two allocations is quite different. Changing from the first best balanced path to the competitive search equilibrium balanced path is equivalent to a 7.94 percent drop in consumption. When the transitional costs of changing the capital stock are taken into account this percentage drops slightly to 7.87.\textsuperscript{25} This large welfare cost is not easy to avoid with correcting policies. For example, regulating the price schedule in a competitive search equilibrium leads only to a negligible welfare improvement equivalent to less than a $10^{-5}$ percent increase in consumption. (This improvement would be zero if the production technology were linear instead of Cobb-Douglas). The large difference in welfare between a competitive search equilibrium and the first best allocation is due to the necessity of financing retail costs with large commercial margins. This inefficiency is unavoidable in the absence of some form of lump-sum taxes which would allow end, we find the commercial margins for Warehouse Clubs and Superstores (0.167), Automotive Dealers (0.175), and Gasoline Stations (0.208). At the upper end, we have the margins for Specialty Food Stores (0.419), Clothing and Footwear (0.426), and Furniture (0.441). In this paper we abstract from the reasons why different goods may trade with different margins, although this is an interesting topic for future research.

\textsuperscript{22}We use 1986 to estimate congestion because the survey on the time individuals spend shopping that we have on hand refers to that year. The data was downloaded in July 1, 2002 from http://stats.bls.gov/data/home.htm (production employees in retail trade: 15.924 million, average hours worked by production employees: 29.2)

\textsuperscript{23}This is the most recent estimate of average time allocated to shopping that we could find. We do not expect that indices of congestion vary dramatically over time.

\textsuperscript{24}The balance growth paths are calculated using the formulas described in the previous sections. The transitional dynamics necessary to calculate the last row are obtained using standard numerical methods.

\textsuperscript{25}In this comparison, we assume that both the equilibrium path and the first best path start with the same capital stock (the one in the first best).
7 Conclusion and Extensions

Search models have been used to study decentralized markets where traders meet bilaterally. These models have been useful to analyse the labor market. Also, with the work of Kiyotaki and Wright (1989 and 1993) they have become the dominant paradigm for the theoretical microfoundations of money. Our paper uses search to capture some important features of the retail sector.

Our key assumption is that matching buyers with the sellers that carry their desired products is costly. We model this cost with the two-sided search technology of Mortensen (1982) and Pissarides (1990). We also assume that buyers are aware of the price schedules of sellers and as a result they direct their search to a subset of sellers with the most desirable combination of prices and congestion. To formalize this idea we incorporate the competitive search concept in Moen (1997) and Shimer (1996). Finally, we assume that the buyers’willingness to pay for a particular product is not observable. In this way, we extend competitive search to a framework with private information.

In this framework, we study the welfare properties of a competitive search allocation by comparing it to the allocation chosen by a central planner. If the planner faces the same informational constraints as the market and the only policy instrument is to regulate market prices, then a competitive search equilibrium coincides with the choice of the planner when the production technology is linear. However, if the planner can use lump-sum taxes to subsidize sales, the planner can improve upon the equilibrium outcome. In fact, the planner can achieve the first best allocation by introducing a linear subsidy on sales. The first best also coincides with a competitive equilibrium if sellers can hide the characteristics of their product and charge the buyers a flat fee to reveal such information.

Even if large commercial margins are unavoidable in the absence of lump-sum taxation, their existence is important for policy design. For example, large commercial margins have a profound effect on the welfare cost of sales taxes, because zero taxes is already a large departure from the first best, so the standard result that small taxes cause negligible dead-weight-losses does not apply. Moreover, because of standard tax equivalence results, the welfare costs of income taxes must also be much larger when commercial margins are taken into account.

For simplicity, we have abstracted from the existence of money by assuming that payments can be made through a central clearing-house. In the absence of this centralized system of payments,
money is a useful device that facilitates trade as in Kiyotaki and Wright (1989 and 1993). Faig (2001) introduces money in a simpler version of the present model where sellers are constrained to make offers that consist of a single quantity-payment pair \((q_t, z_t)\). The main complication of introducing money in the present set-up is that when buyers are lucky to find a seller that carries a variety close to their most preferred variety they would like to spend more money than they choose carry in equilibrium.\(^{26}\) Because this is not possible, they are liquidity constrained. Moreover, these liquidity constraints affect equilibrium price schedules. Despite this complication, the model remains analytically tractable, but the algebraic expressions are longer and harder to interpret than in the present contribution. For this reason, we plan to study the monetary version of the present model in a separate paper.

\(^{26}\)This complication does not apply when the opportunity cost of holding money is zero.
APPENDIX

The proof of Proposition 2 requires the following lemmas:

**Lemma 11** Constraints (28) to (33) imply that there is \( \hat{\varepsilon} \in [0, 1] \) such that \( q_{\varepsilon} = 0 \) and \( v_{\varepsilon} = 0 \) for \( \varepsilon \in [0, \hat{\varepsilon}] \), and \( q_{\varepsilon} > 0 \) and \( v_{\varepsilon} > 0 \) for \( \varepsilon \in (\hat{\varepsilon}, 1] \). The solution to program (27) subject to (28) to (33) must be such that \( \hat{\varepsilon} < 1 \).

**Proof of Lemma 11**

Constraints (32) and (33) immediately imply that there is \( \hat{\varepsilon} \in [0, 1] \) such that \( q_{\varepsilon} = 0 \) for \( \varepsilon \in [0, \hat{\varepsilon}] \), and \( q_{\varepsilon} > 0 \) for \( \varepsilon > \hat{\varepsilon} \). When \( q_{\varepsilon} = 0 \), constraints (30) and (31) imply that \( v_{\varepsilon} = 0 \). With these results, constraint (28) implies that \( q_{\varepsilon} > 0 \) if and only if \( v_{\varepsilon} > 0 \). Finally, if \( \hat{\varepsilon} \) were 1, the optimized value of (27) would be zero, which cannot be a solution to the maximization program because there are many feasible trading assignments that achieve a positive value. For example, \( q_{\varepsilon} = v_{\varepsilon} = 0 \) if \( \varepsilon < 0.5 \), and \( q_{\varepsilon} = \left( \frac{\psi(2\varepsilon-1)}{\mu} \right)^{\frac{1}{2}} \) and \( v_{\varepsilon} = \frac{1}{2} \frac{\sigma}{\mu} [\psi(2\varepsilon-1)]^{\frac{1}{2}} \mu \frac{\sigma-1}{\sigma} \) otherwise.\(^{27}\)

**Lemma 12** Equations (34) to (36) solve program (27) subject to (28) and \( q_{\varepsilon} = v_{\varepsilon} = 0 \) for \( \varepsilon \in [\hat{\varepsilon}, 1] \) where \( \hat{\varepsilon} \) is endogenous. Moreover, the seller’s surplus is

\[
\pi_{\varepsilon} = \hat{\varepsilon} \frac{\mu q_{\varepsilon}}{1 - \sigma} \left( \sigma + \frac{1 - \varepsilon}{\varepsilon - \hat{\varepsilon}} \right).
\]

**Proof of Lemma 12**

Let \( \zeta_{\varepsilon} \) denote the co-state variable associated with the differential equation (28). The current-value Hamiltonian of the program is:

\[
\mathcal{H}(q_{\varepsilon}, v_{\varepsilon}, \zeta_{\varepsilon}, \varepsilon) = (1 - \omega)v_{\varepsilon} + \omega \left( \frac{\varepsilon \psi q_{\varepsilon}^{1-\sigma}}{1 - \sigma} - \mu q_{\varepsilon} - v_{\varepsilon} \right) + \zeta_{\varepsilon} \psi q_{\varepsilon}^{1-\sigma}. \quad (88)
\]

The first order necessary condition with respect to the control variable \( q_{\varepsilon} \) is \( \mathcal{H}_{q_{\varepsilon}} = 0 \):

\[
\omega \left( \psi \varepsilon q_{\varepsilon}^{1-\sigma} - \mu \right) + \zeta_{\varepsilon} \psi q_{\varepsilon}^{1-\sigma} = 0. \quad (89)
\]

The co-state variable must obey \( \zeta_{\varepsilon} = -\mathcal{H}_{\zeta_{\varepsilon}} \):

\[
\dot{\zeta}_{\varepsilon} = -(1 - 2\omega). \quad (90)
\]

\(^{27}\)This is the optimal trading assignment for \( \omega = 1 \).
Finally, the transversality implies\(^{28}\)

\[ \zeta_1 = 0. \quad (91) \]

The value of the co-state variable \( \zeta_\varepsilon \) can be solved for explicitly using conditions (90) and (91):

\[ \zeta_\varepsilon = (2\omega - 1)(\varepsilon - 1). \quad (92) \]

Substituting (92) into (89) and solving for \( q_\varepsilon \), we obtain

\[ q_\varepsilon = \left[ \frac{\psi (3\omega - 1) \varepsilon - (2\omega - 1)}{\mu \omega} \right]^{\frac{1}{2}}. \quad (93) \]

Integrating (28) from \( \hat{\varepsilon} \) to \( \varepsilon \) and noting that \( v_\varepsilon = 0 \), we obtain

\[ v_\varepsilon = \frac{\sigma \omega}{1 - \sigma 3\omega - 1} \mu q_\varepsilon. \quad (94) \]

Both \( q_\varepsilon \) and \( v_\varepsilon \) are well defined and non-negative if and only if \( \varepsilon \geq (2\omega - 1)/(3\omega - 1) \). Therefore, (36) provides the optimal value of \( \hat{\varepsilon} \). For this value, equations (93) and (94) are equivalent to (34) and (35). Using (93) and (36) in the definition in (31), we obtain (87). Finally, the Hamiltonian is strictly concave with respect to \( q_\varepsilon \), and when the Hamiltonian is evaluated at the optimal choice of \( q_\varepsilon \) it is concave with respect to \( v_\varepsilon \). Therefore, (92) to (94) characterize the unique maximum of the program.

\[ \square \]

**Proof of Proposition 2**

When \( \omega > 0.5 \), equations (34) to (36) satisfy all the restrictions of the full set of constraints (28) to (33) of the original program. Because of Lemma 11, the restrictions imposed in the program of Lemma 12 are implied by (28) to (33). Hence, equations (34) to (36) not only solve the simplified program in Lemma 12 but also solve the original program with the full set of constraints (28) to (33).

When \( \omega = 0.5 \), the variable \( v_\varepsilon \) cancels in the objective (27) so the problem becomes separable across types. Ignoring constraints (31) and (32), the first order conditions of the problem yield (37), so (32) is satisfied. Integrating (28) from 0 to \( \varepsilon \) and noting \( v_0 = 0 \), we obtain (38). Using the definition in (31), we obtain \( \pi_\varepsilon = 0 \) for all \( \varepsilon \in [0, 1] \), so (31) is also satisfied. Therefore, for \( \omega = 0.5 \) maximizing (27) is equivalent to maximizing the expected surplus of the buyers subject to

\(^{28}\) The transversality condition is \( \zeta_1 v_1 = 0 \). However, the argument in Lemma 11 that shows \( v_1 > 0 \) applies here as well.
zero expected surplus for the seller. A fortiori, the same equivalence must be true for \( \omega \in [0, 0.5) \), which assigns a lower weight to the seller in the maximized welfare function (27).

Proof of Proposition 5

Equation (64) together with the definitions of \( \theta, m, \) and \( \eta \) in (2), (1), and (48), implies (65) and (68). The labor allocation constraint together with (65) and (68) yields (67). Using (61) to solve the integrals in (63) and (60), we obtain

\[
f'(n) = M_s(b, s) \frac{\sigma^2}{1 - \sigma^2} \left( \frac{\psi}{\mu} \right)^{\frac{1}{\sigma}}, \quad \text{and} \quad (95)
\]

\[
f(n) = M(b, s) \frac{\sigma}{1 + \sigma} \left( \frac{\psi}{\mu} \right)^{\frac{1}{\sigma}}. \quad (96)
\]

The definitions (1), (2), and (48) imply

\[
M_s(b, s) = \frac{1 - \eta(\theta)}{s}. \quad (97)
\]

Equation (66) results from combining (67) with (95) to (97). Finally, combining (61) and (96), we obtain (69).

Proof of Proposition 6

By assumption, the elasticity \( \alpha \) of \( f \) is constant. Solving (54) and (66) we obtain

\[
n = \frac{\alpha(1 - \sigma)}{\alpha(1 - \sigma) + (2 - \gamma)\sigma}, \quad \text{and} \quad (98)
\]

\[
n^* = \frac{\alpha(1 - \sigma)}{\alpha(1 - \sigma) + \sigma}. \quad (99)
\]

Equations (70) to (72) follow from comparing (98) with (99), (55) with (65), and (52) with (67) given that \( \gamma \in [0.5, 1], \eta \in (0, 1), \) and (1), (47), (49) and (99). Combining (40), (10), (11) and (51), we obtain

\[
q_\varepsilon = \begin{cases} 
0 & \text{for } 0 \leq \varepsilon < 1 - \gamma \\
\frac{1 + \sigma}{\gamma m(\theta)} \left( \frac{\varepsilon - (1 - \gamma)}{\gamma} \right)^{\frac{1}{\sigma}} & \text{for } 1 - \gamma \leq \varepsilon \leq 1
\end{cases} \quad (100)
\]

Comparison of (100) with (69) using (70) to (72), (47), (49), and \( m(\theta) = \theta^\eta \) yields:

\[
q_\varepsilon < \frac{(1 + \eta)^{1 + \eta}}{2^{1 + \eta}} q_\varepsilon^*. \quad (101)
\]

Equation (73) follows from (101) because \( \frac{(1 + \eta)^{1 + \eta}}{2^{1 + \eta}} = 1 \) when \( \eta = 1 \), and

\[
\frac{d}{d\eta} \left( \frac{(1 + \eta)^{1 + \eta}}{2^{1 + \eta}} \right) = \left( \frac{(1 + \eta)^{1 + \eta}}{2^{1 + \eta}} \right) \ln \left( \frac{1 + \eta}{2\eta} \right) > 0 \text{ for } \eta \in (0, 1). \quad \blacksquare
Proof of Proposition 7

(a) When buyer types are public information the equilibrium and the first best allocations coincide.

As in Section 3, all bilateral trades must be pairwise efficient in equilibrium and there must be no gain from creating a new submarket. With full information, the efficient quantities \( q_\varepsilon \) are calculated by maximizing the trade surplus, \( \mathcal{U}^b + \mathcal{U}^s \):

\[
q_\varepsilon = \arg \max \left( \frac{\varepsilon \psi q_\varepsilon^{1-\sigma}}{1 - \sigma} - \mu q_\varepsilon \right) \tag{102}
\]

The solution to (102) is:

\[
q_\varepsilon = \left( \frac{\psi \varepsilon}{\mu} \right)^{\frac{1}{\sigma}}. \tag{103}
\]

The expected trade surplus is:

\[
\int_0^1 \left( \frac{\varepsilon \psi q_\varepsilon^{1-\sigma}}{1 - \sigma} - \mu q_\varepsilon \right) \, d\varepsilon = \frac{\sigma^2}{1 - \sigma^2} \psi \frac{1}{\sigma} \mu^{\frac{\sigma - 1}{\sigma}}. \tag{104}
\]

When the market tightness is \( \theta \) and buyers receive a fraction \( \xi \) of the trade surplus, the expected utilities of buyers and sellers are:

\[
\mathcal{V}^b(\theta, \xi) = \xi \frac{m(\theta)}{\theta} \frac{\sigma^2}{1 - \sigma^2} \psi \frac{1}{\sigma} \mu^{\frac{\sigma - 1}{\sigma}}, \text{ and} \tag{105}
\]

\[
\mathcal{V}^s(\theta, \xi) = (1 - \xi) m(\theta) \frac{\sigma^2}{1 - \sigma^2} \psi \frac{1}{\sigma} \mu^{\frac{\sigma - 1}{\sigma}}. \tag{106}
\]

In a competitive search equilibrium, the marginal rates of substitution between \( \theta \) and \( \xi \) of buyers and sellers must coincide. Differentiating (105) and (106), this implies:

\[
\xi = \eta(\theta). \tag{107}
\]

Households allocate \( b \) and \( s \) so \( \mathcal{V}^b(\theta, \xi) = \mathcal{V}^s(\theta, \xi) \). Using (105) and (106), this equality yields (65). Also, households allocate \( s \) and \( n \) so \( \mathcal{V}^s(\theta, \alpha) = \mu f'(n) \). Using (106) and (107) this equality implies:

\[
\mu f'(n) = m(\theta) (1 - \eta(\theta)) \frac{\sigma^2}{1 - \sigma^2} \psi \frac{1}{\sigma} \mu^{\frac{\sigma - 1}{\sigma}}. \tag{108}
\]

Using (103) to calculate the average sales, the resource constraint becomes

\[
f(n) = sm(\theta) \frac{\sigma}{1 + \sigma} \left( \frac{\psi}{\mu} \right)^{\frac{1}{\sigma}}. \tag{109}
\]

Combining (108) and (109), and using (1), (65), and (13), we obtain (66) and (67). From (108),

\[
\left( \frac{\psi}{\mu} \right)^{\frac{1}{\sigma}} = \frac{f(n)}{sm(\theta)} \frac{1 + \sigma}{\sigma}. \tag{110}
\]
which combined with (2) and (103) gives (69). This completes the proof of (a).

(b) When buyer types are private information but sellers charge a lump-sum fee to reveal their variety the competitive search equilibrium and the first best allocations coincide.

Let \( p \) be the fee a seller charges to reveal his variety. After the fee \( p \) has been paid and the variety carried by the seller has been revealed, the trading game between a buyer and a seller is identical to the one in Section 3. Hence, the payment schedule net of the fee \( p \) that implements pairwise efficient trading assignments has still the functional form (39) and the quantities purchased by the buyer are given by (40). The expected utilities of the traders in a market with congestion \( \theta \), price schedule \( \gamma \), and fee \( p \) are

\[
V^b(\theta, \gamma, p) = \frac{m(\theta)}{\theta} \left( \frac{\sigma^2}{1 - \sigma^2} \psi \frac{1}{\mu} \frac{\sigma - 1}{\sigma} \gamma^2 - \lambda p \right), \quad \text{and}
\]

\[
V^s(\theta, \gamma, p) = m(\theta) \left[ \frac{2\sigma^2}{1 - \sigma^2} \psi \frac{1}{\mu} \frac{\sigma - 1}{\sigma} \gamma (1 - \gamma) + \lambda p \right].
\]

In a competitive search equilibrium, the marginal rates of substitution of \( \theta \) for \( \gamma \), and of \( \gamma \) for \( p \) must be equal for buyers and sellers, so

\[
\frac{\eta(\theta) - 1}{\eta(\theta)} = \frac{\gamma}{1 - 2\gamma} \left( \frac{2\sigma^2}{1 - \sigma^2} \psi \frac{1}{\mu} \frac{\sigma - 1}{\sigma} \gamma (1 - \gamma) + \lambda p \right); \quad \gamma = 1.
\]

Substituting (114) in (113) and solving for \( p \),

\[
p = \frac{1}{1 - \psi} \frac{\sigma^2}{\mu} \frac{1}{\mu} \frac{1}{\mu} (1 - \eta(\theta)).
\]

Note that this fee implies \( V^b(\theta, \gamma, p) > 0 \), so buyers are willing to pay the fee to know the seller’s variety.

Households allocate \( b \) and \( s \) so \( V^b(\theta, \gamma, p) = V^s(\theta, \gamma, p) \). Using (114) and (115), this equality yields (65). Also, households allocate \( s \) and \( n \) so \( V^s(\theta, \alpha) = \mu f'(n) \). Using (114) and (115), this equality yields (108). To show that (66), (67) and (69) hold, we use the exact same steps used in the proof of (a).

\[
\text{Proof of Proposition 8}
\]

A type-\( \varepsilon \) buyer chooses

\[
q_\varepsilon = \arg \max \left\{ \frac{\varepsilon q^{1 - \sigma}}{1 - \sigma} - \lambda Z(q) \right\} = \left( \frac{\psi \varepsilon}{\mu \eta} \right)^{\frac{1}{\sigma}}.
\]
Hence, the return of a seller when matched with a type-ε buyer is
\[ \lambda T Z(q_\varepsilon) - \mu q_\varepsilon = (T\eta - 1)\mu q_\varepsilon = \frac{\sigma(1 - \eta)}{1 - \sigma} \left( \frac{\psi_\varepsilon}{\mu \eta} \right)^{1 \over \tilde{\tau}} \mu, \] (117)
where \( \eta = \eta(\theta) \) and \( \theta \) is the first-best level of congestion given by (65). By (65), \( \eta \leq 1 \). Thus, the seller’s individual rationality constraint is satisfied.

When the market tightness is \( \theta \), the expected utilities of the traders are:
\[ V^b(\theta) = \frac{m(\theta)}{\theta} \eta \frac{\sigma^2}{1 - \sigma^2} \left( \frac{\psi}{\mu \eta} \right)^{1 \over \tilde{\tau}} \mu; \] (118)
\[ V^s(\theta) = m(\theta) (1 - \eta) \frac{\sigma^2}{1 - \sigma^2} \left( \frac{\psi}{\mu \eta} \right)^{1 \over \tilde{\tau}} \mu. \] (119)

The household chooses its labor allocation so that \( V^b(\theta) = V^s(\theta) = \mu f'(n) \). The first equality implies that \( \theta \) satisfies (65) so \( \eta = \eta(\theta) \). The second one implies
\[ f'(n) = m(\theta) (1 - \eta) \frac{\sigma^2}{1 - \sigma^2} \left( \frac{\psi}{\mu \eta} \right)^{1 \over \tilde{\tau}}. \] (120)

Using (116) to calculate the average sales, the resource constraint becomes
\[ f(n) = sm(\theta) \frac{\sigma}{1 + \sigma} \left( \frac{\psi}{\mu \eta} \right)^{1 \over \tilde{\tau}}. \] (121)

Combining (120) and (121), and using (1), (65), and (13), we obtain (66) and (67). From (121),
\[ \left( \frac{\psi}{\mu} \right)^{1 \over \tilde{\tau}} = \frac{f(n)}{sm(\theta)} \frac{1 + \sigma}{\sigma} \eta^{1 \over \tilde{\tau}}, \] (122)
which combined with (2) and (116) gives (69).

Proof of Proposition 9
Efficient price schedules must be pairwise incentive efficient, so we can restrict our search to those in (39) for some unknown parameter \( \gamma \). Substituting (40) into (9), solving for the integral, and using (2) to (4) we obtain:
\[ c = \left\{ \frac{sm(\theta)\sigma \gamma}{1 + \sigma} [1 + \sigma (1 - \gamma)] \right\}^{1 \over 1 - \sigma} \left( \frac{\psi}{\mu} \right)^{1 \over \tilde{\tau}}. \] (123)

Denoting \( f(n) = wn \), and using (10), (11), (49), (51), (53), and (54), equation (123) simplifies to:
\[ c = \left( \frac{2\sigma^2}{1 + \sigma} \right)^{1 \over 1 - \sigma} (1 - \sigma) w \gamma (1 - \sigma) m(\theta) \right\}^{1 \over 1 - \sigma}. \] (124)
Since (49) holds, the maximization of \( c \) implies:

\[
(1 - 2\gamma)m(\theta) + \gamma(1 - \gamma)m'(\theta) \frac{1}{2(1 - \gamma)^2} = 0. \tag{125}
\]

Given (48) and (49), (125) implies (47). \( \blacksquare \)

**Proof of Proposition 10**

With a sales subsidy \( T \), the utility of the seller is \( \mathcal{U}^s(q, z) = T\lambda z - \mu q \). Repeating the arguments in Proposition 3, we obtain that an efficient price schedule has the following form:

\[
\mathcal{Z}(q) = \frac{1}{\lambda} \left[ \gamma q + (1 - T\gamma) \frac{\psi q^{1-\sigma}}{1 - \sigma} \right]. \tag{126}
\]

Also, the optimal quantities are

\[
q_\varepsilon = \begin{cases} 
0 & \text{for } 0 \leq \varepsilon < 1 - T\gamma \\
\left( \frac{\psi \varepsilon (1 - T\gamma)}{\mu} \right) \frac{1}{\sigma} & \text{for } 1 - T\gamma \leq \varepsilon \leq 1 
\end{cases} \tag{127}
\]

Given (126) and (127), the expected utilities of the traders are

\[
\mathcal{V}^b(\theta, \gamma) = \frac{m(\theta) (\gamma\sigma)^2}{\theta (1 - \sigma^2 \psi \frac{1}{\sigma} \mu \frac{\sigma - 1}{\sigma} T) \frac{1 + \sigma}{1 - \sigma}}, \text{ and} \tag{128}
\]

\[
\mathcal{V}^s(\theta, \gamma) = \frac{m(\theta) 2\sigma^2 \gamma (1 - T\gamma) \psi \frac{1}{\sigma} \mu \frac{\sigma - 1}{\sigma} T \frac{1 + \sigma}{1 - \sigma}}{1 - \sigma^2}. \tag{129}
\]

Equating the marginal rates of substitution of \( \theta \) for \( \gamma \) of buyers and sellers gives

\[
\gamma = \frac{1 + \eta}{2T}. \tag{130}
\]

The household chooses its labor allocation so that \( \mathcal{V}^b(\theta) = \mathcal{V}^s(\theta) = \mu f'(n) \). This condition implies

\[
\theta = \frac{\gamma}{2(1 - T\gamma)}, \text{ and} \tag{131}
\]

\[
\mu f'(n) = \frac{m(\theta) \sigma^2}{\theta (1 - \sigma^2 \psi \frac{1}{\sigma} \mu \frac{\sigma - 1}{\sigma} T \frac{1 + \sigma}{1 - \sigma})} \psi \frac{1}{\sigma} \mu \frac{\sigma - 1}{\sigma} T \frac{1 + \sigma}{1 - \sigma} \gamma^2. \tag{132}
\]

Substituting (127) in the resource constraint, we obtain

\[
f(n) = sm(\theta) \frac{\sigma}{1 + \sigma} \gamma \left( \frac{\psi}{\mu} \right) \frac{1 + \sigma}{\psi} T \frac{1 + \sigma}{1 - \sigma}, \tag{133}
\]

which, combined with (132), yields (52). Also, when \( f(n) = wn \), (1), (13) and (52) imply

\[
n = \frac{(1 - \sigma) \theta}{(1 - \sigma + \gamma \sigma) \theta + \gamma \sigma}. \tag{134}
\]
Substituting (127) into (9), solving for the integral, and combining with (1), (4), (131), (133) and (52), we obtain

$$c^{1-\sigma} = w^{1-\sigma} \left( \frac{2\sigma^2}{1 - \sigma^2} \right)^\sigma [m(\theta)T\gamma(1 - T\gamma)]^\sigma [1 + \sigma(1 - T\gamma)]^n. \quad (135)$$

When $m(\theta) = \theta^\eta$, using (130), (131), and (134), this expression simplifies to

$$c^{1-\sigma} = \frac{w^{1-\sigma}}{2^{1+\sigma}} \left( \frac{\sigma^2(1 - \eta^2)}{1 - \sigma^2} \right)^\sigma \frac{(2 + \sigma - \sigma\eta)(1 - \sigma)\theta^{\sigma\eta}}{1 - \sigma + \sigma(1 + \theta)(1 - \eta)}. \quad (136)$$

where $\theta = (1 + \eta) / [2T(1 - \eta)]$. The government chooses $T$ to maximize (136). The optimal levels of $\theta$ and $T$ are given by (65) and (76), respectively. From (130), $\gamma = \eta$, so the price schedule is (77). From (134), $n = 1 - \sigma$, and so (53) implies $s = (1 - \eta)\sigma$, which yields the first best allocation of labor. Finally, (127) combined with (133) gives (78). □
REFERENCES

References


[22] Shi, Shouyong, “A Wage-Posting Model of Inequality and Heterogeneous Skills and Skill-Biased Technology,” 2001b, Indiana University manuscript.


Table 1

NUMERICAL EXAMPLE

Production function: \( F(k_t, e_t n_t) = A_1 k_t^\alpha (e_t n_t)^{1-\alpha}, \alpha = 0.36 \)

Depreciation rate: 0.1

Rate of interest: \( r = 0.04 \)

Rate of growth of \( e_t \): \( g = 0.018 \)

Discount factor: \( \beta = (1 + g)/(1 + r) \)

One period utility: \( \ln(c_t) \)

Preference for diversity: \( \sigma = 0.42 \)

Matching technology: \( M(b_t, s_t) = A_0 b_t^\eta s_t^{1-\eta}, \eta = 0.46 \).

<table>
<thead>
<tr>
<th>Competitive Search Equilibrium</th>
<th>First Best Allocation</th>
</tr>
</thead>
</table>

Balanced Growth Path

- Producers (\( n \)): 0.500 0.559
- Sellers (\( s \)): 0.213 0.237
- Buyers (\( b \)): 0.287 0.203
- Capital/Labor Ratio (\( k/n \)): 4.374 4.374
- Marginal Variety Purchased (\( \tilde{\varepsilon} \)): 0.270 0
- Quantity Purchased When \( \varepsilon \geq \tilde{\varepsilon} \) (\( q_\varepsilon \))
  \( 11.23(1.37\varepsilon - 0.37)^{2.38} \quad 10.13\varepsilon^{2.38} \)

Welfare Relative to the First Best

(Equivalent Percentage Change in Consumption)

- Comparison Across Balanced Paths: -0.0794 0
- Comparison With Same Initial Capital: -0.0787 0