One Person, Many Votes:
Divided Majority and Information Aggregation*

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Abstract

In most electoral systems, even small disagreements within the majority can have a dramatic impact on the election outcome. Expressing such divisions is however necessary when voters have contradictory information on the relative merits of the candidates. We propose a novel model of elections and compare the equilibrium properties of three electoral systems when voters have imperfect information. This approach contrasts with the standard assumption that voters preferences are only a function of electoral outcomes. We show that Approval Voting produces full information and coordination equivalence: the alternative that wins the election is the one preferred by a majority of the electorate under full information. This result, which extends those on information aggregation through elections, is shown to be absent in two widely used electoral systems: Plurality and Runoff.

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1 Introduction

In most electoral systems, even small disagreements within the majority can have a dramatic impact on the election outcome. The history of US elections offers many examples, two recent ones being the 1992 and 2000 presidential elections, in which the third candidate, R. Perot in 1992 and R. Nader in 2000, is oftentimes claimed to have stolen the victory of the most deserving contender. The impact of such divisions is equally important in two-round systems. In 2002 in France, divisions among leftist voters led the socialist candidate, Lionel Jospin, to lose the first round by a hair’s breadth to J.-M. Le Pen, an extreme-right candidate with no chance of winning the second round. Another case is D. Ortega, who won the 2006 Nicaraguan election, only due to divisions within the right-wing majority.¹

Expressing such divisions is however necessary when voters have contradictory information on the relative merits of the candidates; it is actually the channel through which elections can aggregate the information dispersed in the population.² But the above examples illustrate that plurality and runoff elections are ill-adapted to aggregate information in multicandidate elections. To prevent the election of the candidate they like least, the majority are constrained to avoid divisions; they should coordinate all their votes on a single “focal” candidate.³ The problem is that, like for any other coordination problem, the selection of “the” focal candidate may itself be quite orthogonal to his or her intrinsic qualities: some information is necessarily lost in such electoral systems.

We propose a model of elections in which voter preferences can be affected by information about the candidates. A majority of the electorate is unified against the minority, but is divided about which of two candidates would best represent them. We compare the voting equilibria produced by three alternative voting systems: Approval Voting emerges as an institution that strictly dominates Plurality and Runoff elections, for at least two reasons. First, it produces a unique equilibrium, which saves voters from the risk of coordination failures. Second, despite contradictory priors among majority voters, the candidate

¹Nicaragua’s system is a runoff in which a candidate wins at the first round if he obtains more than 35% and a 5 point lead over the nearest competitor. D. Ortega (left-wing) won with 38% because the right-wing majority divided their votes between E. Montealegre (28.3%) and J. Rizo (27.1%).


³This is known as Duverger’s Law in political science. See a.o. Duverger (1954) and Cox (1997).
who best represents the majority actually wins with a probability that approaches one as electorate size increases. That is, Approval Voting produces the same outcome as the one that would occur in the other systems, were both information and coordination problems resolved.

In contrast with the standard literature that compares voting rules, our model of elections departs from the classical axiom that voter preferences can directly be expressed in terms of electoral outcomes.\(^4\) That axiom implicitly assumes that voters are perfectly informed about the relative merits of all the candidates; an assumption wildly violated in practice: in most elections, a significant part of the electorate is still undecided a few days before the election. In all elections, many voters are actually trying to understand the positions of each candidate. Expressed differently, while voters may have well-defined preferences over integrity or \textit{policy} outcomes, they rarely have such crisp preferences about \textit{electoral} outcomes – the Watergate scandal and J.-M. Aznar’s attempt to blame ETA for Al Qaeda’s 2004 bombing in Madrid are striking examples in which voters reverted their candidate preference ranking in light of fresh information. Acknowledging that the voters’ preferences over candidates might be sensitive to information is at the center of our results.

The investigation of the properties of Approval Voting (AV) has begun only recently, with the works of Weber (1977, 1995) and Brams and Fishburn (1978, 1983) – more recent advances on the modeling of AV in large electorates can also be found in Myerson (2002) and Laslier (2006). The idea of AV is to allow voters to vote for (or “approve of”) as many candidates as they wish. Each approval counts as one vote, and the candidate alternative with the most votes wins the election. Note that this is the natural way in which we organize appointments when many people are involved; it has also become the business model of Inturico Engineering, with its “Doodle” service.\(^5\) Brams and Fishburn hoped that it would become “the election reform of the twentieth century”. It did, for many academic societies and for the election of the secretary-general of the UN, among others (See Brams and Fishburn 2005). By contrast, it has not pervaded to the most important democratic elections.\(^6\)

\(^4\)This axiom is used in essentially all the literature that compares electoral systems. The most prominent example is Arrow (1951), whose impossibility theorem rests on the premise that voter preferences must be directly expressed in terms of electoral outcomes.

\(^5\)See http://inturico.com/. We found people who use the www.doodle.ch service to choose a city location (among three or four alternatives), to name a new open source software (among 10 alternatives), to organize yoga classes (many possible time schedules), to select among various designs for a website, etc...

\(^6\)This is despite the ease with which voting laws and machines could be adapted. See also Brams
Our results contradict two prejudices against AV that emerged in the literature, and may help explain why it has not been easily implemented in large-scale elections. First, according to traditional analyses, AV would also be displaying a multiplicity of equilibria (see e.g. Myerson and Weber 1993), and may produce inferior equilibria in which the Condorcet winner fails to be elected (see De Sinopoli et al. 2006 and Nuñez 2007). We show that these conclusions are no longer valid when the assumption of fixed preferences over electoral outcomes is relaxed. Second, there is a fear that AV may induce “excessive closeness” of the candidates’ results. Nagel (2007) calls this the “Burr dilemma” of approval voting: voters may end up voting indiscriminately for all the candidates in the majority. Similarly, Myerson and Weber (1993, p106) propose an example in which all candidates obtain the same vote share in equilibrium. According to our results, the evidence produced by Nagel does not extend to large electorates: since voters have opposing ex ante preferences, they always have an incentive to deviate from a strategy of all voting for a same set of candidates. In contrast to Myerson and Weber, we introduce information uncertainty, which weakens voter preferences over electoral outcomes: some voters believe that the best candidate is A; others believe it is B. Yet, all realize that they may be wrong. For this reason, they adopt a strategy that ensures that A has the highest expected vote share when she is the best, and conversely when B is actually the best. Third, our results show that the incumbency advantage no longer exists under AV: leading politicians and parties cannot foreclose entry on the political marketplace. This is because AV makes experimentation easier for the voters, which stiffens up competition and reduces the rents of the main parties. In our view, this in itself helps explain why AV has not pervaded to large-scale elections.

Our modeling of large-scale elections draws on the Condorcet Jury Theorem literature. We rely on extended Poisson games to model a three-candidate election, and compare voting equilibria across electoral systems in the spirit of Myerson and Weber (1993). This generalization of Poisson games was introduced by Myerson (1998a), who also shows that they simplify the analysis of the Condorcet Jury Theorem.7 In Myerson (1998a) as in Austen-Smith and Banks (1996), the electorate wants to select the “best” alternative. Depending on the state of nature, either A or B can be the best, but voters have different prior opinions about these alternatives. One of the main results of the Condorcet Jury (2007), as well as Laslier and Van der Straeten (2007) who ran a large-scale experiment during the 2002 presidential election in France.

7Though based on Poisson games, our results directly extend to multinomial distributions.
Theorem literature is that, at least in a two-candidate setting, there exists an equilibrium strategy such that the best alternative is elected almost surely. That result is robust to changes in the information structure or in the size of the majority required to win – with the notable exception of the unanimity rule (Feddersen and Pesendorfer 1997, 1998).\footnote{See Kim and Fey (2007) and Bhattacharya (2007) for precise necessary conditions on voter preferences.}

In our model, the majority of the electorate knows that they always prefer both $A$ and $B$ to a third alternative, $C$. However, majority-block voters hold opposing beliefs as to which of $A$ and $B$ is most likely to be the best alternative: in the absence of additional information, some prefer $A$ and the others prefer $B$. They also face opposition by a minority who staunchly supports $C$. Hence, the majority may be forced to avoid internal divisions, to prevent $C$ from winning the election. In this setup, we analyze the equilibrium properties of Approval Voting, Plurality and Runoff elections. Only AV produces a unique equilibrium in which the best alternative almost certainly wins.

The intuition is two-pronged. First, by its very design, Approval Voting allows voters to hit two birds with one ballot: they can both vote for their most preferred alternative and lend support to their second-best alternative if they view $C$ as a threat – this is the classical argument in favor of approval voting. Second, we show that the trade-off between dividing the majority and conveying information is drastically different under AV. This is the rationale behind equilibrium uniqueness: when voters know that with some (even tiny) probability, their alternative might be “bad”, they want to avoid that any of the majority-backed alternatives be too much ahead of the other.\footnote{This incentive is absent when voters put a probability zero on their candidate being “bad”. In that case, multiple equilibria can coexist (see Nuñez (2007) and Section 7 in this paper).} This would make her win even when she is bad. Hence, whenever there is an imbalance between the two alternatives, majority-group votes prefer to double vote, \textit{i.e.} vote for both $A$ and $B$. This both reduces the imbalance and ensures that $C$ remains weak. Only when the vote shares are balanced and there is enough double-voting to drag $C$ behind, majority voters can single-vote for their most preferred alternative. This is the channel through which voter preferences generate the information necessary to select the best alternative.

The paper is organized as follows: Section 2 lays out the model. Section 3 identifies actions that are strictly dominated under AV and identifies pivot probabilities for the remaining actions. Section 4 analyzes equilibrium behavior under AV. Section 5 and 6 analyze equilibria under Plurality Voting and Runoff respectively. Section 7 analyzes the case in which voters never reverse their priors. Section 8 concludes.
2 The Model

There are three alternatives, indexed by $P \in \{A, B, C\}$, two states of nature, $\omega \in \{a, b\}$, and three types of voters, $t \in \{t_A, t_B, t_C\}$. Conditional on the state of nature, types $t_A$ and $t_B$ hold identical preferences: they always want to elect the best alternative, which is $A$ in state $a$ and $B$ in state $b$:

$$U(P, t_A, \omega) = U(P, t_B, \omega) = \begin{cases} 1 & \text{if } (P, \omega) = (A, a) \text{ or } (B, b) \\ 0 & \text{if } (P, \omega) = (A, b) \text{ or } (B, a) \\ -1 & \text{if } P = C, \end{cases}$$

(1)

where $U(P, t, \omega)$ denotes the utility of a voter with type $t$ when alternative $P$ is elected and the true state is $\omega$.

From an *ex ante* vantage point, types $t_A$ and $t_B$ have opposite convictions regarding alternatives $A$ and $B$: they hold different beliefs as to which state is most likely. As stated below, a voter with type $t$ believes that the true state is $\omega$ with a probability $q(\omega|t)$. We impose that:

$$\infty > \frac{q(a|t_A)}{q(b|t_A)} > 1 > \frac{q(a|t_B)}{q(b|t_B)} > 0.$$  

(2)

That is, types $t_A$ believe that $A$ is most likely to be the best alternative, whereas types $t_B$ believe it is alternative $B$.

Types $t_C$ are pure partisans: independently of the true state of nature, they always prefer alternative $C$. For the sake of tractability, they are also assumed indifferent between the other two alternatives:

$$U(P, t_C, \omega) = \begin{cases} 1 & \text{if } P = C \\ 0 & \text{if } P \in \{A, B\}, \end{cases}$$

Timing. At the beginning of the game (*time 0*), nature chooses the state $\omega$, which remains unobserved until after the election. The probabilities of states $a$ and $b$ are respectively $q(a)$ and $q(b)$, with $q(a) + q(b) = 1$. At *time 1*, nature selects a random number of voters from a Poisson distribution of mean $n$ and, conditional on the state, assigns them a type $t$ by iid draws.\(^{10}\) The conditional probability that a randomly sampled voter has

\(^{10}\)The main properties of extended Poisson games are discussed in Appendix A1 and in the next section, where we also explain why our results extend to multinomial distributions.
type $t$ is $r(t|\omega)$, with $\sum_t r(t|\omega) = 1$, $\forall \omega$. These probabilities correlate with the true state of nature:

$$
\begin{align*}
    r(t_A|a) &> r(t_A|b), \\
    r(t_B|a) &< r(t_B|b), \\
    r(t_C|a) &= r(t_C|b).
\end{align*}
$$

By (2), $t_A$-voters prior preferences lean in favor of $A$ and conversely for $t_B$-voters. However, this correlation implies that there is additional information dispersed in the whole electorate that, if made available, could overwhelm these priors. To ensure that our results cannot hinge on some type of symmetry across types $t_A$ and $t_B$, we allow types $t_A$ to be potentially more “abundant” than $t_B$:

$$
    r(t_A|a) + r(t_A|b) \geq r(t_B|a) + r(t_B|b).
$$

The election is held at time 2. The probabilities $q(\omega)$ and $r(t|\omega)$ are common knowledge. In contrast, neither the actual state of nature nor the actual number of voters of each type are observed: voters only know their own type, $t$. By Bayesian updating, a voter with type $t$ infers that the probability of state $\omega$ is $q(\omega|t)$:

$$
    q(\omega|t) = \frac{q(\omega) r(t|\omega)}{q(a) r(t|a) + q(b) r(t|b)}.
$$

Clearly, condition (2) imposes restrictions on $q(\omega)$ and $r(t|\omega)$.

Payoffs are realized at time 3: the winning alternative $W \in \{A, B, C\}$ is selected and each voter’s utility $U(W, t, \omega)$ then realizes. In sections 5 and 6, we analyze plurality and runoff elections respectively. Here, we introduce Approval Voting:

**Preference aggregation and electoral rule.** The preferences of the majority depend on the distribution of voters. We focus on the case:

$$
    r(t_C|\omega) < 1/2,
$$

which implies that types $t_C$ are a minority.\footnote{For $r(t_C|\omega) > 1/2$, a majority of the electorate prefers to have $C$ elected, independently of $\omega$. This case is trivial to investigate: since types $t_C$ are a majority, they can elect $C$ with a probability that converges to 1 when population size increases to infinity.} Hence, types $t_A$ and $t_B$ compose the *majority block*, whereas types $t_C$ form the *minority block*. This implies that the majority’s preferred
alternative, \( A \) or \( B \), depends on the state of nature, \( a \) or \( b \), which is unknown at the time of the election.

Under \textit{Approval Voting}, each voter can cast a ballot on as many (or as few) alternatives as she wishes. Each approval counts as one vote: when a voter only approves of \( A \), then only alternative \( A \) is credited with one vote. If the voter approves of both \( A \) and \( B \), then both \( A \) and \( B \) are credited with one vote, and so on. Hence, the voters’ action set is:

\[
\Psi = \{ A, B, C, AB, AC, BC, ABC, \emptyset \},
\]

where, by an abuse of notation, action \( A \) denotes a ballot in favor of \( A \) only, action \( BC \) denotes a joint approval of \( B \) and \( C \), etc. Hence, the difference between approval voting and other, more common, electoral rules is that a voter can cast a single, a double or a triple approval. Single approvals (\( \psi = A, B \) and \( C \)) act as positive votes: an \( A \)-vote can only be pivotal in favor of \( A \), against \( B \) or against \( C \); a \( B \)-vote can be pivotal against \( A \) or \( C \), etc. In our three-candidate setup, double approvals (\( \psi = AB, BC \) and \( AC \)) act as negative votes. For instance, if the voter plays \( AC \), she is acting against \( B \): her ballot can only be pivotal against that alternative, either in favor of \( A \) or of \( C \). Finally, a triple approval (\( ABC \)) can never be pivotal: it is strategically equivalent to abstention.

Letting \( x(\psi) \) denote the number of voters who played action \( \psi \in \Psi \) at time 2, the \textit{total number of approvals} received by alternatives \( A, B, \) and \( C \) are respectively:

\[
\begin{align*}
X(A) &= x(A) + x(AB) + x(AC) + x(ABC), \\
X(B) &= x(B) + x(AB) + x(BC) + x(ABC), \\
X(C) &= x(C) + x(AC) + x(BC) + x(ABC).
\end{align*}
\]

The alternative with the largest total number of approvals wins the election. Ties are resolved by the toss of a fair coin. We will see below that, given a Poisson-distributed total size of the population, each random variable \( x(\psi) \) itself follows a Poisson distribution. This will imply that each voter has a strictly positive probability of being pivotal.

\textbf{Strategy space and equilibrium.} A type \( t \)'s \textit{strategy function} is any mapping \( \sigma(t) : t \to \psi \) that specifies a probability distribution over the set of actions \( \Psi \) for each type \( t \). \( \sigma(\psi|t) \) denotes the probability that a randomly sampled voter of type \( t \) plays action \( \psi \), and the usual constraints apply: \( \sigma(\psi|t) \geq 0 \) and \( \sum_\psi \sigma(\psi|t) = 1 \), \forall t. This strategy function \( \sigma(t) \) reflects the fact that a voter can only condition her strategy on her type \( t \).
When \( \sigma(\psi|t) \) is the strategy played by type-\( t \) voters, then a fraction:

\[
\tau(\psi|\omega) = \sum_t r(t|\omega) \sigma(\psi|t)
\]

of the electorate is expected to play action \( \psi \) in state \( \omega \). We call \( \tau(\psi|\omega) \) the expected share of voters who choose action \( \psi \) in state \( \omega \). Importantly, note that if types \( t_A \) and \( t_B \) play the same strategy \( \sigma(t) \), then vote shares \( \tau(\psi|\omega) \) are identical in the two states of nature. If instead they play different strategies, then expected shares vary with the state of nature.

We analyze symmetric Bayesian Nash equilibria of this voting game for an expected population size \( n \) that becomes infinitely large.\(^\text{12}\) We shall say that:

**Definition 1** An equilibrium produces an informational trap if the expected result of the election is independent of the state of nature:

\[
E_\sigma(X(P)|a) = E_\sigma(X(P)|b), \forall P \in \{A, B, C\}.
\]

In the presence of an informational trap, the outcome of the election cannot reveal anything about the actual state of nature.

### 3 Approval Voting: Elimination of Dominated Strategies

The action set contains eight elements. Identifying strictly dominated strategies, which players never use in equilibrium, allows us to restrict the effective choice set to three elements.

Denoting by \( \Pr(W) \) the probability that alternative \( W \in \{A, B, C\} \) wins the election, the expected utility of a majority-block voter is:

\[
EU(t) = q(a|t) \cdot [\Pr(A|a) - \Pr(C|a)] + q(b|t) \cdot [\Pr(B|b) - \Pr(C|b)], \ t \in \{t_A, t_B\}.
\]

This reads as follows: having observed her type \( t \), the voter anticipates that the true state of nature is \( a \) with probability \( q(a|t) \). In that case, by (1), her utility would be 1 if \( A \) wins, 0 if \( B \) wins, and \(-1\) if \( C \) wins. With probability \( q(b|t) \equiv [1 - q(a|t)] \) the true state is \( b \).

\(^{12}\)Note that the equilibrium mapping \( \sigma(\psi|t) \) must be identical for all voters of a same type \( t \), by the very nature of population uncertainty (see Myerson 1998b, p377, for more detail). Therefore, symmetry is necessarily part of the equilibrium.
In that case, her payoff is 0 if \( A \) wins, 1 if \( B \) wins, and \(-1\) if \( C \) wins. The expected utility of a minority-block voter is:

\[
EU(t_C) = \Pr(C).
\]

The value of each action depends on its probability of affecting the outcome of the election; \textit{i.e.} on its probability of being \textit{pivotal}. A ballot can be pivotal in two cases: when an alternative trails behind the leader by \textbf{exactly one} vote or when the leading alternatives have \textbf{the same} number of votes. It immediately follows that:

\section*{Lemma 1}

For a majority-block voter \( t \in \{t_A, t_B\} \), in equilibrium:

\[
\sigma(A|t) + \sigma(B|t) + \sigma(AB|t) = 1.
\] (8)

For a minority-block voter, playing action \( \psi = C \) is a strictly dominant strategy. Hence, in equilibrium:

\[
\sigma^*(C|t_C) = 1.
\] (9)

The proof is straightforward: consider for instance a majority-block voter and compare actions \( AB \) and \( ABC \). While the latter can never be pivotal, an \( AB \)-ballot can be pivotal against \( C \), either in favor of \( A \) or in favor of \( B \). Both events increase a majority-type’s expected utility. Hence, \( AB \) strictly dominates \( ABC \). All other strict dominance relationships are obtained by performing similar two-by-two comparisons: \( AB \) strictly dominates \( ABC, \varnothing \) and \( C \); \( A \) strictly dominates \( AC \); and \( B \) strictly dominates \( BC \).

Focusing on the three undominated actions, let \( G(\psi|t) \) denote the \textit{expected gain} of action \( \psi = A, B, AB \):

\[
G(A|t) = q(a|t) \left[ \Pr(piv_{AB}|a) + 2 \Pr(piv_{AC}|a) \right] + q(b|t) \left[ \Pr(piv_{AC}|b) - \Pr(piv_{AB}|b) \right],
\] (10)

\[
G(B|t) = q(a|t) \left[ \Pr(piv_{BC}|a) - \Pr(piv_{BA}|a) \right] + q(b|t) \left[ \Pr(piv_{BA}|b) + 2 \Pr(piv_{BC}|b) \right],
\] (11)

and \( G(AB|t) = q(a|t) \cdot \left[ \Pr(piv_{BC}|a) + 2 \Pr(piv_{AC}|a) \right] + q(b|t) \cdot \left[ \Pr(piv_{AC}|b) + 2 \Pr(piv_{BC}|b) \right]. \) (12)

This gain depends on the voter’s \textit{preference}, summarized by \( q(\omega|t) \), and on the strategy function \( \sigma \equiv \{\sigma(A|t), \sigma(B|t), \sigma(AB|t)\} \) of the other majority-block voters. These strategies determine the expected number of votes received by each alternative, and thereby the pivot probabilities \( \Pr(piv_{PQ}|\omega) \).
These pivot probabilities depend on the distribution of the number of voters who play each action. As shown by Myerson (1998a, 1998b, 2000), if the total number of voters is distributed according to a Poisson distribution of mean $n$, then the actual number $x(\psi)$ of voters who choose action $\psi$ also follows a Poisson distribution: $x(\psi) \sim P(n \cdot \tau(\psi|\omega))$, where $\tau(\psi|\omega)$ is the expected fraction of voters playing action $\psi$ in state $\omega$ (see (6) above).

Under approval voting, the number of votes received by alternative $A$ or by alternative $B$ is the sum of two independent Poisson random variables: $X(A) = x(A) + x(AB)$ and $X(B) = x(B) + x(AB)$. A pivot probability is therefore the joint probability of two events:

$$
\Pr(piv_{PQ}|\omega) = \frac{1}{2} \Pr(X(Q) - X(P) \in \{0, 1\} | \omega) \times \Pr(X(R) \leq X(Q) | X(Q) - X(P) \in \{0, 1\}, \omega) \quad (13)
$$

3d alternative, $R$ trails behind

We can exploit the properties proven by Myerson (1998a, 2000) and in Appendix A1 to show that:

**Property 1** For a large electorate ($n$ large), the probability that two alternatives $P$ and $Q \in \{A, B, C\}$ have (almost) the same number of votes converges to zero at an exponential rate called the magnitude of the probability:

$$
\text{mag}(PQ|\omega) \equiv \lim_{n \to \infty} \frac{\log [\Pr(|X(P) - X(Q)| \leq 1|\omega)]}{n}.
$$

The exact form of the different magnitudes $\text{mag}(PQ|\omega)$ is detailed in Appendix A1.

It follows that if two events have a different magnitude, then (Property 3 in Appendix A1):

$$
\lim_{n \to \infty} \frac{\Pr(X(P) = X(Q)|\omega)}{\Pr(X(P) = X(R)|\omega')} = 0 \text{ if and only if } \text{mag}(PQ|\omega) < \text{mag}(PR|\omega'), \quad (14)
$$

with $P, Q, R \in \{A, B, C\}, P \neq Q \neq R$ and $\omega, \omega' \in \{a, b\}$.

The magnitude of a pivot probability $\Pr(piv_{PQ})$ is such that:

$$
\text{mag}(piv_{PQ}|\omega) = \text{mag}(PQ|\omega) \text{ if } \Pr(X(R) \leq X(Q) | X(Q) - X(P) \in \{0, 1\}, \omega) \to 1
$$

$$
< \text{mag}(PQ|\omega) \text{ if } \Pr(X(R) \leq X(Q) | X(Q) - X(P) \in \{0, 1\}, \omega) \to 0
$$

Under Approval Voting, whenever $C$ is expected to rank first or second, the pivot probability between the expected top (resp. bottom) two alternatives has the largest (resp. smallest) magnitude (Property 4 in Appendix A1).
Property 1 summarizes the main results proved in Appendix A1. Equation (14) has been called the magnitude theorem by Myerson (2000). The intuition is that pivot probabilities do not converge towards zero at the same speed. Hence, unless two events have the same magnitude, their likelihood ratio converges either to zero or to infinity. In addition to these classical results, Property 1 tells us that the expected ranking of the pivot probabilities need not correspond to the ranking of vote shares. This is because those voters who double-vote for A and B introduce some correlation between $X(A)$ and $X(B)$ that reduces $mag(piv_{AB}|\omega)$. This correlation is taken care of by computing pivot probabilities on the $x(\psi)$.

These properties are quite general and are not specific to Poisson games. For instance, Myerson (2000, Section 4) shows that pivot probabilities under multinomial distributions are simply a monotone transformation of their Poisson equivalent.\(^{13}\) This is why our results extend directly to the multinomial distribution (the proofs are available upon request).

### 4 Approval Voting: Equilibrium Analysis

Classically, elections with three or more alternatives suffer from information and coordination problems: which is the best alternative is unclear, and one voter’s best response depends on the action profile of the rest of the electorate. In the present setup, and if the information was perfect, one would elect alternative A in state $a$ and alternative B in state $b$. Yet, the voters’ lack of information means that they cannot make their ballot contingent on the true state of nature.

We shall say that:

**Definition 2** Elections satisfy full information and coordination equivalence if, with a probability that converges to 1 as population size increases, they produce the same outcome as the one that would prevail under perfect information.\(^{14}\) To achieve this, equilibrium vote shares must be such that:

\[
\tau(A|a) + \tau(AB|a) > \max \{ \tau(B|a) + \tau(AB|a), \tau(C) \} \quad \text{in state } a, \text{ and}
\]

\[
\tau(B|b) + \tau(AB|b) > \max \{ \tau(A|b) + \tau(AB|b), \tau(C) \} \quad \text{in state } b.
\]

\(^{13}\)Myerson (2000) shows that limits of pivot probabilities under Poisson games are such that $\lim_{n \to \infty} \log(Pr(piv_{PQ})) / n = \mu$. In his Section 4, Myerson shows that, if the distribution is Multinomial instead of Poisson, then $\lim_{n \to \infty} \log(Pr(piv_{PQ})) / n = \log(\mu+1)$, where $\mu$ is the limit under the Poisson distribution. Therefore, the limit likelihood ratio (14) is the same under both distributions.

\(^{14}\)This concept of full information and coordination equivalence is the natural extension of Feddersen and Pesendorfer’s (1997) concept of full information equivalence to elections with more than two candidates.
That is, alternative A’s expected vote share must be the largest one in state a and conversely for alternative B in state b.

Typically, satisfying this constraint is not trivial in a three-candidate setting: first, as seen in the introduction, C may win the election if the majority divide their votes to aggregate information. Second, even if C is only supported by a small minority, there may be multiple equilibria, and hence a coordination problem. Third, the outcome cannot be made contingent on the state of nature if the majority are forced to coordinate on exactly one alternative. Our main contribution is to show that these problems vanish under Approval Voting:

**Theorem 1** Under Approval Voting, the equilibrium is unique and satisfies full information and coordination equivalence: the equilibrium strategies are such that (15) holds.

The fact is that the possibility of double voting, which is built into Approval Voting, deeply modifies the trade-off that is present in other systems. When majority voters can use double-voting to avoid C’s victory, coordinating on one alternative is both unnecessary and undesired. The intuition is that, when A’s victory is threatened in state a, then even types $t_B$ will be willing to lend support to A by double voting, i.e. by playing $AB$. Importantly, this is not only true when A is threatened by C. It is also true when B threatens the victory of A in state a: even types $t_B$ understand that the true state might be a with some probability. Similarly, when B is threatened in state b, then types $t_A$ will be willing to play $AB$. Only when A and B’s vote shares are sufficiently high compared to C’s and balanced with one another, majority voters are willing to divide their votes to aggregate information. There is always room for information aggregation since, by definition, majority voters form a majority of the electorate.

Proving this theorem is the purpose of this section. Each of the next two subsections focuses on one aspect of the proof: first, we prove that there cannot be any informational trap under Approval Voting. Second, we derive the equilibrium strategies: Proposition 3 identifies them and shows that they are unique and induce full information and coordination equivalence. As shown in Section 7, this result would no longer hold if the majority was composed of voters whose preferences do not dependent on information: equilibrium multiplicity is again a concern.
4.1 Absence of Informational Traps

In this section, we prove that there cannot be informational traps in equilibrium, either in pure or in mixed strategy (remember that informational traps arise if all majority types, \(t_A\) and \(t_B\), adopt the same strategy profile in equilibrium). We underline the main trade-off faced by majority-block voters in Proposition 1. Proposition 2 then shows that types \(t_A\) and \(t_B\) necessarily specialize in playing \(A\) and \(B\) respectively.

**Proposition 1** There cannot be an informational trap in which all majority-block voters would use the same strategy with probability 1. That is, none of the three corner strategies:

\[
\{\sigma(A|t), \sigma(B|t), \sigma(AB|t)\} \in \{(1, 0, 0), (0, 1, 0), (0, 0, 1)\}
\]

in which all majority block voters \(t = t_A, t_B\) play an action with probability 1 can be an equilibrium.

**Proof.** See Appendix A2. ■

The intuition is as follows. Imagine that all majority-block voters are expected to vote \(A\). A given type-\(t_B\) would like to have \(B\) winning the election, and therefore would like to play \(B\): this is her preference motive. Yet, she knows that, with a vote share of 0, \(B\) has virtually no chance of winning. So, her strategic motive induces her to give up the idea of playing \(B\), to avoid wasting their ballot: she should vote \(A\) instead. Under approval voting, though, \(t_B\) voters can hit these two birds with one ballot: they can combine their strategic motive (vote \(A\)) together with their preference motive (vote \(B\)) with a joint \(AB\) approval; this deviation is always profitable.\(^{15}\) Hence, under approval voting, voters can never be trapped into a corner strategy of “having to” vote for a lesser liked candidate.

The balance between these two motives is reversed when all other majority voters are expected to double-vote. If majority-block voters are expected to play \(AB\) with probability 1, alternatives \(A\) and \(B\) top the polls with the same expected vote share. In this case, any type \(t_A\) or \(t_B\) would like to deviate and single-vote for her preferred alternative: the other majority-block voters are already taking care of the strategic motive (\(C\) trails behind and

\(^{15}\)This feature is specific to Approval Voting. Consider any other voting rule, in which the voter must withdraw some “voting points” from \(A\) if she wants to also vote for \(B\). In that case, there is a conflict between the preference and strategic motives: the probability that a ballot is pivotal in favor of \(B\) is infinitesimal compared to the pivot probability in favor of \(A\). Hence, any point withdrawn from \(A\) has a cost that is infinitely larger than the benefit of the point(s) given to \(B\). Like in a prisoner’s dilemma, no voter then affords to express her preferences.
therefore has virtually no chance of winning), whereas a single-A or a single-B ballot has a very high probability of making the difference between $A$ and $B$.

Hence, Proposition 1 eliminates three candidate equilibria that could be problematic in elections. The first two candidate equilibria are the game theoretic materialization of Duverger’s Law. In such equilibria, majority-block voters feel compelled to coordinate all their votes on only one alternative, either $A$ or $B$ (we will see in Sections 5 and 6 that these equilibria exist under Plurality and Runoff elections). In this case, new alternatives would face a major barrier to entry: they could be sure to lose the election even when they are perceived as better than both incumbent alternatives by a large fringe of the population. In other words, Proposition 1 shows that the incumbency advantage is notably weakened under Approval Voting. The third candidate equilibrium has been termed the **Burr dilemma** by Nagel (2007). He documents the “[approval] experiment [that] ended disastrously in 1800 with the infamous Electoral College tie between Jefferson and Burr”.

Proposition 1 shows why such a “disaster” cannot happen in large-scale elections: the Electoral College involved few voters, whose behavior was dictated by party discipline.

Even though Proposition 1 eliminates these three candidate equilibria, it does not ensure that equilibrium vote shares are necessarily different in both states of nature. Myerson and Weber (1993), for instance, present an example in which all candidates have the same vote share in equilibrium, which is a weaker version of the Burr dilemma: $A$ and $B$ indeed end up in a tie. As we shall see in the next subsection, this cannot happen in our setup: since majority-block voters $t_A$ and $t_B$ “specialize” into playing $A$ and $B$ respectively, there can never be an informational trap:

**Proposition 2** In equilibrium, we must have: $\sigma (A|t_A) + \sigma (AB|t_A) = 1$ and $\sigma (B|t_B) + \sigma (AB|t_B) = 1$ with $\sigma (A|t_A) > 0$ and $\sigma (B|t_B) > 0$. Hence, majority-block mix between their ‘preferred alternative’ and the joint $AB$ approval.

**Proof.** See Appendix A2. ■

The intuition for the proof is as follows: first, we show that a voter never wants to mix between actions $A$ and $B$. Such a mixed strategy would mean that she is indifferent between the two alternatives. Expressed differently, the voter does not want to choose between them. In that case however, a safer option is to play action $AB$: this action has a higher probability of being pivotal against $C$, and can never be mistakenly pivotal, *e.g.* in favor of $A$ against $B$ when the true state of nature is $b$. 

15
Interestingly, this Proposition also relates to the “swing voter’s curse”. In Feddersen and Pesendorfer (1996), voters with imperfect information abstain to avoid “noisying” the election result. This incentive, which could still be present when there are more than two candidates, is actually absent under Approval Voting, since voters can double vote.

It remains to see why types $t_A$ and $t_B$ respectively play $A$ and $B$ in equilibrium. To understand this, imagine for a moment that no voter plays $\psi = A$. If this were the case, the expected vote shares of alternatives $B$ and $C$ would be exactly identical in the two states of nature. Yet, because of the preference motive, types $t_A$ must play $AB$ more often than types $t_B$, which implies that the expected vote share of $A$ must be strictly larger in state $a$ than in state $b$. Hence, a $B$-ballot is infinitely more likely to be pivotal against $A$ in state $a$ than in state $b$. Since even $t_B$-voters want $A$ to win in state $a$, they strictly prefer to play $AB$. Yet, we have seen from Proposition 1 that it cannot be an equilibrium to have all majority-block voters playing $AB$. By contradiction, some voters must play $\psi = A$ with strictly positive probability in equilibrium. Given the preference motive, one can identify that types $t_A$ are the ones who play $A$ with strictly positive probability (and therefore never play $B$), and conversely for types $t_B$.\footnote{In a setup with fixed preferences, Brams and Fishburn (2007, Theorem 2.1) show that a voter will always include her most preferred alternative in her ballot. One aspect of Proposition 2 is to show how their Theorem extends to voters whose preference ordering is state-dependent.}

### 4.2 Equilibrium Uniqueness

From Proposition 2, we know that all majority-type voters include their \textit{a priori} preferred alternative in their ballot: since they mix between $A$ and $AB$, types $t_A$ necessarily approve of $A$. Types $t_B$ mix between $B$ and $AB$, which always includes $B$. Hence, the strategy of a type $t_A$ influences the vote count of alternative $B$: the more types $t_A$ double-vote, the higher the expected vote share of $B$. Likewise, the strategy of a type $t_B$ influences the expected vote count of alternative $A$.

The vote share of an alternative will thus increase when the incentives of types $t_A$ and $t_B$ become more aligned, \textit{i.e.} when either type feels it must support the other group. We identify two cases in which their incentives are aligned: first, when there is a “major imbalance” between the expected vote shares of $A$ and $B$. Second, when they need to fight alternative $C$.

A major imbalance between the two alternatives occurs when either alternative $A$ or $B$ is much ahead of the other. Imagine for instance that $A$ is expected to receive much
more votes than B. In that case, $t_A$-voters are quite certain that A wins in state $a$, given alternative A’s lead. Instead, they are not quite certain that B wins in state $b$. As responsive voters, they realize that they have to lend support to B as well: this does not threaten A in state $a$, but does give B a chance in state $b$. Hence, they will prefer to play AB if they expect a major imbalance in favor of A.

The fight against C aligns incentives in the same way. Imagine that a vote for B is much more likely to be pivotal against C than against A (this happens when A and B’s vote shares are not sufficiently above C’s). In that case as well, a type $t_A$ prefers to cast a double ballot: it provides additional insurance against the election of C.

These two cases lead to the same conclusion: if B’s vote share is too low, either compared to A’s or to C’s, the incentives of types $t_A$ become aligned with that of types $t_B$ – this is again the materialization of the strategic motive –, which induces them to double-vote with a higher probability. By symmetry, if A’s vote share is too low, then it is types $t_B$ who must lend support to A, and double-vote.

On the other hand, from the previous propositions, we know that the preference motive dominates when sufficiently many majority-block voters double vote. In what follows, we show that there is a unique relationship between the strategy of the types $t_A$ and $t_B$ that prevents major imbalances between A and B, and there is a unique “aggregate level” of double-voting that balances the preference and strategic motives. This is why the equilibrium is unique as well.

Formally, using the expected gain functions (10) – (12), we have:

\[
G(A|t_A) - G(AB|t_A) = q(a|t_A) \left[ \Pr(piv_{AB}|a) - \Pr(piv_{BC}|a) \right] \\
- q(b|t_A) \left[ 2\Pr(piv_{BC}|b) + \Pr(piv_{AB}|b) \right],
\]

(16)

\[
G(B|t_B) - G(AB|t_B) = q(b|t_B) \left[ \Pr(piv_{BA}|b) - \Pr(piv_{AC}|b) \right] \\
- q(a|t_B) \left[ 2\Pr(piv_{AC}|a) + \Pr(piv_{BA}|a) \right].
\]

(17)

A necessary condition to have $G(A|t_A) - G(AB|t_A) \geq 0$ is that $\Pr(piv_{AB}|a)$ is sufficiently large compared to the other three pivot probabilities in (16). Similarly, a necessary condition to have $G(B|t_B) - G(AB|t_B) \geq 0$ is that $\Pr(piv_{BA}|b)$ is sufficiently large compared to the other three pivot probabilities in (17). From Property 1, this requires:

\[
\text{mag}(piv_{AB}|a) \geq \max \{ \text{mag}(piv_{AB}|b), \text{mag}(piv_{BC}|a), \text{mag}(piv_{BC}|b) \},
\]

\[
\text{mag}(piv_{BA}|b) \geq \max \{ \text{mag}(piv_{BA}|a), \text{mag}(piv_{AC}|a), \text{mag}(piv_{AC}|b) \}.
\]

(18)

Let us first focus on the constraint that appears between the vote shares of alternatives A and B. The combination of the two inequalities in (18) imposes that the magnitudes
\( \text{mag}(\text{piv}_{AB}|a) \) and \( \text{mag}(\text{piv}_{BA}|b) \) be equal. Since they must also be larger than all the magnitudes \( \text{mag}(\text{piv}_{PC}|\omega) \), we have by Property 4 in Appendix A1:

\[
\left( \sqrt{r(t_A|a) \cdot \sigma(A|t_A)} - \sqrt{r(t_B|a) \cdot \sigma(B|t_B)} \right)^2 = \left( \sqrt{r(t_A|b) \cdot \sigma(A|t_A)} - \sqrt{r(t_B|b) \cdot \sigma(B|t_B)} \right)^2.
\]

This condition is seen to depend on the two strategy profiles, \( \sigma(A|t_A) \) and \( \sigma(B|t_B) \). Yet, defining:

\[
\rho \equiv \sigma(A|t_A) / \sigma(B|t_B),
\]

one readily sees that condition (19) is satisfied iff:

\[
\left| \sqrt{r(t_A|a) \cdot \rho} - \sqrt{r(t_B|a)} \right| = \left| \sqrt{r(t_B|b)} - \sqrt{r(t_A|b) \cdot \rho} \right|,
\]

which has a unique solution in \( \mathbb{R}^+ \):

\[
\rho^* = \left( \frac{\sqrt{r(t_B|a)} + \sqrt{r(t_B|b)}}{\sqrt{r(t_A|a)} + \sqrt{r(t_A|b)}} \right)^2.
\]

This solution in turn implies: \( \tau(A|a) > \tau(B|a) \) and \( \tau(A|b) < \tau(B|b) \).

Hence, we are now left with one unknown variable: if we find the equilibrium probability \( \sigma(B|t_B) \) with which types \( t_B \) single-vote in equilibrium, the value of \( \sigma(A|t_A) \) immediately follows from (20). The following proposition shows that there is a unique equilibrium solution to \( \sigma(B|t_B) \). This equilibrium value of \( \sigma(B|t_B) \) is the highest one that allows (18) to be satisfied:

**Proposition 3** The equilibrium is unique and such that:

i) \( \sigma(B|t_B) = 1, \sigma(A|t_A) = \rho^* \) iff, for this strategy profile,

\[
\text{mag}(\text{piv}_{AB}|a) = \text{mag}(\text{piv}_{AB}|b) \geq \max_{\omega} \{ \text{mag}(\text{piv}_{AC}|\omega), \text{mag}(\text{piv}_{BC}|\omega) \}.
\]

ii) Otherwise, \( \sigma(B|t_B) = \bar{\sigma}, \sigma(A|t_A) = \rho^* \bar{\sigma} \) with \( \bar{\sigma} \in (0, 1) \) such that:

\[
\text{mag}(\text{piv}_{AB}|a) = \text{mag}(\text{piv}_{AB}|b) = \max_{\omega} \{ \text{mag}(\text{piv}_{AC}|\omega), \text{mag}(\text{piv}_{BC}|\omega) \}.
\]

**Proof.** See Appendix A2. 

As seen above, there is a unique ratio \( \rho^* \) that prevents the vote share of \( A \) from being either too high or too low compared to that of \( B \). Proposition 3 adds that there is a unique value of \( \sigma(A|t_A) \) and \( \sigma(B|t_B) \) that can be an equilibrium. The reason is as follows: whenever \( C \)'s vote share is sufficiently below that of \( A \) and \( B \), the preference motive
dominates: types $t_B$ strictly prefer to single vote for $B$, and so do types $t_A$, who want to single vote for $A$. This increases the gap between $A$ and $B$ in both states of nature. The only obstacle to furthering the difference between $A$ and $B$ is the threat posed by $C$: if there exists a strategy profile for which (21) binds, then the strategic motive starts dominating again, and both types $t_A$ and $t_B$ prefer to double-vote with a sufficiently high probability. The equilibrium is reached when this strategic motive to beat $C$ balances the preference motive, unless a corner solution is reached. The solution is unique because the perceived threat posed by $C$ is monotonically decreasing in the fraction of voters who double vote.

4.3 Numerical Examples

For the sake of simplicity, we focus on symmetric priors: $q(a) = \frac{1}{2} = q(b)$ and a symmetric distribution of types: $r(t_A|a) = r(t_B|b)$. From (20) and Proposition 3, the latter imposes that $\sigma^*(A|t_A) = \sigma^*(B|t_B)$. We shall illustrate the effect of a variation in $r(t_C)$, the size of the minority group in the population, and of a variation in the ratio $r(t_A|a)/r(t_A|b)$, which represents the quality of the information available to the voters.

Let $r(t_C) = 0.4$, $r(t_A|a) = 0.36$ and $r(t_A|b) = 0.24$. With these parameter values, like for the actual cases discussed in the introduction, if majority-group voters divide their votes, we would have: $\tau(C) = 0.4 > \tau(A|a) = \tau(B|b) = 0.36 > \tau(A|b) = \tau(B|a) = 0.24$ and the Condorcet loser, $C$, would asymptotically be sure to win the election. This implies that condition (21) will be binding, and that there must be some double-voting in equilibrium. Indeed, the equilibrium strategy profile is $\sigma(AB|t_A) = 0.57 = \sigma(AB|t_B)$, which implies:

<table>
<thead>
<tr>
<th>Vote shares</th>
<th>state $a$</th>
<th>state $b$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$A$</td>
<td>0.497</td>
<td>0.445</td>
</tr>
<tr>
<td></td>
<td>(first)</td>
<td>(second)</td>
</tr>
<tr>
<td>$B$</td>
<td>0.445</td>
<td>0.497</td>
</tr>
<tr>
<td></td>
<td>(second)</td>
<td>(first)</td>
</tr>
<tr>
<td>$C$</td>
<td>0.4</td>
<td>0.4</td>
</tr>
<tr>
<td></td>
<td>(third)</td>
<td>(third)</td>
</tr>
<tr>
<td>Total</td>
<td>1.342</td>
<td>1.342</td>
</tr>
</tbody>
</table>

Table 1: equilibrium vote shares (left) and magnitudes (right).

<table>
<thead>
<tr>
<th>Magnitudes</th>
<th>state $a$</th>
<th>state $b$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\text{mag}(piv_{AC}</td>
<td>\omega)$</td>
<td>$-0.0052$</td>
</tr>
<tr>
<td>$\text{mag}(piv_{BC}</td>
<td>\omega)$</td>
<td>(small) $-0.0052$</td>
</tr>
<tr>
<td>$\text{mag}(piv_{AB}</td>
<td>\omega)$</td>
<td>$-0.0052$</td>
</tr>
</tbody>
</table>

$^{17}$The pivot probability between the second and third candidates is infinitely lower than the pivot probability between the first and second candidate. In the absence of a closed-form solution for these magnitudes, we cannot compute their exact value.
The possibility to double-vote with some probability allows the majority to “inflate” the expected vote shares of both $A$ and $B$ above the share of $C$. This is why the sum of the three vote shares exceeds 100% of the population: majority-block voters double-vote up to the point in which the magnitude of the pivot probability between $A$ and $B$ is equal to the largest magnitudes against $C$.

The equilibrium propensity to double-vote is directly linked to the size of the minority. If the fraction of types $t_C$ is sufficiently low, majority-group voters do not actually need to double-vote: let $r(t_C) = 0.25$, $r(t_A|a) = 0.45$ and $r(t_A|b) = 0.30$. With these parameter values, the quality of information is the same as in the previous example ($r(t_A|a)/r(t_A|b) = 1.5$) but, majority-group voters divide their votes, we have:

- $\tau (A|a) = \tau (B|b) = 0.45 > \tau (A|b) = \tau (B|a) = 0.30 > \tau (C) = 0.25$, and full information and coordination equivalence obtains. Since $\text{mag}(\text{piv}_{AB}[\omega])$ is strictly larger than the other magnitudes for that strategy profile, this pure-strategy action profile is actually an equilibrium. More generally, in such a symmetric setup, majority-block voters double-vote in equilibrium if and only if $r(t_C) > r(t_A|b) = r(t_B|a)$ and, the higher is $r(t_C)$, the higher is the majority’s propensity to double vote (holding $r(t_A|a)/r(t_A|b)$ constant).

This observation that double-voting vanishes when $r(t_C)$ becomes sufficiently low links directly to Brams and Fishburn’s (2005) case study of the Institute of Electrical and Electronics Engineers (IEEE). In 1986, because of a division within the majority and a significant opposition by a minority, the minority-backed candidate almost won the election for the presidency. This triggered the adoption of Approval Voting. Subsequently, majority divisions and minority size decreased, and the IEEE reverted to Plurality Voting because:

> According to the IEEE executive director [...] ‘few of our members were using [multiple voting...]’ . Brams responded in an e-mail exchange (June 2, 2002) that since “candidates now can get on the ballot with ‘relative ease’ [...] the problem of multiple candidates [...] might actually be exacerbated ... and come back to haunt you [IEEE] some day” (Brams and Fishburn 2005, p16).

Returning to our numerical examples, we now analyze the effect of an improvement in information. Surprisingly, better information induces more double-voting in equilibrium. The rationale is as follows: increasing $r(t_A|a)$ and decreasing $r(t_A|b)$ while holding $r(t_C)$ constant implies that, ceteris paribus, the gap between the first and the second alternative’s vote shares increases. For a given strategy profile, the probability that one vote is pivotal between $A$ and $B$ decreases in magnitude. In comparison, the gap between the first
alternative and $C$ does not increase as fast. Hence, the balance between the strategic and preference motives tilts again in favor of the former: the value of a double vote increases compared to that of a single vote. To illustrate this, set $r(t_A|a) = 0.48$ and $r(t_A|b) = 0.12$ and keep $r(t_C) = 0.4$ as in the first example. In that case, $\sigma(AB|t_A) = 0.8580 = \sigma(AB|t_B)$ in equilibrium:

Table 2: equilibrium vote shares (left) and magnitudes (right).

<table>
<thead>
<tr>
<th>Vote shares</th>
<th>state $a$ (first)</th>
<th>state $b$ (second)</th>
<th>Magnitudes</th>
<th>state $a$ (small)</th>
<th>state $b$ (small)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$A$</td>
<td>0.583</td>
<td>0.532</td>
<td>$\text{mag}(\text{piv}_{AC})$</td>
<td>$-0.0172$</td>
<td>$-0.0172$</td>
</tr>
<tr>
<td>$B$</td>
<td>0.532</td>
<td>0.583</td>
<td>$\text{mag}(\text{piv}_{BC})$</td>
<td>$-0.0172$</td>
<td></td>
</tr>
<tr>
<td>$C$</td>
<td>0.4</td>
<td>0.4</td>
<td>$\text{mag}(\text{piv}_{AB})$</td>
<td>$-0.0172$</td>
<td>$-0.0172$</td>
</tr>
<tr>
<td>Total</td>
<td>1.5144</td>
<td>1.5144</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Compared to the first example, the equilibrium ranking remains the same but there is more double-voting. Pivot magnitudes are also lower, which means that the probability of a mistake, i.e. that $A$ wins in state $b$ or $B$ wins in state $a$, decreases substantially.

5 Plurality Elections

Now that we have analyzed the properties of approval voting, we can compare them with those of other, commonly used, electoral systems. We analyze two such systems: plurality elections in this section, and runoff elections in the next one.

Under plurality, like under approval voting, the alternative receiving the most votes wins the election. The only difference is that voters can only cast a single ballot or abstain. That is, their action set is restricted to: $\Psi_{\text{Plurality}} = \{\emptyset, A, B, C\}$. Otherwise, all pivot probabilities remain the same as in Property 1, with the only difference that, by the definition of $\Psi_{\text{Plurality}}$, we have: $\sigma(AB|t) = 0$, $\forall t$ and hence:

$$
\begin{align*}
X(A) &= x(A),  \\
X(B) &= x(B),  \\
X(C) &= x(C).
\end{align*}
$$

Theorem 2 shows that this single difference between the two electoral procedures is sufficient to induce multiplicity of equilibria. Moreover, as already highlighted by Piketty (2000), many such equilibria fail to produce full information and coordination equivalence:
Theorem 2 Under plurality elections, there are at least three equilibria. The first and second are self-fulfilling equilibria in which all majority types vote for A (resp. B), because they expect the other alternative, B (resp. A) to receive no vote. These equilibria produce an informational trap.

In the third equilibrium, majority types adopt different strategies, hence there is no informational trap. Yet, for \( \tau(C) > 1/[2 + \tau(A|b)/\tau(B|a)] \), equilibrium vote shares are such that:

\[ \tau(C) > \tau(A|a) \simeq \tau(B|b) > \tau(A|b) \simeq \tau(B|a) > 0. \]

In this equilibrium, the dominated candidate C wins with a probability that converges to 1 as \( n \to \infty \).

Proof. See Appendix A3.

6 Runoff Elections

This section analyzes the properties of another commonly used electoral system: plurality runoff elections, also known as two-round elections. In this electoral system, an alternative wins outright if it collects more than 50% of the votes in the first round. If no alternative reaches this 50%-threshold, then a runoff is organized between the two alternatives with the most votes.\(^{18}\) This runoff procedure is often proposed as a solution to the coordination failures that lead to informational traps. Piketty (2000) for instance professes that runoff elections should be able to separate the “communication stage”, in which voters learn which of A and B is best, from the “election stage”. This intuition finds support in Martinelli (2002) who analyses the equilibrium properties of plurality runoff elections with privately informed voters. However, in his analysis, Martinelli (2002) assumes away the risks that are present in the second round: the majority-backed candidate is assumed to win with probability 1. In contrast, we let, in each round, the population follow the same Poisson distribution as under the other electoral systems, which means that the probability of winning is only asymptotically equal to 1. As we show here, this implies that, unless types \( t_C \) represent a very small part of the electorate, runoff elections suffer from the same informational traps as plurality elections.

\(^{18}\)Note that there exists other types of two-round elections in which the threshold for first-round victory is below 50% (for instance in Argentina, Nicaragua, Costa Rica and North Carolina). For an analysis of such two-round elections in Poisson games, see Bouton (2007).
Runoff elections are a two-period game and we want to check whether the first-period strategies \((\sigma(A,t), \sigma(B,t)) \in \{(1,0), (0,1)\}\) for \(t = t_A, t_B\) can be an equilibrium. Hence, solving the game backwards, we are only interested in the subgames in which \(C\) reaches the second round. Let us focus on the subgame that opposes \(A\) to \(C\): in that case, all majority-block voters play \(\psi = A\), and all minority-block voters play \(\psi = C\). The expected utility of a majority type \(t \in \{t_A, t_B\}\) negatively depends on the probability that \(C\) wins the election, \(\Pr(C)\):

\[
\mathbb{E}U(t|A\text{ vs. }C\text{ in 2d round}) = q(a|t) - \Pr(C) \\
= q(a|t) - \left(\frac{\Pr[\tilde{X}(C) = \tilde{X}(A)]}{2} + \Pr[\tilde{X}(C) > \tilde{X}(A)]\right) \\
< q(a|t) - \frac{\Pr[\tilde{X}(C) = \tilde{X}(A)]}{2} = q(a|t) - \Pr(\text{pivot}_{2AC})
\]

where \(\Pr(\text{pivot}_{2AC})\) denotes the second-round pivot probability between \(A\) and \(C\). By Property 2, \(\Pr(\text{pivot}_{2AC})\) is proportional to:

\[
\Pr[\tilde{X}(C) = \tilde{X}(A)] \propto \exp\left[-\left(\sqrt{1 - \tau(C)} - \sqrt{\tau(C)}\right)^2 n\right].
\]

This (whatever small) second-round risk influences the incentives of a majority block voter in the first round: consider the first-round strategy profile \(\sigma(B|t_B) \rightarrow 0\) and \(\sigma(B|t_A) = 0\), for which alternative \(B\)'s expected vote share is vanishingly small. What is a given \(t_B\)-voter’s best response? If she plays \(\psi = A\) and is pivotal to elect \(A\) in the first round, she saves herself from the second-round risk. In comparison, action \(\psi = B\) is valuable if a second round is organized and if her ballot is pivotal in bringing \(B\) to that round.

Comparing the probabilities of each of these events shows that:

**Theorem 3** Under runoff elections, unless the fraction of types \(t_C\) is sufficiently small, there exist two self-fulfilling equilibria in which all majority types play \(\psi = A\) (resp. \(B\)). These equilibria produce an informational trap.

**Proof.** See Appendix A4. ■

The trade-off is self-explanatory. A majority voter has an incentive to abandon a trailing candidate (\(B\) in the above case) if the second-round risk is too high compared to the first-round chance of bringing the trailing candidate to the second round. Typically, the larger is \(C\)'s vote share, the higher is the second-round risk, and the lower is the probability that one vote may bring \(B\) to the second round. The surprising result is that, even though we only focus on a lower bound of that risk (we only compute the probability that
the two candidates tie in the second round to assess it), we find that a vote share of $C$ as low as 6.7% is sufficient to generate such informational traps.$^{19}$

Note still that Theorem 3 does not claim that there is no equilibrium with full information and coordination equivalence. Runoff elections actually feature many equilibria, and some of them do satisfy this equivalence. This is however immaterial to the analysis, for two reasons. First, the equilibrium under Approval Voting is unique. Approval Voting therefore Pareto-dominates Runoff elections. Second, organizing elections is extremely costly. Runoff elections may therefore cost about twice as much as Approval Voting elections, despite its less desirable properties.

7 Equilibria when Preferences do not Depend on Information

To be written

8 Conclusion

We have argued that one must take account of the voter’s sensitiveness to information when studying the properties of electoral systems. Under imperfect information, the voters’ preference ordering is bound to depend, among other things, on fresh information about candidate competence, probity or political preferences.

We proposed a model of elections that captures this information imperfection. Voters in the majority are divided about the candidate they prefer. Each voter supports her candidate but knows that, with some probability, her prior preference is the wrong one. A third candidate, backed by another part of the electorate, runs against the majority. Hence, voters in the majority bear a risk of losing the election altogether if they divide their votes.

In this setup, we studied the asymptotic equilibrium properties of three electoral systems and showed that Approval Voting is ideally suited to aggregate information: it produces a unique equilibrium, in which the candidate who wins the election is actually the best. The other two systems, Plurality rule and Runoff, produce multiple equilibria.

$^{19}$As emphasized in Section 4.1, all our results directly extend to multinomial distributions. In the case of runoff elections, results would even be stronger with such a multinomial distribution. Indeed, the share of $C$ sufficient to generate an informational trap converges to zero as population size increases.
This gives rise to a coordination problem and implies that a bad candidate may be sure to win.

The reason why Approval Voting dominates the other systems is that voters do not need to divide their votes in order to pick the best candidate. They can double vote (that is: approve of their two candidates) both to fight the minority-backed candidate and to balance the support in favor of either majority alternatives. In equilibrium, there always are sufficiently many voters who single vote for their preferred alternative to ensure that the best candidate actually wins the election.

Arguably, the model focused on a simplified baseline case. The trade-offs it highlighted are nevertheless quite general: think for instance of a world with more candidates. If there are \( k \) candidates in the majority and \( l \) candidates in the minority, the trade-off remains identical. As long as their primary objective is to fight one another, both majority-block and minority-block voters will multiple vote for all their candidates. Within the majority, voters will also maintain the balance between all their potentially good candidates, to make sure that the best can win. Indeed, our results have shown that, whenever a candidate trails behind, the other voters in the majority also want to support her with a multiple ballot. Hence, while the analysis would become much more cumbersome given the number of strategies to consider, the main results should remain.

Another simplifying assumption we made is that all voters of a same type have the same preferences and the same information. What would happen if voters had either more general priors or different intensities in their preferences? In a two-candidate setup, Feddersen and Pesendorfer (1997) analyze this issue and show that full information equivalence prevails even if voters have access to different information channels and have heterogeneous preferences. If we extend our model in this direction, the same result should hold under Approval Voting: instead of adopting the same mixed strategy, the voters would adopt cutoff strategies, in which voters with the most intense preferences single vote, and the most moderate double vote.

Finally, we have made the assumption that all voters in the majority are willing to learn and aggregate information. We have also seen that, if none of them is sensitive to fresh information, then multiple equilibria also arise under Approval Voting. These extreme cases suggest that a certain fraction of information-sensitive voters is necessary for our results to hold. Whether the number of such “independent voters” is sufficient in real-world elections will be worth investigating in future research. Similarly, the properties of other voting systems, such as instant runoff, the Borda count or storable votes should
be analyzed.

References


Appendix

Appendix A1 summarizes and extends to approval voting some properties of Poisson Games proven by Myerson (1998a, 1998b, 2000). Appendices A2, A3 and A4 demonstrate the claims made in Sections 4, 5 and 6 respectively.

Appendix A1: Some Properties of Poisson Voting Games

Property 2 (Myerson 2000, Theorem 1 and extension to Approval Voting)
Subject to \( \sum_{\omega \in \{A,B,AB,C\}} \tau(\omega) = 1 \), and for \( \omega \in \{a,b\} \), given expected numbers of votes \( n \tau(\omega) \), the probability that the realized number of votes are \( x = \{x(A), x(B), x(AB), x(C)\} \) is:

\[
\Pr(x|\tau(\omega)) \xrightarrow{n \to \infty} \max_{\lambda} \frac{\exp[mag[x]]}{\prod_{\psi} \sqrt{2\pi x(\psi) + \lambda}},
\]

where: \( mag[x] = \sum_{\psi} \frac{x(\psi)}{n} \left( 1 - \log \left( \frac{x(\psi)}{n \tau(\psi|\omega)} \right) \right) - 1 \quad (\leq 0) \) \quad (23)

For a large electorate (\( n \) large), the probability that two alternatives \( P \) and \( Q \) have (almost) the same number of votes converges to zero at an exponential rate called the magnitude of the probability:

\[
mag(PQ|\omega) = \lim_{n \to \infty} \frac{\log \Pr(|x(P) - x(Q)| \leq 1|\omega)}{n},
\]

where the magnitudes \( mag(PQ|\omega) \) are given by:

\[
\begin{align*}
mag(AB|\omega) &= -\left( \sqrt{\tau(A|\omega)} - \sqrt{\tau(B|\omega)} \right)^2, \quad (24) \\
mag(AC|\omega) &= -\left( \sqrt{\tau(A|\omega) + \tau(AB|\omega)} - \sqrt{\tau(C|\omega)} \right)^2, \quad (25) \\
mag(BC|\omega) &= -\left( \sqrt{\tau(B|\omega) + \tau(AB|\omega)} - \sqrt{\tau(C|\omega)} \right)^2. \quad (26)
\end{align*}
\]

Proof. \( (23) \) is the application of Theorem 1 in Myerson (2000). \( (24), (25) \) and \( (26) \) extend this theorem to Approval Voting. From Theorem 1 in Myerson (2000), the magnitude of the probability that alternatives \( A \) and \( C \) have (almost) the same number of votes is:

\[
\lim_{n \to \infty} \frac{\log \Pr(|x(C) - x(A)| \leq 1|\omega)}{n} = \max_{\lambda} -1 + \sum_{\psi} \frac{x(\psi)}{n} \left( 1 - \log \frac{x(\psi)}{n \tau(\psi|\omega)} \right) \quad (27)
\]

s.t. \( x(A) + x(AB) = x(C) \)

If we denote \( x(A) + x(AB) = x, x(A) = \alpha x, \) and \( x(AB) = (1-\alpha)x \), we find that this is maximized in:

\[
\begin{align*}
\alpha_{AC}^* &= \frac{\tau(A|\omega)}{\tau(A|\omega) + \tau(AB|\omega)}, \\
x_{AC}^* &= n \sqrt{\tau(C) \left[ \tau(A|\omega) + \tau(AB|\omega) \right]}, \\
x(B)_{AC}^* &= n \tau(B|\omega).
\end{align*}
\]
Substituting for $\alpha_{AC}^*, x_{AC}^*$, and $x (B)^*_{AC}$ in (27) thus yields:
\[
\lim_{n \to \infty} \frac{\log[\Pr(|X(C) - X(A)| \leq 1|\omega)]}{n} = -\left(\sqrt{\tau(A|\omega)} + \sqrt{\tau(AB|\omega)} - \sqrt{\tau(C|\omega)}\right)^2.
\]

By analogy, we have that
\[
\lim_{n \to \infty} \frac{\log[\Pr(|X(C) - X(B)| \leq 1|\omega)]}{n} = -\left(\sqrt{\tau(B|\omega)} + \sqrt{\tau(AB|\omega)} - \sqrt{\tau(C|\omega)}\right)^2, \quad \text{and}
\]
\[
\lim_{n \to \infty} \frac{\log[\Pr(|x(B) - x(A)| \leq 1|\omega)]}{n} = -\left(\sqrt{\tau(A|\omega)} - \sqrt{\tau(B|\omega)}\right)^2.
\]

Note the symmetry between $\text{mag}(PQ)$ and $\text{mag}(QP)$:
\[
\lim_{n \to \infty} \frac{\log[\Pr(|X(P) - X(Q)| \leq 1|\omega)]}{n} = \lim_{n \to \infty} \frac{\log[\Pr(|X(Q) - X(P)| \leq 1|\omega)]}{n}.
\]

**Property 3** *(Myerson 2000, Corollary 1)* The relative probability of two events $x$ and $x'$ converges to 0 as population size increases to infinity when the magnitude of $x$ is larger than that of $x'$, and conversely:
\[
\frac{\Pr(x|\tau(\omega))}{\Pr(x'|\tau(\omega))} \xrightarrow{n \to \infty} \infty \text{ if } \text{mag}[x] > \text{mag}[x'] \quad \text{and} \quad \frac{\Pr(x|\tau(\omega))}{\Pr(x'|\tau(\omega))} \xrightarrow{n \to \infty} 0 \text{ if } \text{mag}[x] < \text{mag}[x'].
\]

**Property 4** If $C$ is expected to rank first in state $\omega$, then, for $\tau(A|\omega) > \tau(B|\omega)$, we have:
\[
\text{mag}(\text{piv}_{AC}|\omega) > \text{mag}(\text{piv}_{BC}|\omega) \geq \text{mag}(\text{piv}_{AB}|\omega).
\]
Conversely, for $\tau(A|\omega) < \tau(B|\omega)$, we have $\text{mag}(\text{piv}_{BC}|\omega) > \text{mag}(\text{piv}_{AC}|\omega) \geq \text{mag}(\text{piv}_{AB}|\omega)$. If $C$ is expected to rank second in state $\omega$, then, for $\tau(A|\omega) > \tau(B|\omega)$, we have:
\[
\text{mag}(\text{piv}_{AC}|\omega) > \text{mag}(\text{piv}_{AB}|\omega) \geq \text{mag}(\text{piv}_{BC}|\omega).
\]
Conversely, for $\tau(A|\omega) < \tau(B|\omega)$, we have $\text{mag}(\text{piv}_{BC}|\omega) > \text{mag}(\text{piv}_{AB}|\omega) \geq \text{mag}(\text{piv}_{AC}|\omega)$.

That is, whenever $C$ is expected to rank first or second, the pivot probability between the expected top (resp. bottom) two alternatives has the largest (resp. smallest) magnitude.

**Proof.** From Property 2, we have that the magnitude of the probability that $A$ and $C$ have the same vote shares, i.e.
\[
\lim_{n \to \infty} \frac{\log[\Pr(|X(C) - X(A)| \leq 1|\omega)]}{n} = -\left(\sqrt{\tau(A|\omega)} + \sqrt{\tau(AB|\omega)} - \sqrt{\tau(C|\omega)}\right)^2.
\]
corresponds to the magnitude of the pivot probability between $A$ and $C$ if $x(B)^* < \alpha x^*$, i.e. alternative $B$ trails behind $A$ and $C$ when the pivotability arises. In other words, $\Pr(X(B) \leq X(C)|X(C) - X(A) \in \{0, 1\}, \omega) =$...
1. If this is not the case, i.e. if \( x(B)^* \geq ax^* \), the magnitude of the pivot probability is lower since 
\[
\Pr(X(B) \leq X(C) | X(C) - X(A) \in \{0, 1\}, \omega) < 1:
\]

\[
\text{mag}(piv_{AC}|\omega) \leq \lim_{n \to \infty} \frac{\log[\Pr(|X(C) - X(A)| \leq 1|\omega) - \sqrt{\tau(C|\omega)}]}{n} = - \left( \sqrt{\tau(A|\omega)} + \tau(AB|\omega) - \sqrt{\tau(C|\omega)} \right)^2.
\]

By analogy, it is immediate to check that:

\[
\text{mag}(piv_{BC}|\omega) \leq \lim_{n \to \infty} \frac{\log[\Pr(|X(C) - X(B)| \leq 1|\omega) - \sqrt{\tau(B|\omega)}]}{n} = - \left( \sqrt{\tau(B|\omega)} + \tau(AB|\omega) - \sqrt{\tau(C|\omega)} \right)^2,
\]

and

\[
\text{mag}(piv_{AB}|\omega) \leq \lim_{n \to \infty} \frac{\log[\Pr(|X(C) - X(A)| \leq 1|\omega) - \sqrt{\tau(C|\omega)}]}{n} = - \left( \sqrt{\tau(A|\omega)} - \sqrt{\tau(B|\omega)} \right)^2.
\]

Now, note that, if the constraint is binding, we have: \( x(A) + x(AB) = x(C) = x(B) + x(AB) \),
and the three events: \( piv_{AB} \), \( piv_{AC} \) and \( piv_{BC} \) are therefore identical, which implies that:

\[
\text{mag}(piv_{AC}|\omega) = \text{mag}(piv_{BC}|\omega) = \text{mag}(piv_{AB}|\omega).
\]

We refer to these as restricted magnitudes. The unrestricted magnitudes refer thus to the magnitude of probability that \( P \) and \( Q \) have the same vote shares.

Having observed this, we are now in a position to prove that, if the expected ranking is \( A > C > B \) in state \( \omega \), then:

\[
\text{mag}(piv_{AC}|\omega) > \text{mag}(piv_{AB}|\omega) \geq \text{mag}(piv_{BC}|\omega). \tag{29}
\]

Since the proof proceeds in the same way for all the other possible rankings: \( C > B > A \), \( C > A > B \) and \( B > C > A \), we shall not develop these.

The proof is in 3 steps: first, we compare the unrestricted magnitudes and show that:

\[
\lim_{n \to \infty} \frac{\log[\Pr(|X(C) - X(A)| \leq 1|\omega)]}{n} > \lim_{n \to \infty} \frac{\log[\Pr(|X(B) - X(A)| \leq 1|\omega)]}{n}. \tag{30}
\]

This amounts to showing that:

\[
\tau(A|\omega) + \tau(AB|\omega) > \tau(C|\omega) + \tau(AB|\omega) \tag{31}
\]

implies:

\[
- \left( \sqrt{\tau(A|\omega) + \tau(AB|\omega)} - \sqrt{\tau(C|\omega)} \right)^2 > - \left( \sqrt{\tau(A|\omega)} - \sqrt{\tau(B|\omega)} \right)^2.
\]

Rearranging terms, we find that this inequality holds iff:

\[
\sqrt{\tau(A|\omega) - \sqrt{\tau(B|\omega)} > \sqrt{\tau(A|\omega) + \tau(AB|\omega)} - \sqrt{\tau(C|\omega)},
\]

which is necessarily true. Hence, the ranking (31) indeed implies that (30) is satisfied.

Second, we show that \( \text{mag}(piv_{AC}|\omega) \) is always equal to the unrestricted magnitude. For this, we need to prove that: \( x(A) + x(AB) = x(C) > x(B) + x(AB) \) at the optimum, that is:

\[
a_{AC}^{**}x_{AC}^{**} > x(B)_{AC}^{**}.
\]
Using (28) and performing some manipulations, we see that the latter inequality holds iff:

\[
\sqrt{\frac{\tau(C)}{\tau(A|\omega) + \tau(AB|\omega)}} \geq \frac{\tau(B|\omega)}{\tau(A|\omega)},
\]

in which both fractions are smaller than one. The latter implies that: 

\[
\frac{\tau(B|\omega)}{\tau(A|\omega)} \leq \frac{\tau(B|\omega) + \tau(AB|\omega)}{\tau(A|\omega) + \tau(AB|\omega)} \leq \sqrt{\frac{\tau(C)}{\tau(A|\omega) + \tau(AB|\omega)}},
\]

and by (31), the last member of this inequality is always smaller than \(\sqrt{\frac{\tau(C)}{\tau(A|\omega) + \tau(AB|\omega)}}\), which proves that \(mag(piv_{AC}|\omega)\) is always unrestricted. Hence \(mag(piv_{AC}|\omega)\) is always larger than \(mag(piv_{AB}|\omega)\), be the latter restricted or not.

Third, to complete the proof that (29) always holds under the expected ranking (31), it remains to demonstrate that \(mag(piv_{AB}|\omega) \geq mag(piv_{BC}|\omega)\). To this end, we prove that \(mag(piv_{BC}|\omega)\) is always the restricted magnitude \(mag(piv_{BC}^*|\omega)\).

Mutatis mutandis, the derivation of the critical values \(\alpha_{BC}^{**}, x_{BC}^{**}\), and \(x(A)^{**}_{BC}\) is identical to that of \(\alpha_{AC}^{**}, x_{AC}^{**}\), and \(x(B)^{**}_{AC}\) in (28), which yields:

\[
\begin{align*}
\alpha_{BC}^{**} &= \frac{\tau(B|\omega)}{\tau(B|\omega) + \tau(AB|\omega)}, \\
x_{BC}^{**} &= n\sqrt{\tau(C)|\tau(B|\omega) + \tau(AB|\omega)}}, \\
x^{**}_{BC} &= n\tau(A|\omega)
\end{align*}
\]

and the magnitude \(mag(piv_{BC}|\omega)\) would be unrestricted iff, in these points:

\[
\alpha_{BC}^{**}x_{BC}^{**} > x^{**}_{BC}
\]

To show that the latter inequality can never hold, we take the equivalent to (32) and show that:

\[
\sqrt{\frac{\tau(C)}{\tau(B|\omega) + \tau(AB|\omega)}} \leq \frac{\tau(A|\omega)}{\tau(B|\omega)},
\]

in which both fractions are larger than one. The latter implies that: 

\[
\frac{\tau(A|\omega)}{\tau(B|\omega)} \geq \frac{\tau(A|\omega) + \tau(AB|\omega)}{\tau(B|\omega) + \tau(AB|\omega)} \geq \sqrt{\frac{\tau(C)}{\tau(B|\omega) + \tau(AB|\omega)}},
\]

which proves that \(mag(piv_{BC}|\omega)\) is always restricted. This implies that \(mag(piv_{BC}|\omega)\) is equal to \(mag(piv_{AB}|\omega)\) if the both are restricted, and strictly smaller than \(mag(piv_{AB}|\omega)\) if the latter is not restricted. ■

**Property 5** (*Myerson 2000, Theorem 2*) The probability that two alternatives, \(P, Q \in \{A, B, C\}\), receive a number of votes that differs by a constant \(c (c << n)\) in state of the nature \(\omega \in \{a, b\}\), is:

\[
\lim_{n \to \infty} \Pr(x(P) = x(Q) + c|\omega, \tau(P|\omega), \tau(Q|\omega)) = \frac{\tau(P|\omega)}{\tau(Q|\omega)} c^2 \exp\left[-2\sqrt{\tau(P|\omega)} \cdot \sqrt{\tau(Q|\omega)} \cdot n\right].
\]
Appendix A2: Proofs for Section 4

Lemma 2

\[ G(A|t) \geq G(AB|t) \iff \frac{q(b|t)}{q(a|t)} \leq \frac{1}{M_1} = \frac{\Pr(\text{ piv}_{AB}|a) - \Pr(\text{ piv}_{BC}|a)}{\Pr(\text{ piv}_{AB}|b) + 2\Pr(\text{ piv}_{BC}|b)} \]  
\[ G(B|t) \geq G(AB|t) \iff \frac{q(a|t)}{q(b|t)} \leq M_2 = \frac{\Pr(\text{ piv}_{BA}|b) - \Pr(\text{ piv}_{AC}|b)}{\Pr(\text{ piv}_{BA}|a) + 2\Pr(\text{ piv}_{AC}|a)} \]

Proof. Immediate from (10) – (12). ■

Proof of Proposition 1.

Conjecture the following strategy functions: \( \sigma(t_A) = \sigma(t_B) = \{1, 0, 0\} \). That is, all majority types play \( \psi = A \) with probability 1. These strategies imply that \( \tau(\psi|a) = \tau(\psi|b), \forall \psi \). Therefore: \( \Pr(\text{ piv}_{PQ}) = \Pr(\text{ piv}_{PQ}|a) = \Pr(\text{ piv}_{PQ}|b) \). Now, we show that playing \( \psi = AB \) is a best response to \( \sigma(t) \) for a type \( t_B \):

\[ G(AB|t) - G(A|t) = q(a|t) \{ \Pr(\text{ piv}_{BC}) - \Pr(\text{ piv}_{AB}) \} + q(b|t) \{ 2\Pr(\text{ piv}_{BC}) + \Pr(\text{ piv}_{AB}) \} = (1 + q(b|t)) \Pr(\text{ piv}_{BC}) + (q(b|t) - q(a|t)) \Pr(\text{ piv}_{AB}) . \]

Since \( q(b|tB) > q(a|tB) \), all terms in (36) are strictly positive, which proves that a type \( t_B \) always wants to deviate from \( \sigma(t_A) = \sigma(t_B) = \{1, 0, 0\} \). By symmetry, \( \sigma(t_A) = \sigma(t_B) = \{0, 1, 0\} \) cannot be an equilibrium either.

It remains to show that \( \sigma(t_A) = \sigma(t_B) = \{0, 0, 1\} \) cannot be an equilibrium. That is, all majority types will never play \( \psi = AB \) with probability 1. To see this, note again that, by Properties ?? and ??:

\[ \lim_{n \to \infty} \frac{\Pr(\text{ piv}_{BC})}{\Pr(\text{ piv}_{AB})} = \lim_{n \to \infty} \frac{\Pr(\text{ piv}_{AC})}{\Pr(\text{ piv}_{BA})} = 0, \]

since alternatives \( A \) and \( B \) are expected to lead the election, with the same vote share.\(^{20}\) Hence:

\[ \lim_{n \to \infty} \frac{G(AB|t) - G(A|t)}{\Pr(\text{ piv}_{AB})} = q(b|t) - q(a|t) , \]
\[ \lim_{n \to \infty} \frac{G(AB|t) - G(B|t)}{\Pr(\text{ piv}_{BA})} = q(a|t) - q(b|t) . \]

The former value is strictly positive for types \( t_A \) and the latter is strictly positive for types \( t_B \). Hence, both types strictly prefer to deviate from a pure \( AB \) vote, and single-vote for their preferred alternative. ■

Proof of Proposition 2.

From Proposition 1, we know that majority-block voters never play the same action in pure strategy. It thus remains to show that majority block voters never play the same mixed strategy in

\( \tau(AB|\omega) = (1 - \tau(t_C)) > \tau(C) = \tau(t_C) \) and \( \tau(A|\omega) = \tau(B|\omega) = 0 \). Applying Property 1 yields the result.\(^{20}\) We have two strategies being played: minority types play \( C \) and majority types play \( AB \). Hence:
equilibrium. We begin by showing that $\sigma(A|t) > 0$ implies $\sigma(B|t) = 0$ and conversely, for any $t \in \{t_A, t_B\}$. We use a proof by contradiction.

We know that equilibrium strategies lie on the simplex $\{\sigma(A|t), \sigma(B|t), \sigma(AB|t)\}$. A necessary condition for $A$ and $B$ to be played with positive probability in equilibrium is that, for some $t \in \{t_A, t_B\}$:

$$G(A|t) = G(B|t) \geq G(AB|t), \quad (37)$$

and, from Lemma 2 (in this Appendix), $G(A|t), G(B|t) \geq G(AB|t)$ require $\Pr(piv_{AB}|a) > \Pr(piv_{BC}|a)$ and $\Pr(piv_{BA}|b) > \Pr(piv_{AC}|b)$.

Using (10) and (11), a necessary condition for $G(A|t) = G(B|t)$ is:

$$\frac{q(a|t)}{q(b|t)} = \frac{\Pr(piv_{BA}|b) - \Pr(piv_{AC}|b) + \Pr(piv_{AB}|b) + 2\Pr(piv_{BC}|b)}{\Pr(piv_{AB}|a) - \Pr(piv_{BC}|a) + \Pr(piv_{BA}|a) + 2\Pr(piv_{AC}|a)}. \quad (38)$$

Now, we prove that (37) can never hold: using Lemma 2, we identify a lower bound for $M_1$ and an upper bound for $M_2$. Then, we show that this lower bound for $M_1$ is strictly larger than the upper bound for $M_2$, whereas condition (37) requires:

$$M_1 \leq M_2, \quad (39)$$

hence the contradiction.

$$M_1 = \frac{\Pr(piv_{AB}|b) + 2\Pr(piv_{BC}|b)}{\Pr(piv_{AB}|a) - \Pr(piv_{BC}|a)}$$

is strictly increasing in $\Pr(piv_{BC}|a)$ and $\Pr(piv_{BC}|b)$. A lower bound to $M_1$ is thus found by setting these two pivot probabilities equal to 0. Similarly, an upper bound to $M_2$ is found by setting $\Pr(piv_{AC}|a)$ and $\Pr(piv_{AC}|b)$ equal to zero. This establishes that:

$$\frac{\Pr(piv_{AB}|b)}{\Pr(piv_{AB}|a)} < M_1 \text{ and } M_2 < \frac{\Pr(piv_{BA}|b)}{\Pr(piv_{BA}|a)}, \quad (40)$$

and hence that a necessary condition for (39) is that:

$$\frac{\Pr(piv_{AB}|b)}{\Pr(piv_{BA}|a)} < 1.$$

Using Property 5 (in Appendix A1), the left-hand side of this expression is equal to:

$$\sqrt{\frac{\tau(A|a)\tau(B|b)}{\tau(A|b)\tau(B|a)}},$$

which cannot be smaller than 1. Indeed, by (38), types $t_A$ must vote for $A$ with a higher probability than types $t_B$, since $\frac{q(a|t_A)}{q(b|t_A)} > \frac{q(a|t_B)}{q(b|t_B)}$. Hence, in equilibrium:

$$\frac{\tau(A|a)}{\tau(A|b)} \geq 1 \text{ and } \frac{\tau(B|b)}{\tau(B|a)} \geq 1. \quad (41)$$

It follows that $G(A|t) = G(B|t)$ implies $G(AB|t) > G(A|t)$, and therefore that a strict mixture between $A$ and $B$ is a strictly dominated strategy: $\sigma(A|t) > 0$ implies $\sigma(B|t) = 0$ and conversely.

It remains to prove that $\sigma(A|t_A)$ and $\sigma(B|t_B)$ are strictly positive in equilibrium. To this end, we show that:

$$\sigma(B|t_B) > 0 \text{ and } \sigma(A|t_A) = 0 \quad (42)$$
leads to a contradiction. Indeed, (42) implies \( \tau (A|\omega) = 0 \) in both states. Hence, by Property ??:

\[
\text{mag}(\text{piv}_{BA}|\omega) = \tau (B|\omega).
\]

By (41), we have: \( \tau (B|a) < \tau (B|b) \), which implies that \( \lim_{n \to \infty} \text{Pr}(\text{piv}_{BA}|b) / \text{Pr}(\text{piv}_{BA}|a) = 0 \) and therefore that \( \lim_{n \to \infty} M_2 \leq 0 \) in Lemma 2. Instead, \( \sigma (B|t_B) > 0 \) imposes that \( M_2 \) be strictly positive. This shows that \( \sigma (A|t_A) = 0 \) contradicts the possibility that \( \sigma (B|t_B) > 0 \). By symmetry, we cannot either have: \( \sigma (A|t_A) > 0 \) and \( \sigma (B|t_B) = 0 \).

Together with Proposition 1 and (41), this proves that, in equilibrium, we must have \( \sigma (A|t_A) > 0 \) and \( \sigma (B|t_B) > 0 \). From the first part of this proof, this also implies that: \( \sigma (B|t_A) = 0 = \sigma (A|t_B) \). ■

**Proof of Proposition 3.**

To prove that there is a unique equilibrium, we proceed in two steps. First, we show that \( \sigma (A|t_A) = \rho^* \sigma (B|t_B) \) is the unique best response of types \( t_A \) given the strategy of types \( t_B \).

Second, we prove that there is a unique equilibrium strategy \( \sigma^* (B|t_B) \).

From (18) and (20), we must have in equilibrium:

\[
\text{mag}(\text{piv}_{AB}|a) = \text{mag}(\text{piv}_{AB}|b) \geq \max \{ \text{mag}(\text{piv}_{BC}|a), \text{mag}(\text{piv}_{BC}|b), \text{mag}(\text{piv}_{AC}|a), \text{mag}(\text{piv}_{AC}|b) \}.
\]

(43)

We can check that types \( t_A \) never want to deviate from \( \sigma (A|t_A) = \rho^* \sigma (B|t_B) \): for any \( \sigma (A|t_A) < \rho^* \sigma (B|t_B) \), we have \( \sigma (AB|t_A) > 1 - \rho^* \sigma (B|t_B) \). This implies that the expected share of alternative \( B \) increases in both states and hence that: \( \text{mag}(\text{piv}_{AB}|a) \) increases above \( \text{mag}(\text{piv}_{AB}|b) \), whereas \( \text{mag}(\text{piv}_{BC}|a) \) and \( \text{mag}(\text{piv}_{BC}|b) \) decrease.

Using Lemma 2 and (43), this implies:

\[
\frac{q (b|t_A)}{q (a|t_A)} < \lim_{n \to \infty} \frac{1}{M_1} = \frac{\text{Pr}(\text{piv}_{AB}|a) - \text{Pr}(\text{piv}_{BC}|a)}{\text{Pr}(\text{piv}_{AB}|b) + 2 \text{Pr}(\text{piv}_{BC}|b)} = \infty,
\]

and hence: \( G(A|t_A) > G(AB|t_A) \). Therefore, \( \sigma (A|t_A) < \rho^* \sigma (B|t_B) \) cannot be true in equilibrium.

For any \( \rho \sigma (B|t_B) < 1 \), we also have to check that \( \sigma (A|t_A) > \rho^* \sigma (B|t_B) \) cannot be either an equilibrium. Following the same procedure as above, one can check that \( \sigma (A|t_A) > \rho^* \sigma (B|t_B) \) implies:

\[
\frac{q (b|t_A)}{q (a|t_B)} > \lim_{n \to \infty} \frac{1}{M_1} = \frac{\text{Pr}(\text{piv}_{AB}|a) - \text{Pr}(\text{piv}_{BC}|a)}{\text{Pr}(\text{piv}_{AB}|b) + 2 \text{Pr}(\text{piv}_{BC}|b)} < 0,
\]

which in turn implies \( G(A|t) < G(AB|t) \). Hence, \( \sigma (A|t_A) > \rho^* \sigma (B|t_B) \) cannot be true in equilibrium. Therefore, when (43) holds, \( \sigma^* (A|t_A) = \rho^* \sigma (B|t_B) \) is the unique best response of types \( t_A \) to \( \sigma (B|t_B) \).

It remains to prove that there is a unique equilibrium strategy \( \sigma^* (B|t_B) \), which will always imply (43). Two cases must be considered:

Case 1: \( G(B|t_B) - G(AB|t_B) \geq 0 \) in \( \sigma (B|t_B) = 1, \sigma (A|t_A) = \rho \).
In that case, $\sigma(B|t_B) = 1$ is the only possible best response for types $t_B$. Indeed, $\sigma(B|t_B) < 1$ would imply $\sigma(AB|t_B) > 0$. This induces an increase in the expected vote share of alternative $A$ in both states of nature and hence that: $\text{mag}(\text{piv}_{BA}|b)$ increases above $\text{mag}(\text{piv}_{BA}|a)$, whereas $\text{mag}(\text{piv}_{AC}|a)$ and $\text{mag}(\text{piv}_{AC}|b)$ decrease. Using Lemma 2 and (43), this implies:

$$\frac{q(a|t_B)}{q(b|t_B)} < \lim_{n \to \infty} M_2 = \frac{\Pr(\text{piv}_{BA}|b) - \Pr(\text{piv}_{AC}|b)}{\Pr(\text{piv}_{BA}|a) + 2\Pr(\text{piv}_{AC}|a)} = \infty,$$

and hence $G(B|t_B) > G(AB|t_B)$. Therefore, $\sigma(B|t_B) = 1$ is the unique best response to $\sigma(A|t_A) = \rho$.

It remains to show that types $t_B$ would deviate from any $\{\sigma(A|t_A), \sigma(B|t_B)\} = \{\rho\sigma, \sigma\}$ if $\sigma < 1$. To this end, we need to show that

$$\lim_{n \to \infty} \frac{G(B|t_B) - G(AB|t_B)}{\Pr(\text{piv}_{AB}|a)} = q(b|t_B) \frac{\Pr(\text{piv}_{BA}|b)}{\Pr(\text{piv}_{AB}|a)} - q(a|t_B) \frac{\Pr(\text{piv}_{BA}|a)}{\Pr(\text{piv}_{AB}|a)} > 0, \quad (44)$$

for any $\{\sigma(A|t_A), \sigma(B|t_B)\} = \{\rho\sigma, \sigma\}, \sigma < 1$.

The strategy of the types $t_A$ implies:

$$\lim_{n \to \infty} \frac{G(A|t_A) - G(AB|t_A)}{\Pr(\text{piv}_{AB}|a)} = q(a|t_A) - q(b|t_A) \frac{\Pr(\text{piv}_{BA}|b)}{\Pr(\text{piv}_{AB}|a)} = 0$$

$$\implies \frac{\Pr(\text{piv}_{BA}|b)}{\Pr(\text{piv}_{AB}|a)} = \frac{q(a|t_A)}{q(b|t_A)}.$$ 

By Myerson’s offset theorem: $\Pr(\text{piv}_{BA}|\omega) = \Pr(\text{piv}_{AB}|\omega) \sqrt{\frac{\tau(A|\omega)}{\tau(B|\omega)}}$. Hence, (44) can be rewritten as:

$$\frac{q(b|t_B)}{q(a|t_B)} \frac{q(a|t_A)}{q(b|t_A)} > \sqrt{\frac{\tau(A|\omega)}{\tau(B|\omega)}},$$

By (3), the left-hand side of this inequality is equal to: $\frac{\tau(A|\omega) \cdot \tau(B|\omega)}{\tau(B|\omega) \cdot \tau(B|\omega)} > 1$, which proves that (44) holds.

Case 2: $G(B|t_B) - G(AB|t_B) < 0$ in $\sigma(B|t_B) = 1$, $\sigma(A|t_A) = \rho$.

In this case, there must exist a $\tilde{\sigma} \in (0,1)$ such that, for $\{\sigma(A|t_A), \sigma(B|t_B)\} = \{\rho\tilde{\sigma}, \tilde{\sigma}\}$, we have: $G(B|t_B) - G(AB|t_B) = 0$. Indeed, by Proposition 1, $G(B|t_B) - G(AB|t_B) > 0$ for $\{\sigma(A|t_A), \sigma(B|t_B)\} = \{0,0\}$. The existence of $\tilde{\sigma}$ immediately follows from the continuity of the $G$ function.

This value of $\tilde{\sigma}$ is unique and such that:

$$\text{mag}(\text{piv}_{AB}|a) = \text{mag}(\text{piv}_{AB}|b) = \max\{\text{mag}(\text{piv}_{BC}|a), \text{mag}(\text{piv}_{BC}|b), \text{mag}(\text{piv}_{AC}|a), \text{mag}(\text{piv}_{AC}|b)\}. \quad (45)$$

Indeed, any $\sigma < \tilde{\sigma}$ implies that the total expected vote shares of alternatives $A$ and $B$ increase. Since (45) implies that $C$ is third in both states, the magnitudes $\text{mag}(\text{piv}_{PC}|\omega)$ must decrease, for any $P \in \{A, B\}$ and $\omega \in \{a, b\}$. In contrast, the magnitudes $\text{mag}(\text{piv}_{AB}|\omega)$ must increase, since:

$$\text{mag}(\text{piv}_{AB}|a) = \text{mag}(\text{piv}_{AB}|b) = \left(\sqrt{r(t_A|a) \cdot \rho \tilde{\sigma} - \sqrt{r(t_B|a) \cdot \sigma}}\right)^2$$

$$= \left(\sqrt{r(t_A|a) \cdot \rho - \sqrt{r(t_B|a)}}\right)^2 \sigma.$$
is strictly increasing in $\sigma$. Hence (43) holds with a strict inequality for any $\sigma < \bar{\sigma}$. This implies that (44) holds, and hence that $G(B|t_B) - G(AB|t_B) > 0$ for any $\{\sigma (A|t_A), \sigma (B|t_B)\} = \{\rho\sigma, \sigma\}$, $\sigma < \bar{\sigma}$.

Similarly, one can check that (43) is violated for any $\sigma > \bar{\sigma}$ which implies $G(B|t_B) - G(AB|t_B) < 0$ for any $\{\sigma (A|t_A), \sigma (B|t_B)\} = \{\rho\sigma, \sigma\}$, $\sigma > \bar{\sigma}$. This proves that (45) must hold at $\{\sigma (A|t_A), \sigma (B|t_B)\} = \{\rho\bar{\sigma}, \bar{\sigma}\}$, as well as the uniqueness of $\bar{\sigma}$. ■

Appendix A3: Proof for Section 5

Proof of Theorem 2.

1) First, we prove that, for all majority types $t \in \{t_A, t_B\}$, $G(A|t) - G(B|t)$ is strictly positive if $\tau(B|\omega) \to 0$. This proves that, if $B$ is expected to receive too few votes, all majority types strictly prefer to vote for $A$. By symmetry, it also proves that all majority types vote for $B$ if they expect $A$ to receive too few votes.

For any strategy profile, we have:

$$G(A|t) - G(B|t) = q(a|t) \left\{ 2 \Pr(piv_{AC}|a) + \Pr(piv_{AB}|a) + \Pr(piv_{BA}|a) - \Pr(piv_{BC}|a) \right\}$$

$$+ q(b|t) \left\{ \Pr(piv_{AC}|b) - \Pr(piv_{AB}|b) - \Pr(piv_{BA}|b) - 2 \Pr(piv_{BC}|b) \right\}. \tag{46}$$

By (6), for $\tau(B|\omega) \to 0$ we have: $\tau(A|\omega) \to 1 - r(t_C)$. Hence, by Properties ?? and 2, for any given $\omega = a, b$ we have:

$$\lim_{n \to \infty} \frac{\Pr(piv_{BC}|\omega)}{\Pr(piv_{AC}|\omega)} = \lim_{n \to \infty} \frac{\Pr(piv_{AB}|\omega)}{\Pr(piv_{AC}|\omega)} = \lim_{n \to \infty} \frac{\Pr(piv_{BA}|\omega)}{\Pr(piv_{AC}|\omega)} = 0.$$

Hence:

$$\lim_{\tau(B|\omega)\to 0} \frac{G(A|t) - G(B|t)}{\Pr(piv_{AC}|a)} = 2q(a|t) + q(b|t) \frac{\Pr(piv_{AC}|b)}{\Pr(piv_{AC}|a)}.$$

which is strictly positive. This proves the existence of the two “sunspot” equilibria.

2) Second, we show the existence of the third equilibrium. Following Theorem 2 of Myerson 1998a, if a type $t \in \{t_A, t_B\}$ adopts a strictly mixed strategy, then the other type $t' \neq t$, $t' \in \{t_A, t_B\}$ votes for “his” candidate with probability 1. The reason is that $q(a|t_A) > q(a|t_B)$, which implies $G(A|t_A) - G(B|t_A) > G(A|t_B) - G(B|t_B)$ for any expected voting profile.

Having noted this, we know that a necessary condition for majority-types voters to adopt a different strategy is that:

$$G(A|t_A) - G(B|t_A) \geq 0,$$

$$G(A|t_B) - G(B|t_B) \leq 0. \tag{47}$$

Next, remark that: a) pivot probabilities are continuous in the voters’ propensity to cast their ballot on $A$ and on $B$, and b) payoffs are bounded. Therefore, the difference $G(A|t) - G(B|t)$ is continuous in the voters’ propensity to vote for $A$, and we can apply Kakutani’s fixed point theorem.
Now, consider a strategy profile \( \bar{\sigma} \) such that \( \tau(A|a) = \tau(B|b) \equiv \bar{\tau} \). If voters marginally increase their propensity to vote \( A \) above \( \bar{\sigma} \), we have: \( \tau(A|a) > \tau(B|b) > \tau(A|b) > \tau(B|a) \). By Property ??, for any such strategy profile, we have:

\[
G(A|t) - G(B|t) > 0 \text{ for both } t \in \{t_A, t_B\}, \text{ if } \tau(C) < \bar{\tau},
\]

\[
G(A|t) - G(B|t) < 0 \text{ for both } t \in \{t_A, t_B\}, \text{ if } \tau(C) > \bar{\tau},
\]

and the inequalities are reversed if the voters’ propensity to vote for \( A \) decreases below \( \bar{\tau} \). By the continuity of the payoff functions, (47) must hold in a neighborhood of \( \bar{\sigma} \).

Now, we show that, for \( \tau(C) > 1/\left[2 + r(t_A|b)/r(t_A|a)\right] \), the following strategy profile is an equilibrium:

\[
\begin{align*}
\sigma(\emptyset|t_A) &= 0 = \sigma(\emptyset|t_B), \\
\sigma(B|t_B) &= 1, \\
\sigma(A|t_A) &= \frac{r(t_B|b) + r(t_A|b)}{r(t_A|a) + r(t_A|b)} \text{ and } \sigma(B|t_A) = 1 - \sigma(A|t_A).
\end{align*}
\]

For that strategy profile, we have \( \tau(A|a) \simeq \tau(B|b) \equiv \bar{\tau} \) and: \( \tau(C) > \bar{\tau} > \tau(A|b) \simeq \tau(B|a) \). By Property ??, this implies:

\[
\lim_{n \to \infty} \frac{\Pr(piv_{BC}|a)}{\Pr(piv_{AC}|a)} = \lim_{n \to \infty} \frac{\Pr(piv_{AC}|b)}{\Pr(piv_{BC}|b)} = 0.
\]

Finally, since alternative \( A \) and \( B \)'s vote shares are second and third in both states of nature, by Property 4 in Appendix A1, we have:

\[
\lim_{n \to \infty} \max \left\{ \frac{\Pr(piv_{AB}|a)}{\Pr(piv_{AC}|a)}, \frac{\Pr(piv_{BA}|a)}{\Pr(piv_{AC}|a)} \right\} = \lim_{n \to \infty} \max \left\{ \frac{\Pr(piv_{AB}|b)}{\Pr(piv_{BC}|b)}, \frac{\Pr(piv_{BA}|b)}{\Pr(piv_{BC}|b)} \right\} = 0.
\]

It results that, in \( \bar{\sigma} \):

\[
\lim_{n \to \infty} \frac{G(A|t) - G(B|t)}{\Pr(piv_{AC}|a)} = 2 \left[ q(a|t) - q(b|t) \frac{\Pr(piv_{BC}|b)}{\Pr(piv_{AC}|a)} \right],
\]

and, by Kakutani’s fixed point theorem, there must exist a strategy profile \( \sigma(A|t_A) \) in the neighborhood of \( \frac{r(t_B|b)+r(t_A|b)}{r(t_A|a)+r(t_A|b)} \) such that: \( \lim_{n \to \infty} \frac{G(A|t_A) - G(B|t_A)}{\Pr(piv_{AC}|a)} = 0 \). It remains to prove that abstention is strictly dominated. To this end, it can be checked that: \( G(A|t_A) > 0 \) and \( G(B|t_B) > 0 \), which can be compared to the value of abstention: zero. \( \blacksquare \)

**Appendix A4: Proof for Section 6**

**Proof of Theorem 3.** The probability that \( A \) is elected from the first round, with a majority of the votes is:

\[
\Pr[X(A) \geq X(B) + X(C) + 1].
\]

For \( \sigma(A|t_A) = 1 \) and \( \sigma(A|t_B) \to 1 \), we have \( \tau(A|\omega) \to 1 - r(t_C) \) and \( \tau(B|\omega) \to 0 \). The magnitude of this probability is therefore:

\[
\lim_{\tau(B|\omega) \to 0} \text{mag}(piv_{AC}^1|\omega) = -\left(\sqrt{1-r(t_C)} - \sqrt{r(t_C)}\right)^2, \forall \omega \in \{a, b\},
\]

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where \( \text{piv}^1_{AC} \) denotes the event that a ballot is pivotal in electing \( A \) in the first round. In contrast, the probability that a \( B \) ballot is pivotal in bringing \( B \) to a second round is given by:

\[
\frac{1}{2} \Pr \left[ \max \{ X(A), X(B), X(C) \} \leq \frac{X(A)+X(B)+X(C)}{2} \cap \min \{ X(A), X(C) \} - X(B) \in [0, 1] \right].
\]

When alternative \( B \)'s vote share approaches zero, the magnitude of this joint event converges to \(-1\).

However, if \( X(A) = X(B) + X(C) \), a ballot for \( A \) would be pivotal to elect \( A \) in the first round. Similarly, if \( X(A) = X(B) + X(C) + 1 \), a \( B \)-ballot would be pivotal in forcing the organization of a second round. Hence, when a voter compares the two options, she values the \( A \)-ballot only in proportion to the second-round risk:

\[
G(A|t) > \Pr(\text{piv}^1_{AC}) \Pr(\text{piv}^2_{AC}),
\]

where \( \Pr(\text{piv}^2_{AC}) \) denotes the second-round pivot probability. Yet, the two probabilities, \( \Pr(\text{piv}^1_{AC}) \) and \( \Pr(\text{piv}^2_{AC}) \) are identical. Hence:

\[
G(A|t) > \Pr(\text{piv}^1_{AC})^2.
\]

Taking logarithms and dividing by \( n \):

\[
\log \left[ \frac{\Pr(\text{piv}^1_{AC})^2}{n} \right] \rightarrow -2 \left( \sqrt{1 - r(t_C)} - \sqrt{r(t_C)} \right)^2,
\]

which must be compared to the magnitude of the probability that a \( B \) ballot is pivotal in bringing \( B \) to a second round. That magnitude is equal to \(-1\). Hence:

\[
-2 \left( \sqrt{1 - r(t_C)} - \sqrt{r(t_C)} \right)^2 \geq -1
\]

is a sufficient condition for \( G(A|t) > G(B|t) \). Solving it in \( r(t_C) \) yields: \( r(t_C) \geq 0.06699 \). Hence, for any \( r(t_C) \geq 0.06699 \), there exists an informational trap equilibrium with \( \sigma(A|t) = 1, t \in \{t_A, t_B\} \).

By symmetry, there exists another equilibrium with \( \sigma(B|t) = 1, t \in \{t_A, t_B\} \).