# EMBEDDING AN ANALYTIC EQUIVALENCE RELATION IN THE TRANSITIVE CLOSURE OF A BOREL RELATION

EDWARD J. GREEN

Abstract. The transitive closure of a reflexive, symmetric, $\mathbf{\Sigma}^1_1$ relation is a $\mathbf{\Sigma}^1_1$ equivalence relation. Does some smaller class contain the transitive closure of every reflexive, symmetric, closed relation? An essentially negative answer is provided here. The Baire space, $\mathcal{N}$, is homeomorphic to an open subset of itself, $X$, that has an open complement. It is shown that, for any $\mathbf{\Sigma}^1_1$ equivalence relation $E \subseteq \mathcal{N} \times \mathcal{N}$, its homeomorphic image in $X \times X$ is the intersection with $X \times X$ of the transitive closure of a reflexive, symmetric, closed relation $R \subseteq X \times X$. Specifically, $R$ can be constructed as the union of two closed equivalence relations.

## 1. Introduction

This note answers a question in descriptive set theory that arises in the context of the Bayesian theory of decisions and games. It concerns the notion of common knowledge, formalized by Robert Aumann [1976]. For an event $A$ that is represented as a subset of a measurable space $\Omega$, Aumann defines the event that an agent *knows* $A$ to be the event $A \setminus [\Omega \setminus A]_\Pi$, where $\Pi$ is the agent's *information partition* of $\Omega$.[1] If $\Pi$ is the meet of individual agents' information partitions (in the lattice of partitions where $\Pi \leq \Pi' \iff \Pi'$ refines $\Pi$), then Aumann defines

$$A \setminus [\Omega \setminus A]_\Pi \tag{1.1}$$

to be the event that $A$ is common knowledge among the agents.[2]

Aumann restricts attention to the case that $\Omega$ is countable (or that the Borel $\sigma$-algebra on $\Omega$ is generated by the elements of a countable partition), so that measurability issues do not arise. But, otherwise, the passage from information partitions to a common-knowledge partition is very badly behaved, as is the passage from an information partition $\Pi$ and an event $A$ to the related event that $A$ is known according to $\Pi$. For example, let $X$ be an arbitrary subset of $(0,1)$, and let $\Omega = [0,2]$. Consider two agents, whose information partitions are $\Pi_1 = \{\{\omega, \omega+1\} \mid \omega \in X\} \cup \{\{\omega\} \mid \omega \notin X\}$ and $\Pi_2 = \{\{\omega\} \mid \omega < 1\} \cup \{[1,2]\}$. Then $X \cup [1,2]$ is the

*Date*: 2012.02.22.

2010 *Mathematics Subject Classification.* 03E02.

I am grateful to the Human Capital Foundation, (`http://www.hcfoundation.ru/en/`) and particularly to Andrei Vavilov, for support for this research. Also I would like to acknowledge the hospitality of CEMFI (`http://www.cemfi.es`) where I have conducted the research.

[1] $[A]_\Pi$ denotes $\bigcup\{\pi \mid \pi \in \Pi \text{ and } \pi \cap A \neq \emptyset\}$, the saturation of $A$ with respect to $\Pi$. Aumann's definition corresponds to the truth condition for $\Box A$ in Kripke [1959].

[2] Aumann sketches a proof—reminiscent of a general principle in proof theory (cf. Pohlers [2009, Lemma 6.4.8, p. 89])—that this definition is equivalent to the intuitive, recursive definition of common knowledge: that $A$ has occurred and that, for all $n \in \mathbb{N}$, both agents know. . . that both agents know ($n$ times) that $A$ has occurred.

block of the common-knowledge partition that includes the block $[1, 2]$ in $\Pi_2$. From information partitions composed of the simplest events—singletons, pairs, and a closed interval—we have passed to a common-knowledge partition with a block that, depending on what is $X$, might even be outside the projective hierarchy. Knowledge of an event by individual agent is likewise problematic. In the present example, the event that agent 1 knows $(0, 1)$ is $(0, 1) \setminus X$.

These measurability problems dictate that information partitions should be represented as equivalence relations. If $E_1$ and $E_2$ are $\mathbf{\Sigma}^1_1$ (that is, analytic) equivalence relations, then the meet of the partitions that they induce is induced by the transitive closure of their union, which is also a $\mathbf{\Sigma}^1_1$ equivalence relation.[3] Moreover, if an information partition is represented by a $\mathbf{\Sigma}^1_1$ relation and an event is $\mathbf{\Pi}^1_1$ (that is, co-analytic), then knowledge of the event according to the information partition is also a $\mathbf{\Pi}^1_1$ event.[4] This observation implies that knowledge of a Borel event is a universally measurable event.

In applications to Bayesian decision theory and game theory, it is reasonable to specify each agent's information as a $\mathbf{\Delta}^1_1$ (that is, Borel) equivalence relation, or even as a smooth or closed Borel relation.[5] Thus it may be asked: if the graphs of $E_1$ and $E_2$ are in $\mathbf{\Delta}^1_1$ or in some smaller class, then how is the graph of the transitive closure of $E_1 \cup E_2$ restricted?

It will be shown here that no significant restriction of the common-knowledge partition is implied by such restriction of agents' information partitions. For $R \subseteq \Omega^2$, let $R^{(1)} = R$ and $R^{(n+1)} = RR^{(n)} \subseteq \Omega^2$ (that is, the composition of relations $R$ and $R^{(n)}$). Denote the transitive closure of $R$ by $\overline{R} = \bigcup_{n \in \mathbb{N}} R^{(n)}$. If $\Omega$ is a Polish space and $E_0 \subset \Omega \times \Omega$ is a $\mathbf{\Sigma}^1_1$ equivalence relation, then there are $\mathbf{\Delta}^1_1$ equivalence relations $E_1$ and $E_2$ and a $\mathbf{\Delta}^1_1$ subset $Z$ of $\Omega$, such that $\overline{E_1 \cup E_2} \restriction (Z \times Z)$ is Borel equivalent to $E_0$. If $\Omega$ is the Baire space, then $E_1$ and $E_2$ can be taken to be closed, $Z$ can be taken to be open, and the Borel equivalence can be taken to be a homeomorphic equivalence.

## 2. Statement of the theorem

As the Polish space $\Omega$ to be studied, take $\mathcal{N} = \mathbb{N}^{\mathbb{N}}$.[6] Define subsets $X$ and $Y$ of $\mathcal{N}$ by $X = \{\alpha | \alpha_0 > 0\}$ and $Y = \{\alpha | \alpha_0 = 0\}$. $X$ and $Y$ are both homeomorphic to $\mathcal{N}$, and homeomorphisms $f \colon X \to Y$ and $g \colon Y \times Y \times Y \to Y$ are routine to construct.[7] Each of $X$ and $Y$ is both open and closed in $\mathcal{N}$. It follows that, if $Z$ is either $X$ or $Y$, then $A \subseteq Z$ is open (resp. closed, Borel, $\mathbf{\Sigma}^1_1$) as a subset of $A$ iff it is open (resp. closed, Borel, $\mathbf{\Sigma}^1_1$) as a subset of $Z$. This invariance to the ambient space extends to product spaces. (For example a subset of $X \times Y$ is closed in $X \times Y$

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[3] Composition is defined with a single existential quantifier, and thus takes a pair of $\mathbf{\Sigma}^1_1$ relations to a $\mathbf{\Sigma}^1_1$ relation. The countable union of $\mathbf{\Sigma}^1_1$ relations is $\mathbf{\Sigma}^1_1$. Cf. Moschavakis [2009, Theorem 2B.2, p. 54].

[4] This is equivalent, by (1.1), to the fact that the saturation of a $\mathbf{\Sigma}^1_1$ set with respect to a $\mathbf{\Sigma}^1_1$ equivalence relation is $\mathbf{\Sigma}^1_1$. This latter fact is true because the saturation is defined with a single extential quantifier.

[5] Smoothness (also called tameness) and closedness are co-extensive for equivalence relations on standard Borel spaces. Cf. Harrington et al. [1990, proof of Theorem 1.1, p. 920].

[6] $\mathbb{N} = \{0, 1, \ldots\}$. $\mathcal{N}$ is topologized as the product of discrete spaces.

[7] Since $Y$ is homeomorphic with $\mathcal{N}$, $g$ can be constructed from the function described by Moschavakis [2009, p. 31].

iff it is closed in $\mathcal{N}\times\mathcal{N}$.) In subsequent discussions, subsets of these subspaces will be characterized (for example, as being closed) without mentioning the subspace.

**Theorem 2.1.** *If $E \subseteq X \times X$ is a $\mathbf{\Sigma}^1_1$ equivalence relation, then there are equivalence relations $I$ and $J$ on $\mathcal{N}\times\mathcal{N}$, each of which has a closed graph, such that $E = \overline{I \cup J} \cap (X \times X)$.*

Before proceeding to the proof of this theorem, note that $I \cup J$ is a closed, reflexive, symmetric relation. Thus, theorem 2.1 implies the following proposition.

**Corollary 2.2.** *If $E \subseteq X \times X$ is a $\mathbf{\Sigma}^1_1$ equivalence relation, then there is a closed, reflexive, symmetric relation $R$ on $\mathcal{N}\times\mathcal{N}$, such that $E = \overline{R} \cap (X \times X)$.*

Theorem 2.1 and corollary 2.2 would be equivalent if every closed, reflexive, symmetric relation were the union of two closed equivalence relations, but that is not the case.

**Example 2.3.** Let $\alpha \in \mathcal{N}$. Define $R = D \cup (\{\alpha\}\times\mathcal{N}) \cup (\mathcal{N}\times\{\alpha\})$, and define

$$\mathcal{E} = \bigcup \{(D \cup \{(\alpha,\beta),(\beta,\alpha)\}) | \beta \in \mathcal{N}\setminus\{\alpha\}\}\,.$$

$R = \bigcup \mathcal{E}$; every $E \in \mathcal{E}$ is an equivalence relation; $R$ is closed, reflexive, and symmetric; and $2^{\aleph_0}$ is the cardinality of $\mathcal{E}$. There is no other set $\mathcal{F}$ of equivalence relations such that $R = \bigcup \mathcal{F}$. Thus, $R$ is not a union of fewer that $2^{\aleph_0}$ equivalence relations.

The assertions regarding $\mathcal{E}$ are obvious from its construction. To obtain a contradiction from supposing that $\mathcal{E}$ were not unique, suppose that $R$ were also the union of a set $\mathcal{F} \neq \mathcal{E}$ of equivalence relations. Not $\mathcal{F} \subsetneq \mathcal{E}$. So, there must be some $E \in \mathcal{F}\setminus\mathcal{E}$. By symmetry, there must be three distinct points, $\alpha,\beta,\gamma$ such that $\{(\beta,\alpha),(\alpha,\gamma)\} \subseteq E$. Since $E$ is transitive, $(\beta,\gamma) \in E \setminus R$, contrary to $R = \bigcup \mathcal{F}$.

## 3. Proof of the theorem

Denote the diagonal (that is, identity) relation in $\mathcal{N}\times\mathcal{N}$ by $D = \{(\alpha,\alpha)|\alpha \in \mathcal{N}\}$. $D$ is closed.

If $1 \le i < j \le k$ and $\vec{\alpha} = (\alpha_1,\alpha_2,\ldots,\alpha_k) \in \mathcal{N}^k$, then a transposition mapping is defined by $t_{ij}(\vec{\alpha}) = (\alpha_1,\ldots,\alpha_{i-1},\alpha_j,\alpha_{i+1},\ldots,\alpha_{j-1},\alpha_i,\alpha_{j+1},\ldots,\alpha_k)$.[8] The abbreviation $\widetilde{A} = t_{12}(A) = \{t_{12}(\alpha)|\alpha \in A\}$ will sometimes be used. Each $t_{ij}$ is a homeomorphism of $\mathcal{N}^k$ with itself. Note that $t_{ij} \restriction X^k$ and $t_{ij} \restriction Y^k$ map $X^k$ and $Y^k$ homeomorphically onto themselves.

Recall that a relation $E \subseteq X \times X$ is $\mathbf{\Sigma}^1_1$ iff there is a set $F$ such that

$$F \subseteq X \times X \times \mathcal{N} \text{ is closed, and } (\alpha,\beta) \in E \iff \exists\gamma\,(\alpha,\beta,\gamma) \in F. \tag{3.1}$$

**Lemma 3.1.** *If $E \subseteq X \times X$ is symmetric, then $E$ is $\mathbf{\Sigma}^1_1$ iff there is a closed, $t_{12}$-invariant set $F \subset X \times X \times X$ that satisfies (3.1).*

*Proof.* Let $F_0$ satisfy (3.1). Let $h$ be a homeomorphism from $\mathcal{N}$ to $X$, and define $F_1 \subseteq X \times X \times X$ by $(\alpha,\beta,\gamma) \in F_0 \iff (\alpha,\beta,h(\gamma)) \in F_1$. $F_1$ also satisfies (3.1), then, and it is closed. By symmetry of $E$, $\widetilde{F_1}$ is another closed set that satisfies (3.1). Consequently, $F = F_1 \cup \widetilde{F_1}$ is a $t_{12}$-invariant closed set that satisfies (3.1). $\square$

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[8] A sub-sequence of subscripted alphas distinct from $\alpha_i$ and $\alpha_j$ having subscripts that are not increasing, which occurs if $i = 1$ or $j = i + 1$ or $j = k$, denotes the empty sequence.

Let $i$ denote the identity function on $\mathcal{N}$. If $K, L, M, N$ are any sets, and $p\colon K \to L$ and $q\colon M \to N$, then denote the product mapping by $p \times q\colon K \times M \to L \times N$.

The two closed equivalence relations that theorem 2.1 asserts to exist are defined from the homeomorphisms $f$ and $g$ introduced in section 2, and the closed, $t_{12}$-invariant set $F$ guaranteed to exist by lemma 3.1, as follows.

$$j(\alpha,\beta,\gamma) = g(f(\alpha), f(\beta), f(\gamma)).$$

$$\begin{aligned} G &= \{(\alpha, j(\alpha,\beta,\gamma)) | (\alpha,\beta,\gamma) \in F\} \subseteq X \times Y; \\ H &= \{(j(\alpha,\beta,\gamma), j(\beta,\alpha,\gamma)) | (\alpha,\beta,\gamma) \in X \times X \times X\}; \\ I &= D \cup G \cup \widetilde{G} \cup \widetilde{G}G; \\ J &= D \cup H. \end{aligned} \tag{3.2}$$

**Lemma 3.2.** *$D$, $G$, $\widetilde{G}$, $H$, and $J$ are closed.*

*Proof.* $D$ is closed because $\mathcal{N}$ is a metric space.

The function $i \times j$ is a homeomorphism from $X \times X \times X \times X$ to $X \times Y$. Being a homeomorphism, it is an open mapping (which takes closed sets to closed sets). $G = [i \times j](((D \restriction (X \times X) \times X \times X)) \cap (X \times F))$. $(D \restriction (X \times X)) \times X \times X$ and $X \times F$ are both closed subsets of $X \times X \times X \times X$, so $G$ is closed. $\widetilde{G}$ is closed, as the image of $G$ under $t_{12}$, a self-homeomorphism of $\mathcal{N} \times \mathcal{N}$.

$j \times j$ is a homeomorphism from $(X \times X \times X) \times (X \times X \times X)$ to $Y \times Y$. The image under $j \times j$ of a closed subset of its domain is therefore closed in its range. $\{((\alpha,\beta,\gamma),(\beta,\alpha,\gamma)) | (\alpha,\beta,\gamma) \in X \times X \times X\}$ is $t_{23} \circ t_{25}((D \restriction (X \times X)) \times (D \restriction (X \times X)) \times (D \restriction (X \times X))$, which is closed. $H$, the image of this set under $j \times j$, is therefore closed.

$J$, the union of two closed sets, is closed. □

**Lemma 3.3.** *$G\widetilde{G} = D \restriction (X \times X)$. $\widetilde{G}G = \{(j(\alpha,\beta,\gamma), j(\alpha,\delta,\epsilon)) | (\alpha,\beta,\gamma) \in F$ and $(\alpha,\delta,\epsilon) \in F\}$. $H = \widetilde{H}$. $H^{(2)} = D \restriction (Y \times Y)$. $GH = \{(\alpha, j(\beta,\alpha,\gamma)) | (\alpha,\beta,\gamma) \in F\}$. $GH\widetilde{G} = E$. $\widetilde{G}G$ and $I$ are closed.*

*Proof.* All assertions except the one regarding closedness of $\widetilde{G}G$ and $I$ are verified by straightforward calculations. That $F$ is invariant under $t_{12}$ is used to show That $H = \widetilde{H}$ and that $E \subseteq GH\widetilde{G}$.

The proof that $\widetilde{G}G$ is closed is parallel to the proof that $H$ is closed. According to the first part of this lemma, $\widetilde{G}G = [j \times j](t_{24}((D \restriction (X \times X)) \times X \times X \times X \times X) \cap (F \times F))$.

$I$, the union of four closed sets, is closed. □

**Lemma 3.4.** *$I$ and $J$ are equivalence relations.*

*Proof.* These relations are reflexive and symmetric, so their transitive closures are equivalence relations. Thus, the lemma is equivalent to the assertion that $I = \overline{I}$ and $J = \overline{J}$. For any relation $K$, $K^{(2)} = K$ is sufficient for $K = \overline{K}$. In the following calculations of $I^{(2)}$ and $J^{(2)}$, composition of relations is distributed over unions. Terms that evaluate by identities that were calculated in lemma 3.3 to a previous term or its sub-relation, are omitted from the expansion by terms in the pentultimate step of each calculation.

$$
\begin{aligned}
I^{(2)} &= (D \cup G \cup \widetilde{G} \cup \widetilde{G}G)(D \cup G \cup \widetilde{G} \cup \widetilde{G}G) \\
&= (D \cup G \cup \widetilde{G} \cup \widetilde{G}G) \cup (G \cup G\widetilde{G} \cup G\widetilde{G}G) \cup (\widetilde{G} \cup \widetilde{G}G \cup \widetilde{G}\widetilde{G} \cup \cup \widetilde{G}\widetilde{G}G) \\
&\quad \cup (\widetilde{G}G \cup \widetilde{G}GG \cup \widetilde{G}G\widetilde{G} \cup \widetilde{G}G\widetilde{G}G) \\
&= D \cup G \cup \widetilde{G} \cup \widetilde{G}G \\
&= I. \qquad (3.3)
\end{aligned}
$$

$$
\begin{aligned}
J^{(2)} &= (D \cup H)(D \cup H) \\
&= (D \cup H) \cup (H \cup H^{(2)}) \\
&= D \cup H \\
&= J.
\end{aligned}
$$

□

*Proof of theorem 2.1.* Lemmas 3.2–3.4 show that the each of the relations $I$ and $J$ on $\mathcal{N} \times \mathcal{N}$, is an equivalence relation that has a closed graph. It remains to be shown that that $E = \overline{I \cup J} \cap (X \times X)$. Note that, since $D \subseteq I \cup J$, $I \cup J \subseteq (I \cup J)^{(2)} \subseteq (I \cup J)^{(3)} \subseteq \ldots$ Hence, if $(I \cup J)^{(n)} = (I \cup J)^{(n+1)}$, then $(I \cup J)^{(n)} = \overline{I \cup J}$.

The following calculation shows that $(I \cup J)^{(5)} = (I \cup J)^{(6)}$. The calculation is done recursively, according to the following recipe at each stage $n > 1$:

(1) Begin with the equation $(I \cup J)^{(n+1)} = (I \cup J)(I \cup J)^{(n)}$.
(2) Rewrite $(I \cup J)$ as $D \cup G \cup \widetilde{G} \cup \widetilde{G}G \cup H$ according to (3.2), rewrite $(I \cup J)^{(n)}$ according to the result of the previous step, and then distribute composition of relations over union in the resulting equation.
(3) For each identity stated in lemma 3.3, and for each identity that, for some $K \in \{G, \widetilde{G}, H\}$, equates a composition $KD$ or $DK$ of $K$ and $D$ (or a restriction of $D$ to a product set of which $K$ is a subset) to $K$, do as follows: Going from left to right, apply the identity wherever possible.[9] Repeat this entire step (consisting of one pass per identity) until no further simplifications are possible.
(4) Delete compositions of relations that include terms $KL$ such that the range of $K$ and the domain of $L$ (viewed as correspondences) are disjoint, in which case the term denotes the empty relation. Delete $D \restriction (X \times X)$ (occurring as a term by itself), of which $D$ is a superset.
(5) Delete each term of form $[K]\widetilde{G}[L]$ (resp. $[K]G[L]$) from a union in which the corresponding term for its superset, $[K]\widetilde{G}E[L]$ (resp. $[K]EG[L]$) also appears. (One or both of the bracketed sub-terms may be absent from both terms in the pair.) Delete $D$ (occurring as a term by itself) from every union that contains both $D \restriction (Y \times Y)$ and $E$, since $D \subseteq D \restriction (Y \times Y) \cup E$.

[9] Let $P = D \restriction (X \times X)$ and $Q = D \restriction (Y \times Y)$. Identities are applied in the following order at each stage of the recursion: $DD = D$, $DE = E$, $DG = G$, $D\widetilde{G} = \widetilde{G}$, $DH = H$, $DP = P$, $DQ = Q$, $ED = E$, $EE = E$, $EP = E$, $GD = G$, $G\widetilde{G} = P$, $GH\widetilde{G} = E$, $GQ = G$, $\widetilde{G}D = \widetilde{G}$, $\widetilde{G}P = \widetilde{G}$, $HD = H$, $HH = Q$, $HQ = H$, $PD = P$, $PE = E$, $PG = G$, $PP = P$, $QD = Q$, $Q\widetilde{G} = \widetilde{G}$, $QH = H$, $QQ = Q$.

(6) Reorder terms lexigraphically, in the order $D < D \restriction (Y \times Y) < E < G < \widetilde{G} < H$. Delete repeated terms.

$$
\begin{aligned}
(I \cup J) &= D \cup G \cup \widetilde{G} \cup \widetilde{G}G \cup H \\[2ex]
(I \cup J)^{(2)} &= D \cup D \restriction (Y \times Y) \cup G \cup GH \cup \widetilde{G} \cup \widetilde{G}G \cup \widetilde{G}GH \\
&\quad \cup H \cup H\widetilde{G} \cup H\widetilde{G}G \\[2ex]
(I \cup J)^{(3)} &= D \restriction (Y \times Y) \cup E \cup EG \cup GH \cup \widetilde{G}E \cup \widetilde{G}EG \cup \widetilde{G}GH \\
&\quad \cup H \cup H\widetilde{G} \cup H\widetilde{G}G \cup H\widetilde{G}GH
\end{aligned}
$$

$$
\begin{aligned}
(I \cup J)^{(4)} &= D \restriction (Y \times Y) \cup E \cup EG \cup EGH \cup \widetilde{G}E \cup \widetilde{G}EG \cup \widetilde{G}EGH \\
&\quad \cup H \cup H\widetilde{G}E \cup H\widetilde{G}EG \cup H\widetilde{G}GH
\end{aligned} \tag{3.4}
$$

$$
\begin{aligned}
(I \cup J)^{(5)} &= D \restriction (Y \times Y) \cup E \cup EG \cup EGH \cup \widetilde{G}E \cup \widetilde{G}EG \cup \widetilde{G}EGH \\
&\quad \cup H \cup H\widetilde{G}E \cup H\widetilde{G}EG \cup H\widetilde{G}EGH \\[2ex]
(I \cup J)^{(6)} &= D \restriction (Y \times Y) \cup E \cup EG \cup EGH \cup \widetilde{G}E \cup \widetilde{G}EG \cup \widetilde{G}EGH \\
&\quad \cup H \cup H\widetilde{G}E \cup H\widetilde{G}EG \cup H\widetilde{G}EGH \\
&= (I \cup J)^{(5)}
\end{aligned}
$$

Thus $\overline{I \cup J} = (I \cup J)^{(5)}$. Note that $D \restriction (Y \times Y)$, $G$, $\widetilde{G}$, $H$ and all relations of form or $\widetilde{G}Q$ or $HQ$ or $QG$ or $QH$ (where variable $Q$ ranges over compositions of $G$, $\widetilde{G}$, $H$, and $E$), are disjoint from $X \times X$. Therefore, from the calculation in (3.4) of $(I \cup J)^{(5)}$ as a union of $E$ with such relations, it follows that $\overline{I \cup J} \cap (X \times X) = E$. □

## 4. The general case of an uncountable Polish space

**Corollary 4.1.** *Let $\Omega$ be an uncountable Polish space, and let $E_0 \subseteq \Omega \times \Omega$ be a $\mathbf{\Sigma}^1_1$ equivalence relation. There are $\mathbf{\Delta}^1_1$ equivalence relations $E_1$ and $E_2$, a $\mathbf{\Delta}^1_1$ subset $Z$ of $\Omega$, and a Borel isomorphism $e\colon \Omega \to Z$ such that $E_0$ is Borel equivalent to $(\overline{E_1 \cup E_2}) \restriction (Z \times Z)$ by $(\omega, \omega') \in E_0 \iff (e(\omega), e(\omega')) \in (\overline{E_1 \cup E_2})$.*

*Proof.* There is a Borel isomorphism $k\colon \Omega \to \mathcal{N}$.[10] Setting $x(\alpha_0, \alpha_1, \ldots) = (\alpha_0 + 1, \alpha_1, \ldots)$ specifies a homeomorphism from $\mathcal{N}$ to the open set $X$ of theorem 2.1. Define $e = k^{-1} \circ x \circ k$ and $Z \subseteq \Omega$ by $Z = e(\Omega)$. If $E \subset X \times X$ is defined by $(\omega, \omega') \in E_0 \iff (x \circ k(\omega), x \circ k(\omega')) \in E$, then $E$ is a $\mathbf{\Sigma}^1_1$ equivalence relation.[11] Let $I$ and $J$ be the closed equivalence relations defined in (3.2), and define $(\omega, \omega') \in E_1 \iff (k(\omega), k(\omega')) \in I$ and $(\omega, \omega') \in E_2 \iff (k(\omega), k(\omega')) \in J$. Now the corollary follows immediately from theorem 2.1. □

[10]Moschavakis [2009, Theorem 1G.4, p. 41; note 4, p. 46].

[11]Moschavakis [2009, Theorem 2B.2, p. 54].

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Department of Economics, The Pennsylvania State University, University Park, PA 16802, USA

*E-mail address*: eug2@psu.edu