The Effect of Corruption on Bidding Behavior in First-Price Auctions*

Leandro Arozamena[†] and Federico Weinschelbaum[‡]

November 2004 (First version: April 2004)

Abstract

If the owner of an object sells it through an auction run by an agent of hers, there is scope for corruption. We analyze the effect of a particular form of corruption on bidding behavior in a single-object, private-value sealed-bid auction with risk-neutral bidders. Bidders believe that, with a certain probability, the auctioneer has reached an agreement with one of the bidders by which, after receiving all bids, (i) she will reveal to that bidder all of her rivals' bids, and (ii) she will allow that bidder to change her original bid upwards or downwards. In a second-price auction, the possibility of this form of corruption has no effects. In a first price auction, however, honest bidders can become more or less aggressive than they would be without corruption, or their behavior can remain unchanged. We identify sufficient conditions for each of the three possibilities. Finally, we analyze the consequences of this form of corruption on efficiency, bidders' welfare and expected revenue. Our results are useful as well for the case, unrelated to corruption, where one of the bidders is granted a right of first refusal.

Keywords: Auctions; Corruption, Right-of-first-refusal.

JEL classification: C72, D44, L14.

*We are grateful to Walter Cont, participants at various seminars and especially to Paul Klemperer for their comments on a previous version of this paper, and to Maximiliano Appendino for excellent research assistance.

[†]Universidad Torcuato Di Tella, Business School and Economics Dept. E-mail: larozamena@utdt.edu .

[‡]Universidad de San Andrés, Economics Department. E-mail: fweinsch@udesa.edu.ar .

1 Introduction

Most auctions are not organized and run by the owner of the object on sale, but are usually in charge of an agent of hers. This separation between the owner and the auctioneer generates the scope for corruption. The auctioneer may be tempted to enter into an agreement with one of the bidders to tilt the auction in her favor.

In this paper, we examine a particular form of corruption in a single-object, private-value sealed-bid auction with risk-neutral bidders. We focus on the case where corrupt dealings between the auctioneer and any bidder consist of revealing information on how much other participants in the auction have bid. Specifically, we assume that bidders believe that the auctioneer has reached, with a certain probability, an agreement with one of them by which, after receiving all bids, (i) she will reveal to that bidder all of her rivals' bids, and (ii) she will allow that bidder to change her original bid upwards or downwards if she wishes to do so. Bidders believe that it is possible, then, that bids are not submitted simultaneously but sequentially. We are interested in the consequences that the possibility that such a deal exists has on how bidders –particularly those not involved in the corrupt agreement– behave and on the auction's welfare and revenue properties.

Our exercise can be interpreted in more than one way. In this paper, we are vague about how the corrupt agreement between the auctioneer and one bidder has come to exist, but concetrate on its influence on the auction itself. We assume that the agreement is made before the auction takes place. We believe this analysis is especially relevant in those cases where the prior agreement is part of a long-term relationship, or one that exceeds the context of the individual auction under study. This is particularly relevant for the case of public procurement,[1] where officials in charge of the auction may have been captured by parties interested not just in one but in a series of public contracts. Alternatively, we could consider our setup as part of a more complete, two-stage model of corruption, where, first, who the favored bidder will be and how she will compensate the auctioneer are determined, and, second, the auction takes place. Our analysis is useful for any such model. We discuss further the interpretation of our

[1] Although we model the auction as a selling mechanism, all of our results apply as well to procurement auctions.

setup in Section 2.

We are particularly interested in the effect of the possible existence of this form of corruption on how honest bidders behave in the auction. We want to ascertain how an honest bidder will react when the probability that her bid be revealed to a rival grows; i.e., whether she will bid more or less aggressively. This will turn out to be crucial for the effects of this form of corruption on the auction's result. Needless to say, in answering that question we will determine how the behavior of honest bidders when corruption is possible compares to their behavior in standard auctions without corruption.

In a second-price auction, since bidding her own valuation is a weakly dominant strategy for every bidder, this form of corruption has no effect on bidding behavior or on the auction's result. In a first-price auction, however, the situation is more complex. We show below that an honest bidder can become more or less aggressive or her behavior can remain unchanged when she perceives as more likely that a rival is colluding with the auctioneer. In Proposition 1, we provide sufficient conditions for each of the three possibilities. Namely, if F is the cumulative distribution function of a bidder's valuation for the object being auctioned and f is the corresponding density, if the ratio F/f is strictly convex (respectively, strictly concave, linear) in the valuation, the honest bidder will become more aggressive (respectively, less aggressive, equally aggressive) as corruption becomes more probable. Furthermore, we establish the extent to which the most commonly used distributions satisfy one of those conditions. In passing, we find a result that may be of independent interest, unrelated to corruption: the curvature of the ratio F/f determines as well whether bidding functions in standard, first-price auctions are strictly concave, linear or strictly convex.

We then move on to assess the effect of this form of corruption on efficiency, revenue and bidders' welfare. If the ability to bid last knowing her rival bids is conferred to a bidder with positive probability, the first-price auction will not be efficient. In addition, we show that if any of our sufficient conditions holds, honest bidders' expected utilities are strictly decreasing in the probability of corruption (Proposition 2). The coalition between the auctioneer and the favored bidder always generates a positive surplus to split between the two by reaching a corrupt deal (Proposition 3). However, whether this surplus falls or grows with the probability that honest bidders attach to the existence of the deal depends on whether the latter become more or less aggressive, as described above. Finally, expected revenue cannot be higher with than without corruption in "regular" cases, but we provide an example of a "nonregular" case where corruption raises the expected price received by the seller.

Our general setup includes, as a particular case, a relevant application that is unrelated to corruption. If it is common knowledge that there is a favored bidder and every party to the auction knows who that favored bidder is, that bidder publicly enjoys the right to match the highest bid that any of her rivals may submit. This is exactly the arrangement known as *right of first refusal*, usually employed, for example, by firms to favor preferred suppliers with whom they have long-term relationships, in the purchase of a partnership by one of the current parners, and in professional sports. In this alternative context, our analysis attempts to determine when bidders respond more or less aggressively when facing a rival that enjoys this privilege, and its consequences on expected revenue.[2]

Other papers have dealt with related forms of corruption. In particular, Jones and Menezes (1995), Lengwiler and Wolfstetter (2000), Burguet and Perry (2002) and Menezes and Monteiro (2003) consider cases where a corrupt arrangement, just as in this paper, implies revealing to one of the bidders what her rivals have bid. However, their analyses differ from the one we present in several respects. In Jones and Menezes (1995), bidders are not aware of the possibility of corruption when choosing their bids, so bidding behavior remains unaltered by assumption.[3] Lengwiler and Wolfstetter (2000) and Menezes and Monteiro (2003) consider situations where the auctioneer approaches the winning bidder offering the chance to lower her bid (while still winning the auction) in exchange for a bribe. In this paper, on the contrary, who the auctioneer may conspire with is independent of the auction's result. That is, their agreement is reached before the auction takes place. In addition, the favored bidder is allowed to raise or lower her bid according to her interests. Burguet and Perry (2002) is closest to our analysis. They study

[2] The right of first refusal has been studied in Burguet and Perry (2003), Bichchandani et al. (2003) and Choi (2003).

[3] In addition, they consider a setting where bidders draw their valuations from uniform distributions. We prove below that, for such distributions, even if bidders were aware of the possibility of corruption their behavior would remain unaltered.

several variants as to the exact form corruption may take. They are particularly interested in the effect of the bargaining game between the auctioneer and the dishonest bidder on the auction's result. Only one of the variants they deal with involves allowing the favored bidder to freely revise her bid upwards or downwards, but, as in all the remaining cases, they only focus on the two-bidder case when corruption is certain to all parties. Furthermore, they do not concentrate on the effect corruption has on equilibrium bids. Here, we allow for the existence of corruption to be uncertain, extend the analysis to the general, N-bidder case and are able to provide a fuller characterization of the effect of corruption on bidding behavior.[4] Finally, Compte et al. (forthcoming), consider the situation where the auctioneer reveals the winning bid to all participants and allows them to compete for the chance to resubmit their bids. Their focus, then, is on bribing competition and its effects.

While our interest here limits to the case where a single-dimensional object is auctioned, another related strand in the literature focuses on the possibility of corruption in multidimensional procurement auctions. Specifically, in addition to the price, the object being procured has a quality dimension that affects the procurer's welfare. The procurer delegates the assessment of quality on an agent, and the scope for corruption is thereby created. Celentani and Ganuza (2002) and Burguet and Che (2004) examine different forms of corruption in that procurement environment.

The paper is organized as follows, in Section 2 below, we present the auctioning context and provide sufficient conditions to characterize the effect of corruption on honest bidders' behavior. Then, we ascertain the extent to which standard distributions satisfy each of those sufficient conditions. In Section 3 we study the consequences of corruption on welfare and revenue. Finally, we conlcude in Section 4.

[4] However, they consider the case where bidder's valuations are distributed asymmetrically, while we concetrate on a symmetric environment.

2 The Model

The owner of a single, indivisible object is selling it through an auction[5] organized and run by an agent of hers. There are N bidders whose valuations v_i ($i = 1, ...N$) for the object are distributed identically and independently according to the c.d.f. F with support on the interval $[\underline{v}, \overline{v}]$ and a density f that is positive on the whole support. The context is, then, one of independent private values. We will assume below that F is logconcave.[6] For future use, let $\alpha(v_i) = F(v_i)/f(v_i)$. Note that the logconcavity of F means that α is increasing. Both bidders, the auctioneer and the owner are risk neutral.

We will focus below on sealed-bid auctions with no reserve prices. Given the fact that the owner of the object being sold and the auctioneer are not the same, there is scope for corruption. The auctioneer may tilt the auction in favor of one of the bidders[7] in exchange for a compensation. The exact form this collusion may take is open to multiple possibilities. Here, we concentrate on one particular case. We assume that before the auction takes place the auctioneer may have reached an agreement with one bidder before the auction takes place whereby she will reveal to that bidder all of her rivals' bids[8], and then allow her to modify her original bid upwards or downwards if she wishes to do so.

In what follows, then, any given bidder may be in one of two situations. If she is colluding with the auctioneer, she will be sure that none of her rivals is colluding at the same time — we rule out the possibility that the auctioneer colludes with more than one bidders. If she is not colluding with the auctioneer, i.e. if she is "honest", she believes that one of her rivals is colluding with probability p. Furthermore, we will consider a situation that is completely symmetric among honest bidders. Any honest bidder believes, then, that with probability p she

[5] All of our results, however, are applicable as well to the case of procurement auctions.

[6] Logconcavity of the c.d.f. function holds for most well known distributions, such as the uniform, normal, logistic, extreme value, chi-squared, chi, exponential, Laplace, Pareto and any truncation of these distributions. For details see Bagnoli and Bergstrom (1989).

[7] We are not allowing for the possibility of collusion between the bidders (see Hendricks and Porter, 1989, for a general analysis of this phenomenon), but only between one bidder and the auctioneer.

[8] It is immaterial whether the auctioneer reveals to the favored bidder all or only the highest of her rivals' bids.

is facing one dishonest bidder and $(N-2)$ honest bidders, all of whom are in the exact same situation as she is —each of them believes that, with probability p, there is a favored rival. And she believes that, with probability $(1-p)$ she is facing $(N-1)$ rivals in her exact same situation.

Several interpretations of this setup are possible. We could think of the prior agreement between the auctioneer and the favored bidder as part of a long-term relationship, or one that exceeds the context of the individual auction under study. This is particularly relevant for the case of public procurement, where officials in charge of the auction may have been captured by parties interested not just in one but in a series of public contracts. Examples where public officials have –or are perceived to have– long-standing relationships with specific firms in the private sector are abundant.

Alternatively, our analysis may be viewed as part of a more complete study of how corruption affects the result of an auction. A two-stage model could be constructed where, first, who the favored bidder may be and what kind of agreement she will reach with the auctioneer are determined. Then, the auction would take place. Within this, more general, approach, we concentrate on the second stage. We do not model how the auctioneer and the favored bidder bargain when deciding if they will collude. Still, one of the most relevant considerations in any such bargaining game will be what would happen if honest bidders attached a given probability to the fact that one rival will be favored, i.e. if the dishonest bidder would face more or less aggressive competitors. So our results below will be crucial in any study of this form of corruption. In particular they determine which is the total surplus that this coalition will have to split. Furthermore, examining how the auctioneer and a bidder bargain is open to many modelling alternatives. Different assumptions could be made, for instance, on what knowledge the auctioneer has of the bidder's valuation, or on how much bargaining power each party has. Our own results will be relevant to any such specification. We will assume, though, that the auctioneer and the colluding bidder bargain efficiently: they will reach an agreement whenever they can both gain by doing so.

A related question is whether honest bidders who face a high probability of corruption would stay in the auction. In our analysis, participating in the auction will be costless, so there is no loss of generality in assuming they will stay. If participating were costly, it would still be crucial to know what would happen if all honest bidders stayed. Our results should provide an answer to such a question.

Our setup could be understood in a more specific and straightforward way as well. We may assume that, before the auctioneer may approach any of them, all bidders believe that the auctioneer is corrupt —and then expect her to make an approach to one of the bidders— with probability s. Then, if a bidder has not been approached, this may mean that the auctioneer is not corrupt or that the auctioneer has approached a rival. The updated probability that an honest bidder attaches to the fact that the auctioneer be corrupt is then $p = \frac{(N-1)s}{N-s}$.

Independently of the exact interpretation of our model one may prefer, though, all that will matter below is that each honest bidder believes there is a favored bidder with probability p. Our main object of analysis will be how an honest bidder, who believes that this form of corruption happens with probability p changes her behavior as p varies. That is, we are interested in whether honest bidders become more or less aggressive when they perceive an increase in the probability of corruption. Reaching such a conclusion, of course, would let us as well compare, as a particular case, the differences in honest bidders' behavior between the cases where $p > 0$ and $p = 0$, the latter being a standard auction without corruption.

The effect of corruption on bidding behavior will clearly differ according to the sealed-bid auction format. We consider the second- and the first-price auction in what follows.

2.1 The Second-Price Auction

In a second price auction, bidding her own valuation is a weakly dominant strategy for every bidder without corruption, and, of course, remains so once the possibility of corruption appears. Thus, bidding behavior remains unaltered for all honest bidder. As for a dishonest bidder, she will compare the highest rival bid to her own valuation. If the former is higher, she will have no interest in winning the auction. If the latter is higher, she will then submit a bid high enough to beat all honest rivals and win the object at a price equal to the highest honest bid. But this is exactly the same result she would generate by originally submitting a bid equal to her own valuation and leaving it unaltered: bidding her valuation remains optimal for the dishonest bidder.

Hence, the result of the auction will not change. It remains true that the bidder with the highest valuation wins in equilibrium, and she pays the highest valuation among her rivals', exactly as occurs when $p = 0$. The form of corruption we are examining has no effect in the case of the second-price auction.[9]

This result can be useful to analyze, as we will next, the first-price auction. Given that the second price auction is unaffected by the value of p, and given that —by the Revenue (and Payoff) Equivalence Theorem— the two auction formats are equivalent when $p = 0$, it follows that, for any $p > 0$, comparing the first- and the second-price auctions for that value of p is exactly the same as comparing the first-price auction for that value of p with the first-price auction without corruption (i.e. with $p = 0$).

2.2 The First-Price Auction

In a first-price auction, however, the possibility of corruption has a significant effect. If a bidder colludes with the auctioneer and learns her rivals' bids, she may have an incentive to change her original bid.

Remark 1 *When a first-price auction is used, a particular case of our model can be given a different interpretation. If $p = 1$, then all honest bidders are certain that there is a favored rival. In addition suppose that the identity of the favored bidder is known to all.*[10] *Then, it is common knowledge that one specific bidder will have the right to match the highest bid made by any of her rivals. This situation describes exactly what is usually called "right-of-first-refusal." Although it is not related to corruption, all our results below apply in this particular case as well.*

Let d be the dishonest bidder, and let b^h be the highest bid submitted by an honest bidder (i.e. $b^h = \max_{i \neq d} b_i$). Let b_d be d's revised bid. If, according to her original bid, d is winning

[9] These conclusions could change if we allowed for corrupt coalitions where the auctioneer and more than one bidder could be involved. See Lengwiler and Wolfstetter (2000) for an examination of this issue.

[10] In our symmetric context, it is irrelevant whether the identity of the favored bidder is public or secret.

the auction, then she will revise her bid downwards and set $b_d = b^h + \varepsilon$.[11] If she is losing the auction, there are two possibilities. If $v_d > b^h$, then she will again set $b_d = b^h + \varepsilon$ and win the auction. If $b^h > v_d$, then she will submit a revised bid $b_d < b^h$ (she may leave her original bid unchanged), and lose.

Viewing the auction from the standpoint of an honest bidder that may face a colluding rival, this means that the former will have to bid above the latter's valuation to win. In other words, she will be competing against the dishonest rival's valuation instead of competing against her bid.

Let $b_i^p : [\underline{v}, \overline{v}] \longrightarrow \mathbb{R}$ be bidder i's bidding function when i is honest and believes that there is a favored rival with probability p. For convenience, we will use the inverse of her bidding function, $\phi_i^p(.)$. We will look for an equilibrium in the first-price auction that is symmetric among honest bidders. That is, we will look for a Bayesian equilibrium where $b_i^p(v) = b^p(v)$ for all i such that i is honest. An honest bidder i with valuation v_i who faces a dishonest rival with probability p and all of whose honest rivals bid according to the inverse bidding function $\phi^p(.)$ that is strictly increasing will choose her bid by solving the following expected utility maximization problem

$$\max_b \, (v_i - b) \left[pF^{(N-2)}(\phi^p(b))F(b) + (1-p)F^{(N-1)}(\phi^p(b)) \right] \tag{1}$$

By bidding b, an honest bidder will defeat any honest rival with probability $F^{(N-2)}(\phi^p(b))$, and she will defeat the dishonest bidder (if there is one) with probability $F(b)$. Then, the expression in brackets is the probability of winning the auction by bidding b for an honest bidder. The first-order condition resulting from (1) is[12]

$$v_i - b = \frac{pF(\phi^p)F(b) + (1-p)F^2(\phi^p)}{p\left[(N-2)\,F(b)f(\phi^p)\phi^{p\prime} + F(\phi^p)f(b)\right] + (1-p)\,(N-1)\,F(\phi^p)f(\phi^p)\phi^{p\prime}}$$

In a symmetric equilibrium we will have $v_i = \phi^p(b)$, and the first-order condition becomes

$$\phi^p(b) - b = \frac{pF(\phi^p)F(b) + (1-p)F^2(\phi^p)}{p\left[(N-2)\,F(b)f(\phi^p)\phi^{p\prime} + F(\phi^p)f(b)\right] + (1-p)\,(N-1)\,F(\phi^p)f(\phi^p)\phi^{p\prime}} \tag{2}$$

[11] We may assume that $b_d = b^h$ and, in the event of a tie, the auctioneer chooses the winner. Therefore, she will always choose the bidder she is trying to favor.

[12] To economize on notation, we will omit the argument of the inverse bidding function ϕ^p where possible.

This differential equation (2) characterizes the inverse bidding function $\phi^p(b)$. Note that $\phi^0(b)$ is the standard inverse bidding function in a first-price auction without corruption. In addition, as mentioned in Remark 1, $\phi^1(p)$, the inverse bidding function with the certainty that there is a favored bidder, is the relevant one for the particular case of the right-of-first-refusal.

It is straightforward that, as usual in the first-price auction, $\phi^p(\underline{v}) = \underline{v}$ for all $p \in [0,1]$. Our main objective is to establish how the function $\phi^p(b)$ changes with p. That is, we want to find out whether it is possible to say, when the probability attached by honest bidders to the existence of an honest rival goes from p_0 to p_1 (with $p_1 > p_0$), that honest bidders become uniformly more aggressive ($\phi^{p_1}(b) < \phi^{p_0}(b)$ for all $b > \underline{v}$ in their common support), uniformly less aggressive ($\phi^{p_0}(b) < \phi^{p_1}(b)$ for all $b > \underline{v}$ in their common support) or keep their behavior unaltered ($\phi^{p_0}(b) = \phi^{p_1}(b)$ for all b). In the remainder of this section, we provide sufficient conditions for each of these three possibilities.

As we mentioned above, given the logconcavity of the cdf F, we know that $\alpha(v)$ is increasing. We will show in what follows that the key fact to ensure that a higher perception of corruption generates more, equal or less aggressivenes is whether $\alpha(v)$ is strictly convex, linear or strictly concave. The following Lemma is a first step towards that result.

Lemma 1 *If $\alpha(v)$ is strictly convex (linear, strictly concave) then $\alpha(b) < \frac{\alpha(\phi^p)}{\phi^{p\prime}}$ (respectively, $\alpha(b) = \frac{\alpha(\phi^p)}{\phi^{p\prime}}$, $\alpha(b) > \frac{\alpha(\phi^p)}{\phi^{p\prime}}$) for all $b > \underline{v}$ and for all $p \in [0,1]$.*

The proof is provided in the Appendix.

Remark 2 *Lemma 1 is useful to establish a result that is different but related to our main question. Let $\widetilde{\phi}(b)$ be the inverse bidding function of an honest bidder who is certain that all of her rivals will learn her bid and then be allowed to rebid*[13]*. Thus, such a bidder would have to bid higher than the highest among her rivals' valuations to win the auction. It is straightforward to show that the analogue of (2) in this case is:*

$$\widetilde{\phi}(b) - b = \frac{F(b)}{(N-1)f(b)} = \frac{\alpha(b)}{N-1} \tag{3}$$

[13] This case is equivalent to ours only if $N = 2$ and $p = 1$.

Corollary 1 *If $\alpha(v)$ is strictly convex (linear, strictly concave) then $\widetilde{\phi}(b) < \phi^p(b)$ (respectively, $\widetilde{\phi}(b) = \phi^p(b)$, $\widetilde{\phi}(b) > \phi^p(b)$) for all $b > \underline{v}$ in their common supports, and for all $p \in [0,1]$.*[14]

Proof. The result follows immediately by comparing (2) and (3) and using Lemmas 1 and 2 in the Appendix. ■

We are now ready to state the main result of this section. Proposition 1 provides sufficient conditions to determine the effect of a change in the probability attached to corruption on the behavior of an honest bidder.

Proposition 1 *If $\alpha(v)$ is strictly convex (linear, strictly concave) then $\phi^{p_0}(b) < \phi^{p_1}(b)$ (respectively, $\phi^{p_0}(b) = \phi^{p_1}(b)$, $\phi^{p_0}(b) > \phi^{p_1}(b)$) for all $b > 0$ in their common supports and for any p_0, p_1 such that $0 \leq p_0 < p_1 \leq 1$.*

We prove this proposition in the Appendix. It follows that $\alpha(v)$ being strictly concave (convex) determines that corruption generates more (less) aggressiveness, while $\alpha(v)$ being linear makes the possibility of corruption irrelevant for the behavior of an honest bidder.

It turns out, in addition, that the curvature of $\alpha(v)$ determines as well the curvature of bidding functions in standard, symmetric first price auctions in the absence of corruption. This result, which –being unrelated to corruption– may be of independent interest, is established in the following Corollary, proved in the Appendix.

Corollary 2 *If $\alpha(v)$ is strictly convex (linear, strictly concave) then $b^0(v)$, the bidding function in a standard, symmetric first-price auction, is strictly concave (respectively, linear, strictly convex).*

Let us use Corollary 2 to gain a better understanding of the results in Proposition 1. To simplify, assume N=2 and $\underline{v} = 0$, and compare $\phi^0(b)$ with $\phi^1(b)$. From (2), $\phi^0(b) - b = \frac{\alpha(\phi^0(b))}{\phi^{0\prime}(b)}$ and $\phi^1(b) - b = \alpha(b)$. Certainty of corruption shifts the rival's valuation that the honest bidder faces marginally (the valuation she ties with by bidding b) from $\phi^0(b)$ to b, which is lower.

[14] Porter and Shoham (forthcoming), independently proved this result for the particular case where $p = 0$.

Whether the honest bidder will then have incentives to be more or less aggressive depends on the relationship between $\frac{\alpha(\phi^0(b))}{\phi^{0\prime}(b)}$ and $\alpha(b)$. From the logconcavity of $F(v)$, we know that $\alpha(b) < \alpha(\phi^0(b))$. That is, if the dishonest bidder's inverse bidding function had a slope at b which equaled $\phi^{0\prime}(b)$, then the honest bidder would necessarily become more aggressive with corruption. However, there is a counteracting effect: with corruption, the dishonest bidder bids her true valuation, so her inverse bidding function has a slope equal to 1. Since $\phi^{0\prime}(b) > 1$, this change in the marginal behavior of her rival provides the honest bidder with incentives to become less aggressive.

Suppose $\alpha(v)$ is linear. Corollary 2 tells us, then, that ϕ^0 is linear, so $\frac{\alpha(\phi^0)}{\phi^{0\prime}} = \frac{\alpha(\phi^{0\prime}b)}{\phi^{0\prime}}$. From the linearity of $\alpha(v)$, the last expression equals $\alpha(b)$. That is, the ratio $\alpha(\phi^0)/\alpha(b)$ is constant and equal to the (constant) slope of the inverse bidding function, $\phi^{0\prime}$. The two effects mentioned above exactly offset each other in this case. Proposition 1, thus, tells us that the first effect outweighs the second in the case where $\alpha(v)$ is strictly convex, and the opposite result obtains when $\alpha(v)$ is strictly concave. It also tells us that this generalizes to any number of bidders and any probabilities p_0, p_1 such that $p_1 > p_0$.

We have provided sufficient conditions to characterize how bidding behavior is affected by the possibility of a corrupt arrangement between the auctioneer and one of the bidders. As noted, how an honest bidder's behavior will be influenced by the possible existence of corruption will be determined by the concavity or convexity of $\alpha(v)$. The next natural question is, of course, if most of the commonly used distribution functions imply that $\alpha(v)$ is concave or convex. The next subsection briefly deals with this issue.

2.3 The effect of corruption for commonly used distributions

We have shown above that how a bidder who does not take part in any arrangement with the auctioneer will alter her behavior in the presence of corruption will depend crucially on the convexity or concavity of $\alpha(v) = \frac{F(v)}{f(v)}$.

It is easy to verify that $\alpha(v)$ being linear is equivalent to $F(v)$ being a power function distribution, i.e. $F(v) = \frac{(v-\underline{v})^k}{(\overline{v}-\underline{v})^k}$[15] for some $k > 0$. Bidding functions are given by $b^p(v) =$

[15] In this case, $\alpha(v) = \frac{(v-\overline{v})}{k}$. Naturally, a particular and very usual example of this family is the uniform

$\frac{1}{k(N-1)+1}\left[\underline{v}+k(N-1)v\right]$ for all $p \in [0,1]$. We can fully characterize, then, the family of distributions where our sufficient condition for corruption not to have any effect on honest bidding behavior applies.

Note that

$$\alpha'(v) = 1 - \frac{F(v)}{f(v)}\frac{f'(v)}{f(v)}$$

If F is strictly logconcave (logconvex), then $F(v)/f(v)$ will be strictly increasing (decreasing). By the same token, if f is strictly logconcave (logconvex), then $f'(v)/f(v)$ will be strictly decreasing (increasing). Recall that we have assumed that F is logconcave. We can combine these possibilities to provide simple sufficient conditions for the concavity or convexity of $\alpha(v)$.

Remark 3 *(a) If $f(v)$ is logconcave and decreasing ($f'(v) < 0$), then $\alpha(v)$ is convex.*

(b) If $f(v)$ is logconvex and increasing ($f'(v) > 0$), then $\alpha(v)$ is concave.

The exponential distribution is an example of part (a) in the Remark 3, while $F(v) = \frac{e^{-\lambda}}{1-e^{-\lambda}}\left(e^{\lambda v}-1\right)$ is an example of (b). In addition, it can be easily verified as well if $\alpha(v)$ is concave or convex for any of the standard distributions. Straightforward calculations show that in the cases of the logistic, Laplace and Pareto distributions, for instance, $\alpha(v)$ is strictly convex. Hence, for all of them the existence of corruption makes an honest bidder more aggressive. As for the normal distribution, however, $\alpha(v)$ is not strictly concave or convex on all of its support. Still, it is only strictly concave more than six standard deviations below its mean, and it is strictly convex in the remainder of the support. Hence, most truncations of the normal distribution would generate a strictly convex $\alpha(v)$ and more aggressiveness with more probable corruption.

3 Efficiency and Welfare

We have analyzed above how a specific form of corruption affects the behavior of honest bidders in first-price auctions. Assuming that the auctioneer may reach an agreement with one of the bidders by which the latter will be shown all of her rival's bids and will be allowed to resubmit

distribution.

her bid accordingly, we have provided sufficient conditions to assess how honest bidders adjust their bids when facing a rival that is possibly dishonest. Those conditions determine whether an honest bidder will behave more, equally or less aggressively than in the absence of corruption. Furthermore, we have evaluated the extent to which most commonly used distributions satisfy one of these conditions.

Let us emphasize again that, even though we have been very precise in terms of the advantages that corruption confers to a dishonest bidder in our analysis, we have been vague when referring to the negotiations between such a bidder and the auctioneer that lead to a corrupt arrangement. Our results, then, can be regarded as relevant to any specific model for such negotiation in the context of sealed-bid auctions.

In this section we analyze the effect of this form of corruption on efficiency, seller's revenue, and bidders' welfare. As mentioned in Subsection 2.1 above, the form of corruption we study has no effects in the case of a second-price auction. The second-price auction, then, preserves all its welfare properties. In particular, it is efficient, as happens in the case where $p = 0$, since the winner is always the bidder with the highest valuation. The seller's expected revenue and the expected utility of each of the bidders (honest or dishonest) are invariant to changes in p.

In what follows, hence, we limit our analysis to the case of the first-price auction. It will be useful, for expositional purposes, to split the total effect that an increasee in the value of p[16] has on the different welfare measures into two effects, the "direct effect" and the "perception effect". The direct effect is the one that an increase in the value of p has on the expected utility of any party to the auction holding the bidding functions of all honest bidders constant. It reflects, then, the fact that the existence of a dishonest bidder is more likely, but does not include any variation in optimal strategies. The perception effect, on the other hand, is the change in the expected utility of any party that is exclusively due to the fact that honest bidders change their behavior when they perceive that p grows. Notice that when $\alpha(v)$ is linear, the perception effect vanishes, since there is no change in how honest bidders behave.

[16] We use p as an indicator for the overall probability of corruption, although it is the probability that an honest bidder attaches to it. Recall from one of the interpretations in Section 2 that we could write p as a function of s, the probability that the auctioneer be corrupt ($p = \frac{(N-1)s}{N-s}$).

Let us start with efficiency. In the absence of corruption, and keeping all our other assumptions, it is well known that, just as the second-price auction, the first-price auction is efficient. A natural consequence of our previous analysis, however, is that the first-price auction will be inefficient when corruption is possible. Every honest bidder will shade her bid (i.e. her equilibrium bid will be lower than her actual valuation). Let v_d be the dishonest bidder's valuation, as above, and $v^h = \max_{i \neq d} v_i$. Given that honest bidders have increasing symmetric equilibrium strategies $b^p(v)$, the highest honest bid will be $b^h = b^p(v^h)$. As Figure 1 shows, if the dishonest bidder is allowed to examine bids and resubmit her own by the auctioneer and $v_d \in [b^p(v^h), v^h]$, she will win the auction in spite of the fact that her valuation is not the highest.

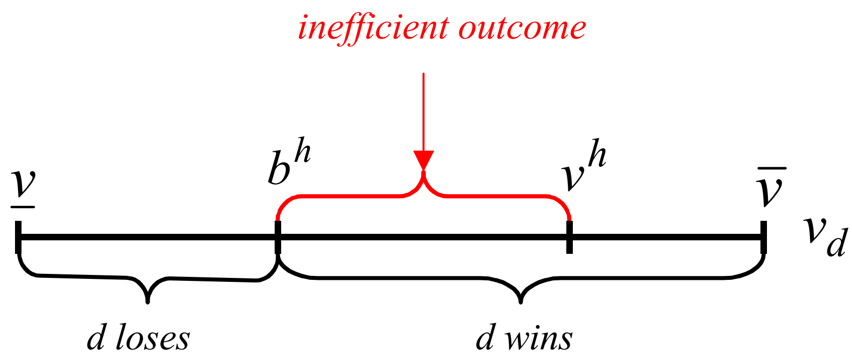

Figure 1

This means as well that the first-price auction is worse in efficiency terms than the second-price auction in the presence of corruption.

How does efficiency change when p varies? The direct effect on efficiency is always negative, for any initial value of p: the probability that a bidder wins despite not having the highest valuation grows with the probability of corruption, if we keep the behavior of honest bidders constant. The size of the inefficiency, however, for a given probability of corruption, is an increasing function of the difference between the valuation and the bid of the honest bidder that values the object the most, $(v^h - b^h)$. The perception effect affects the inefficiency only by changing this difference. From Proposition 1, we know that $(v^h - b^h)$ is constant in p if $\alpha(v)$ is linear, it grows with p if $\alpha(v)$ is strictly concave, and it falls with p if $\alpha(v)$ is strictly convex. Therefore, the perception effect on efficiency is zero in the first case, negative in the second and positive in the third. We thereby know, combining both effects, that inefficiency grows with p in the first two cases, but the sign of the total effect is not clear in the third.

Let us turn now to bidders' welfare. Regarding the expected utility of honest bidders, we know that the direct effect is always negative: given the behavior of honest rivals, a higher probability of corruption means that an honest bidder is more likely to lose in some cases where he would have won. This is a consequence of the fact that a dishonest bidder is allowed by the auctioneer to revise her original offer upwards to match the highest original bid, and it is monotone in p. As for the perception effect, it is not positive in the cases where honest bidders do not become less aggressive in the presence of corruption (that is, if $\alpha(v)$ is linear or convex). So in these cases, combining both effects, it is certain that corruption is detrimental to the expected utility of an honest bidder in the auction. With corruption, any honest bidder wins with a lower probability and, when she wins, she has to pay a (weakly) higher price.

When $\alpha(v)$ is concave the perception effect is nonnegative (positive for the case of strictly concave), since all honest rivals does not bidd more agressively (bid less agressively). We therefore have two effects of opposite sign. To assess the the total effect, we can use standard mechanism design tools. Let us construct the direct revelation mechanism that is equivalent to the first-price auction with probability of corruption (as believed by honest bidders) p. Let $x(v)$ be the probability with which an honest bidder that announces a valuation v wins, and let $T(v)$ be the expected price she has to pay. Then, the expected utility of an honest bidder with valuation v is $U(v) = vx(v) - T(v)$. By the envelope theorem, $U'(v) = x(v)$, so $U(v) = U(\underline{v}) + \int_{\underline{v}}^{v} x(s)ds$. Since $U(\underline{v}) = 0$, it follows that

$$U(v) = \int_{\underline{v}}^{v} x(r)dr = \int_{\underline{v}}^{v} \left[(1-p)F^{(N-1)}(r) + pF^{(N-2)}(r)F(b^p(r))\right] dr \tag{4}$$

When $\alpha(v)$ is concave, $b^p(v)$ does not increase with p (uniformly falls for $\alpha(v)$ strictly concave). Furthermore, $b^p(v) < v$. Then, the integrand of this expression is decreasing in p. The perception effect can never compensate the direct effect. We have thus proved the following proposition.

Proposition 2 *The expected utility of honest bidders' is monotonically decreasing in p when $\alpha(v)$ is either concave, linear or convex on $[\underline{v}, \overline{v}]$.*

For the cases where $\alpha(v)$ does not belong to any of the three groups (i.e. it is strictly concave for some values of v and strictly convex for others) we know that the honest bidders are always worse off with $p > 0$ than with $p = 0$. This follows directly from (4) and the fact that $b^p(v) < v$ for all $p > 0$ and $v > \underline{v}$.[17]

As regards the coalition between a dishonest bidder and the auctioneer, how they will distribute any gains they may reap from their agreement is undetermined in our analysis. So we will talk of their surplus as the utility of the coalition. Suppose first that the coalition exists, i.e. there is corruption. We may then ask how the surplus of the coalition varies with the probability that honest bidders attach to its existence —in other words, we may look purely at the perception effect. The answer follows straightforwardly from Proposition 1: the coalition's utility grows (falls) when $\alpha(v)$ is strictly concave (strictly convex), since honest bidders respond by bidding less aggressively (more agressively) when they percieve an increase in p; needless to say, the coalition's surplus is invariant to p when $\alpha(v)$ is linear.

A related question is whether there will be a positive surplus for the coalition, i.e. if its expected utility is larger when they reach an agreement (and honest bidders attach a probability p to its existence) than if they reach no agreement and the auction is corruption-free (so that $p = 0$). Here, the direct and perception effects are again combined. Clearly, the coalition is better off whenever honest bidders do not become more aggressive with corruption (that is, if $\alpha(v)$ is linear concave or strictly concave). The dishonest bidder wins with a higher probability for any valuation (the direct effect is positive) and never pays a higher price (the perception effect is also positive). For the case of $\alpha(v)$ strictly convex, the perception effect is negative, and its absolute value grows with p. Is it possible that honest bidders become so much more aggressive with corruption that there is no gain for the coalition in reaching an agreement? The following proposition provides a negative answer.

Proposition 3 *The coalition always has larger utility when there is corruption (for any value of p) than without corruption.*[18]

[17] Although we believe that the monotonicity result also holds for these cases, we have not been able to prove it yet.

[18] Another interesting question may be whether reaching an agreement with the auctioneer is better for a

Proof. Let $\widehat{U}^p(v)$ be the expected utility of the coalition when the dishonest bidder's valuation is v and the probability that honest bidders attach to the existence of corruption is p. Then,

$$\widehat{U}^p(v) = \int_{\underline{v}}^{\phi^p(v)} (v - b^p(r))F^{(N-2)}(r)f(r)dr$$

But

$$\int_{\underline{v}}^{\phi^p(v)} (v - b^p(r))F^{(N-2)}(r)f(r)dr > \int_{\underline{v}}^{v} (v - r)F^{(N-2)}(r)f(r)dr$$

where the last expression is the expected utility of a dishonest bidder when all honest rivals bid their own valuations, and the inequality follows from the fact that $b^p(r) < r$ for all r and $\phi^p(v) > v$. Since all honest bidder shade their bids for any value of p, the coalition, when the dishonest bidder's valuation is v, is better off than facing honest bidders that bid their valuations. But the right-hand side is then the expected utility of the coalition in a second-price auction, for any value of p —in particular, for $p = 0$. When $p = 0$, the expected utility of the coalition in a second-price auction is just the expected utility of any bidder with valuation v in a second-price or a first-price auction without corruption, since both auctions are payoff-equivalent in that case. We conclude that for any valuation v of the dishonest bidder, and for any value of p that honest bidder may have if coalition actually forms, there is a positive surplus from corruption for the coalition. ■

Note that Proposition 3 applies in all cases, even in those where $\alpha(v)$ is not always linear, strictly concave or strictly convex on $[\underline{v}, \overline{v}]$.

Finally, let us examine the consequences of corruption on expected revenue. The direct effect is always negative and monotone in p. The dishonest bidder may choose to revise her bid upwards or downwards. When the former occurs, only the identity of the winner changes, but not the price paid. When the latter occurs, however, the price paid to the owner falls. Leaving the behavior of honest bidders unchanged, an increase in p raises the likelihood of such a revision. The perception effect is zero when $\alpha(v)$ is linear, and negative if $\alpha(v)$ is strictly concave. So we certainly know that an increase in p will lower expected revenue in those two cases.

---

bidder than a situation where another bidder becomes the favored one. The combination of Propositions 2 and 3 provides an affirmative answer.

When $\alpha(v)$ is strictly convex, however, honest bidders become more agressive, so the two effects go in opposite directions. In particular, when the change in honest bidders' behavior is large enough, it may be the case that the seller's expected revenue grows with p. We do know, however, that expected revenue for any $p > 0$ has to be lower than expected revenue with $p = 0$ in the cases that are usually called "regular," that is, in those cases where a bidder's "virtual" valuation $v - [(1 - F(v)) / f(v)]$ is increasing in her "true" valuation v. In regular cases, the standard first-price auction is revenue-maximizing within the class of mechanism where the object is always sold. Within this class, to maximize revenue the object should be awarded to the the bidder with the highest virtual valuation. For any regular case, that is just the bidder with the highest valuation, and that is the bidder who wins in a standard first-price auction. When $p > 0$, with positive probability the winner does not have the highest real of virtual valuation, so expected revenue has to be lower.

This result, though, does not hold for nonregular cases. Moreover, honest bidders can become so much more aggressive in the presence of corruption that expected revenue may grow even if compared to a situation where $p = 0$, as we show in the following example.

Example 1 *Suppose $N = 2$ and valuations are distributed according to the Pareto distribution with parameters 1 and 0.75 (i.e. $F(v) = 1 - \left(\frac{1}{v}\right)^{0.75}$), with support on $[1, +\infty)$. Without corruption, the seller's revenue is 3, whereas it is 3.7431 when corruption is certain (i.e. when $p = 1$).*

4 Conclusion

We have analyzed above how a specific form of corruption affects the behavior of honest bidders in sealed-bid auctions. Assuming that the auctioneer may have reached an agreement with one of the bidders by which the latter will be shown all of her rival's bids and will be allowed to resubmit her bid accordingly, we have shown that this has no effects in the case of the second-price auction. For the first-price auction, we have provided sufficient conditions to assess how honest bidders adjust their bids when facing a rival that is possibly dishonest. Those conditions determine whether honest bidders will behave more, equally or less aggressively when the probability they attach to the existence of a corrupt agreement grows. In any case, the first-price auction becomes inefficient with corruption. We have established as well that an increased probability of corruption hurts honest bidders, and that the coalition of the auctioneer and the favored bidder always has a surplus to divide between the two if they reach an agreement –although how that surplus varies with the probability with which corruption is believed to exist depends on whether honest bidders react more or less aggressively. Finally, corruption is detrimental to expected revenue in "regular" cases, though not necessarily so in "nonregular" ones.

Let us emphasize again that, even though we have been very precise in terms of the advantages that corruption confers to a dishonest bidder in our analysis, we have been vague when referring to the negotiations between such a bidder and the auctioneer that lead to a corrupt arrangement. Our results, then, can be regarded as relevant to any specific model for such negotiation in the context of sealed-bid auctions. It is relevant as well, as explained above, to the situation where one of the bidders is granted a right of first refusal, a particular case of the way we have modeled corruption.

Our analysis could be extended in several directions. First, it would be instructive to allow for the possibility that bidders' valuations be distributed asymmetrically, or to explore the case where bidders are risk averse. Second, the bargaining game between the auctioneer and the favored bidder could be included in the analysis, as well as the possibility that honest bidders facing a positive probability of corruption decide to exit the market if participation is costly. Finally, we have studied a case where corruption takes a very specific form. The effects of many other forms of corruption on behavior in auctions remain unexplored.

References

[1] Bagnoli, M. and T. Bergstrom, "Log-concave Probability and its Applications," Discussion Paper 89-23, University of Michigan, 1989.

[2] Bikhchandani, S., S. Lippman and R. Ryan, "On the Right-of-first-refusal," mimeo, UCLA, 2003.

[3] Burguet, R. and Y.Che, “Competitive Procurement with Corruption,” *RAND Journal of Economics*, 35, 50-68, 2004.

[4] Burguet, R. and M. Perry, “Bribery and Favoritism by Auctioneers in Sealed-Bid Auctions,” mimeo, Institut d’Analisi Economica and Rutgers University, 2002.

[5] Burguet, R. and M. Perry, “Preferred Suppliers and Vertical Integration in Auction Markets,” mimeo, Institut d’Analisi Economica and Rutgers University, 2003.

[6] Celentani, M. and J.Ganuza, “Competition and Corruption in Procurement,” *European Economic Review*, 43, 1273-1303, 2002.

[7] Choi, A., “A Rent Extraction Theory of Right of First Refusal,” mimeo, University of Virginia, 2003.

[8] Compte, O., A.Lambert-Mogiliansky and T.Verdier, “Corruption and Competition in Procurement,” *RAND Journal of Economics*, forthcoming.

[9] Hendricks, K. and R.Porter, “Collusion in Auctions,” *Annales d’Economie et de Statistique*, 15/16, 1415-1444, 1989.

[10] Jones, C. and F.Menezes, “Auctions and Corruption: How to Compensate the Auctioneer,” mimeo, Australian National University, 1995.

[11] Lengwiler, Y. and E. Wolfstetter, “Auctions and Corruption,” mimeo, Humboldt-Universität, 2000.

[12] Menezes, F. and P. Monteiro, “Corruption and Auctions,” mimeo, Australian National University and EFGE/FGV, 2003.

[13] Porter, R.W. and Y.Yoham “On Cheating in Sealed-Bid Auctions,” Decision Support Systems, forthcoming.

Appendix

Lemma 2 *$\alpha(b) \leq \frac{\alpha(\phi^p)}{\phi^{p\prime}}$ if and only if*

$$\frac{\alpha(b)}{N-1} \leq \phi^p - b \leq \frac{\alpha(\phi^p)}{\phi^{p\prime}(N-1)}$$

Naturally, an analogous condition applies reverting the inequalities.

Proof. Notice that, from (2),

$$\begin{aligned} & \phi^p - b - \frac{\alpha(b)}{N-1} \\ = & \frac{pF(\phi^p)F(b) + (1-p)F^2(\phi^p)}{p\left[(N-2)F(b)f(\phi^p)\phi^{p\prime} + F(\phi^p)f(b)\right] + (1-p)(N-1)F(\phi^p)f(\phi^p)\phi^{p\prime}} - \frac{F(b)}{f(b)(N-1)}, \end{aligned}$$

which is nonnegative if and only if

$$\begin{aligned} & \left[pF(\phi^p)F(b) + (1-p)F^2(\phi^p)\right] f(b)(N-1) \\ \geq & \left\{p\left[(N-2)F(b)f(\phi^p)\phi^{p\prime} + F(\phi^p)f(b)\right] + (1-p)(N-1)F(\phi^p)f(\phi^p)\phi^{p\prime}\right\} F(b) \end{aligned}$$

or

$$\left[(1-p)(N-1)F(\phi^p) + p(N-2)F(b)\right]\left[F(\phi^p)f(b) - f(\phi^p)\phi^{p\prime}F(b)\right] \geq 0$$

This is true if and only if $\alpha(b) \leq \frac{\alpha(\phi^p)}{\phi^{p\prime}}$.

In addition,

$$\begin{aligned} & \frac{\alpha(\phi^p)}{\phi^{p\prime}(N-1)} - (\phi^p - b) \\ = & \frac{F(\phi^p)}{f(\phi^p)\phi^{p\prime}(N-1)} - \frac{pF(\phi^p)F(b) + (1-p)F^2(\phi^p)}{p\left[(N-2)F(b)f(\phi^p)\phi^{p\prime} + F(\phi^p)f(b)\right] + (1-p)(N-1)F(\phi^p)f(\phi^p)\phi^{p\prime}} \end{aligned}$$

is nonnegative if and only if

$$\begin{aligned} & F(\phi^p)\left\{p\left[(N-2)F(b)f(\phi^p)\phi^{p\prime} + F(\phi^p)f(b)\right] + (1-p)(N-1)F(\phi^p)f(\phi^p)\phi^{p\prime}\right\} \\ \geq & f(\phi^p)\phi^{p\prime}(N-1)\left[pF(\phi^p)F(b) + (1-p)F^2(\phi^p)\right] \end{aligned}$$

or

$$pF(\phi^p)\left[F(\phi^p)f(b) - F(b)f(\phi^p)\phi^{p\prime}\right] \geq 0$$

Again, this is true if and only if $\alpha(b) \leq \frac{\alpha(\phi^p)}{\phi^{p\prime}}$. ■

Lemma 3 *For any $b > \underline{v}$, and holding the inverse bidding function $\phi^p(.)$ constant, the ratio*

$$\frac{pF(\phi^p)F(b) + (1-p)F^2(\phi^p)}{p\left[(N-2)\,F(b)f(\phi^p)\phi^{p\prime} + F(\phi^p)f(b)\right] + (1-p)\,(N-1)\,F(\phi^p)f(\phi^p)\phi^{p\prime}}$$

is (weakly) decreasing in p if and only if $\alpha(b) \leq \frac{\alpha(\phi^p)}{\phi^{p\prime}}$.

Proof. Holding $\phi^p(.)$ constant and differentiating with respect to p, the derivative thereby obtained will be nonpositive if and only if

$$\begin{aligned} & \left[F(\phi^p)F(b) - F^2(\phi^p)\right]\left[p\left[(N-2)\,F(b)f(\phi^p)\phi^{p\prime} + F(\phi^p)f(b)\right] + (1-p)\,(N-1)\,F(\phi^p)f(\phi^p)\phi^{p\prime}\right] \\ \leq\ & \left[pF(\phi^p)F(b) + (1-p)F^2(\phi^p)\right]\left[(N-2)\,F(b)f(\phi^p)\phi^{p\prime} + F(\phi^p)f(b) - (N-1)\,F(\phi^p)f(\phi^p)\phi^{p\prime}\right] \end{aligned}$$

or, after a few steps of algebra,

$$F^2(\phi^p)\left[F(b)f(\phi^p)\phi^{p\prime} - F(\phi^p)f(b)\right] \leq 0.$$

But this is nonpositive if and only if $\alpha(b) \leq \frac{\alpha(\phi^p)}{\phi^{p\prime}}$. ■

Proof of Lemma 1: We only present the proof for the case where $\alpha(v)$ is strictly convex, since the remaining two cases are analogous. We proceed in two steps:

1. We show first that if $\alpha(\widehat{b}) = \frac{\alpha(\phi^p(\widehat{b}))}{\phi^{p\prime}(\widehat{b})}$ for some $\widehat{b} > \underline{v}$, then $\alpha(b) < (>) \frac{\alpha(\phi^p(b))}{\phi^{p\prime}(b)}$ for b close to but larger (smaller) than $\widehat{b}$. This means that if these two functions cross at $\widehat{b}$, $\alpha(b)$ crosses $\frac{\alpha(\phi^p(b))}{\phi^{p\prime}(b)}$ from above and $\alpha(b) < \frac{\alpha(\phi^p(b))}{\phi^{p\prime}(b)}$ for all $b > \widehat{b}$. From Lemma 2 in the Appendix, we know that $\alpha(\widehat{b}) = \frac{\alpha(\phi^p(\widehat{b}))}{\phi^{p\prime}(\widehat{b})}$ if and only if $\frac{\alpha(\widehat{b})}{N-1} = \phi^p(\widehat{b}) - \widehat{b}$. Then, combining these two equalities yields,

$$\phi^{p\prime}(\widehat{b}) - 1 = \frac{\alpha(\phi^p(\widehat{b}))}{\alpha(\widehat{b})} - 1 = \frac{\alpha(\phi^p(\widehat{b})) - \alpha(\widehat{b})}{\left[\phi^p(\widehat{b}) - \widehat{b}\right](N-1)}$$

By the strict convexity of $\alpha(v)$, hence,

$$\phi^{p\prime}(\widehat{b}) - 1 > \frac{\alpha'(\widehat{b})}{N-1}$$

Then, $\phi(b) - b$ is growing faster than $\frac{\alpha(b)}{N-1}$ at $b = \widehat{b}$. Resorting to Lemma 2 once more, our claim follows.

2. It is immediate that $\alpha(\underline{v}) = \frac{\alpha(\phi^p(\underline{v}))}{\phi^{p\prime}(\underline{v})} = 0$[19]. Our second step proves that, for b close to $\underline{v}$, it has to be true that $\alpha(b) < \frac{\alpha(\phi^p)}{\phi^{p\prime}}$. Given our first step, this claim would establish the Lemma.

   Assume, towards a contradiction, that $\alpha(b) \geq \frac{\alpha(\phi^p)}{\phi^{p\prime}}$ for all $b \in (\underline{v}, \delta)$ for some $\delta > 0$ arbitrarily small. Take any b_0 in that interval. Without loss of generality, we can concentrate on the case where $\alpha(b_0) > \frac{\alpha(\phi^p(b_0))}{\phi^{p\prime}(b_0)}$, since if $\alpha(b_0) = \frac{\alpha(\phi^p(b_0))}{\phi^{p\prime}(b_0)}$, step 1 above implies that $\alpha(b) > \frac{\alpha(\phi^p(b))}{\phi^{p\prime}(b)}$ for b smaller but close to b_0. From Lemma 2, $\frac{\alpha(b_0)}{N-1} > \phi^p(b_0) - b_0$. Then, as $\frac{\alpha(\underline{v})}{N-1} = \phi^p(\underline{v}) - \underline{v} = 0$, there has to exist a bid $b_1 \leq b_0$ such that $\frac{\alpha(b_1)}{N-1} > \phi^p(b_1) - b_1$ and $\frac{\alpha'(b_1)}{N-1} > \phi^{p\prime}(b_1) - 1$. Since $\alpha(b_1) > \frac{\alpha(\phi^p(b_1))}{\phi^{p\prime}(b_1)}$, Lemma 2 implies as well that $\phi^{p\prime}(b_1) > \frac{\alpha(\phi^p(b_1))}{[\phi^p(b_1) - b_1](N-1)}$. Therefore,

$$\frac{\alpha'(b_1)}{N-1} > \frac{\alpha(\phi^p(b_1)) - [\phi^p(b_1) - b_1](N-1)}{[\phi^p(b_1) - b_1](N-1)}$$

   Furthermore, from the strict convexity of $\alpha(v)$,

$$\frac{\alpha(\phi^p(b_1)) - \alpha(b_1)}{[\phi^p(b_1) - b_1](N-1)} > \frac{\alpha'(b_1)}{N-1}$$

   For these last two inequalities to hold it has to be the case that $\alpha(b_1) < [\phi^p(b_1) - b_1](N-1)$. We thereby conclude that such a b_1 cannot exist. ∎

Proof of Proposition 1: Lemma 1 implies that $\alpha(b) < \frac{\alpha(\phi^p(b))}{\phi^{p\prime}(b)}$ for any $b > \underline{v}$ and for any

[19] It can be easily verified that, for any $p \in [0,1]$, $\phi^{p\prime}(0) = \frac{N}{N-1}$.

$p < 1$ ($p \leq 1$ if $N > 2$). Hence, if $\phi^{p_0}(b) \leq \phi^{p_1}(b)$,

$$
\begin{aligned}
& \frac{p_0 F(\phi^{p_0})F(b) + (1-p_0)F^2(\phi^{p_0})}{p_0\left[(N-2)F(b)f(\phi^{p_0})\phi^{p_0\prime} + F(\phi^{p_0})f(b)\right] + (1-p_0)(N-1)F(\phi^{p_0})f(\phi^{p_0}))\phi^{p_0\prime}} \\
\leq \; & \frac{p_1 F(\phi^{p_1})F(b) + (1-p_1)F^2(\phi^{p_1})}{p_1\left[(N-2)F(b)f(\phi^{p_1})\phi^{p_1\prime} + F(\phi^{p_1})f(b)\right] + (1-p_1)(N-1)F(\phi^{p_1})f(\phi^{p_1})\phi^{p_1\prime}} \\
< \; & \frac{p_0 F(\phi^{p_1})F(b) + (1-p_0)F^2(\phi^{p_1})}{p_0\left[(N-2)F(b)f(\phi^{p_1})\phi^{p_1\prime} + F(\phi^{p_1})f(b)\right] + (1-p_0)(N-1)F(\phi^{p_1})f(\phi^{p_1})\phi^{p_1\prime}}
\end{aligned}
\tag{5}
$$

where the last inequality follows from Lemma 3.

Examining the extreme left and right-hand sides of (5), a few steps of algebra yield

$$
\begin{aligned}
& p^2(N-2)F^2(\phi^{p_1})\left[F(\phi^{p_0})f(b) - F(b)f(\phi^{p_0})\right] + \\
& p(1-p)F(\phi^{p_0})F(b)\left[\begin{array}{c} (N-2)\left(\frac{F(\phi^{p_0})F(\phi^{p_1})f(b)}{F(b)} - \frac{F(\phi^{p_1})F(b)f(\phi^{p_0})}{F(\phi^{p_0})}\right) + F(\phi^{p_0})f(\phi^{p_1}) \\ -F(\phi^{p_1})F(b) + (N-1)\left(F(\phi^{p_1})f(b) - F(\phi^{p_1})f(\phi^{p_0})\right) \end{array}\right] + \\
& (1-p)^2(N-1)F(\phi^{p_0})F(b)\left(F(\phi^{p_0})f(b) - f(\phi^{p_0})F(b)\right) \\
< \; & 0
\end{aligned}
\tag{6}
$$

From Lemma 1, the first and third terms on the left-hand side of (6) are positive. The second term can be re-expressed as

$$
\begin{aligned}
& (N-2)F(\phi^{p_1})\left[\frac{F(\phi^{p_0})f(b)}{F(b)} - f(\phi^{p_0}) - \left(\frac{f(\phi^{p_0})F(b)}{F(\phi^{p_0})} - f(b)\right)\right] \\
& + \left[F(\phi^{p_0})f(\phi^{p_1}) - F(\phi^{p_1})f(\phi^{p_0})\right] - \left[F(b)f(\phi^{p_1}) - F(\phi^{p_1})f(b)\right]
\end{aligned}
$$

Using Lemma 1 once again, it is straightforward that this expression (and then all the left-hand side of (6)) cannot be negative unless

$$
\frac{\alpha(\phi^{p_0})}{\phi^{p_0\prime}(b)} < \frac{\alpha(\phi^{p_1})}{\phi^{p_1\prime}(b)} \tag{7}
$$

The proof now proceeds in two steps that are analogous to the ones followed in the proof of Lemma 1 above.

First, note that if $\phi^{p_0}(\widehat{b}) = \phi^{p_1}(\widehat{b})$ for some $\widehat{b} > \underline{v}$, then (7) implies that $\phi^{p_0\prime}(\widehat{b}) > \phi^{p_1\prime}(\widehat{b})$. This means that if the two bidding functions cross at some $\widehat{b} > 0$, then $\phi^{p_0}(b)$ crosses $\phi^{p_1}(b)$ from below and $\phi^{p_0}(b) > \phi^{p_1}(b)$ for all $b > \widehat{b}$.

Second, let $\delta > 0$ be an arbitrarily small number. We want to show that

$\phi^{p_0}(b) > \phi^{p_1}(b)$ for all $b \in (\underline{v}, \delta)$. Assume, towards a contradiction, that there exists a bid $b_0 \in (\underline{v}, \delta)$ such that $\phi^{p_0}(b_0) \leq \phi^{p_1}(b_0)$. Then, (5) holds again, which implies

$$\frac{d}{db}\left[\frac{F(\phi^{p_1}(b))}{F(\phi^{p0}(b))}\right] < 0$$

for all $b \in (\underline{v}, b_0)$It follows that

$$\int_0^{b_0} \frac{d}{db}\left[\frac{F(\phi^{p_1}(b))}{F(\phi^{p0}(b))}\right] db < 0$$

or

$$\frac{F(\phi^{p_1}(b_0))}{F(\phi^{p0}(b_0))} - \lim_{b \to 0} \frac{F(\phi^{p_1}(b))}{F(\phi^{p0}(b))} < 0$$

The second term on the right-hand side of this expression, using L'Hôpital's rule, becomes

$$\frac{f(\underline{v})\phi^{p_1\prime}(\underline{v})}{f(\underline{v})\phi^{p_0\prime}(\underline{v})} = 1$$

since $\phi^{p_0\prime}(\underline{v}) = \phi^{p_1\prime}(\underline{v})$, as can be easily verified. Therefore, it must be true that

$$\frac{F(\phi^{p_1}(b_0))}{F(\phi^{p0}(b_0))} < 1,$$

which contradicts our assumption that $\phi^{p_0}(b_0) \leq \phi^{p_1}(b_0)$. ■

Proof of Corollary 2: From (2), we know that

$$\phi^{0\prime} = \frac{\alpha(\phi^0)}{(N-1)\left(\phi^0 - b\right)} \tag{8}$$

Differentiating this expression,

$$\phi^{0\prime\prime} = \frac{\alpha'(\phi^0)\phi^{0\prime}(\phi^0 - b) - \alpha(\phi^0)(\phi^{0\prime} - 1)}{(N-1)\left(\phi^0 - b\right)^2}$$

Substituting according to (8), the numerator of this expression becomes

$$\frac{\alpha'(\phi^0)\alpha(\phi^0)}{N-1} - \alpha(\phi^0)(\phi^{0\prime} - 1) = \frac{\alpha(\phi^0)}{N-1}\left[\alpha'(\phi^0) - (\phi^{0\prime} - 1)(N-1)\right] \tag{9}$$

From (8), we know that

$$(N-1)(\phi^{0\prime}-1)=\frac{\alpha(\phi^0)-(N-1)(\phi^0-b)}{\phi^0-b}$$

Proposition 1 implies that if $\alpha(v)$ is linear, then $(N-1)(\phi^0-b)=\alpha(b)$. Then,

$$(N-1)(\phi^{0\prime}-1)=\frac{\alpha(\phi^0)-\alpha(b)}{\phi^0-b}=\alpha'(\phi^0)$$

where the last equality follows from the linearity of $\alpha(v)$. It follows that the expression in (9) equals zero, and we conclude that $\phi^{0\prime\prime}=0$, and so does $b^{0\prime\prime}$. Analogoustly, if $\alpha(v)$ is strictly convex (strictly concave), then, from Proposition 1, $(N-1)(\phi^0-b)>(<)\alpha(b)$, so

$$(N-1)(\phi^{0\prime}-1)<(>)\frac{\alpha(\phi^0)-\alpha(b)}{\phi^0-b}<(>)\alpha'(\phi^0)$$

where the last inequality follows from the strict convexity (strict concavity) of $\alpha(v)$. Therefore, the expression in (9) is positive (negative), so $\phi^{0\prime\prime}>(<)0$. Then, $b^{0\prime\prime}<(>)0$. ■