Rivalry, Exclusion and Coalitions

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Abstract

We analyze a situation where individuals and coalitions can obtain effective property rights over a resource by means of an exclusion contest. Players face a trade-off when they decide to incorporate new members: Big coalitions control the resource more likely but their members have more diluted property rights. It turns out that the grand coalition is stable if players are committed to minimize deviators’ payoffs but not necessarily if players play best responses. In such case, there is a strong tendency towards two-sided conflicts, the symmetry related with the effectiveness of conflict investments. Moreover, under non-cooperative exploitation of the resource, conflict may be socially efficient.

1 Introduction

Suppose a society to fall into such want of all common necessaries that the utmost frugality and industry cannot preserve the greater number from perishing, and the whole from extreme misery: It will readily be admitted that the strict laws of justice are suspended in such pressing emergence, and give place to the motives of necessity and self-preservation...”

David Hume (1751), An Enquiry Concerning the Principles of Morals.
1.1 Motivation and overview

Individuals often face situations in which they perceive the presence of others as potentially dangerous or harmful: The exploitation of natural resources or markets are economic examples where the interaction with increasing numbers provokes a decline in individual payments. In these settings, we say that individuals are rivals in nature.

Suppose now that the members of a given population can predict in advance the "tragic" result of their economic interaction. Will not they be tempted to invest part of their time or effort in avoiding such ending by non-economic means? It seems natural to conclude that they will divert part of their productive endowments into appropriative activities, aiming to reduce the number of individuals they will have to deal with finally. In fact, throughout human history, fights for the control over or access to resources have been a main root of conflict among individuals and states.

Rivalry leads to competition, but competition may lead to cooperation: Even if individuals are rivals in nature, it is not obvious that they will remain in the state of "the war of all against all". Sooner or later they realize that by joining with others individuals and agreeing on a peaceful arrangement, groups may face external hostilities in a much better position. A natural question is: Does this clustering process eventually lead to an universal agreement or to social fragmentation?

We have real world examples of this phenomenon For example, in 1998 the Project on Environmental Scarcities, State Capacity and Civil Violence of the University of Toronto concluded that resource scarcity has triggered predatory behavior by elite groups in Indonesia, China and India (among other countries). These groups try to influence on the government in order to change property rights and obtain monopolistic access to the resources. The immediate consequence is a defensive reaction by weaker groups that also depend on them.

This paper investigates the formation of groups or coalitions when individuals may undertake conflictive activities in order to exclude others from a resource. We explore a general equilibrium model where, once a coalition structure has formed, members of coalitions allocate their endowments into conflict effort (effort henceforth) and productive activities (labor henceforth). Agents are budget constrained by their initial endowments. The profile of effort determines both the probability of success of each coalition in the contest and the productive possibilities in case of victory. Moreover, we incorporate rivalry into production (see below), so the more the players one joins with the more likely is to obtain control but the potentially lower is the share of the final output. Therefore, coalitions face a trade-off when they decide to incorporate or not a new member.

\(^1\)See also Homer-Dixon (1994).
The coalition formation game induces externalities in non-members: When two individuals merge they agree on not to fight each other. Consequently, their exclusion effort changes. This affects the winning probabilities across coalitions, and payoffs therefore. We are interested on what coalition structures arise endogenously in this game.

We first analyze the joint production case where the members of the winning coalition agree on sharing the total final output equally. This model captures the essential points of coalitional interaction in an exclusion contest. We show that in the unique interior equilibrium of the game, low-elasticity production technologies lead to higher total levels of conflict; and that sufficiently effective conflict technologies make coarser coalition structures induce higher levels of overall conflict. Under the assumption that players are committed to inflict as much harm as possible to possible deviators the grand coalition turns out to be stable. Moreover, the static concepts of $\alpha$ and $\beta$ core coincide. When only best responses are employed some partial results are provided, showing that the game has a strong tendency towards two-sided conflicts that are more or less symmetric depending upon the effectivity of effort.

Next we deal with the case of individual production: That is, once a coalition obtains control over the resource, its members exploit it non-cooperatively. In particular, we address one of the most clear examples of economic rivalry: The ’tragedy of the commons’. Recall that this phenomenon occurs when, due to negative externalities, a resource of common use becomes overexploited. For instance, when herdsmen put the individually optimal amount of cattle in the pasture they do not take into account that this decreases the available pasture for other herdsmen’s cattle. Inefficiency increases as the number of individuals who exploit the resource grows.

A possible reply is, if the good is of common property, by definition, it cannot be owned by a few! Grossman (2000) points out the evident difference between effective and formal property rights: To say that an agent has an effective property right over an object means that this agent controls its allocation and distribution. On their side, formal property rights are those stated by legal ownership. The latter may not confer control rights by themselves. In fact, sufficiently strong control rights are the main step for the recognition of formal ones if they are previously undefined. For example, in the 1960s, oil and gas were found under the North Sea. Several countries contested for the exploitation rights. Although the United Nations’ Law of the Sea claims that resources in the seabed are ”the common heritage of all mankind”, Britain and Norway finally obtained such rights because they were able to impose the ”smallest distance to the coast” criterion. This was also the case of the English Enclosure during the 18th century when land, traditionally of common property, was privatized through the political
initiative of the upper classes.

Normative approaches to the problem of the commons may be then too naive if appropriative activities (that lead to the creation of effective property rights) are an alternative for individuals or groups. Under this possibility it turns out that the formation of the grand coalition is much more difficult than in the case above because the very nature of the common goods problem makes rivalry stronger. In fact, conflict can act sometimes as a discipline device that deters players from devoting too much labor in the exploitation of the resource: This perverse effect provokes that coalitions may prefer to expel members and "suffer" higher hostilities in order to avoid overexploitation of the resource; the coincidence of the $\alpha$ and $\beta$ core holds only for sufficiently small populations and not very effective conflict technologies.

The paper is structured as follows: In the subsection below the present work is related with the literature. In Section 2 we give some basic notation and assumptions. In Section 3 we deal with the joint production case for a given coalition structure. Section 4 addresses coalition formation for this case. In Section 5 we carry out the same analysis for the common pool case. In Section 6 we provide some lines for further research. All proofs are in the Appendix.

1.2 Relationship with the literature

This paper is related with three different strands of the economic literature: Economic models of conflict, coalition formation games with externalities and common property resources.

Economic models of conflict date back to Bush and Meyer (1974) and have received important contributions by Skaperdas (1992), Hirshleifer (1995) and Neary (1996). These models are closely related to rent-seeking models. In fact, both belong to the more general class of models of rivalry\(^2\). The basic idea behind is that if individuals can undertake non-economic or appropriative activities in order to achieve their goals, the allocations resulting from economic interactions may not be exclusively those derived from productivity but also from relative performance in a conflict stage: Agents first decide how much they spend in productive activities; labor is transformed into output. It is redistributed by force in the second stage where players decide how much they devote to appropriation. The probability of winning in the conflict stage can be interpreted thus as the proportion of the total output allocated to each agent.

This canonical model however have been criticized because it can be interpreted only as a theory of the right of access to common property, but it fails to account for the creation of private property rights\(^3\). Grossman

\(^2\)We refer the reader to Neary (1997) for a nice exposition of these issues.

and Kim (1995) and Muthoo (2000) deal with the enforcement of the right to enjoy one’s fruits of his own labor. The contest is then over individual productions not over some aggregate.

All these models ignore the issue of coalition formation. The reason behind this fact is that conflict models have focused on struggles over objects rather than over rights\(^4\): Whereas the former case would be appropriate for pre-modern conflicts, victory in present-day contests (as the empirical evidence cited above shows) implies the control (or effective property right) over some contested object. Under this simple modification the role of coalitions is clear and may be considered\(^5\). Then, the main trade-off is no longer between productive and unproductive activities but between a higher chance of success and shared property rights. Skaperdas (1996), Tan and Wang (2000) and Esteban and Sakovics (2000) have dealt with these issues. However, rather than taking the general equilibrium approach in the fashion of conflict models they assume exogenous prizes like the rent-seeking models\(^6\).

The issue of coalition formation in common-pool resources in the absence of conflict for control has been explored by Funaki and Yamato (1999) and Meinhardt (1999). If players are able to communicate, they can form groups in order to exploit the common. This partially solves the externality problem because members internalize the negative effects on other members. Although the grand coalition is the most desirable outcome, the presence of external effects may prevent its formation: Funaki and Yamato (1999), in a partition function approach, show that the core of their game is non-empty if players have the most pessimistic expectations but not if they have the most optimistic ones. Meinhardt (1999) addresses the issue by means of the \(\alpha\) and \(\beta\) characteristic functions\(^7\): They turn out to coincide and define therefore a game in characteristic form that is convex and whose core coincides with the Von Neumann-Morgenstern’s stable set.

Our model would be therefore complementary to these two because we all address, from a positive point of view, situations in which the tragedy of the commons may not be an irreversible outcome.

Third, the present work adds up to the existing literature on coalition formation games with externalities, surveyed in Yi (1999), that depart from

\(^4\)To the best of our knowledge, the only exception is Skaperdas and Syropoulos (1998), where two agents fight for the right to access to some fixed factor that they can use in production in case of victory.

\(^5\)On the models of enforcement of private property rights, the simple existence of more than two players may lead to inconsistencies: If an agent challenges \(k > 1\) outsiders he may loose \(k\) times his individual production!

\(^6\)In fact, these models mainly focus on coalition formation under continuing conflict: once a coalition has won the contest its members engage in conflict once more until only one player is left. On the contrary, ours is one-shot.

\(^7\)These concepts were introduced by Aumann (1959). We will explain them carefully later on.
traditional characteristic form games in the fact that coalitional payoffs depend on outsiders actions. With the exception of Tan and Wang (2000) appropriative activities have been ignored as generators of externalities. We also try to provide foundations to the different coalition structures.

Finally, our model is somehow related with some works in the field of sociobiology as an instance of the competitive exclusion principle\(^8\) that states that two species cannot coexist indefinitely under a limited amount of resource. Anyway, we assume implicitly this principle rather than proving it.

## 2 The model

Consider a set \( N = \{1, 2, ..., n\} \) of identical players that try to gain control over a resource by excluding the rest of individuals. Each of them possesses one unit of initial endowment that can be transformed into effort in the exclusion contest or in labor (the labor choice has effect only if the agent finally obtains the control of the resource). We denote these investments by \( r_i \) and \( l_i \) respectively, subject to the constraint \( r_i + l_i \leq 1 \).

Players may form coalitions in order to get the control of the resource. Let \( \mathcal{N} \) denote the set of all nonempty coalitions. For each coalition \( S \in \mathcal{N} \), let us denote its cardinality by \( s \). A coalition structure \( \pi \) is a collection of disjoint coalitions \( \{S_k\}_{k \in K} \) whose union is \( N \) (i.e., a partition of \( N \)). The structure \( \pi \) is said to be a coarsening of \( \pi' \) if \( \pi \) can be obtained from \( \pi' \) by merging coalitions in \( \pi' \).

We will address the coalition formation process from two perspectives. First, by means of a static approach that simplifies the existence of externalities: Outsiders are assumed to follow a fixed behavior. This approach defines an orthogonal cooperative game where the coalitional worth is independent of others’ actions. In the second one coalitions form sequentially and we impose coalitions to play only best responses. The game that arises is in partition function, that is, the description of the payoff incorporates the coalition structure the coalition belongs to.

Once a coalition structure \( \pi \) has formed, individuals engage in the exclusion contest. Let us denote by \( r(\pi) = (r^{S_1}, r^{S_2}, ..., r^{S_K}) \), where \( r^{S_k} = \sum_{i \in S_k} r_i \), the vector of coalitional efforts (we will denote individuals by subscripts and coalitions by superscripts). The result of the contest among coalitions is driven by the conflict technology that maps \( r(\pi) \) to a vector \( p = \{p^{S_k}\}_{k \in K} \) of coalitional winning probabilities (with probability \( p^S \) the coalition \( S \) attains the control of the resource and so on).

We employ the well-known ratio form for the conflict technology, that characterizes the coalitional winning probability as proportional to the coalitions.
tional outlay, \((r^S)^m\), where \(r^S\) is the sum of individual efforts of its members, and the exponent \(m \geq 0\) represents the returns to scale or effectivity of conflict effort. Hence, coalition \(S\) gains the access to the resource with probability

\[
p^S(r) = \frac{(r^S)^m}{(r^S)^m + r^{-S}}
\]

where \(r^{-S} = \sum_{S_k \in \pi \setminus \{S\}} (r^{S_k})^m\), the sum of all coalitional outlays across \(\pi\).

Notice that the particular \(\pi\) we are considering makes a difference: The total of coalitional efforts may be the same for two different coalition structures but, for \(m \neq 1\), they lead to different levels of total coalitional outlays. The limit case \(m = 0\) is an equal chance lottery among the elements of \(\pi\); the case \(m = \infty\) is a first-price auction\(^9\) where the coalition with the highest aggregate effort wins the contest with probability 1.

Once a coalition attains control of the resource, members exploit it. Production is carried through the production function \(f(L)\), where \(L = \sum_{i \in S} l_i\), satisfying \(f(0) = 0\). This technology is continuous and concave in labor and satisfies that \(f'(0) > c_l\), where \(c_l\) is the unit cost of labor. This formulation can be found, among others, in Cornes and Sandler (1983) and Meinhardt (1999). We adopt it in order to avoid the labor-leisure trade-off, as for instance in Roemer (1996), and to focus just on the trade-off between exclusionary and productive activities.

The elasticity of production with respect to labor

\[
\varepsilon = \frac{f'(L)L}{f(L)},
\]

is a useful proxy for scarcity; by concavity \(\varepsilon \leq 1\). Finally, technology \(f\) is said to dominate technology \(g\) if and only if \(\varepsilon_f > \varepsilon_g\) for any \(L\).

We will consider two possibilities of exploitation. Both coincide in that effort is privately determined but differ in the way in which members of the winning coalition exploit the resource: In the joint production case players are pre-committed to share the final output equally. Therefore, the solution concept used is just Nash equilibrium, there is no actual decision at the production stage. In the individual production case there is a real decision in that stage: Players exploit the resource non-cooperatively and their share over the final total output is determined by their own labor decision. Therefore, in this case the solution concept employed is the Subgame Perfect Equilibrium.\(^9\)

\(^9\)The value of the parameter \(m\) can be related with the tolerance of a society or politicians towards influence activities.
3 Joint exploitation

Then, let us describe first the game individuals play given a coalition structure: Individuals decide simultaneously and non-cooperatively how much effort they devote to the success of their coalition. Hence, the strategy for a player \(i \in S\) is a level of effort \(0 \leq r_i \leq 1\). This choice yields in aggregate a co-\(r^S = \sum_{i \in S} r_i\) and coalition \(S\) attains control with probability \(p^S\). Output is shared equally. If players form the grand coalition, all of them accede to the resource without contest.

The winning coalition accedes to the resource and its members exploit it by using their remaining endowments. The members of \(S\) decide jointly the amount of labor \(0 \leq l_i \leq 1\) in order to maximize payoff per-capita. Notice that due to joint production, if \(c_l > 0\) coalitions never enter in the negative returns zone so formed. Then, without loss of generality, throughout all this Section we will assume the this cost is zero.

Then, decisions for a given player \(i \in S\) come from the maximization of the expected individual payoff, that is,

\[
u_i^S(r_i, r_{-i}) = \frac{(r^S)^m}{(r^S)^m + r^S}\left[\frac{1}{s}f(t^S)\right], \tag{2}\]

subject to \(r_i + l_i \leq 1\). Given that the function is concave there exist certain rivalry in production as coalitions become large. Inspection of (2) reveals that the \(p^S\) is like a privately provided public good: Its level is determined according to voluntary contributions but it is enjoyed by all the members. This feature will generate strong incentives to free-ride in the individual production case.

Notice that the entire endowment is consumed in both activities in any optimal decision: Suppose that at the beginning of stage 2 a member of the winning coalition that has invested \(r\) in the previous stage finds that his optimal level of labor is \(\hat{l} < 1 - r\). Then, the choice in the first stage is not optimal because given that winning probability is increasing in own effort he could have enjoyed a higher expected payoff by investing \(1 - \hat{l}\) instead of \(r\). Given this fact \(t^S = s - r^S\) and one can rewrite (2) as

\[
u_i^S(r_i, r_{-i}) = \frac{(r^S)^m}{(r^S)^m + r^S}\left[\frac{1}{s}f(s - r^S)\right]. \tag{3}\]

Let us now define the best reply of an agent: Denote by \(r(\pi)\setminus r_i\) the strategy profile under the unilateral deviation of player \(i\) from the strategy profile \(r(\pi)\).

Definition 1 (Best reply for an individual) Given a coalition structure \(\pi\), the set of individual best replies, denoted by \(B_i^S(r_{-i})\), of agent \(i \in S\)
to the strategy profile \( r_{-i} = \{ r_j \}_{i \neq j} \), chosen by the rest of members of \( S \) (if any) and the outsiders is

\[
B^S_i(r_{-i}) = \{ r_i \in [0, 1] \mid u^S_i(r(\pi)) \geq u^S_i(r(\pi) \setminus r_i) \}.
\]

In the Nash Equilibrium of this game all players are playing their best response \( r_i(r_{-i}) \) to the strategy profile \( r_{-i} = (r^{S \setminus i}, r^{-S}) \), where \( r^{S \setminus i} \) is the vector of strategies played by members of \( S \) different from \( i \). More formally:

**Definition 2 ((Nash Equilibrium at individual level))** A profile of effort choices \( (r_1, \ldots, r_n) \) is a Nash Equilibrium at individual level of the Joint production case under the coalition structure \( \pi \) if \( u^S_i(r(\pi)) \geq u^S_i(r(\pi) \setminus r_i) \) \( \forall i \in N \).

By the same token, given a coalition structure \( \pi \), one can define a non-cooperative game at coalitional level where coalitions act as individuals: They maximize

\[
u^S(r^S, r^{-S}) = \frac{(r^S)^m}{(r^S)^m + r^{-S}[f(s - r^S)]}, \tag{4}\]

subject to \( r^S \leq s \) and take \( r^{-S} \) as given.

Therefore, the definitions above can be easily extended to coalitions:

**Definition 3 ((Best reply for a coalition))** Given a coalition structure \( \pi \), the set of coalitional best replies, denoted by \( B^S(r^{-S}) \), of coalition \( S \in \pi \) to the strategy profile \( r^{-S} = \{ r^S_k \}_{S_k \neq S} \), chosen by the rest of coalitions belonging to \( \pi \) is

\[
B^S_S(r^{-S}) = \{ r^S \in [0, s] \mid r^S = \arg\max u^S(r^S, r^{-S}) \}.
\]

**Definition 4 ((Nash Equilibrium at coalitional level))** A profile of effort choices \( (r^S_1, r^S_2, \ldots, r^S_K) \) is a Nash Equilibrium at coalitional level of the Joint production case under the coalition structure \( \pi \) if \( r^S_k \in B^S_k(r^{-S}) \) \( \forall S_k \in \pi \).

One important result is that, whereas there are multiple Nash equilibria at individual level, all of them yield the same profile of equilibrium coalitional efforts and moreover this profile constitutes the unique Nash equilibrium of the non-cooperative game at the coalitional level.

**Proposition 5** Under any coalition structure \( \pi = \{ S_k \}_{k \in K} \)
(i) The joint production game at coalitional level has a unique interior Nash Equilibrium characterized by the following system of equations

\[
\frac{m(rS_k)^{m-1}r-S_k}{(rS_k)^m + r-S_k} \left[ f(s_k - rS_k) \right] = \frac{(rS_k)^m}{(rS_k)^m + r-S_k} \left[ f'(s_k - rS_k) \right], \quad \forall k = 1, ..., K
\]

(ii) The non-cooperative game at individual level has multiple equilibria. However all of them yield the profile of coalitional efforts that satisfies the system of equations defined by (5)

Our target now is to investigate the effect of different productive and conflict technologies, parametrized by \( \varepsilon \) and \( m \) respectively on the agents’ optimal and equilibrium choices.

**Proposition 6** The best response effort has the following properties:

(i) \( \frac{\partial e^g}{\partial r} > 0 \) and \( \frac{\partial e^g}{\partial r} < 0 \).

(ii) \( \frac{\partial e^\varepsilon}{\partial r} \leq 0 \).

and the equilibrium effort

(i') is higher under \( g \) than under \( f \) provided that \( f \) dominates \( g \).

(ii') is increasing in \( m \) if

\[
\sum_{k \in \pi \setminus S} (rS_k)^m \left( \ln \frac{rS_k}{r^S} \right) \geq 0.
\]

Part (i) of the proposition implies that exclusion contests induce exclusion races indeed: An increment in effort by outsiders makes more likely the exclusion of a player and less costly therefore the allocation of additional units to conflict activities. Part (ii) shows that members’ efforts are strategic substitutes.

Third, conflict is linked with scarcity: In a world of constant returns to labor conflict makes less sense. But, when the productivity of labor is low (low \( \alpha \) for (F2), for instance) exclusion efforts may be advantageous.

Unfortunately, part (ii') only allow us to extract partial conclusions: Results depend on both the relative position of coalitions and the previous coalitional efforts. Symmetric coalition structures (formed by coalitions of the same size) induce higher conflict expenditures in equilibrium when the conflict technology improves. This conclusion may extend to other "balanced" coalition structures where no coalition is in a situation of special advantage due to size.

For the rest of the Section we will consider two families of production functions:
(F1) $f(L) = aL - bL^2$ that can be parametrized through $\theta = \frac{a}{b}$ as measure of linearity.

(F2) $f(L) = L^\alpha$ where $\alpha \leq 1$ that satisfies constant elasticity of labor ($\varepsilon = \alpha$) and constant measure of concavity $(-\frac{(s-r^2)f''}{f'}) = 1 - \alpha$.

**Example 1:** Let us illustrate the results above with the following example. Take $N = \{a, b, c\}$ and, in order to obtain readable numbers, suppose that agents have 35 units of initial endowment. Let $f(l) = l^{0.4}$ and $m = 1$. The coalition structure we assume that has previously form is $\pi = \{\{a, b\}, \{c\}\}$.

In Figure 1 we plot the effect of different technologies, both of production and exclusion: In the vertical axis we project the best reply coalitional effort of $\{a, b\}$ and of $\{c\}$ in the horizontal one. The intersection of these reaction functions constitute the Nash equilibrium of the coalitional game.

In panel 1a, the dashed lines are for the case when $f(l) = l^{0.8}$. As stated above, players have less incentives to put effort given that returns to scale of labor increase and rivalry in production has been alleviated.

In panel 1b, the dashed lines correspond to the case when $m = 2$. As pointed out in Proposition 6, the effect of this change is ambiguous: For low values of outsiders’ effort, best replies lie below. However, the equilibrium takes place at higher investments in exclusion.

### 4 Coalition Formation under Joint production

#### 4.1 Static approach

The main difficulty of the non-orthogonal games of coalition formation, in contrast with standard characteristic form games, is that outsiders’ actions affect coalitional payoffs. Static approaches simplify this issue by assuming a
specific pattern of behavior for the rest of players that pins down a coalitional payoff.

For our purposes, it will be important to define an indirect payoff function. Denote by $r^S(r^{-S})$ the maximizer of expression (4).

**Definition 7** Let $r^S(r^{-S})$ be the maximizer of the payoff of coalition $S$. Then, we define the **indirect payoff function** as

$$u^*(r^{-S}) = u^S(r^S(r^{-S}), r^{-S}) = \max_{r^S} u^S(r^S, r^{-S})$$

(6)

Now we can define the $\alpha$ and $\beta$ characteristic functions:

**Definition 8** The $\alpha$-characteristic function, $v_\alpha$, in the exclusion game is defined by:

$$v_\alpha(S) = \max_{r^S} \min_{r^{-S}} u^S(r^S, r^{-S}) = \max_{r^S} u^S(r^S, \hat{r}^{-S}) = u^S(r^S(\hat{r}^{-S}), \hat{r}^{-S}) = u^*(\hat{r}^{-S})$$

(7)

where $\hat{r}^{-S}$ is the minimizer of the expected coalitional payoff. This expression corresponds to the indirect payoff function (6) when the outsiders have chosen the action (and therefore a partition) that minimizes the coalitional payoff: It is the minimum payoff that coalition $S$ can guarantee to itself.

The beta notion defines the payoff coalition cannot prevented from for any choice of outsiders:

**Definition 9** The $\beta$-characteristic function $v_\beta$ in the exclusion game is defined by:

$$v_\beta(S) = \min_{r^{-S}} \max_{r^S} u^S(r^S, r^{-S}) = \min_{r^{-S}} u^S(r^S(\hat{r}^{-S}), r^{-S}) = \min_{r^{-S}} u^*(r^{-S})$$

(8)

Notice that both characteristics forms will coincide if and only if $\hat{r}^{-S} = \min_{r^{-S}} u^*(r^{-S})$. Under joint production this holds.

**Proposition 10** For the joint exploitation case of the exclusion game the indirect payoff function is decreasing in $r^{-S}$. Therefore the $\alpha$ and $\beta$ characteristic functions coincide, i.e. $v_\alpha(S) = v_\beta(S)$.

Then, the minimizer of the coalitional payoff is the same regardless of if players react passively to outsider’s “best” punishment or if they wait for it after employing best responses. Hence we obtain the coincidence result also obtained for Common-Pool games (Meinhardt, 1999) and Cournot games (Zhao, 1999).

The next question is if, under the assumptions about outsiders’ behavior, there is room for cooperation, i.e. if the grand coalition can be blocked or not by some coalition $S \subset N$.\[12\]
**Definition 11** The $\alpha$-core ($\beta$-core) of the exclusion game is nonempty if there is no coalition $S \in \mathcal{N}$ such that $v_\alpha(S) > v_\alpha(N)$ ($v_\beta(S) > v_\beta(N)$).

Scarf (1971) showed that the $\alpha$-core of a NTU game is non-empty if the strategy space for each player is compact and convex and payoff functions are all continuous and quasiconcave. This conditions are satisfied by our game. Then, Proposition 10 implies the next result.

**Corollary 12** Under joint production of the exclusion game of coalition formation the $\alpha$-core and $\beta$-core are nonempty. Moreover, they coincide.

Given that the result is important we briefly give its proof: Simple inspection of (4) show us that the worst case scenario for $S$ when they are waiting for the choice of their rivals occurs when $r^{-S}$ attains its maximum. When $m \geq 1$ the coalition $N \setminus S$ must form and all its members must put their entire endowment; then $\hat{r}^{-S} = (n-s)^m$. When $m < 1$, the complement of $S$ must form singletons, put also $r_i = 1$ and then $\hat{r}^{-S} = (n-s)$. Let us suppose that we are in the former case (the other is analogous). Then, the alpha characteristic function is just the best response to $(n-s)^m$.

We know by Proposition 10 that the indirect characteristic function is decreasing in $r^{-S}$. This ensures that for any coalition $u^*(r^{-S})$ attains its minimum when $r^{-S} = (n-s)^m$. So finally we have:

$$v_\alpha(S) = u^S(r^S((n-s)^m), (n-s)^m) = u^*((n-s)^m) = \min_{r^{-S}} u^*(r^{-S}) = v_\beta(S)$$

Hence, if individuals are committed to inflict as much harm as possible to potential deviators an universal agreement prevails: The severe punishment suffered by deviating coalitions outweighs the potential gains derived from smaller numbers to share the resource with.

### 4.2 Sequential approach

The static approach to coalition formation allows non optimal reactions by outsiders; in our framework, agents in $N \setminus S$ may not be able to commit to total warfare in case of deviation because it is never optimal for them to invest the entire endowment in conflict. Hence, the complement of $S$ should be allow to use only best responses: Coalitions will therefore attach a payoff to each structure that may arise and predict the reply of outsiders to their movements.

There is no a unique approach to tackle this issue. Several procedures of coalition formation has been proposed. Here, we will follow Bloch (1996), where coalitions form if and only if all members agree to do it à la Rubinstein: The first player in a pre-determined protocol makes a proposal for a
coalition; the players in this proposed coalition decide sequentially to accept or not that proposal. The process stops when all members accept or one rejects. In the former case, the coalition finally forms; in the latter, the rejector must make another proposal. Bloch (1996) shows that this game yields the same stationary subgame perfect equilibrium coalition structure as the much simpler "Size Announcement game": First player proposes a coalition of size \( s_1 \) that immediately forms. Then the \((s_1 + 1)\)-th player in the protocol proposes a coalition \( s_2 \) and so on, until the player set is exhausted. The game is solved through backward induction and has generally a unique subgame perfect equilibrium.

However, we face a new difficulty: Contrary to the existing literature on coalition formation games with externalities, the payoff function cannot be characterized by the number of coalitions in \( \pi \). We can only provide partial results then. A first step to characterize the equilibrium of Bloch’s game in our context would be to explore when players will decide to add a new member to their coalition in spite of the "congestion" it may produce.

**Lemma 13** Fixed \( r^{-S} \), the individual indirect payoff function is non-decreasing in size if

\[
f'(s - r^S) \geq \frac{1}{s} f(s - r^S)
\]

Notice that for a linear technology this condition always holds, i.e. there is no congestion. Then the cost of absorbing enemies is zero and the grand coalition is always stable.

For \((F1)\) and \((F2)\) condition (9) reduces to

\[
(F1) \ t^S \leq \sqrt{s} r^S \text{ and } \\
(F2) \ t^S \leq s \alpha.
\]

Yet, as a coalition becomes bigger, \( r^{-S} \) does not remain constant: When two coalitions merge they absorb one rival but may fuel effort by outsiders if the new coalition is more aggressive than the sum of the two previous ones. Therefore, we should know how \( r^{-S} \) changes as \( \pi \) becomes coarser (or finer). Notice that this also defines the direction of externalities across coalitions: Games with positive externalities are those where mergers of coalitions produce positive effects on non-members; with negative externalities, the effect is the opposite.

Then, given that the indirect payoff function is decreasing in \( r^{-S} \), we can distinguish between the two cases:

**Definition 14** The exclusion game is of **positive (negative) externalities** if given two coalition structures \( \pi \) and \( \pi' \) it happens that \( r^{-S}(\pi') < (>) r^{-S}(\pi) \), where \( \pi' \setminus \{S\} \) can be obtained by merging coalitions in \( \pi \setminus \{S\} \).
We know that the coalitional outlay, \((r^S)^m\), is non-decreasing in size. However, this monotonicity is not sufficient for the superadditivity we need, that is, \(r^S + r^T \leq r^{S\cup T}\). We should look for its convexity or concavity in order to discern how the overall level of hostility changes when coalition structures become coarser.

**Proposition 15** The coalition formation game under joint production (i) is of negative externalities if \(m \geq 2\) (ii) and of positive externalities under the technologies \((F_1)\) and \((F2)\) if \(m \leq 1\).

This allows us to distinguish between situations where a merger leads or not to a higher level of overall conflict.

**Example 2:** Assume that \(N = 4\), the players have 35 units of initial endowment. Let \(f(l) = 20L - \frac{1}{5}L^2\). We allow \(m\) to be 1 or 2. Expected payoffs are displayed in the following tables.

\[
\begin{array}{ccccccc}
\pi & u_a(\pi) & u_b(\pi) & u_c(\pi) & u_d(\pi) & u_a(\pi) & u_b(\pi) & u_c(\pi) & u_d(\pi) \\
m = 1 & & & & & & & \\
a \cdot b \cdot c \cdot d & 83 & 83 & 83 & 83 & 60 & 60 & 60 & 60 \\
ab c \cdot d & 144 & 144 & 85 & 85 & 151 & 151 & 50 & 50 \\
abc \cdot d & 184 & 184 & 184 & 90 & 202 & 202 & 202 & 42 \\
abcd & 200 & 200 & 200 & 200 & 200 & 200 & 200 & 200 \\
m = 2 & & & & & & & \\

\end{array}
\]

It is easy to see what are the stable Bloch coalitions structures: For player \(a\) (the first in the protocol) it is dominant to announce \(\{N\}\) in the first case and to announce \(\{3\}\) when \(m = 2\).

Their impact on efficiency is different: Notice that when \(m = 1\) individual payoffs are aligned with efficiency (by cooperative production, \(\{N\}\) is always the most efficient structure). However, when \(m = 2\) the three-player coalition can weaken so much the remaining player that the individual payoff becomes the largest possible.

5 **Individual production under common property**

The tragedy of the commons develops in this way. Picture a pasture open to all [...] each herdsman will try to keep as many cattle as possible. Such an arrangement may work satisfactorily for centuries because tribal wars, and disease keep the numbers of man and beasts below the carrying capacity of the land. Finally, however, comes the day [...] when the long-desired goal of social stability becomes a reality. At this point, the inherent logic of the commons remorselessly generates tragedy.

Garrett Hardin (1968), *The tragedy of the Commons.*
A common good is an object that is owned by nobody or, equivalently, by everybody. This Section will be concerned with the creation of effective property rights over common goods (and therefore about exclusion from them) in the absence of contracts over their exploitation. So suppose now that our "aggressive" individuals depend on what they can extract from a common property resource: a fishery, a pasture... If they can predict the final result of their interaction but cannot communicate enough to internalize the social costs, Will the tragedy "remorselessly" occur?

The exclusion game in the presence of a common pool resource is essentially the same as in the joint production case. The main difference lies at the specification of the payoffs in the second stage. So let us first describe briefly the main features of the commons problem.

Each individual $i$ decides non-cooperatively how much labor he employs. The production function is strictly concave, depends only on the total labor input $l^S$ and it is assumed that $f'(0) > c_l$. Then, when an individual makes his optimal labor choice he takes as given the labor used by the rest of players, $l^S - l_i$. It is assumed that a unit of time is equally productive for all of them. Hence, the individual payoff is simply $\frac{1}{s}f(l^S) - c_l l_i$.

Notice first that the efficiency condition would require that $f'(l^S) = c_l$. However, in the unique symmetric Nash Equilibrium the total labor input when $s$ players are exploiting the resource satisfies

$$\frac{1}{s}f'(l^S) + \frac{s - 1}{s}f(l^S) = c_l. \quad (10)$$

This means that the total labor input is a weighted average between the efficiency level (achieved when only one agent entries) and the equalization to the average productivity, where the resource is overexploited. Moreover, the equilibrium payoff is decreasing in $s$ because inefficiency becomes more severe as $s$ grows; as $s \to \infty$ individual payoff approaches zero.

So, now, let us incorporate this feature to our joint production case. As mentioned above, the equilibrium concept for the individual production case, given a coalition structure, is the Subgame Perfect Equilibrium. However, the two stages may embodied in one expression for a given player. The expected payoff of player $i$ when he is a member of coalition $S$ given his optimal choices at each stage of the game is

$$u^S_i = \frac{(r^S)^m}{(r^S)^m + r - S} [l^S_i f'(l^S) - c_l l_i] \quad (11)$$

There are clear differences with respect to (??): Now we cannot define an equivalent game at coalitional level. Coalitional and individual payoffs

\[10\] See for instance Cornes and Sandler (1983) or Funaki and Yamato (1999).
are no longer aligned. In fact, exploitation by the members of the winning coalition may not be efficient: The payoff function in the production stage presents always a zone of negative returns (for sufficiently high levels of total labor input) and non-cooperative exploitation allows to enter in it. Rivalry in production becomes stronger because coalitions will try to "capture" the positive returns part. Consequently it is much more difficult for the grand coalition to be stable.

By the discussion above, we know that in any equilibrium of this game, players in the second stage will employ all their remaining endowments. So, one can rewrite expression (11) and obtain

\[ u^S_i(r^S, r^{-S}) = \frac{(r^S)^m}{(r^S)^m + r^{-S}(s - r^S)} - c_i(1 - r_i)]. \] (12)

Players therefore maximize (12) subject to \(0 \leq r_i \leq 1\). Notice also that now there is a reason for free riding on effort: The share of final production is decreasing in \(r_i\).

Proposition 16 Given a coalition structure \(\pi\), the symmetric, that is, \(r_i = r_j \forall i, j \in S\) and \(\forall S \in \pi\) constitutes a Subgame Perfect Equilibrium of the individual production case.

The symmetric equilibrium is characterized by the following system of equations \(\forall k = 1, ..., K\):

\[
\frac{m(s_k r_k)^{m-1} - s_k}{[(s_k r_k)^m + r^{-S_k}]^2 [f(s_k - s_k r_k) - c_k(s_k - s_k r_k)]} \frac{1 - r_i}{s - r^S} \left[ f(s - r^S) - c_i(1 - r_i) \right].
\] (13)

Furthermore, the level of effort is bounded from below for all coalitions: When \(s\) symmetric players exploit a common pool resource separately and without contest, the total labor input, denoted by \(l^s(s)\), satisfies (10). Then, the optimal labor choice \(l^*\) in the second stage, provided that a coalition \(S\) has won the contest and their members spent \(r^*\) in exclusion, is simply:

\[
l^* = \begin{cases} 
  l^*(s)/s & \text{if } 1 - r^* \geq l^*(s)/s \\
  1 - r^* & \text{if } 1 - r^* < l^*(s)/s
\end{cases}
\] (14)

Then, it can be easily shown that, as a consequence of both symmetry and the playing at Stage 2, we can delete an interval of strictly dominated strategies\(^{11}\):

\(^{11}\)This is also true for the benchmark case under the type \((F1)\) of functional forms.
Claim 17 No agent belonging to a coalition of size $s$ will put less conflict effort than $1 - \frac{l^*(s)}{s}$ nor 1.

On the contrary, suppose that the at $1 - \frac{l^*(s)}{s}$ this agent decides to withdraw a $\epsilon$ on labor. The remaining labor input is now $\frac{l^*(s)}{s} + \epsilon$. The first consequence is that coalitional winning probability decreases. Moreover, if he employs this $\epsilon$ as labor input, the increase in both total production and in his share do not compensate its cost (further increments of labor put beyond $l^*(s)$ have this effect). Hence, his individual payoff would go down for sure. The second part of the statement is straightforward.

This results sets a ground floor of effort $r_{\text{min}}(s) = 1 - \frac{l^*(s)}{s}$ which will be positive whenever optimal solutions to the production stage are interior. This lower bound, $r(s)_{\text{min}}$, will vary according to the size of the coalition and the elasticity of labor:

Lemma 18 The lower bound of effort $r(s)_{\text{min}}$

(i) is increasing in size for any $s \geq 2$,

(ii) given $f$ and $g$, such that $f$ dominates $g$, $r_g(s)_{\text{min}} > r_f(s)_{\text{min}}$ for any $s$.

Point (i) shows that the 'lower-bound’ effect works against free-riding on effort: We will see that from a given a critical mass and on, individual efforts are increasing in size. Part (ii) of the Lemma asserts that when elasticity of labor is low, the opportunity cost of effort is small and we should expect greater levels of conflict than under nearly linear technologies.

As the exclusion contest is considered, results do not depart much from those of the joint production case. Parts (i) and (i)’-(ii)’ of Proposition 6 hold for the common pool case.

With respect to size, there are two effects of size in agents’ optimal choices: First, the winning probability induces free-riding in effort as size increases. Second, the lower bound of effort, $1 - \frac{l^*(s)}{s}$, is increasing for $s \geq 2$. The following Lemma shows that, from a threshold size $s^*$ and on the latter effect will push upwards the effort choice, avoiding a systematic free-riding. Therefore, for sufficiently large coalitions, the lower bound effect will dominate the free-riding incentives.

Lemma 19 Fixed $r^{-S}$, there exists a threshold size $s^*$ such that for any $s \geq s^*$ the individual optimal choice of effort is increasing in size.

From a critical mass and on, free riding vanishes because big coalitions are very attached to $r_{\text{min}}(s)$, that is increasing in size. Notice that in such cases we cannot assert that coarser coalition structures lead to higher levels of conflict.
5.1 Coalition formation

The coincidence result of the $\alpha$ and $\beta$ characteristic functions no longer holds, as the following Lemma illustrates.

**Lemma 20** The indirect payoff function of a player $i \in S$ is strictly decreasing in $r^{-S}$ if and only if

$$s \geq (m + 1) \frac{1 - \varepsilon}{1 - (c_l(s - r^S)/f(s - r^S))} - m. \quad (15)$$

Why is the effect of $r^{-S}$ indeterminate? Notice first that if a coalition is underexploiting, $f'(L) > c_l$, condition (15) holds because the right hand side is negative. However this is not necessarily true when the coalition is overexploiting the resource. In such cases, conflict has a perverse effect: It is a discipline device that prevents agents from spending too much in exploiting the resource. Then, higher external threats reduce their labor and all are better off in the second stage.

**Corollary 21** In the exclusion game of coalition formation under individual production, $v_\alpha(S) = v_\beta(S)$ if for any $S \subset N$

$$c_l < f'(s - r^S).$$

Then the $\alpha$-core and $\beta$-core are nonempty and coincide.

If we can be sure that no coalition will overexploit the resource then we can be sure of in which direction should the punishment go: If $n - 1$ players do not overexploit (enough) the resource, they are indifferent between reacting passively to the outsiders’ choice that best punishes the coalition and waiting for the punishment when they employ best responses\(^\text{12}\).

The perverse effect of overexploitation has a big impact when players employ best responses: As long as this coalition underexploits a resource (because it is relatively small and faces high levels of hostility) it finds a double incentive to grow, both higher winning probability and payoff in the second stage. However, at some point, the winning coalition starts overexploiting the resource (the existence of outsiders is not enough to prevent them from putting too much labor). By adding new members, $r^{-S}$ decreases and the total labor input in the common increases. The improvement in the winning probability may not be enough to outweigh the payoff reduction.

\(^{12}\)Corollary 24 does not imply that the $\beta$ core is empty. In fact, if $\hat{r}^{-S} = (n - s)^m$ is the actual minimizer of $u^*_\beta$, the exclusion game satisfies all the conditions posed in Theorem 1 in Zhao (1999) for the non-emptiness of the $\beta$ core. The real problem is that it is not possible in this framework to compare a corner solution with possible interior minimizers.
So big coalitions ”need” the presence of outsiders. Not too many but not too few.

We illustrate these points in the following example.

**Example 3:** The initial data of this game are taken from Meinhardt (1999). Note that, by symmetry, they are equivalent to those used in Example 2. This allows to compare the three models, Meinhardt’s and joint and individual production.

So assume that \( N = 4 \); the players have 35 units to spend in production or exclusion and the unit cost of labor is 3. The production function takes the form \( f(L) = 23L - \frac{1}{8}L^2 \). Again, \( m \) can be equal to 1 or 2.

First, we compare the alpha (and beta, because they coincide for the independent production case too) characteristic functions. The characteristic form game of Meinhardt (1999) is convex, that is, individual contributions to coalitional worth are greater the bigger the coalition the player joins with. However, this does not hold for our exclusion games

\[
\begin{align*}
\text{Meinhardt (1999)} & & 95 & 126 & 162 & 200 \\
\text{Joint production} (m = 1) & & 45 & 96 & 143 & 200 \\
\text{Individual production} (m = 1) & & 45 & 80 & 109 & 128
\end{align*}
\]

where \( v_\alpha(s) \) is the value generated by a coalition of size \( s \) when outsiders behave in the \( \alpha \) fashion. This table shows that the joint production case is intermediate between coalitional Meinhardt’s and individual production. Anyway, values for the exclusion games are always below Meinhardt’s ones. It also suggests that economic punishments (overharvesting of the fishery, for instance) are less severe that non-economic ones (exclusion and expropriation).

Let us now assume that coalitions play best responses. We compute the partition function for Meinhardt (1999):

\[
\begin{array}{cccc}
\pi & u_a(\pi) & u_b(\pi) & u_c(\pi) & u_d(\pi) \\
\text{a b c d} & 128 & 128 & 128 & 128 \\
\text{ab c d} & 100 & 100 & 200 & 200 \\
\text{abc d} & 118 & 118 & 118 & 335 \\
\text{ab cd} & 177 & 177 & 177 & 177 \\
\text{abcd} & 200 & 200 & 200 & 200
\end{array}
\]

Notice that this game is of positive externalities: When players merge they reduce their labor input because the internalize part of the social costs. Outsiders take advantage from it: They have now a bigger share of a higher overall production. Then, players are reluctant to form coalitions: They want others to do so. Consequently, the grand coalition prevails as the subgame perfect equilibrium of the size announcement game.

Things change dramatically for the individual production case:
Note first that, in contrast with the joint production case, this game is of positive externalities for both values of $m$: When $m = 1$, it is dominant for $a$ to announce a two-player coalition because for $c$ it will optimal to form $\{cd\}$. On the other side, when $m = 2$ it is dominant for $a$ to form $\{abc\}$. The reason for this difference lies at the fact that when $m = 1$ the three players coalition is overexploiting: The coalition $\{abc\}$ is alone-stable but the presence of a fourth player make it break up. The expulsion of $c$ is worthy because they can use conflict as a discipline device. The payoff in the second stage increases and the winning probability does not decrease enough.

Observe that in both cases $\{ab \mid cd\}$ is the most efficient structure. Whereas, in the former individual payoffs are aligned with efficiency, when $m = 2$, players in $\{abc\}$ can take again advantage of the weak position of $d$: When forces are even, conflict can be socially good in ex-ante terms. But, when conflict technology is too effective large coalitions become profitable because it is relatively cheap to keep outsiders away. Furthermore, notice that conflict is always two-sided.

Finally let us compare the stable structures of the three models so far considered:

<table>
<thead>
<tr>
<th></th>
<th>$m = 1$</th>
<th>$m = 2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$a \mid b \mid c \mid d$</td>
<td>$\pi$</td>
<td>$u_a(\pi)$</td>
</tr>
<tr>
<td>$ab \mid c \mid d$</td>
<td>83</td>
<td>83</td>
</tr>
<tr>
<td>$abc \mid d$</td>
<td>152</td>
<td>152</td>
</tr>
<tr>
<td>$ab \mid cd$</td>
<td>177</td>
<td>177</td>
</tr>
<tr>
<td>$abcd$</td>
<td>128</td>
<td>128</td>
</tr>
</tbody>
</table>

This results suggest that even under the possibility of enough communication among agents, very effective conflict technologies make a difference. On the other side, by accepting the possibility of conflict in non-cooperative environments, the ‘tragedy of the commons’ is partially alleviated: The expected production is closer to the joint production of the resource, the best case scenario.

### 6 Further research

The present work opens some questions and lines of research:

Our game is one shot. Once a coalition has won the contest, its member agree on not to fight again. However, in spite of the existence of alone-stable
coalitions, they may have incentives to re-open hostilities. This scenario of continuing conflict is addressed by Esteban and Sakovics (2000), Skaperdas (1996) and Tan and Wang (2000). In these models, conflict is assumed to be fought to the finish, that is, until only one player prevails and obtains the prize. A straightforward extension of the model is to consider a repeated game. At the beginning of each period players of the previous winning coalition form coalitions again. Given that, contrary to these models, ours is a general equilibrium model, investments and payoff should be made each period. Under this extensive game, the key question is which coalition structures are conflict-proof, that is, immune to the re-opening of conflict.

In the models above, winning probabilities are exogenous. That is why they mainly focus on obtaining conditions under which weak players join against the strong ones. On the contrary, in our model agents are identical. This is because we are mainly interested on exploring the validity of conflict as a mechanism that individuals may use to increase their payoffs. However, by assuming inequality of endowments or strengths allocation resting on conflict (or power relationship) becomes straightforward. Nevertheless, our analysis should be complemented by accepting the possibility of differences in endowments or conflict skills. Given the complexity of the model the easiest way would be to assume the existence of two types of players.

A third extension of the model would be to relax the assumption that the losing players face “death”. This may be a source of additional conflict investments. In Skaperdas and Syropoulos (1996) victory means that the winners trade with the losers in a dominant position. In these line, It could be assumed that the winning coalition also gains the power to hire labor (or take it freely) from the losers. Then, if the payoff after exclusion is not zero or the winners care about the left over endowments of the losers (because they become essentially their slaves) exclusion races might be alleviated and conflict less fierce.

Finally let us make a remark on the role of conflict as a discipline device: This results should be regarded as a confirmation of Hardin’s quotation. Social conflict can avoid resource overexploitation. However, it seems less valid from the coalition formation viewpoint because it implies that one of the motivations of the agents for forming coalitions is to maintain a high enough level of conflict. This suggests that other approaches to coalition formation should be considered in order to isolate this perverse effect.

References


### Appendix

#### Proof of Proposition 5.

First we show that the objective function in both cases is quasiconcave (it is enough to prove it in one case): With functions of one variable, it is sufficient to proof that the function is strictly monotonic or first strictly increasing and then strictly decreasing. In order to prove it we need first the following auxiliary Lemma:

**Lemma 22 (A1)** If \( \frac{\partial u^S}{\partial r} \leq 0 \), then the objective function is strictly concave (i.e., \( \frac{\partial^2 u^S}{\partial r^2} < 0 \))

**Proof.** If \( \frac{\partial u^S}{\partial r} \leq 0 \) then
\[
\frac{m(r^S)^{-1}r^{-S}}{(r^S)^m + r^{-S}}[f(s - r^S)] \leq f'(s - r^S).
\]

Multiply now both sides by \(-\frac{2m(r^S)^{-1}r^{-S}}{(r^S)^m + r^{-S}}\) then
\[
-2\frac{m^2(r^S)^{m-2}r^{-2}}{(r^S)^m + r^{-S}r} [f(s - r^S)] > -2\frac{m(r^S)^{m-1}r^{-S}}{(r^S)^m + r^{-S}}[f'(s - r^S)].
\]

On its side, second derivative is:
\[
\frac{\partial u^S}{\partial^2 r} = -\frac{m(m+1)(r^S)^{2m-2}r^{-S} + m(m-1)(r^S)^{m-2}(r^{-S})^2}{(r^S)^m + r^{-S}r} [f(s - r^S)] \tag{17}
\]
\[
-2\frac{m(r^S)^{m-1}r^{-S}}{(r^S)^m + r^{-S}r} [f'(s - r^S)] + \frac{(r^S)^m}{(r^S)^m + r^{-S}} [f''(s - r^S)].
\]

If we substitute (16) we get
\[
\frac{\partial u^S}{\partial^2 r} < -\frac{m(m+1)(r^S)^{m-2}r^{-S}}{(r^S)^m + r^{-S}r} [f(s - r^S)] + \frac{(r^S)^m}{(r^S)^m + r^{-S}} [f''(s - r^S)] < 0. \tag{18}
\]

Next, notice that at the coalitional level quasi-concavity conditions hold because \( u^S(0, r^{-S}) = u^S(s, r^{-S}) = 0 \) and the derivative of the objective function at \( r^S = 0 \) is strictly positive. This implies that there exists at least one level \( r^S \in (0, s) \) such that the derivative of the objective function is zero. But in such case, given the Lemma A1, this point is a maximum.

For uniqueness, it is sufficient to show that the Jacobian Matrix of the best responses has a dominant diagonal (thus, the mapping is a contraction): A matrix of size \( K \times K \) has a dominant diagonal when there exists a vector \( (r_1, \ldots, r_K) > 0 \) such that for all \( i = 1, \ldots, K \) it is true that \( |r_i a_{ii}| > \sum_{j \neq i} |r_j a_{ij}| \).
We show it, without loss of generality, for the first coalition $S_1$. Notice that its corresponding row in the Jacobian matrix consists first of expression (17) and all the remaining consist of $(\partial u_{S_1}/\partial r_{-S_1})$, for all $k = 2, \ldots, K$.

\[
\partial u_{S_1}/\partial r_{S_1} = \frac{m(r_{S_1})^{m-1}(r_{S_1})^{m-r_{S_1}}}{(r_{S_1})^{m+r_{S_1}}}[f(s_1 - r_{S_1})]
\]

\[
+ \frac{(r_{S_1})^{m}}{(r_{S_1})^{m+r_{S_1}}}[f'(s_1 - r_{S_1})]
\]

\[
= \frac{m(r_{S_1})^{2m-1}}{(r_{S_1})^{m+r_{S_1}}}[f(s_1 - r_{S_1})],
\]

where the last equality follows from the first order condition (5). This also shows that best response effort is increasing in $r^{-S}$, given that, by the implicit function theorem $\partial u_{S_1}/\partial r_{S_1} = -\partial u_{S_1}/\partial r_{S_1}/\partial r_{-S_1}$. Therefore, if one employs the vector $(r_{S_1}, r_{S_2}, \ldots, r_{S_K})$ the right hand side of the dominant diagonal condition states

\[
\sum_{k \neq 1} r_{S_k} \frac{\partial u_{S_1}}{\partial r_{S_1}} \frac{\partial -r_{S_1}}{\partial r_{-S_1}} \frac{\partial r_{S_k}}{\partial r_{S_k}} = \frac{m(r_{S_1})^{2m-1}r_{S_1}}{(r_{S_1})^{m+r_{S_1}}}[f(s_1 - r_{S_1})]
\]

After some manipulation one can obtain that

\[
\frac{\partial u}{\partial r^{-S}} < -m(m+1)(r^{-S})^{2m-2}r^{-S} - m^2(r^{-S})^{m-2}(r^{-S})^2[f(s - r^{-S})].
\]

So then

\[
|r_{S_1} \frac{\partial u_{S_1}}{\partial r_{S_1}}| = -r_{S_1} \frac{\partial u_{S_1}}{\partial r_{S_1}} > \frac{m(m+1)(r_{S_1})^{2m-1}r_{S_1} + m(r_{S_1})^{m-1}(r_{S_1})^2}{(r_{S_1})^{m+r_{S_1}}}[f]
\]

\[
> \frac{m^2(r_{S_1})^{2m-1}r_{S_1}}{(r_{S_1})^{m+r_{S_1}}}[f] = \sum_{k \neq 1} r_{S_k} \left| \frac{\partial u_{S_1}}{\partial r_{S_1}} \right| |\frac{\partial r_{S_k}}{\partial r_{S_1}}| |\frac{\partial r_{S_k}}{\partial r_{S_k}}|
\]

Then, the Jacobian matrix has a dominant diagonal and there is a unique Nash equilibrium of the non-cooperative game at coalitional level.

At the individual level, players take as given both $r^{S,i}$ and $r^{-S}$. It is clear that if $r^{-S}$ is the equilibrium coalitional effort given $r^{S,i}$ it also maximizes individual payoff. Therefore the best response individual effort is $r_i = \min \{1, r^{-S} - r^{S,i}\}$. Thus, multiple equilibria exists because the final share of $r^S$ may vary. But in all of them this level of coalitional effort will be achieved.

**Proof of Proposition 6.** Statement $(i)$ : From the previous proposition we know that the sign of the derivative of the best reply with respect to $r^{-S}$ is finally positive. So then

\[
\frac{\partial r^{-S}/r^{-S}}{\partial r^{-S}} = \frac{m(r^{-S})^{m-1}}{(r^{-S})^{m+r^{-S}}}[f(s - r^{-S})]
\]

\[
\frac{m(m+1)(r^{-S})^{2m-2}r_{-S}}{(r^{-S})^{m+r^{-S}}}[f(s - r^{-S})] - f''(s - r^{-S})
\]

26
Which is clearly decreasing in \( r^{-S} \).

The result of Statement (ii) is straightforward from the discussion above.

Statement (iii) : Condition (5) can be rewritten in the following way

\[
\frac{m(r^S)^{-1}r^{-S}}{(r^S)^m + r^{-S}(s - r^S)} - \varepsilon = 0
\]

(19)

Then it is easy to see that \(-\frac{\partial h(r^S, r^{-S})}{\partial \varepsilon} = 1\), so the effort is inversely related with the elasticity of output with respect to labor.

By total differentiation

\[
\frac{dr^S}{d\varepsilon} = \frac{\partial r^S}{\partial \varepsilon} + \frac{\partial r^S}{\partial r^{-S}} \frac{dr^{-S}}{d\varepsilon},
\]

So, given that the second term is also negative, in equilibrium, effort decreases with elasticity.

Statement (iv) : Rearranging:

\[
\frac{(r^S)^m}{(r^S)^m + r^{-S}[r^{-S}(f(s - r^S)) - f'(s - r^S)]} = 0
\]

Recall first that \( r^{-S} = \sum_{k \in \pi \setminus S}(r^S_k)^m \). Then, by the implicit function theorem and given that \( \frac{\partial r^S}{\partial m} = -\frac{\partial h}{\partial m} \), the relationship between best reply and conflict effectiveness is driven by the sign of

\[
-\frac{\partial h}{\partial m} = -\left[ f(s - r^{-S}) \right] \left( \frac{(r^S)^m(r^S)^{-1}}{(r^S)^m + r^{-S}} \right) [r^{-S} + \frac{m(r^S)^m \sum_{k \in \pi \setminus S}(r^S_k)^m \ln r^{S_k} - \ln r^S)}{(r^S)^m + r^{-S})]
\]

\[
= -\left[ f(s - r^{-S}) \right] \left( \frac{(r^S)^m}{(r^S)^m + r^{-S}} \right) [r^{-S} + \frac{m \sum_{k \in \pi \setminus S}(r^S_k)^m \ln r^{S_k}}{(r^S)^m + r^{-S})}]
\]

\[
= -\left[ f(s - r^{-S}) \right] \left( \frac{(r^S)^m}{(r^S)^m + r^{-S}} \right) \sum_{k \in \pi \setminus S}(r^S_k)^m [1 + \frac{\ln r^{S_k}}{r^{-S}}]
\]

For symmetric coalition structures \( \ln \frac{\bar{s}_k}{r^{-S}} = 0 \) for any \( S_k \in \pi \) (or if no coalition enjoys a very advantageous position) and then \( \frac{\partial r^S}{\partial m} > 0 \). Again, by total differentiation:

\[
\frac{dr^S}{dm} = \frac{\partial r^S}{\partial m} + \frac{\partial r^S}{\partial r^{-S}} \frac{dr^{-S}}{dm}
\]

Under these conditions, the second term is positive as well.

\[
\text{Proof of Proposition 10.} \quad \text{As stated in the text, both characteristic functions coincide if and only if } \hat{\gamma}^{-S} = \Min_{r^{-S}} u^*(r^{-S}). \text{ Then, to prove that the indirect payoff function is decreasing in } r^{-S} \text{ is a sufficient condition.}
\]

\[
u^* = \frac{(r^S(r^{-S}))^m}{(r^S(r^{-S}))^m + r^{-S}[f(s - r^S(r^{-S})]}
\]

(20)
Let \( r' \) be the short hand notation of \( \frac{\partial r^S(r^S)}{\partial r^S} \). Then, by the envelope theorem

\[
\frac{\partial u^*_s}{\partial s} = \frac{\partial u^*_i}{\partial s} + \frac{\partial u^*_i}{\partial r^S} \frac{\partial r^S}{\partial s} + \frac{\partial u^*_i}{\partial r^S} \frac{\partial r^S}{\partial s}
\]

where the second term vanishes. Therefore:

\[
\frac{\partial u^*_i}{\partial s} = \frac{(r^S)^m}{(r^S)^m + r^S} \frac{f'(s - r^S) - f(s - r^S)}{s} 
\]

On the other side:

\[
\frac{\partial r^S(r^S)}{\partial s} = \frac{m^2(r^S)^2 - (r^S)^2}{(r^S)^m + r^S} \frac{f(s - r^S) - f(m - r^S)}{s}
\]

Hence \( \frac{\partial r^S}{\partial s} < 0 \) because there is a coalition the new member of \( S \) comes from. Then, given that \( \frac{\partial u^*_i}{\partial r^S} \), the condition stated in the text is enough for the individual payoff to be increasing in size. ■

**Proof of Proposition 15.** Let us define \( r = r^S/s \). Then \( \frac{\partial r^S}{\partial s} = r + sr' \). Monotone average coalitional outlay is sufficient for superadditivity. And one can rewrite the average coalitional outlay as a function of \( r \), i.e. \( \tau = s^{m-1}r^m \), and its derivative:

\[
\frac{\partial \tau}{\partial s} = s^{m-2}r^m (m - 1) + \frac{s}{r} \frac{\partial \tau}{\partial s}
\]

Then, given that \( \frac{\partial \tau}{\partial s} > 0 \), \( r' > -\frac{\tau}{s^2} \). Then \( \frac{\partial \tau}{\partial s} > s^{m-2}r^m (m - 2) \) and hence that is positive for any \( m \geq 2 \). On the other side, concavity of the coalitional outlay implies that it is sub-additive.

\[
\frac{\partial (r^S)^m}{\partial^2 s} = m(r^S)^{m-1} \frac{m - 1}{r^S} \frac{\partial r^S}{\partial s} + \frac{\partial^2 r^S}{\partial^2 s}
\]

For the families \( (F1) \) and \( (F2) \) second derivatives are:

\[
\frac{\partial^2 r^S}{\partial^2 s} = \frac{2m^2(r^S)^2 - (r^S)^2}{(r^S)^m + r^S} \frac{0}{(r^S)^m + r^S} \leq 0
\]
\[ \frac{\partial^2 r^S}{\partial^2 s} = \frac{2^m(r^S)^{2-r-S} \left( \frac{m}{r^m} + r^{m-r-S} \right) - (m+1)\alpha (1-\alpha)(s-r^S)^{2\alpha-3}}{[m(m+1)(r^S)^{2-r-S}(s-r^S)\alpha + \alpha(1-\alpha)(s-r^S)^{\alpha-2}]^2} \leq 0. \]

Then, \( m \leq 1 \) is sufficient for concavity in size. \( \blacksquare \)

**Proof of Proposition 19.** Quasiconcavity of the objective function can be easily stated.

\[ u_i^S(0, r-i) = \frac{(r^S)^m}{(r^S)^m + r^{-S}} \left[ \frac{1}{s-r^S} f(s-r^S) - c_i \right] > 0 = u_i^S(1, r-i) \]

Finally, as it is shown below, when the first order condition equals zero, the second order condition is negative. Finally, it remains to check that the symmetric strategy profile constitutes the equilibrium: Take the new first order condition. When \( r \to 0 \) the it converges to \( \infty \) because the second term vanishes and \( m \geq 0 \). When \( r \to 1 \) the expression converges to \(-\frac{m^m}{m^m+1} f'(0) < 0 \). Then, there is a value for \( r \) that makes the derivative equal to zero. \( \blacksquare \)

**Proof of Lemma 21.** Let \( h(s, c_l, \varepsilon) = f'(l^*(s)) + (s-1)f(l^*(s))/l^*(s) - sc_l = 0 \). By the implicit function theorem:

\[ \frac{\partial l^*(s)}{\partial s} = \frac{\partial h(s, c_l, \varepsilon)/\partial s}{\partial h(s, c_l, \varepsilon)/\partial l^*(s)} \]

By simple calculations

\[ \frac{\partial h(s, c_l, \alpha)}{\partial l^*(s)} = f''(l^*(s)) + \frac{(s-1)}{l^*(s)} \left( f'(l^*(s)) - f(l^*(s)) / l^*(s) \right) < 0 \]

by concavity

It is clear that the derivative of \( h(s, c_l, \alpha) \) with respect to \( s \) and \( c_l \) are positive and negative respectively. Hence \( \frac{\partial l^*(s)}{\partial s} > 0 \)

\[ \frac{\partial_{\text{min}}}{\partial s} = -\left[ \frac{\partial l^*(s)/\partial s}{\partial l^*(s)} \right] = -\left[ \frac{l^*(s)}{s^2} \right] \text{ some manipulation yields} \]

\[ = -\left[ \frac{l^*(s)}{s^2} \right] \left[ s \left( f(l^*(s))/l^*(s) - c_l \right) - \frac{f(l^*(s))/l^*(s) - c_l}{l^*(s)} \right] \frac{-f(l^*(s))/l^*(s)}{l^*(s)} \]

by condition (10) we know that

\[ s[f(l^*(s))/l^*(s) - c_l] = f(l^*(s))/l^*(s) - f'(l^*(s)) \]

and then

\[ \frac{\partial_{\text{min}}}{\partial s} = -\left[ \frac{l^*(s)}{s^2} \right] \left[ (s-1)[f(l^*(s))/l^*(s) - f'(l^*(s))] \right] > 0 \]

For any \( s \geq 2 \) the numerator is greater than the denominator and the first term in brackets is smaller than one. Then the overall expression is positive.

Finally if we rearrange condition (10), one obtains:

\[ h(s, c_l, \varepsilon) = (s + \varepsilon - 1) \frac{f(L)}{L} - sc_l = 0 \]
and
\[
\frac{\partial^2 h}{\partial s \partial r} = \frac{1}{s} \frac{\partial h(s,c_t,s)}{\partial s} < 0
\]

However, we can only establish a partial ordering because we cannot compare situations in which elasticities’ rankings change as total labor input varies. 

**Proof of Lemma 22.** By the Implicit Function Theorem \(\frac{\partial r}{\partial s} = \frac{1 - \partial h(s)/\partial s}{\partial h(s)/\partial r}\), where \(h(s)\) is given by (13).

\[
h(s, r) = \frac{m(sr)^{m-1}r-S}{(sr)^m + r-S^2} \left[ \frac{1}{s} f(s - sr) - c_1(1 - r) \right] - \frac{m}{(sr)^m + r-S^2} \left[ \frac{1}{s} f'(s - sr) + \frac{s - 1}{s} f(s - sr) - c_1 \right].
\]

By first order conditions we can reduce both expressions and one finally obtains

\[
\frac{\partial h}{\partial r} = \frac{m - s - 1}{s} \left[ f'' + \frac{s - 1}{s} f'(s - sr) - \frac{m + 1}{s} f(s - sr) \right] - \frac{m}{(sr)^m + r-S^2} \left[ f'' + \frac{s - 2}{s} f'(s - sr) - \frac{m}{s} f(s - sr) \right].
\]

By first order conditions we can reduce both expressions and one finally obtains

\[
\frac{\partial h}{\partial s} = \frac{-r}{s} + \frac{1}{s} \left[ f'' + \frac{s - 1}{s} f'(s - sr) - \frac{m + 1}{s} f(s - sr) \right] + \frac{m}{sr} \left[ f - c_1(s - sr) \right] + f''
\]

Then \(\frac{\partial h}{\partial s} > 0\) if and only if

\[
\frac{m(1-p)}{sr} [m - 1 - \frac{m - 1}{s} (p - 2)] \frac{1}{1 - r} + m - 1 \frac{1}{m} [f - c_1(s - sr)] + (1-r)f'' < 0
\]

A sufficient condition when \(s \geq 2\) is that

\[
2 - \frac{m}{s} (1 - p) - (s - 2) \frac{1}{1 - r} + \frac{m - 1}{s} < 0
\]

It is evident that this expression is decreasing in \(s\). The question is if it there exists a size that makes it negative. It turns out that for values above

\[
s \geq 2 + m(1 - p); \frac{(1 - r)}{r} \left(1 + \frac{\sqrt{1 + \frac{r}{1 - r} m + 1}}{m}\right)
\]
the whole derivative is negative. Then $s^*$ is below this value. ■

Proof of Lemma 23. Let $r'$ be the short hand notation of $\frac{\partial r_i}{\partial r^S}$ and $r(r^S)$ the best response strategy of the member of $S$. Then indirect payoff function and its derivative with respect to $r^S$ are

$$u_i^*(r^S) = \frac{(sr(r^S))^m}{(sr(r^S))^m + r^S-1} f(s - sr(r^S)) - c_i(1 - r(r^S)).$$

$$\frac{\partial u_i^*(r^S)}{\partial r^S} = \frac{\partial u_i^*}{\partial r} r' + \frac{\partial u_i^*}{\partial r^S} = sp^S [r^S - 1 \frac{1}{1 - r} - \frac{1}{(sr)^m + r^S-1}] [1 - f - c_i(1 - r)]$$

where:

$$\frac{\partial r^S}{\partial r^S} = \frac{m(s^r)^{m-1}}{(sr)^m + r^S} \left[ \frac{1}{(sr)^m + r^S-1} \left( \frac{1}{sr} + \frac{1}{r^S-1} \right) \right] [\frac{1}{r'} f - c_i(1 - r) - f''(\frac{1}{sr} + \frac{1}{r^S-1}) + \frac{1}{r'}]$$

that has no clear sign given that $r' > 0$. The sign of $\frac{\partial u_i^*(r^S)}{\partial r^S}$ is equal to the sign of

$$\frac{-m(s^r)^{m-1}}{(sr)^m + r^S} \left[ \frac{m+1}{r^S-1} \left( \frac{1}{sr} + \frac{1}{r^S-1} \right) \right] [\frac{1}{r'} f - c_i(1 - r) - f''(\frac{1}{sr} + \frac{1}{r^S-1}) + \frac{1}{r'}]$$

Unfortunately we should restrict to conditions that ensure the negative sign of this derivative. It is sufficient to show that

$$\frac{m+1}{r^S-1} - \frac{1}{r} > 0$$

Finally, first order condition allows us to rewrite this condition in terms of production and conflict technology only. ■

Proof of the Corollary 24. Given that $v_\alpha(N) = v_\beta(N)$ always coincide we need just to check (15) for $n-1$. Then $s \geq (m+1) \frac{f(s - sr)/sr - f'}{f(s - sr)/sr - c_i - m}$ to ensure that for any coalition $u_i^*(r^S)$ attains its minimum also when $r^S = (n - s)^m$. So finally we have:

$$v_\alpha(S) = u_i^S((n - s)^m), (n - s)^m) = u_i^*(n - s)^m) = \min_{r^S} u_i^*(r^S) = v_\beta(S)$$

■