Estimation of Dynamic Discrete Games when Players' Beliefs are not in Equilibrium

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Abstract

This paper deals with the econometrics of a class of dynamic games where players are rational, in the sense that they maximize expected payoffs given their beliefs about other players' actions, but their beliefs may not be in equilibrium, i.e., they are not self-fulfilling. In this econometric model, players' beliefs are probability distributions defined over the space of other players' actions and conditional on common knowledge state variables. These distributions are nonparametrically specified and they are treated as incidental parameters that, together with the structural parameters of the game, determine the stochastic process followed by players' actions. This general model contains as a particular case the model where beliefs are in equilibrium. We study identification and estimation of structural parameters and beliefs. We show that under level-2 rationality (i.e., players are rational and they know that the other players are rational too), a exclusion restriction and a large-support condition on one of the exogenous explanatory variables are sufficient for point-identification of both structural parameters and players' beliefs. We propose two estimation methods. The first estimator, that has similarities with some matching estimators used in the treatment effect literature, imposes the restriction of point-identification of the structural parameters in the sample. The main advantage of this estimator is its relative computational simplicity, but it has several limitations. The estimator relies critically on the large-support conditions on some explanatory variables and the point-identification result follows an *at-infinity* argument. In finite samples, imposing point-identification may induce significant biases. The second estimator deals with these issues and recognizes the possibility of having only set identification, but not point identification, in the sample or even in the population. We propose a method that provides a parameter set that minimizes a penalty function based on moment inequalities that summarize the restrictions of the model. Using a representation of best response functions from Aguirregabiria and Mira (2007), we show that the system of moment inequalities that describes the model restrictions can be represented as a linear-in-parameters system. We estimate the identified set using an efficient linear programming algorithm for the solution of linear systems of inequalities. We apply this model and methods to actual data in a dynamic game of store location by retail chains.

Keywords: Identification; Partial Identification; Dynamic games; Rational behavior; Rationalizability; Estimation with conditional moment inequalities.

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1 Introduction

The principle of *revealed preference* (Samuelson, 1938) is a cornerstone in the structural empirical analysis of decision models, either static or dynamic, single-agent decision problems or games. Under the principle of *revealed preference*, agents maximize expected payoffs, and this implies that their actions reveal information on the structure of payoff functions. This simple but powerful concept has allowed econometricians to use data on agents' decisions to identify important structural parameters for which there is very limited information from other sources. Agents' degree of risk aversion, intertemporal rates of substitution, market entry costs, adjustment costs and switching costs, consumer willingness to pay, preference for a political party, or the cost of a merger, are just some examples of the type of structural parameters that have been estimated under the principle of revealed preference. In the context of empirical games of incomplete information, either static or dynamic, expected payoffs depend on players' beliefs on the behavior of other players. In this literature, every empirical study has combined the principle of revealed preference with the assumption that players' beliefs are in equilibrium. There are several reasons why the assumption of equilibrium beliefs is very useful in the estimation of games. First, equilibrium restrictions have identification power. Imposing these restrictions contributes to improve asymptotic and finite sample properties of estimators (Aguirregabiria and Mira, 2007, Kasahara and Shimotsu, 2008, Aradillas-Lopez and Tamer, 2008). Second, in games with multiple equilibria, the assumption of equilibrium beliefs is key to have point identification of structural parameters and beliefs, and for the implementation of relatively simple methods of estimation. That is the case in games of incomplete information under the assumption that, for a given value of the exogenous explanatory variables, all the observations in the data come from the same equilibrium. Under this assumption, it is possible to estimate players' beliefs consistently using a nonparametric estimator of the distribution of players' actions. This nonparametric estimator of beliefs can be used to construct players' expected payoffs and to obtain an estimator of structural parameters that optimizes a sample criterion function based on players' best responses to the estimated beliefs from the data. This simple two-step approach for identification and estimation cannot be applied when players beliefs are not in equilibrium. Third, one of the most attractive features of structural models is that they can be used to study the effects of counterfactual changes in structural parameters or in public policies. Models where agents' beliefs are endogenously determined in equilibrium are particularly attractive because they take into account how these beliefs will change in the counterfactual scenario.

Despite these important and attractive implications of the assumption of equilibrium beliefs, there are empirical applications of games where the assumption is not realistic and it is of interest to relax it. The following are several examples.

EXAMPLE 1. Structural change or policy change within the sample period. Suppose that we want to estimate a dynamic game of competition in an oligopoly industry. The set of decision variables in the empirical game includes firms' choice to adopt or not a new technology. There is a cost of adopting the new technology, but once adopted the new technology implies lower marginal costs of production. In this game, firms' adoption decisions are strategic complements and the game has multiple stable equilibria. In particular, there is an equilibrium where firms have a low probability of adoption, and there is other equilibrium with high probabilities of adoption. We have panel data of firms in this industry over several periods of time (and perhaps over several local markets). Suppose that an important policy change occurred in the middle of the sample period, e.g., a new government subsidy that tries to encourage the adoption of the new technology. It seems realistic to consider that, after this policy change, it will take some time for firms to learn about the new strategies of competitors. Learning about the new strategies of other players may be particularly complicated in the case of multiple equilibria. Firms may wonder whether the new policy has just increased firms' probability of adoption but within the same equilibrium type (i.e., the low probability equilibrium type), or if the new policy has induced a change in beliefs and in the equilibrium type (e.g., a jump from the low probability to the high probability equilibrium type). Firms' learning about the new equilibrium can take some time during which firms' beliefs will be out of equilibrium. Imposing the assumption of equilibrium beliefs during the periods just after the policy change seems unreasonable. To deal with this issue, the researcher may choose to ignore the observations during these periods of "transition}, or he might be willing to propose a structural model that explicitly specifies the process of learning towards the new equilibrium. Alternatively, the researcher might prefer to use a more robust approach that imposes minimum assumptions on the evolution of players' beliefs during the transition period but that uses these observations for estimation. This latter approach is the one that we consider in this paper.

EXAMPLE 2: Heterogeneity in players' beliefs on other players' behavior. There is significant empirical evidence from laboratory experiments showing that players in these experiments tend to play heterogeneous strategies and that they also have very heterogenous beliefs on other players' strategic behavior (Camerer, 2003). There is some but still very scarce non-experimental evidence on this issue. An exception is the recent paper by Goldfarb and Xiao (2009) that studies entry decisions in the US local telephone industry and finds significant heterogeneity in firms' beliefs about other firms' strategic behavior. Our paper provides an approach to identify and estimate heterogeneity in players' beliefs. It is important to emphasize that, in contrast to the theoretical and empirical literature on behavioral game theory (e.g., Camerer, Ho and Chong, 2004, and Goldfarb and Xiao, 2009), our approach does not replace the assumption of equilibrium or rational beliefs by other type of assumption on beliefs. We do relax the assumption of rational beliefs. Our model contains as a particular case the model where beliefs are in equilibrium, as well as many other models for the determination of beliefs. To this respect, our approach is very different to the old econometric literature on disequilibrium models (see Quandt, 1988) and to the recent empirical literature on behavioral game theory.

In this paper we study identification, estimation, and inference in dynamic discrete games of incomplete information when we relax the assumption of equilibrium beliefs. The paper contains several contributions: identification results, an estimation method, a test of equilibrium beliefs, and an empirical application that illustrates these methodological contributions.

In the class of econometric models that we consider, players' beliefs are probability distributions over the set of other players' actions. These distributions are nonparametrically specified and they are treated as incidental parameters that, together with the structural parameters of the game, determine the stochastic process followed by players' actions. When players' beliefs are not in equilibrium, they are different to the actual distribution of players' actions in the population. Therefore, beliefs cannot be identified and estimated by simply using a nonparametric estimator of the distribution of players' actions. However, we show that under level-2 rationality (i.e., players are rational and they know that other players are also rational), an *exclusion restriction* and a *large-support condition* on one of the exogenous explanatory variables are sufficient for point-identification of structural parameters and players' beliefs. The exclusion restriction is an exogenous (or predetermined) observable variable that has a direct effect on the own player's payoff but not on the other players' payoffs (though it has an indirect effect on other players' payoffs through their beliefs about the player's expected behavior). For instance, in a dynamic game of firms' capital investment in an oligopoly industry, a firm's current profit does not depend directly on other firms' capital stocks at the beginning of the period, but it depends indirectly through the current investment decisions of other firms. The *large-support condition* establishes that the variables of the exclusion restriction have unbounded support on the real line.

We propose two estimation methods. The first estimator has similarities with some matching estimators used in the treatment effect literature (Abadie and Imbens, 2006). It imposes the restriction of point-identification of the structural parameters in the sample. The main advantage of this estimator is its relative computational simplicity, but it has several limitations. The estimator relies critically on the large-support conditions on some explanatory variables and the point-identification result follows an *at-infinity* argument. In finite samples, imposing point-identification may induce significant biases. The second estimator deals with these issues and recognizes the possibility of having only set identification, but not point identification, in the sample or even in the population. We propose a method that provides a parameter set that minimizes a penalty function based on moment inequalities that summarize the restrictions of the model.

Our set-estimator of structural parameters minimizes a penalty function based on moment inequalities that summarize the restrictions of the model. The optimization of criterion function with respect to a set can be a computationally very demanding task, and this computational cost may restrict importantly the class of models that we can estimate. In this respect, the paper presents several results that simplify very significantly the estimation of this class of games. Based of those results, the estimation method that propose has similar computational cost as the estimation of the same model under the assumption of equilibrium beliefs. First, we show that under the assumption of weak submodularity of payoff functions with respect to players' actions, the model restrictions on players' choice probabilities can be derived using a simple sequential procedure. Second, we use a representation of players' best response functions based on Aguirregabiria and Mira (2007). In the context of this paper, this representation has several useful implications. Given a player's beliefs, it provides a consistent approximation to a player's best response to that beliefs, and it does not require one to solve the player's dynamic programming problem, or repeatedly solve this problem for different trial values of the structural parameters. Furthermore, if the one-period payoff function is linear in structural parameters, there is a representation of players' value functions that is also linear in structural parameters. This result implies that the model restrictions implied by level-1 rationality can be represented using a linear system of inequalities. Our estimator of the identified set exploits an efficient but very simple linear programming algorithm for the solution of linear systems of inequalities: a version of the so called *relaxation method* or *perceptron algorithm* (Agmon, 1954, Goffin, 1980, and Dunagan and Vempala, 2008). For levels of rationality greater than 1, we propose a recursive method that maintains the linearity in the structural parameters. We also propose different tests for the null hypothesis that players' beliefs are in equilibrium.

We apply these inference methods to a dynamic game of store location by McDonalds (MD) and Burger King (BK) using data for United Kingdom during the period 1991-1995. The dataset is a panel of 422 local markets (districts) and five years with information on the stock of stores and the flow of new stores of MD and BK in each local market, as well as local market characteristics such as population, density, age distribution, average rent, income per capita, local retail taxes, and distance to the headquarters of the firm in UK. The main empirical question that we want to analyze in this application is whether the beliefs of MD and BK about the strategic behavior of other competitor are consistent with the actual behavior of the competitor.

This paper is related to different literatures on the econometrics of empirical games which have received attention during recent years. It builds an extends the literature on estimation of dynamic games of incomplete information, with recent methodological contributions by Aguirregabiria and Mira (2007), Bajari, Benkard and Levin (2007), Pakes, Ostrovsky and Berry (2007), Pesendorfer and Schmidt-Dengler (2008), or Kasahara and Shimotsu (2009), and empirical applications by Ryan (2008), Collard-Wexler (2008), Sweeting (2007), Dunne, Roberts, and Xi (2009), Xu (2008), or Aguirregabiria and Mira (2009), among others. All the papers in this literature have assumed that the data come from a Markov Perfect Equilibrium. We relax that assumption. Our approach is related to the one in Aradillas-Lopez and Tamer (2008). These authors study the identification power of the assumption of equilibrium beliefs in the context of static discrete games of complete and incomplete information. Our paper extends their analysis in different directions. We study dynamic games. There are different aspects in which the assumption of equilibrium beliefs and identification issues are substantially different in dynamic games than in static ones (Magnac and Thesmar, 2002, Aguirregabiria, 2010). We study inference problems and propose estimators and tests. Our paper is also related to recent studies on partially identified models and estimation using moment inequalities, such as Chernozhukov, Hong, and Tamer (2007), Ciliberto and Tamer (2009), Pakes, Porter, Ho, and Ishii (2007), Andrews, Berry, and Jia (2005), Galichon and Henry (2008), Beresteanu, Molchanov, and Molinari (2008), and Aradillas-Lopez (2009).

The rest of the paper includes the following sections. Section 2 presents the model and basic assumptions. In section 3, we derive the restrictions that rationality imposes on players' best response probabilities. Section 4 presents our identification results. Section 5 describes the set-estimator based on moment inequalities. The empirical application is described in section 6. We summarize and conclude in section 7.

2 Model

This section presents a dynamic game of incomplete information where two players make binary choices over an infinite horizon. The assumption of only two players and two choice alternatives is made for notational simplicity, and we show how the main results in the paper extend to dynamic games with more than two players or choice alternatives. We use the indexes $i \in \{1,2\}$ and $j \in \{1,2\}$ to represent a player and his opponent, respectively. Time is discrete an indexed by $t \in \{1,2,...\}$. Every period t, players choose simultaneously and non-cooperatively between alternatives 0 and 1. Let $Y_{it} \in \{0,1\}$ represent the choice of player i at period t. Each player makes this decision to maximize his expected and discounted payoff, $E_t(\sum_{s=0}^{\infty} \delta_i^s \Pi_{i,t+s})$, where $\delta_i \in (0,1)$ is player i's discount factor, and Π_{it} is his payoff at period t. The one-period payoff function has the following structure:[1]

$$\Pi_{it} = z(\mathbf{W}_i, X_{it}, X_{jt}, Y_{it}, Y_{jt}) \; \boldsymbol{\theta}_i - Y_{it} \; \varepsilon_{it} \tag{1}$$

$\boldsymbol{\theta}_i$ is a column vector of structural parameters, and $\boldsymbol{\theta} \equiv (\boldsymbol{\theta}_1, \boldsymbol{\theta}_2)$ is the vector with the parameters of all players. $z()$ is a vector-valued function that depends on players' actions (Y_{it}, Y_{jt}), on players' state variables (X_{it}, X_{jt}), and on a vector $\mathbf{W}_i$ of time-invariant exogenous characteristics of the player. Structural parameters, the vector-valued function $z()$, players' state variables (X_{it}, X_{jt}), and the characteristics $(\mathbf{W}_i, \mathbf{W}_j)$ are common knowledge to the two players. The variable ε_{it} is private information of firm i at period t. A firm has uncertainty on the current value of his opponent's ε, and on future values of both his own and his opponent's $\varepsilon' s$.

The state variable X_{it} is an endogenous 'stock' variable for player i that evolves over time according to a transition that can be either stochastic or deterministic. For instance,

$$X_{it+1} = \min\{0 \;, \; X_{it} + Y_{it} - \xi_{i,t+1}\} \tag{2}$$

$\xi_{i,t+1} \in \{0,1\}$ is a Bernoulli random variable that captures exogenous depreciation in the stock, and it is i.i.d. with parameter $\lambda \equiv \Pr(\xi_{i,t+1} = 1)$. The set of possible values for these stock variables is $\mathcal{X} \equiv \{0,\, 1,\, 2,\, ...,\, K-1\}$ where $K - 1 \geq 1$ is a natural number that represents the maximum level of the stock. The variables ε_{1t} and ε_{2t} are independent of $(\mathbf{W}_1, \mathbf{W}_2)$, independent of each other, and independently and identically distributed over time. Their distribution functions, G_1 and G_2, are absolutely continuous and strictly increasing with respect to the Lebesgue measure on $\mathbb{R}$.

EXAMPLE 3 (Capacity Investment in an Oligopoly Industry). Consider a dynamic game of capacity

[1] The linearity of the payoff function in the structural parameters facilitates significantly the estimation of the model.

investment between two firms competing in an oligopoly industry of an homogeneous product.[2] The demand function is $Q_t = S_t(b_0 - b_1 P_t)$, where b_0 and b_1 are parameters, Q_t represents aggregate demand, S_t is the exogenous market size, and P_t is the product price. There are two firms that may operate in the industry. Every period t, these firms compete in quantities a la Cournot (static game), and choose whether to invest to increase their capacity (dynamic game). Production costs are linear in the quantity produced, i.e., $C_{it} = MC_{it}\ q_{it}$, where MC_{it} is the marginal cost, and q_{it} represents output. Marginal cost declines with installed capacity, i.e., $MC_{it} = c_i - d(X_{it} + Y_{it})$, where $c_i > 0$ and $d > 0$ are parameters, X_{it} is the installed capacity at the beginning of period t, and $Y_{it} \in \{0, 1\}$ represents capacity investment, that is a binary choice. It is simple to show that the Cournot equilibrium variable profit of firm i is:

$$VP_{it} \quad = \quad 1\{X_{it} + Y_{it} > 0\}\ \frac{S_t}{b_1}\left(\frac{b_0 + MC_{jt} - MC_{it}}{3}\right)^2 \tag{3}$$

where $1\{.\}$ is the indicator function. We can represent this variable profit function as linear function of structural parameters:

$$\begin{aligned} VP_{it} \quad &= \quad \theta_{0i}^{VP}\ S_t\ 1\{X_{it} + Y_{it} > 0\} \\ &+ \quad \theta_{1i}^{VP}\ S_t\ 1\{X_{it} + Y_{it} > 0\}(X_{it} + Y_{it} - X_{jt} - Y_{jt}) \\ &+ \quad \theta_{2i}^{VP}\ S_t\ 1\{X_{it} + Y_{it} > 0\}(X_{it} + Y_{it} - X_{jt} - Y_{jt})^2 \end{aligned} \tag{4}$$

θ_{0i}^{VP}, θ_{1i}^{VP}, and θ_{2i}^{VP} are structural parameters that are known functions of the 'deep' parameters b_0, b_1, c_i, c_j, and d, i.e., $\theta_{0i}^{VP} \equiv (b_0 + c_j - c_i)^2$, $\theta_{1i}^{VP} \equiv 2d(b_0 + c_j - c_i)$, and $\theta_{2i}^{VP} \equiv d^2$. Here we concentrate on the identification and estimation of the parameters $(\theta_{0i}^{VP}, \theta_{1i}^{VP}, \theta_{2i}^{VP} : i = 1, 2)$ together with the parameters in fixed costs.[3] The set of possible capacity levels is {0, 1, 2, ..., $K - 1$} where $K - 1 \geq 1$ is a natural number that represents the maximum level of capacity. A firm's capacity evolves over time according to the transition rule $X_{it+1} = X_{it} + Y_{it}$. The firm's total profit function is:

$$\Pi_{it} = VP_{it} - \theta_{0i}^{FC}\ 1\{Y_{it} + X_{it} > 0\} - \theta_{1i}^{FC}(Y_{it} + X_{it}) - \theta_{2i}^{FC}(Y_{it} + X_{it})^2 - Y_{it}\ \varepsilon_{it} \tag{5}$$

where θ_{0i}^{FC}, θ_{1i}^{FC} and θ_{2i}^{FC} are parameters in the fixed cost function. θ_{0i}^{FC} is a lump-sum cost associated with having some positive capacity, and it can be interpreted as an entry cost. The function $\theta_{1i}^{FC}(Y_{it} + X_{it}) + \theta_{2i}^{FC}(Y_{it} + X_{it})^2$ takes into account that fixed operation may increase with

[2] See Besanko and Doraszelski (2004), or Ryan (2009) for related dynamic games of firm capacity.

[3] It is simple to verify that given a value of the vector of parameters $(\theta_{0i}^{VP}, \theta_{1i}^{VP}, \theta_{2i}^{VP} : i = 1, 2)$, we can over-identify the 'deep' structural parameters d, b_0, and the cost differential $c_j - c_i$.

capacity with a linear or quadratic form. The variable ε_{it} is a private information shock in the firm's investment cost, and it is normally distributed. In this example, the vector of structural parameters for firm i is

$$\boldsymbol{\theta}_i \equiv (\theta_{0i}^{VP},\ \theta_{1i}^{VP},\ \theta_{2i}^{VP},\ \theta_{0i}^{FC},\ \theta_{1i}^{FC},\ \theta_{2i}^{FC})' \tag{6}$$

And the vector-valued function $z()$ is:

$$\begin{aligned} z_i(X_{it}, X_{jt}, Y_{it}, Y_{jt}) \quad \equiv \quad & \{\ S_t 1\{X_{it} + Y_{it} > 0\},\ S_t 1\{X_{it} + Y_{it} > 0\}(X_{it} + Y_{it} - X_{jt} - Y_{jt}), \\ & S_t 1\{X_{it} + Y_{it} > 0\}(X_{it} + Y_{it} - X_{jt} - Y_{jt})^2 \\ & -1\{X_{it} + Y_{it} > 0\},\ -(X_{it} + Y_{it}),\ -(X_{it} + Y_{it})^2 \} \end{aligned} \tag{7}$$

In our empirical application in section 6, we consider a version of this model to study the industry of fast-food burger restaurants. The two companies are McDonalds and Burger King. A local market is a district. A firm's capacity, X_{it}, is the number of stores that the firm operates in the local market. Y_{it} is the decision to open a new store in the local market. The specification of marginal costs, $MC_{it} = c_i - d(X_{it} + Y_{it})$ with $d > 0$, tries to capture economies of scope in the variable costs of running a number of stores. Alternatively, an estimate of $d < 0$ might be capturing 'cannibalization' effects between stores of the same chain at the same retail market. During the sample period of our analysis (1991-1995), these firms did not close any existing store. This is why there is not an exit decision in the model. The model assumes that the decision to open a new store is completely irreversible.

The recent literature on estimation of dynamic discrete games typically assumes that the data comes from a Markov Perfect Equilibrium (MPE). This equilibrium concept incorporates three main assumptions.

ASSUMPTION 1 ('Payoff Relevant State Variables'): Players' strategy functions depend only on payoff relevant state variables.

ASSUMPTION 2 ('Rational Beliefs on Own Future Behavior'): Players are forward looking, maximize expected intertemporal payoffs, and have rational expectations on their own behavior in the future.

ASSUMPTION 3 ('Rational or Equilibrium Beliefs on other Players' actions): Strategy functions are common knowledge, and players' have rational expectations on the current and future behavior of other players. That is, players beliefs about other players' behavior are equilibrium (self-fulfilling) beliefs.

For the moment, consider that we impose only Assumption 1. Let $\mathbf{X}_t$ be the vector with the payoff-relevant, common knowledge state variables, i.e., $\mathbf{X}_t \equiv (X_{1t}, X_{2t})$.[4] At period t, players observe $\mathbf{X}_t$ and choose their respective actions Y_{it}. The payoff-relevant information set of player i is $\{\mathbf{X}_t, \varepsilon_{it}\}$. Let $\sigma_i(\mathbf{X}_t, \varepsilon_{it})$ be a strategy function for player i. This is a function from the support of $(\mathbf{X}_t, \varepsilon_{it})$ into the binary set $\{0,1\}$, i.e., $\sigma_i : \mathcal{X} \times \mathbb{R} \rightarrow \{0,1\}$, where $\mathcal{X} = \{0,1,...,K-1\} \times \{0,1,...,K-1\}$. Given any strategy function σ_i, we can define a choice probability function $P_i(\mathbf{X}_t)$ that represents the probability of $Y_{it} = 1$ conditional on $\mathbf{X}_t$ given that player i follows strategy σ_i. That is,

$$P_i(\mathbf{X}_t) \equiv \int 1\left\{\sigma_i\left(\mathbf{X}_t, \varepsilon_{it}\right) = 1\right\}\ dG_i\left(\varepsilon_{it}\right) \tag{8}$$

where $1\{.\}$ is the indicator function. It is convenient to represent players' behavior and beliefs using these *Conditional Choice Probability* (CCP) functions. Note that, given that the variables in $\mathbf{X}_t$ have a discrete support, we can represent the CCP function $P_i(.)$ using a finite dimension vector $\mathbf{P}_i \equiv \{P_i(\mathbf{X}_t) : \mathbf{X}_t \in \mathcal{X}\}$. Throughout the paper, we use either the function $P_i(.)$ or the vector $\mathbf{P}_i$ to represent the *actual behavior* of player i.

Without imposing Assumption 3 ('Equilibrium Beliefs'), a player's beliefs about other players' behavior do not represent necessarily the actual behavior of other players. Therefore, we need a function other than $P_j(.)$ to represent players i' beliefs about the behavior of player j. We use the function $B_i(\mathbf{X}_t)$ to represent player i's beliefs about the probability that player j chooses $Y_{jt} = 1$ when the current state is $\mathbf{X}_t$. Again, we can represent the function $B_i(.)$ using a finite-dimensional vector $\mathbf{B}_i \equiv \{B_i(\mathbf{X}_t) : \mathbf{X}_t \in \mathcal{X}\}$. Note that players' beliefs are consistent with our Markovian assumptions. That is, the beliefs function $B_i(.)$ incorporates the assumption that player i beliefs that player j will behave now and in the future following the same dynamic decision rule. It is important to note that this assumption does not mean that we impose the restriction that players beliefs do not change over time. Our approach allows players' beliefs to vary over time. However, we assume that, every period, players belief that other players' behavior is Markovian.

Now, suppose that we impose both Assumptions 1 and 2, but not Assumption 3. Our next step is to characterize the *rational* behavior or optimal response of a player. We say that a strategy function $\sigma_i(.)$ (and the associated CCP function $P_i()$) is *rational* if for every possible value of $(\mathbf{X}_t, \varepsilon_{it})$ the action $\sigma_i(\mathbf{X}_t, \varepsilon_{it})$ maximizes player i's expected and discounted value given his beliefs on the opponents' strategies. A player's best response is the optimal decision rule of a Markov dynamic programming (DP) problem. Given beliefs $\mathbf{B}_i$, define the expected one-period payoff

[4] For notational simplicity, we omit the time-invariant variables $\mathbf{W}_1$ and $\mathbf{W}_2$ from the vector $\mathbf{X}_t$. These variables are implicitly in this vector or in the primitives of the model.

function:

$$\Pi_i^{\mathbf{B}}(Y_{it}, \mathbf{X}_t) = (1 - B_i(\mathbf{X}_t))\ z_{it}(Y_{it}, 0)\ \boldsymbol{\theta}_i + B_i(\mathbf{X}_t)\ z_{it}(Y_{it}, 1)\ \boldsymbol{\theta}_i \tag{9}$$

where, for notational simplicity, we use $z_{it}(Y_{it}, Y_{jt})$ to represent $z(\mathbf{W}_i, X_{it}, X_{jt}, Y_{it}, Y_{jt})$. And define player i's beliefs on the conditional choice transition probability of $\mathbf{X}_t$:[5]

$$f_i^{\mathbf{B}}(\mathbf{X}_{t+1}|Y_{it}, \mathbf{X}_t) = 1\{X_{it+1} = X_{it} + Y_{it}\}\ B_i(\mathbf{X}_t)^{1\{X_{jt+1}=X_{jt}+1\}}\ (1 - B_i(\mathbf{X}_t))^{1\{X_{jt+1}=X_{jt}\}} \tag{10}$$

Let $V_i^{\mathbf{B}}(\mathbf{X}_t)$ be the integrated value function for player i's DP problem.[6] By Bellman's principle, the value function $V_i^{\mathbf{B}}$ is the unique fixed point of the Bellman equation $V = \Gamma_i^{\mathbf{B}}(V)$, where $\Gamma_i^{\mathbf{B}}$ is the *integrated Bellman operator*:

$$\Gamma_i^{\mathbf{B}}(V)(\mathbf{X}) \quad = \quad \int \max_{Y_i \in \{0,1\}} \left\{ \Pi_i^{\mathbf{B}}(Y_i, \mathbf{X}) - Y_i\ \varepsilon_i + \delta_i \sum_{\mathbf{X}'} f_i^{\mathbf{B}}(\mathbf{X}'|Y_i, \mathbf{X})\ V(\mathbf{X}') \right\} dG_i(\varepsilon_i) \tag{11}$$

The form of this *integrated Bellman operator* depends on the distribution of the private information shock ε_{it}. For instance, if ε_{it} has a extreme value distribution (i.e., DP logit model), we have that:

$$\Gamma_i^{\mathbf{B}}(V)(\mathbf{X}_t) \quad = \quad \log \left(\sum_{Y_i=0}^{1} \exp \left\{ \Pi_i^{\mathbf{B}}(Y_i, \mathbf{X}) + \delta_i \sum_{\mathbf{X}'} f_i^{\mathbf{B}}(\mathbf{X}'|Y_i, \mathbf{X})\ V(\mathbf{X}') \right\} \right) \tag{12}$$

If ε_{it} has a standard normal distribution (i.e., DP probit model), we have:

$$\begin{aligned}
\Gamma_i^{\mathbf{B}}(V)(\mathbf{X}_t) \quad = & \quad \Pi_i^{\mathbf{B}}(0, \mathbf{X}) + \delta_i \sum_{\mathbf{X}'} f_i^{\mathbf{B}}(\mathbf{X}'|0, \mathbf{X})\ V(\mathbf{X}') \\
+ & \quad \Phi \left(\tilde{\Pi}_i^{\mathbf{B}}(\mathbf{X}) + \delta_i \sum_{\mathbf{X}'} \tilde{f}_i^{\mathbf{B}}(\mathbf{X}'|\mathbf{X}) V(\mathbf{X}') \right) \left(\tilde{\Pi}_i^{\mathbf{B}}(\mathbf{X}) + \delta_i \sum_{\mathbf{X}'} \tilde{f}_i^{\mathbf{B}}(\mathbf{X}'|\mathbf{X}) V(\mathbf{X}') \right) \\
+ & \quad \phi \left(\tilde{\Pi}_i^{\mathbf{B}}(\mathbf{X}) + \delta_i \sum_{\mathbf{X}'} \tilde{f}_i^{\mathbf{B}}(\mathbf{X}'|\mathbf{X}) V(\mathbf{X}') \right)
\end{aligned} \tag{13}$$

where $\tilde{\Pi}_i^{\mathbf{B}}(\mathbf{X}) \equiv \Pi_i^{\mathbf{B}}(1, \mathbf{X}) - \Pi_i^{\mathbf{B}}(0, \mathbf{X})$; $\tilde{f}_i^{\mathbf{B}}(\mathbf{X}'|\mathbf{X}) \equiv f_i^{\mathbf{B}}(\mathbf{X}'|1, \mathbf{X}) - f_i^{\mathbf{B}}(\mathbf{X}'|0, \mathbf{X})$; and ϕ and Φ are the PDF and the CDF of the standard normal, respectively.

The optimal response function of player i with beliefs $\mathbf{B}_i$ is the optimal decision rule of the previous DP problem. That is, the optimal response function, in the action space, is:

$$\{Y_{it} = 1\} \ \text{ iff } \ \left\{ \varepsilon_{it} \le \tilde{\Pi}_i^{\mathbf{B}}(\mathbf{X}_t) + \delta_i \sum_{\mathbf{X}_{t+1}} \tilde{f}_i^{\mathbf{B}}(\mathbf{X}_{t+1}|\mathbf{X}_t)\ V_i^{\mathbf{B}}(\mathbf{X}_{t+1}) \right\} \tag{14}$$

[5] Again, for notational simplicity, we consider the deterministic transition rule $X_{it+1} = X_{it} + Y_{it}$ instead of the stochastic transition $X_{it+1} = \min\{0\ ,\ X_{it} + Y_{it} - \xi_{i,t+1}\}$. It is simple to verify that all the results extend to the stochastic transition.

[6] As defined in Rust (1994), this integrated value function is the integral of the original value function $J_i^{\mathbf{B}}(\mathbf{X}_t, \varepsilon_{it})$ over the distribution of ε_{it}: i.e., $V_i^{\mathbf{B}}(\mathbf{X}_t) \equiv \quad J_i^{\mathbf{B}}(\mathbf{X}_t, \varepsilon_{it}) dG_i(\varepsilon_{it})$. It is convenient to use $V_i^{\mathbf{B}}$ instead of $J_i^{\mathbf{B}}$ to describe the solution of the DP problem because its lower dimension. In particular, for discrete $\mathbf{X}_t$ and continuous ε_{it}, $J_i^{\mathbf{B}}$ lives in an infinite-dimension space while $V_i^{\mathbf{B}}$ lives in a finite-dimension Euclidean space.

And the optimal response probability function is $G_i(\tilde{\Pi}_i^{\mathbf{P}}(\mathbf{X}_t) + \delta_i \sum_{\mathbf{X}_{t+1}} \tilde{f}_i^{\mathbf{P}}(\mathbf{X}_{t+1}|\mathbf{X}_t)\ V_i^{\mathbf{P}}(\mathbf{X}_{t+1}))$. Therefore, under Assumptions 1 and 2 and given beliefs $\mathbf{B}_i$, the actual behavior of player i, as represented by the CCP $P_i(.)$, is:

$$P_i(\mathbf{X}_t) = G_i\left(\tilde{\Pi}_i^{\mathbf{B}}(\mathbf{X}_t) + \delta_i \sum_{\mathbf{X}_{t+1}} \tilde{f}_i^{\mathbf{B}}(\mathbf{X}_{t+1}|\mathbf{X}_t)\ V_i^{\mathbf{B}}(\mathbf{X}_{t+1})\right) \tag{15}$$

where $V_i^{\mathbf{B}}$ is implicitly defined by the Bellman equation $V_i^{\mathbf{B}} = \Gamma_i^{\mathbf{B}}(V_i^{\mathbf{B}})$. These equations summarize all the restrictions that Assumptions 1 and 2 impose on players' choice probabilities.

The concept of Markov Perfect Equilibrium (MPE) is completed with Assumption 3 ('Equilibrium Beliefs'). Under this assumption, players' beliefs are in equilibrium such that $B_i(.) = P_j(.)$ for every i and j. A MPE can be described as a set of CCP functions, one for each player, such that each player function is the best response to the other players' functions: i.e., $P_i(\mathbf{X}_t) = G_i(\tilde{\Pi}_i^{\mathbf{P}}(\mathbf{X}_t) + \delta_i \sum_{\mathbf{X}_{t+1}} \tilde{f}_i^{\mathbf{P}}(\mathbf{X}_{t+1}|\mathbf{X}_t)\ V_i^{\mathbf{P}}(\mathbf{X}_{t+1}))$, where $\tilde{\Pi}_i^{\mathbf{P}}$, $\tilde{f}_i^{\mathbf{P}}$, and $V_i^{\mathbf{P}}$ are constructed using the actual CCPs of other players. To represent a MPE of the model in a compact form, we use $\Psi(\mathbf{P})$ to represent the mapping with best response probabilities for every player and every value of the state variables:

$$\Psi(\mathbf{P}) \equiv \left\{ G_i(\tilde{\Pi}_i^{\mathbf{P}}(\mathbf{X}_t) + \delta_i \sum_{\mathbf{X}_{t+1}} \tilde{f}_i^{\mathbf{P}}(\mathbf{X}_{t+1}|\mathbf{X}_t) V_i^{\mathbf{P}}(\mathbf{X}_{t+1})) : i = 1,2;\ \mathbf{X}_t \in \mathcal{X} \right\} \tag{16}$$

Therefore, a MPE is a vector of CCPs $\mathbf{P}$ such that $\mathbf{P} = \Psi(\mathbf{P})$.

3 Bounds on Choice Probabilities

For the rest of the paper, we maintain Assumptions 1 and 2 but we relax Assumption 3 on 'Equilibrium Beliefs'. Our approach is agnostic about the formation of players' beliefs. We replace the assumption of 'Equilibrium Beliefs' by the following much weaker assumption.

ASSUMPTION 4 ('Level 2 Rationality'): Players are rational in the sense that their strategies maximize expected and discounted payoffs given their beliefs on other players' behavior. Furthermore, players know that their opponents are also rational. Therefore, players' beliefs are consistent with other players' having rational strategies.

Assumptions 1, 2, and 4 impose restrictions on players' vectors of actual CCPs, $\mathbf{P}_i$. In this section, we show that under these assumptions, we have that the vector $\mathbf{P}_i$ lies in a compact set that is strictly contained within the hypercube $[0,1]^{|\mathcal{X}|}$.[7] That is, $\mathbf{P}_i \in C_i^{(2)} \subset [0,1]^{|\mathcal{X}|}$ where

[7] Note that the support of $\mathbf{X}_t$ is $\mathcal{X} = \{0,1,...,K-1\} \times \{0,1,...,K-1\}$. Therefore, the dimension of the vector $\mathbf{P}_i$ is $|\mathcal{X}| = K^2$.

$C_i^{(2)}$ is the set of best response probability vectors that are consistent with level-2 rationality. To abbreviate, we call it *level-2 best response set*. This set can be calculated given the parameters of the model. Similarly, for any level of rationality $R \geq 1$, we use $C_i^{(R)}$ to represent the *best response set* under *level-R rationality*. In this section, we describe the derivation of these best response sets. We start with a definition of these sets in terms of primitives of the model. Then, we prove that the restrictions imposed by the assumption of rationality level-1 and level-2 are informative in the sense that the sets of possible best responses is smaller than $[0,1]^{|\mathcal{X}|}$. Finally, we propose a simple recursive method for the computation of these sets.

Define the *threshold value function*:

$$v_i^{\mathbf{B}}(\mathbf{X}_t) \equiv \tilde{\Pi}_i^{\mathbf{B}}(\mathbf{X}_t) + \delta_i \sum_{\mathbf{X}_{t+1}} \tilde{f}_i^{\mathbf{B}}(\mathbf{X}_{t+1}|\mathbf{X}_t)\ V_i^{\mathbf{B}}(\mathbf{X}_{t+1}) \tag{17}$$

We call it *threshold value function* because it represents the threshold value of ε_{it} that makes player i indifferent between the choice of alternative 1 and the choice of 0. A player's best response function can be represented as $P_i(\mathbf{X}_t) = G_i(v_i^{\mathbf{B}}(\mathbf{X}_t))$. Define the best response mapping of player i, $\Psi_i(\mathbf{B}_i)$, from $[0,1]^{K^2}$ into $[0,1]^{K^2}$:

$$\Psi_i(\mathbf{B}_i) \equiv \begin{bmatrix} G_i(v_i^{\mathbf{B}}(\mathbf{X}^{(1)}) \\ G_i(v_i^{\mathbf{B}}(\mathbf{X}^{(2)}) \\ \\ G_i(v_i^{\mathbf{B}}(\mathbf{X}^{(|\mathcal{X}|)}) \end{bmatrix} \tag{18}$$

where $\mathbf{X}^{(1)}$, $\mathbf{X}^{(2)}$, ..., $\mathbf{X}^{(|\mathcal{X}|)}$ are the different values of $\mathbf{X}_t$ in its support $\mathcal{X}$. Let *image*(Ψ_i on D) represent the image set of the mapping Ψ_i when the domain is D, e.g., *image*(Ψ_i on $[0,1]^{|\mathcal{X}|}$) $\equiv$ $\{\Psi_i(\mathbf{B}_i) : \mathbf{B}_i \in [0,1]^{|\mathcal{X}|}\}$. By definition, the set of possible best responses of a player that is level-1 rational is just the image set Ψ_i when the domain is $[0,1]^{|\mathcal{X}|}$:

$$C_i^{(1)} = \textit{image}(\Psi_i \text{ on } [0,1]^{|\mathcal{X}|}) \tag{19}$$

And the set of possible best responses of a player that is level-2 rational is the image set Ψ_i when the domain is $C_j^{(1)}$:

$$C_i^{(2)} = \textit{image}(\Psi_i \text{ on domain } C_j^{(1)}) \tag{20}$$

In general, for level-R rationality, we have that $C_i^{(R)} = \textit{image}(\Psi_i \text{ on domain } C_j^{(R-1)})$. Since Ψ_i is a continuous mapping and the domain $[0,1]^{|\mathcal{X}|}$ is closed, it is clear that *image*(Ψ_i on $[0,1]^{|\mathcal{X}|}$) is a closed set (i.e., continuous functions map closed sets into closed sets). Furthermore, *image*(Ψ_i on $[0,1]^{|\mathcal{X}|}$) is compact because it is bounded. Therefore, the set $C_i^{(1)}$ is equal to its *compact closure*. Similarly, using a recursive argument, it is straightforward to show that $C_i^{(R)}$ is also a compact set.

The assumption of rationality (or of level-R rationality) implies informative bounds on players' behavior only if the effect of beliefs $\mathbf{B}_i$ on the threshold value function $v_i^{\mathbf{B}}(\mathbf{X}_t)$ is bounded with probability one. Otherwise, the best response probability of an arbitrarily pessimistic (optimistic) rational player would be zero (one) with probability one. Proposition 1 establishes that the assumption of rationality imposes informative restrictions, and the higher the level of rationality, the stronger the restrictions.

PROPOSITION 1: For any finite vector of the structural parameters $\boldsymbol{\theta}$, and discount factors strictly smaller than 1, the best response sets are such that $C_i^{(R)} \subset C_i^{(R-1)} \subset ... \; C_i^{(2)} \subset C_i^{(1)} \subset [0,1]^{|\mathcal{X}|}$.

Without further restrictions, the derivation of the best response sets $C_i^{(R)}$ can be a very complicated task. *** EXPLAIN WHY. IMPORTANT ***** The following condition plays a key role in our approach to calculate best response sets $C_i^{(R)}$ using a simple sequential method.

ASSUMPTION 5 (Weak Submodularity): The set of possible values of the structural parameters, Θ, is such that the one-period profit function $\Pi_i(Y_i, Y_j, X_i, X_j)$ (i.e., $z(X_{it}, X_{jt}, Y_{it}, Y_{jt})\boldsymbol{\theta}_i$) is submodular in (Y_i, Y_j) and submodular in (X_i, X_j). That is, for any value of (X_i, X_j),

$$\Pi_i(1,1,X_i,X_j) - \Pi_i(0,1,X_i,X_j) \leq \Pi_i(1,0,X_i,X_j) - \Pi_i(0,0,X_i,X_j) \tag{21}$$

And for any value of (Y_i, Y_j, X_i, X_j):

$$\Pi_i(Y_i,Y_j,X_i+1,X_j+1) - \Pi_i(Y_i,Y_j,X_i,X_j+1) \leq \; \Pi_i(Y_i,Y_j,X_i+1,X_j) - \Pi_i(Y_i,Y_j,X_i,X_j) \tag{22}$$

The model of capacity investment in the Example of section 2.1 satisfies this assumption of weak submodularity for any value of the structural parameters. Under Assumption 5, the threshold value function $v_i^{\mathbf{B}}(\mathbf{X}_t)$ is decreasing with respect to any probability in the vector of beliefs $\mathbf{B}_i$. Proposition 2 establishes this property and shows that it facilitates very significantly the derivation of best response sets.

PROPOSITION 2: Under Assumption 5, our dynamic game is such that best response probability functions are strictly decreasing in the vector of beliefs B_i. For any two values of the vector of state variables, say $\mathbf{X}_A$ and $\mathbf{X}_B$, we have that $\dfrac{\partial \Psi_i(\mathbf{X}_A, \mathbf{B}_i)}{\partial B_i\,(\mathbf{X}_B)} \leq 0$. This property implies the following (sharp) bounds on best response probabilities. Let $L_i^{(R)}(\mathbf{X})$ and $U_i^{(R)}(\mathbf{X})$ be the lower and upper bounds of the best response probability $\Psi_i(\mathbf{X}, \mathbf{B}_i)$ under level-R rationality, such that $L_i^{(R)}(\mathbf{X}) \leq \Psi_i(\mathbf{X}, \mathbf{B}_i) \leq U_i^{(R)}(\mathbf{X})$ for any $(\mathbf{X}, \mathbf{B}_i)$. And let $\mathbf{L}_i^{(R)}$ and $\mathbf{U}_i^{(R)}$ be the corresponding *vectors with the bounds for every value of* $\mathbf{X}$*. These bounds can be obtained using the following recursive formulas. For* $R \geq 1$,

$$\begin{array}{rcl} \mathbf{L}_i^{(R)} & = & \Psi_i(\mathbf{U}_j^{(R-1)}) \\ \mathbf{U}_i^{(R)} & = & \Psi_i(\mathbf{L}_j^{(R-1)}) \end{array} \tag{23}$$

with $\mathbf{U}_j^{(0)} = \mathbf{1}$ *and* $\mathbf{L}_j^{(0)} = \mathbf{0}$.

Proposition 2 shows that the best response sets $C_i^{(R)}$ are hyper-rectangles within the hyper-cube $[0,1]^{|\mathcal{X}|}$. That is, $C_i^{(R)} = \left\langle \mathbf{L}_i^{(R)}, \mathbf{U}_i^{(R)} \right\rangle$. Furthermore, the vertices of these hyper-rectangles can be obtained recursively by solving single-agent DP problems. For instance, $\mathbf{L}_i^{(1)}$ is the vector of optimal CCPs in a DP model where player i beliefs that his opponent will always choose alternative $Y_{jt} = 1$ at any possible state and with probability one. Similarly, $\mathbf{U}_i^{(1)}$ is the vector of optimal CCPs in a DP model where player i beliefs that his opponent will always choose $Y_{jt} = 0$. In the next section, we show that, for the estimation of the model, the derivation of the vectors of bounds $\mathbf{L}_i^{(R)}$ and $\mathbf{U}_i^{(R)}$ is in fact much simpler. It is possible to use a value function representation in Aguirregabiria and Mira (2007) to obtain these bounds without having to solve the DP problems.

4 Identification

Suppose that the researcher has a random sample of many (infinite) independent realizations of the game, e.g., many local markets. We use the subindex m to represent markets. For every market m in the sample, we observe a realization of the variables $\{Y_{imt}, X_{imt}, \mathbf{W}_{im} : i = 1,2;\ t = 1,2,...,T\}$. The number of sample periods T is small, and in fact it can be as small as $T = 2$. The unobservable variables $\{\varepsilon_{imt}\}$ are assumed to be independently and identically distributed across markets and over time. We want to use this sample to estimate the vector of structural parameters $\boldsymbol{\theta}$.[8]

Let $\{\mathbf{P}_{imt}^0 : i = 1,2\}$ be the vectors in $[0,1]^{|\mathcal{X}|}$ with the true (population) conditional probability functions $\Pr(Y_{imt} = 1|\mathbf{X}_{mt})$ in market m at period t. Let $\{\mathbf{B}_{imt}^0 : i = 1,2\}$ be the vectors with players' beliefs in market m at period t. And let $\boldsymbol{\theta}^0$ be the true value of $\boldsymbol{\theta}$ in the population under study. Assumption 6 summarizes our conditions on the Data Generating Process (DGP).

ASSUMPTION 6: (A) For every player i, $\mathbf{P}_{imt}^0$ *is the best response of player* i *given his beliefs* $\mathbf{B}_{imt}^0$ *and the vector of structural parameters* $\boldsymbol{\theta}^0$*, i.e.,* $\mathbf{P}_{imt}^0 = \Psi_i(\mathbf{B}_{imt}^0, \boldsymbol{\theta}^0)$*. (B) Players' beliefs may change over time (in an unrestricted way) but they are constant across markets, i.e., for every*

[8] This framework can be extended to incorporate unobservable state variables for the econometrician which are common knowledge to players and have a distribution with finite support (see Kasahara and Shimotsu, 2008b).

market m, $\mathbf{B}^0_{imt} = \mathbf{B}^0_{it}$.[9]

Assumption 6 implies that choice probabilities describing players' actual behavior (CCPs) do not vary across markets: i.e., for any market m, $\mathbf{P}^0_{imt} = \Psi_i(\mathbf{B}^0_{it}, \boldsymbol{\theta}^0) = \mathbf{P}^0_{it}$. It also implies that we can identify these probabilities nonparametrically from the data. For any player i, any period t, and any value of $\mathbf{X} \in \mathcal{X}$, we have that $P^0_{it}(\mathbf{X}) = E(Y_{imt}|\mathbf{X}_{mt} = \mathbf{X})$, and this conditional expectation can be estimated consistently using data on Y_{imt} and $\mathbf{X}_{mt}$ for a random sample of markets. For instance, given that $\mathbf{X}_{mt}$ is a vector of discrete random variables, the frequency or 'cell' estimator $\sum_{m=1}^{M} Y_{imt} 1\{\mathbf{X}_{mt} = \mathbf{X}\} / \sum_{m=1}^{M} 1\{\mathbf{X}_{mt} = \mathbf{X}\}$ is a consistent estimator of $P^0_{it}(\mathbf{X})$. For notational simplicity, we omit the time subindex for the rest of this section, but it should be implicit that the identification results allow beliefs to vary over time in an unrestricted way.

Under level-R rationality, the restrictions of the model can be summarized by the expression $\mathbf{L}^{(R)}_i(\boldsymbol{\theta}^0) \leq \mathbf{P}^0_i \leq \mathbf{U}^{(R)}_i(\boldsymbol{\theta}^0)$. More specifically, for level-1 rationality, we have the following restrictions:

$$G_i(v_i^{\mathbf{B=1}}(\mathbf{X}_t, \boldsymbol{\theta}^0)) \leq P^0_i(\mathbf{X}_t) \leq G_i(v_i^{\mathbf{B=0}}(\mathbf{X}_t, \boldsymbol{\theta}^0)) \tag{24}$$

where $v_i^{\mathbf{B=1}}(\mathbf{X}_t, \boldsymbol{\theta})$ is the threshold value function when player i beliefs that player j will choose $Y_j = 1$ at any state with probability 1, and $v_i^{\mathbf{B=0}}(\mathbf{X}_t, \boldsymbol{\theta})$ is the threshold value function when player i beliefs that player j will choose $Y_j = 0$ at any state with probability 1. Similarly, for level-2 rationality, we have the following restrictions:

$$G_i(v_i^{\mathbf{B=U}_j^{(1)}}(\mathbf{X}_t, \boldsymbol{\theta}^0)) \leq P^0_i(\mathbf{X}_t) \leq G_i(v_i^{\mathbf{B=L}_j^{(1)}}(\mathbf{X}_t, \boldsymbol{\theta}^0)) \tag{25}$$

where $L^{(1)}_j(\mathbf{X}, \boldsymbol{\theta}^{\mathbf{0}}) = G_j(v_j^{\mathbf{B=1}}(\mathbf{X}, \boldsymbol{\theta}^0))$ and $U^{(1)}_j(\mathbf{X}, \boldsymbol{\theta}^{\mathbf{0}}) = G_j(v_j^{\mathbf{B=0}}(\mathbf{X}, \boldsymbol{\theta}^0))$.

To study identification, as well as for the implementation of our estimation methods, we use the representation of best response functions and threshold value functions proposed by Aguirregabiria and Mira (2002 and 2007). This representation has the following form:

$$v_i^{\mathbf{B}}(\mathbf{X}_t, \boldsymbol{\theta}) \quad = \quad \widetilde{\mathbf{Z}}^{\mathbf{P},\mathbf{B}}_{it} \boldsymbol{\theta}_i - \tilde{e}^{\mathbf{P},\mathbf{B}}_{it} \tag{26}$$

where $\widetilde{\mathbf{Z}}^{\mathbf{P},\mathbf{B}}_{it} \equiv \mathbf{Z}^{\mathbf{P},\mathbf{B}}_{it}(1) - \mathbf{Z}^{\mathbf{P},\mathbf{B}}_{it}(0)$, and $\mathbf{Z}^{\mathbf{P},\mathbf{B}}_{it}(Y_{it})$ is the expected and discounted sum of current and future vectors $\{z_{it+s}(Y_{it+s}, Y_{jt+s}) : s = 0, 1, 2, ...\}$ which may occur along all possible histories originating from the choice of Y_{it} in state $\mathbf{X}_t$ when player i behaves in the future according to the choice probabilities in $\mathbf{P}_i$ and the other player behaves now and in the future according to the

[9] Our sampling design with large M and small T is the standard case in applications of empirical games in Industrial Organization. Alternatively, if the sampling design is such that the number of periods T is large and the number of markets M is small, then we can allow beliefs to vary over markets but being constant over time.

choice probabilities in $\mathbf{B}_i$. Similarly, $\tilde{e}_{it}^{\mathbf{P},\mathbf{B}} \equiv e_{it}^{\mathbf{P},\mathbf{B}}(1) - e_{it}^{\mathbf{P},\mathbf{B}}(0)$, and $e_{it}^{\mathbf{P},\mathbf{B}}(Y_{it})$ is the expected and discounted sum of realizations of $\{\varepsilon_{it+s} Y_{it+s} : s = 0, 1, 2, ...\}$ originating from the choice of Y_{it} in state $\mathbf{X}_t$, when player i behaves in the future according to the choice probabilities in $\mathbf{P}_i$ and the other player behaves now and in the future according to the choice probabilities in $\mathbf{B}_i$. Given the vectors $\mathbf{P}_i$ and $\mathbf{B}_i$, it is possible to obtain $\mathbf{Z}_{it}^{\mathbf{P},\mathbf{B}}(Y_{it})$ and $e_{it}^{\mathbf{P},\mathbf{B}}(Y_{it})$ by solving a system of linear equations with dimension $|\mathcal{X}|$. We describe the details of this derivation in Appendix B. Using this representation of best response probability functions, we can present the model restrictions as follows. Under level-R rationality,

$$\widetilde{\mathbf{Z}}_{it}^{\mathbf{P}^0,\mathbf{U}_j^{(R-1)}} \boldsymbol{\theta}_i^0 - \tilde{e}_{it}^{\mathbf{P}^0,\mathbf{U}_j^{(R-1)}} \quad \leq \quad G_i^{-1}\left(P_i^0(\mathbf{X}_t)\right) \quad \leq \quad \widetilde{\mathbf{Z}}_{it}^{\mathbf{P}^0,\mathbf{L}_j^{(R-1)}} \boldsymbol{\theta}_i^0 - \tilde{e}_{it}^{\mathbf{P}^0,\mathbf{L}_j^{(R-1)}} \tag{27}$$

where $G_i^{-1}(.)$ is the inverse function of the CDF G_i.

Suppose that we knew the true players beliefs $\mathbf{B}_i^0$. Then, it is straightforward to show that $\boldsymbol{\theta}_i^0$ is point-identified.

PROPOSITION 3: Consider that Assumption 6 holds and the moment matrices $E(\mathbf{Z}_{it}(0,0)\mathbf{Z}_{it}(0,0)')$, $E(\mathbf{Z}_{it}(1,0)\mathbf{Z}_{it}(1,0)')$, $E(\mathbf{Z}_{it}(0,1)\mathbf{Z}_{it}(0,1)')$, *and* $E(\mathbf{Z}_{it}(1,1)\mathbf{Z}_{it}(1,1)')$ *have full-column rank. If the researcher knows the true values of players' beliefs,* $\mathbf{B}_i^0$, *then the vector of structural parameters* $\boldsymbol{\theta}_i^0$ *is point identified under level-1 rationality (or under any level* $R > 0$*). In particular,* $\boldsymbol{\theta}_i^0 = E(\widetilde{\mathbf{Z}}_{it}^{\mathbf{P}^0,\mathbf{B}^0\prime}\widetilde{\mathbf{Z}}_{it}^{\mathbf{P}^0,\mathbf{B}^0})^{-1}E(\widetilde{\mathbf{Z}}_{it}^{\mathbf{P}^0,\mathbf{B}^0\prime}[G_i^{-1}(P_i^0(\mathbf{X}_t)) + \tilde{e}_{it}^{\mathbf{P}^0,\mathbf{B}^0}])$.

Of course, the assumption that the researcher knows, ex ante, players' beliefs is very unrealistic. Now, consider the more relevant case where the researcher does not know players beliefs. Let $\Theta^{(R)}$ be the identified set of parameters for level-R rational players. By definition:

$$\Theta_I^{(R)} = \left\{ \begin{array}{l} \boldsymbol{\theta} \in \Theta : \quad \text{for any } (i, \mathbf{X}_t) \\ \\ \widetilde{\mathbf{Z}}_{it}^{\mathbf{P}^0,\mathbf{U}_j^{(R-1)}} \boldsymbol{\theta}_i - \tilde{e}_{it}^{\mathbf{P}^0,\mathbf{U}_j^{(R-1)}} \leq G_i^{-1}\left(P_i^0(\mathbf{X}_t)\right) \leq \widetilde{\mathbf{Z}}_{it}^{\mathbf{P}^0,\mathbf{L}_j^{(R-1)}} \boldsymbol{\theta}_i - \tilde{e}_{it}^{\mathbf{P}^0,\mathbf{L}_j^{(R-1)}} \end{array} \right\} \tag{28}$$

It is possible to show that for any level of rationality R, $\Theta^{(R)} \subset \Theta^{(R-1)}$. However, without further assumptions, it is not possible to show that level-R rationality provides point identification.

Proposition 4 below shows that under standard exclusion restrictions and large support conditions, the vector of structural parameters $\boldsymbol{\theta}_i^0$ and the discount factor δ_i are point-identified. To prove point identification one should establish that for any vector $\boldsymbol{\theta} \neq \boldsymbol{\theta}^0$ there are values of $\mathbf{X}_t$ with positive probability mass such that one of the inequalities in expression (27) does not hold.

PROPOSITION 4: Suppose that one of the components of the vector $z_{it}(Y_{it}, Y_{jt})$ *is equal to* $Y_{it}W_i$, *where* W_i *is a observable exogenous variable that has sample variation over players and over markets. Let* $\alpha_i \subset \theta_i$ *be the structural parameter associated with the term* $Y_{it}W_i$. *Suppose that* $\alpha_i^0 \neq 0$

and the random variable W_i conditional on W_j has unbounded support. Then, under level-1 rationality α_i^0 is point-identified, and under level-2 rationality the whole vector θ_i^0 and the discount factor δ_i are point-identified.

The point-identification result in Proposition 4 is based on an *exclusion restriction* and a *large-support condition* on one of the exogenous explanatory variables, W_i. The exclusion restriction is an exogenous (or predetermined) observable variable that has a direct effect on the own player's payoff but not the other players' payoffs. Exclusion restrictions may appear naturally in many dynamic games. For instance, in a dynamic game of firms' capital investment in an oligopoly industry, a firm's current profit does not depend directly on other firms' capital at previous period, but it depends indirectly through the current investment decisions of other firms. The *large-support condition* establishes that the variables of the exclusion restriction have unbounded support on the real line. The formal proof of Proposition 4 is in the Appendix. We show that for any vector $\boldsymbol{\theta} \neq \boldsymbol{\theta}^0$ there are values of $(W, \mathbf{X}_t)$ with positive probability mass such that one of the inequalities in expression (27) does not hold. The following example provides a more informal discussion and some intuition for the identification result in Proposition 4.

EXAMPLE 4. Consider a model where the best response probability mapping has the following form: $P_i(\mathbf{X}_{mt}) = G_i(\theta_{i0} + \theta_{i1} W_{im} + \theta_{i2} X_{imt} + \theta_{i3} B_i(\mathbf{X}_{mt}))$, where θ_{i0}, θ_{i1}, θ_{i2}, and θ_{i3} are structural parameters. Under level-2 rationality, we have that:

$$P_i(\mathbf{X}_{mt}) = G_i\left(\theta_{i0} + \theta_{i1} W_{im} + \theta_{i2} X_{imt} + \theta_{i3} G_j\left(\theta_{j0} + \theta_{j1} W_{jm} + \theta_{j2} X_{jmt} + \theta_{i3} B_j(\mathbf{X}_{mt})\right)\right) \quad (29)$$

Suppose that conditional on W_{im} the state variable W_{jm} has unbounded support on the real line, and the same property applies to W_{jm} conditional on W_{im}. Without loss of generality suppose that $\theta_{i1} > 0$ and $\theta_{j1} > 0$. Let W_j^L be a value of W_{jm} small enough such that for any $W_{jm} < W_j^L$, the probability $G_j(\theta_{j0} + \theta_{j1} W_{jm} + \theta_{j2} X_{jmt} + \theta_{i3} B_j(\mathbf{X}_{mt}))$ is arbitrarily close to zero. Therefore, conditional on $\{W_{jm} < W_j^L\}$, we have that $G_i^{-1}(P_i^0(\mathbf{X}_{mt})) = \theta_{i0}^0 + \theta_{i1}^0 W_{im} + \theta_{i2}^0 X_{imt}$, and it is clear that θ_{i0}^0, θ_{i1}^0, and θ_{i2}^0 are point identified. Now, let W_j^U be a value of W_{jm} large enough such that for any $W_{jm} > W_j^U$, the probability $G_j(\theta_{j0} + \theta_{j1} W_{jm} + \theta_{j2} X_{jmt} + \theta_{i3} B_j(\mathbf{X}_{mt}))$ is arbitrarily close to one. Therefore, conditional on $\{W_{jm} > W_j^U\}$, we have that $G_i^{-1}(P_i^0(\mathbf{X}_t)) = \theta_{i0}^0 + \theta_{i1}^0 W_{im} + \theta_{i2}^0 X_{imt} + \theta_{i3}^0$, and this implies that θ_{i3}^0 is also point identified. We can apply the same argument to identify the structural parameters for player j. We can use also this example to illustrate how relaxing the large support condition implies that structural parameters are only set identified. Suppose that for $W_{jm} < W_j^L$, the probability $G_j(\theta_{j0} + \theta_{j1} W_{jm} + \theta_{j2} X_{jmt} + \theta_{i3} B_j(\mathbf{X}_{mt}))$ is not arbitrarily close to zero but to a function $\lambda_L(\mathbf{X}_{mt})$ that is strictly greater than zero. Therefore, conditional on $\{W_{jm} < W_j^L\}$, we have that $G_i^{-1}(P_i^0(\mathbf{X}_t)) = \theta_{i0}^0 + \theta_{i1}^0 W_{im} + \theta_{i2}^0 X_{imt} + \theta_{i3}^0 \lambda_L(\mathbf{X}_{mt})$. Similarly, conditional on $\{W_{jm} > W_j^U\}$, we have that $G_i^{-1}(P_i^0(\mathbf{X}_t)) = \theta_{i0}^0 + \theta_{i1}^0 W_{im} + \theta_{i2}^0 X_{imt} + \theta_{i3}^0 \lambda_U(\mathbf{X}_{mt})$, where $\lambda_U(\mathbf{X}_{mt})$ is a function greater than $\lambda_L(\mathbf{X}_{mt})$ but strictly smaller than 1. In this case, without further restrictions, we cannot point identify the structural parameters.

5 Estimation and Inference

5.1 Estimation

We propose two estimation methods. The first estimator has similarities with some matching estimators used in the treatment effect literature (Abadie and Imbens, 2006). It imposes the restriction of point-identification of the structural parameters in the sample. The main advantage of this estimator is its relative computational simplicity, but it has several limitations. The estimator relies critically on the large-support conditions on some explanatory variables and the point-identification result follows an *at-infinity* argument. In finite samples, imposing point-identification may induce significant biases. The second estimator deals with these issues and recognizes the possibility of having only set identification, but not point identification, in the sample or even in the population. We propose a method that provides a parameter set that minimizes a penalty function based on moment inequalities that summarize the restrictions of the model.

5.1.1 Estimator based on extreme values of explanatory variables

We start describing this estimator in the context of the simple static game in Example 4. Under level-2 rationality, the best response probability function is described by equation (29). For $q \in (0,1)$, let $W_j^{(q)}$ be the q-quantile in the population distribution of W_{jm}, i.e., $\Pr(W_{jm} \leq W_j^{(q)}) = q$. The best response probability of player j given $W_j^{(q)}$ goes to 0 as q goes to zero. Therefore, given $W_j^{(q)}$ and $q \to 0$, the best response of player i goes to $G_i(\theta_{i0} + \theta_{i1} W_{im} + \theta_{i2} X_{imt})$. Similarly, given $W_j^{(q)}$ and q going to one, the best response of player i goes to $G_i(\theta_{i0} + \theta_{i1} W_{im} + \theta_{i2} X_{imt} + \theta_{i3})$. Based on this result, we can construct a consistent estimator of the structural parameters. Let $q_M \in (0,1)$ be such that q_M goes to zero and $M q_M$ goes to infinite as the sample size M goes to infinite. And let $\hat{W}_j^{(q_M)}$ be the q_M-quantile in the *sample* distribution of W_{jm}. Then, we construct the following regression-like equation: for any observation (m,t) such that $\{W_{jm} \leq \hat{W}_j^{(q_M)}$ OR $W_{jm} \geq \hat{W}_j^{(1-q_M)}\}$,

$$G_i^{-1}(P_i^0(\mathbf{X}_{mt})) = \theta_{i0} + \theta_{i1} W_{im} + \theta_{i2} X_{imt} + \theta_{i3}\ 1\{W_{jm} \geq \hat{W}_j^{(1-q_M)}\} \tag{30}$$

where 1{.} is the indicator function. Based on this expression, we can implement a simple two-step estimator. In the first step, we estimation nonparametrically the CCP function $P_i^0(.)$, and the quantiles $W_j^{(q_M)}$ and $W_j^{(1-q_M)}$. In the second step, we run the linear regression in equation (30) using the subsample of observations with $\{W_{jm} \leq \hat{W}_j^{(q_M)}\ OR\ W_{jm} \geq \hat{W}_j^{(1-q_M)}\}$ and replacing P_i^0 by the estimator $\hat{P}_i^0$. This estimator is root-M consistent and asymptotically normal.

Now, we extend this estimator to the dynamic game model. Under level-2 rationality, the best response probability function is:

$$P_i(\mathbf{X}_{mt}) = G_i\left(\widetilde{\mathbf{Z}}_{imt}^{\mathbf{P}_i,\mathbf{G}_j(\mathbf{B}_j)}\boldsymbol{\theta}_i - \tilde{e}_{imt}^{\mathbf{P}_i,\mathbf{G}_j(\mathbf{B}_j)}\right)$$

where $\mathbf{G}_j(\mathbf{B}_j)$ is a compact form to represent the best response probabilities of player j given his beliefs $\mathbf{B}_j$. Given $W_j^{(q)}$ and q going to zero, the best response probability of player j goes to $\mathbf{G}_j(\mathbf{B}_j) = 0$, and therefore the best response of player i goes to $G_i(\widetilde{\mathbf{Z}}_{imt}^{\mathbf{P}_i,\mathbf{0}}\boldsymbol{\theta}_i - \tilde{e}_{imt}^{\mathbf{P}_i,\mathbf{0}})$. Similarly, given $W_j^{(q)}$ and q going to one, the best response of player i goes to $G_i(\widetilde{\mathbf{Z}}_{imt}^{\mathbf{P}_i,\mathbf{1}}\boldsymbol{\theta}_i - \tilde{e}_{imt}^{\mathbf{P}_i,\mathbf{1}})$. As in the static game, we can write the following regression-like equation: for any observation (m, t) such that $\{W_{jm} \leq \hat{W}_j^{(q_M)}\ OR\ W_{jm} \geq \hat{W}_j^{(1-q_M)}\}$,

$$\begin{aligned}&G_i^{-1}(P_i^0(\mathbf{X}_{mt})) + 1\{W_{jm} \leq \hat{W}_j^{(q_M)}\}\tilde{e}_{imt}^{\mathbf{P}_i^0,\mathbf{0}} + 1\{W_{jm} \geq \hat{W}_j^{(1-q_M)}\}\tilde{e}_{imt}^{\mathbf{P}_i^0,\mathbf{1}} \\ &= \left[1\{W_{jm} \leq \hat{W}_j^{(q_M)}\}\widetilde{\mathbf{Z}}_{imt}^{\mathbf{P}_i^0,\mathbf{0}} + 1\{W_{jm} \geq \hat{W}_j^{(1-q_M)}\}\widetilde{\mathbf{Z}}_{imt}^{\mathbf{P}_i^0,\mathbf{1}}\right]\boldsymbol{\theta}_i\end{aligned} \tag{31}$$

Based on this expression, we can implement a root-M consistent two-step estimator of $\boldsymbol{\theta}_i$ as described for the static game.

5.1.2 Estimator based on moment inequalities

This section is based on the inference methods for models with moment inequalities in Chernozhukov, Hong, and Tamer (CHT) (2007). We present an estimator of the set of identified parameters Θ_I, and an estimator of a confidence region for that set. The estimated confidence region applies both to the case when $\boldsymbol{\theta}^0$ is point-identified and when it is only set-identified.

Our model is defined in terms of the moment restrictions:

$$\widetilde{\mathbf{Z}}_{imt}^{\mathbf{P}^0,\mathbf{U}_j}\boldsymbol{\theta}_i^0 - \tilde{e}_{imt}^{\mathbf{P}^0,\mathbf{U}_j} \quad \leq \quad G_i^{-1}\left(P_i^0(\mathbf{X}_{mt})\right) \quad \leq \quad \widetilde{\mathbf{Z}}_{imt}^{\mathbf{P}^0,\mathbf{L}_j}\boldsymbol{\theta}_i^0 - \tilde{e}_{imt}^{\mathbf{P}^0,\mathbf{L}_j} \tag{32}$$

where, for notational simplicity, we have omitted the super-index for the level of rationality (R).

Our inference methods are based on the following population criterion function:

$$Q_0(\boldsymbol{\theta}) = \sum_{\mathbf{X}_{mt}\in\mathcal{X}} \sum_{i=1}^{2} \max\left\{ \widetilde{\mathbf{Z}}_{imt}^{\mathbf{P}^0,\mathbf{U}_j} \boldsymbol{\theta}_i - \tilde{e}_{imt}^{\mathbf{P}^0,\mathbf{U}_j} - G_i^{-1}\left(P_i^0(\mathbf{X}_{mt})\right) \ ; \ 0 \right\}^2 \Pr(\mathbf{X}_{mt})$$
$$+ \sum_{\mathbf{X}_{mt}\in\mathcal{X}} \sum_{i=1}^{2} \min\left\{ \widetilde{\mathbf{Z}}_{imt}^{\mathbf{P}^0,\mathbf{L}_j} \boldsymbol{\theta}_i - \tilde{e}_{imt}^{\mathbf{P}^0,\mathbf{L}_j} - G_i^{-1}\left(P_i^0(\mathbf{X}_{mt})\right) \ ; \ 0 \right\}^2 \Pr(\mathbf{X}_{mt}) \tag{33}$$

Given this criterion function, we can define the identified set Θ_I as:

$$\Theta_I = \{\boldsymbol{\theta} \in \Theta : Q_0(\boldsymbol{\theta}) = 0\} \tag{34}$$

To estimate this identified set, we consider the following set-estimator:

$$\hat{\Theta}_I = \left\{ \boldsymbol{\theta} \in \Theta : Q_M(\boldsymbol{\theta}) \leq \frac{b_M}{M} \right\} \tag{35}$$

b_M is such that $b_M \to \infty$ and $b_M/M \to 0$ as the sample size M goes to infinite. The function Q_M is the sample counterpart of the population criterion Q_0:

$$Q_M(\boldsymbol{\theta}) = \frac{1}{2MT} \sum_{m=1}^{M} \sum_{t=1}^{T} \sum_{i=1}^{2} \max\left\{ \widetilde{\mathbf{Z}}_{imt}^{\hat{\mathbf{P}}^0,\mathbf{U}_j} \boldsymbol{\theta}_i - \tilde{e}_{imt}^{\hat{\mathbf{P}}^0,\mathbf{U}_j} - G_i^{-1}\left(\hat{P}_i^0(\mathbf{X}_{mt})\right) \ ; \ 0 \right\}^2$$
$$+ \frac{1}{2MT} \sum_{m=1}^{M} \sum_{t=1}^{T} \sum_{i=1}^{2} \min\left\{ \widetilde{\mathbf{Z}}_{imt}^{\hat{\mathbf{P}}^0,\mathbf{L}_j} \boldsymbol{\theta}_i - \tilde{e}_{imt}^{\hat{\mathbf{P}}^0,\mathbf{L}_j} - G_i^{-1}\left(\hat{P}_i^0(\mathbf{X}_{mt})\right) \ ; \ 0 \right\}^2 \tag{36}$$

where $\hat{\mathbf{P}}^0$ is a nonparametric estimator of the population CCP function $\mathbf{P}^0$, e.g., a frequency estimator.

Under the assumptions of the model, it is possible to show that $\sup |Q_M(\boldsymbol{\theta}) - Q_0(\boldsymbol{\theta})| = O_p(1/\sqrt{M})$ and $Q_M(\boldsymbol{\theta}) = O_p(1/M)$ for every $\boldsymbol{\theta} \in \Theta_I$. Therefore, we can apply Theorem 3.1 in CHT to show that $\hat{\Theta}_I$ is a consistent estimator of Θ_I: i.e., $d_H(\hat{\Theta}_I, \Theta_I) = o_p(1)$, where $d_H(A, B)$ represents the Haussdorff distance between sets A and B.[10]

To compute the set estimator $\hat{\Theta}_I$ we use a linear programming algorithm. Consider first the case for level-1 rationality, and where the scalar b_M is zero. In this case, $\hat{\Theta}_I$ is the set of values $\boldsymbol{\theta}$ that satisfy the restriction $Q_M(\boldsymbol{\theta}) = 0$. This set is equivalent to the set of values $\boldsymbol{\theta}$ that solve the linear system of inequalities $\widetilde{\mathbf{Z}}_{imt}^{\hat{\mathbf{P}}^0,\mathbf{1}} \boldsymbol{\theta}_i - \tilde{e}_{imt}^{\hat{\mathbf{P}}^0,\mathbf{1}} - G_i^{-1}(\hat{P}_i^0(\mathbf{X}_{mt})) \leq 0$ and $G_i^{-1}(\hat{P}_i^0(\mathbf{X}_{mt})) - \widetilde{\mathbf{Z}}_{imt}^{\hat{\mathbf{P}}^0,\mathbf{0}} \boldsymbol{\theta}_i - \tilde{e}_{imt}^{\hat{\mathbf{P}}^0,\mathbf{0}} \leq 0$ for every observation (i, m, t) in the sample. We can represent this system of inequalities in a compact form as:

$$A\ \boldsymbol{\theta} + b \leq \mathbf{0} \tag{37}$$

[10] The Hausdorff distance is defined as $d_H(A, B) \equiv \max\ \sup_{a\in A} \inf_{b\in B} \|a - b\|\ ,\ \sup_{b\in B} \inf_{a\in A} \|a - b\|\ $, where $\|.\|$ is the Euclidean distance.

where A and b are a matrix and a vector, respectively, with a number rows equal to $4\min\{MT, |\mathcal{X}|\}$. To solve this system of inequalities we use a recent version of a well-known algorithm in linear programming called *relaxation method* or *perceptron algorithm* (see Agmon, 1954, Goffin, 1980, and Dunagan and Vempala, 2008). The relaxation method is an algorithm to obtain values of $\boldsymbol{\theta}$ that satisfy this system of linear restrictions. It was introduced by Augman (1954) and it is one of most commonly applied algorithms to solve this class of problems. Its main advantages are its simplicity, that each iteration is very fast, and that it always converges to a solution. Though in the worst-case scenario it is an exponential time algorithm (i.e., in the worst-case type of problems, CPU time increases exponentially with the number of inequalities in the system), it is well-known that this is over-pessimistic and that in an average-case scenario it is a polynomial time algorithm (Goffin, 1980). Furthermore, recent extensions on this algorithm, such as the randomized version proposed by Dunagan and Vempala (2008), are polynomial time in a worst-case scenario.

Our estimated set $\hat{\Theta}_I$ is based on the condition $Q_M(\boldsymbol{\theta}) \leq b_M/M$, with $b_M > 0$. This is equivalent to find the set of solutions of the linear system of inequalities $A\ \boldsymbol{\theta} + b - c_M\mathbf{1} \leq \mathbf{0}$, where $c_M > 0$ is a scalar constant, and $\mathbf{1}$ is a vector of ones. The scalar c_M should be such that the implicit b_M implied by c_M should satisfy the conditions $b_M \to \infty$ and $b_M/M \to 0$. These conditions hold if $c_M \to 0$ and $Mc_M \to \infty$.

For levels of rationality greater than $R = 1$, we have that probability bounds $\mathbf{U}_j$ and $\mathbf{L}_j$, and therefore $\widetilde{\mathbf{Z}}_{imt}^{\hat{\mathbf{P}}^0,\mathbf{U}}$, $\widetilde{\mathbf{Z}}_{imt}^{\hat{\mathbf{P}}^0,\mathbf{L}}$, $\tilde{e}_{imt}^{\hat{\mathbf{P}}^0,\mathbf{U}}$, and $\tilde{e}_{imt}^{\hat{\mathbf{P}}^0,\mathbf{L}}$, depend on structural parameters. This implies that the system of inequalities is no longer linear in the structural parameters. Linearity in $\boldsymbol{\theta}$ is very convenient for the estimation of $\hat{\Theta}_I$. To maintain linearity we implement the following recursive method. First, we start estimating $\hat{\Theta}_I$ under the assumption of level-1 rationality, i.e., $\hat{\Theta}_I^{(1)}$. Given this set we obtain the infimum of the probability bound $\mathbf{L}_j(\boldsymbol{\theta})$ within $\hat{\Theta}_I^{(1)}$, and the supremum of the probability bound $\mathbf{U}_j(\boldsymbol{\theta})$ within $\hat{\Theta}_I^{(1)}$:

$$\begin{aligned} \inf \mathbf{L}_j^{(1)} &= \inf_{\in\hat{\Theta}_I^{(1)}} \mathbf{L}_j^{(1)}(\boldsymbol{\theta}) \\ \sup \mathbf{U}_j^{(1)} &= \sup_{\in\hat{\Theta}_I^{(1)}} \mathbf{U}_j^{(1)}(\boldsymbol{\theta}) \end{aligned} \tag{38}$$

Then, we use $\inf \mathbf{L}_j^{(1)}$ and $\sup \mathbf{U}_j^{(1)}$ to calculate $\widetilde{\mathbf{Z}}_{imt}^{\hat{\mathbf{P}}^0,\mathbf{U}}$, $\widetilde{\mathbf{Z}}_{imt}^{\hat{\mathbf{P}}^0,\mathbf{L}}$, $\tilde{e}_{imt}^{\hat{\mathbf{P}}^0,\mathbf{U}}$, and $\tilde{e}_{imt}^{\hat{\mathbf{P}}^0,\mathbf{L}}$. These bounds are not as sharp as the true ones, $\mathbf{L}_j^{(1)}(\boldsymbol{\theta}^0)$ and $\mathbf{U}_j^{(1)}(\boldsymbol{\theta}^0)$, but they are much sharper than the bounds under level-1 rationality, i.e., $\mathbf{L}_j^{(0)} = \mathbf{0}$ and $\mathbf{U}_j^{(0)} = 1$. The main advantage of this approach is that we can compute the estimator $\hat{\Theta}_I^{(2)}$ using efficient but simple algorithms for the solution of linear systems of inequalities.

5.2 Tests of Equilibrium Beliefs

In this subsection, we present four different approaches to test for the null hypothesis of equilibrium beliefs. The first test is a standard Lagrange Multiplier (LM) or Score test based on the constrained maximum likelihood estimation (MLE) of structural parameters and beliefs. The second test is a Likelihood Ratio test that takes into account that the unconstrained estimator may only partially identify structural parameters and beliefs.

This test is standard but of very limited applicability given the computational problems to estimate by maximum likelihood dynamic games with multiple equilibria.

The second test is also an LM test but it is less standard because it is based on the (constrained) Nested Pseudo Likelihood (NPL) estimator proposed in Aguirregabiria and Mira (2007). This estimator also imposes the equilibrium restrictions but it is much simpler to compute than the constrained MLE. Finally, the third test is in the spirit of a likelihood ratio (LR) test but it takes into account that the unconstrained estimator may only partially identify structural parameters and beliefs.

LM test based on constrained MLE. Define the log-likelihood function:

$$l(\boldsymbol{\theta}, \mathbf{P}) \equiv \sum_{m=1}^{M} \sum_{t=1}^{T} \sum_{i=1}^{2} \log \Psi_i(\mathbf{X}_{mt}, \boldsymbol{\theta}, \mathbf{P}) \tag{39}$$

where $\Psi_i(\mathbf{X}_{mt}, \boldsymbol{\theta}, \mathbf{P})$ is the best response function $G_i(\widetilde{\mathbf{Z}}_{imt}^{\mathbf{P}_i,\mathbf{P}_j} \boldsymbol{\theta}_i - \tilde{e}_{imt}^{\mathbf{P}_i,\mathbf{P}_j})$. The constrained MLE is defined as a vector $(\hat{\boldsymbol{\theta}}_{MLE}, \hat{\mathbf{P}}_{MLE})$ such that:

$$\begin{array}{rcl} (\hat{\boldsymbol{\theta}}_{MLE}, \hat{\mathbf{P}}_{MLE}) & = & \arg\max\limits_{(\ ,\mathbf{P})} l(\boldsymbol{\theta}, \mathbf{P}) \\ & & \text{subject to:} \quad \mathbf{P} = \Psi(\boldsymbol{\theta}, \mathbf{P}) \end{array} \tag{40}$$

We want to test the null hypothesis $\mathbf{P} = \Psi(\boldsymbol{\theta}, \mathbf{P})$, that consists of $2|\mathcal{X}|$ constrains on $(\boldsymbol{\theta}, \mathbf{P})$. The standard LM statistic for testing this null hypothesis is:

$$LM_{(MLE)} = \frac{\partial l(\hat{\boldsymbol{\theta}}_{MLE}, \hat{\mathbf{P}}_{MLE})}{\partial (\boldsymbol{\theta}, \mathbf{P})'} \left[\frac{\partial^2 l(\hat{\boldsymbol{\theta}}_{MLE}, \hat{\mathbf{P}}_{MLE})}{\partial (\boldsymbol{\theta}, \mathbf{P}) \partial (\boldsymbol{\theta}, \mathbf{P})'} \right]^{-1} \frac{\partial l(\hat{\boldsymbol{\theta}}_{MLE}, \hat{\mathbf{P}}_{MLE})}{\partial (\boldsymbol{\theta}, \mathbf{P})} \tag{41}$$

Under the null hypothesis, this statistic is distributed as a chi-square with $2|\mathcal{X}|$ degrees of freedom.

LM test based on NPL estimator. The NPL estimator is a vector $(\hat{\boldsymbol{\theta}}_{NPL}, \hat{\mathbf{P}}_{NPL})$ that satisfies the conditions:

$$\begin{array}{rcl} \hat{\boldsymbol{\theta}}_{NPL} & = & \arg\max l(\boldsymbol{\theta}, \hat{\mathbf{P}}_{NPL}) \\ \text{and} & & \\ \hat{\mathbf{P}}_{NPL} & = & \Psi(\hat{\boldsymbol{\theta}}_{NPL}, \hat{\mathbf{P}}_{NPL}) \end{array} \tag{42}$$

In the case that there are multiple vectors of $(\boldsymbol{\theta}, \mathbf{P})$ satisfying these conditions, the NPL estimator is defined as the vector that provides the maximum value of the likelihood $l(\boldsymbol{\theta}, \mathbf{P})$. Under the assumption that players beliefs are in equilibrium, this estimator is consistent, asymptotically normal, but not asymptotically efficient. However, it is more efficient and it has better finite sample properties than two-step estimators that do not impose equilibrium restrictions in the sample. In contrast to the constrained MLE, the NPL estimator is such that the component of the score associated with $\boldsymbol{\theta}$ is zero by construction, i.e., $\partial l(\hat{\boldsymbol{\theta}}_{NPL}, \hat{\mathbf{P}}_{NPL})/\partial\boldsymbol{\theta} = 0$. In general, the other component of the score, $\partial l(\hat{\boldsymbol{\theta}}_{NPL}, \hat{\mathbf{P}}_{NPL})/\partial\mathbf{P}$, is not zero. As in MLE estimation, it is possible to show that, under the null hypothesis, the score vector $\partial l(\hat{\boldsymbol{\theta}}_{NPL}, \hat{\mathbf{P}}_{NPL})/\partial\mathbf{P}$ is asymptotically normal with variance $E[-\partial^2 l(\boldsymbol{\theta}^0, \mathbf{P}^0)/\partial\mathbf{P}\partial\mathbf{P}']$. Therefore, the LM statistic for this second test is:

$$LM_{(NPL)} = \frac{\partial l(\hat{\boldsymbol{\theta}}_{NPL}, \hat{\mathbf{P}}_{NPL})}{\partial\mathbf{P}'} \left[\frac{\partial^2 l(\hat{\boldsymbol{\theta}}_{NPL}, \hat{\mathbf{P}}_{NPL})}{\partial\mathbf{P}\partial\mathbf{P}'}\right]^{-1} \frac{\partial l(\hat{\boldsymbol{\theta}}_{NPL}, \hat{\mathbf{P}}_{NPL})}{\partial\mathbf{P}} \tag{43}$$

Under the null hypothesis, this statistic is distributed as a chi-square with $2|\mathcal{X}|$ degrees of freedom.

Likelihood Ratio test with partial identification of the unconstrained model. Consider first the case that we use ML estimation. The LR test is not affected by the set identification of the unrestricted model, i.e., the LR statistic is $2(l_U - l_R)$ where l_R and l_U are the maximum values of the log-likelihood function for the restricted and the unrestricted models, respectively. However, as mentioned above, ML estimation of the restricted model is very costly computationally. Furthermore, set estimation of the unrestricted model is also very challenging and it cannot be represented in terms of a system of linear inequalities. Therefore, we consider a modified LR test: $LR = 2(\hat{l}_U - l(\hat{\boldsymbol{\theta}}_{NPL}, \hat{\mathbf{P}}_{NPL}))$, where the unrestricted log-likelihood $\hat{l}_U$ is defined as

$$\hat{l}_U \equiv \max_{(\ ,\mathbf{P})} \left\{ l(\boldsymbol{\theta}, \mathbf{P}) \quad \text{subject to: } (\boldsymbol{\theta}, \mathbf{P}) \in \hat{\Theta}_I \times \hat{\Pi}_I \right\}$$

where $\hat{\Theta}_I$ are $\hat{\Pi}_I$ estimated sets of structural parameters and beliefs, respectively, based on the estimation procedure described in the previous sub-section.

6 Empirical Application

To illustrate the application of our model and method, we estimate a dynamic game of store location by McDonalds (MD) and Burger King (BK) using data for United Kingdom during the period 1991-1995. The main empirical question that we want to analyze in this application is whether the beliefs of MD and BK about the strategic behavior of other competitor are consistent with the actual behavior of the competitor.

The dataset comes from the paper Toivanen and Waterson (2005).[11] It is a panel of 422 local markets (districts) and five years with information on the stock of stores and the flow of new stores of MD and BK in each local market, as well as local market characteristics such as population, density, age distribution, average rent, income per capita, local retail taxes, and distance to the headquarters of the firm in UK.

Table 1 presents descriptive statistics on the evolution of the number of stores for the two firms. In 1990, MD had more than three times the number of stores of BK, and it was active in more than twice local markets than BK. Conditional on being active in a local market, MD had significantly more stores per market than BK. These differences between MD and BK have not declined significantly over the sample period 1991-1995. While BK have entered in more new local markets than MD (69 new markets for BK and 48 new markets for MD), MD has open more stores (143 new stores for BK and 166 new stores for MD).

Table 2 presents estimates of reduced form Probit models for the decision to open a new store. We obtain separate estimates for MD and BK. The set of explanatory variables includes the indicator of own presence in the market at previous year, the number of own stores and previous year, the indicator of the competitor presence in the market at previous year, the number of the competitor stores at previous year, and local market characteristics such as population, population between ages 15 and, population density, average rent, and distance to the own headquarters. To deal with (permanent) unobserved market heterogeneity, we include county-dummies[12] and a variable that captures the initial conditions of the firm in the local market at year 1990. This variable is the ratio between the number of stores the firm has in the local market (in 1990) and the population between ages 15 and 29. A high value of this variable represents a "good match" between the firm and the local market unobserved characteristics. For each explanatory variable we report the estimated parameter and the estimated marginal effect calculated at the mean values of all the explanatory variables. We do not report estimates for time dummies and county dummies but both are jointly significant. All the statistically significant effects have the expected signs. The own presence in the market and the own number of stores have significant negative effects on entry. One additional store reduces the probability of entry between 1 and 2 percentage points (the probability of entry at the average is 2.3% for BK and 5.5% for MD). Population between ages 15 and 29 has a strong and significant effect on entry: doubling the size of this population increases the probability of entry between 15 and 16 percentage points for both firms. The variable that captures initial conditional

[11] We would like to thank Otto Toivanen and Michael Waterson for generously sharing their data with us.

[12] In our sample, there are 422 districts and 62 counties.

or unobserved firm-market characteristics has also a significant and positive effect.

The most surprising and puzzling result in the probit estimates of Table 2 is that the effect of the competitor presence, or the competitor number of stores, is positive, though not statistically significant in most of the cases. This result is fully consistent with the ones reported by Toivanen and Waterson (2005). Despite that the products sold by the two firms are clearly substitutes, it seems "as if", the firms' entry decisions were ignoring these substitution or competitive effects. There are different possible interpretations of this puzzling finding. Perhaps, the simplest interpretation is that the estimated coefficients for these variables are upward biased due to permanent unobserved market heterogeneity. This seems to be a factor. Controlling for county fixed effects and for initial conditions reduced significantly the value of these estimates. However, if we include district fixed effects the competitive effect is still not significant in this reduced form Probit models. Toivanen and Waterson argue that this empirical evidence is consistent with firms' incomplete information about some market characteristics and with a model where firms can learn about these characteristics from the entry decisions of the opponent. Here we explore two other explanations. First, firms forward looking behavior may explain this apparent absence of competitive effects. Once we take into account forward looking behavior in a dynamic game of entry, the estimated competitive effects may appear statistically significant. The second hypothesis that we explore is that firms beliefs about the behavior of the opponent are not in equilibrium, i.e., do not represent the actual behavior of the competing firm.

The specification of the model is the one that we have presented above in Example 3. X_{imt} represents the number of installed stores of firm i in market m at the beginning of the year. The maximum value of X_{imt} in the sample is 13, and we consider that the set of possible values of X_{imt} is $\{0, 1, ..., 15\}$. Therefore, the state space $\mathcal{X}$ is $\{0, 1, ..., 15\} \times \{0, 1, ..., 15\}$ that has 256 grid points. Y_{imt} is the binary indicator of the event "firm i opens a new store in market m at year t". We consider that market characteristics are constant over time. The measure of market size S_m is total population in the district. For some specifications, we allow the cost of investment to depend on market characteristics such as average rent, retail taxes, population density, or average income. Therefore, each market has its own vector of players' CCPs. The dimension of the vectors $\mathbf{P}_i$ in this model is equal to $108,032$, i.e., 422 markets times 256 states $\mathbf{X}$.

Tables 3 and present estimates of the structural model both under the assumption that firms are myopic, $\beta = 0$, and under the assumption that firms are forward looking, $\beta = 0.95$. We report two different sets of point estimates: estimates using a simple two-step method Pseudo

Maximum Likelihood method where the estimator of (equilibrium) players' beliefs in the first step is a nonparametric frequency estimator; and estimates using the Nested Pseudo Likelihood (NPL) method proposed in Aguirregabiria and Mira (2007). The NPL method imposes the equilibrium restrictions in the sample (i.e., the estimated beliefs should be equal to the estimated best response probabilities), while the two-step method only satisfies the equilibrium restrictions asymptotically. The NPL estimator has smaller asymptotic variance and finite sample bias than the two-step method. The parameters that represent time discount factors have been estimated using a discrete-grid method. There are very substantial differences between two models, very particularly in the estimates of the parameters that capture cannibalization and competition effects. While these effects have the 'wrong' sign in the myopic model, the signs are the expected ones in the forward looking model. All the parameter estimates in the forward looking model have the expected signs and have reasonable magnitudes. Therefore, it seems that forward looking behavior explains part of the puzzle in the reduced form estimates. [MORE ON THIS]

Table 5 presents our estimates of parameter confidence intervals using the estimator described in section 5. For the sake of comparison, we also include in this table a column with the NPL estimates.

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APPENDIX A: PROOFS OF PROPOSITIONS

TBW

APPENDIX B.

Define the value vector $\mathbf{W}_{\mathbf{Z}i}^{\mathbf{P},\mathbf{B}}(\mathbf{X}_t) \equiv (1-P_i(\mathbf{X}_t))\mathbf{Z}_{it}^{\mathbf{P},\mathbf{B}}(0) + P_i(\mathbf{X}_t)\mathbf{Z}_{it}^{\mathbf{P},\mathbf{B}}(1)$, and the scalar value $W_{ei}^{\mathbf{P},\mathbf{B}}(\mathbf{X}_t) \equiv (1-P_i(\mathbf{X}_t))e_{it}^{\mathbf{P},\mathbf{B}}(0) + P_i(\mathbf{X}_t)e_{it}^{\mathbf{P},\mathbf{B}}(1)$. By definition of the present values $\mathbf{Z}_{it}^{\mathbf{P},\mathbf{B}}(Y_{it})$ and $\mathbf{W}_{\mathbf{Z}i}^{\mathbf{P},\mathbf{B}}(\mathbf{X}_t)$, it is simple to show that:

$$\mathbf{Z}_{it}^{\mathbf{P},\mathbf{B}}(Y_{it}) = (1-B_i(\mathbf{X}_t))\ \mathbf{Z}_{it}(Y_{it},0) + B_i(\mathbf{X}_t)\ \mathbf{Z}_{it}(Y_{it},1) + \beta \sum_{\mathbf{X}_{t+1}} f_i^{\mathbf{B}}(\mathbf{X}_{t+1}|Y_{it},\mathbf{X}_t)\ \mathbf{W}_{\mathbf{Z}i}^{\mathbf{P},\mathbf{B}}(\mathbf{X}_{t+1}) \tag{B.1}$$

Similarly,

$$e_{it}^{\mathbf{P},\mathbf{B}}(Y_{it}) = \beta \sum_{\mathbf{X}_{t+1}} f_i^{\mathbf{B}}(\mathbf{X}_{t+1}|Y_{it},\mathbf{X}_t)\ W_{ei}^{\mathbf{P},\mathbf{B}}(\mathbf{X}_{t+1}) \tag{B.2}$$

Define the matrix of values $\mathbf{W}_{\mathbf{Z}i}^{\mathbf{P},\mathbf{B}} \equiv \{\mathbf{W}_{\mathbf{Z}i}^{\mathbf{P},\mathbf{B}}(\mathbf{X}) : \mathbf{X} \in \mathcal{X}\}$ and the vector of values $\mathbf{W}_{\mathbf{e}i}^{\mathbf{P},\mathbf{B}} \equiv \{W_{ei}^{\mathbf{P},\mathbf{B}}(\mathbf{X}) : \mathbf{X} \in \mathcal{X}\}$. By definition, the matrix $\mathbf{W}_{\mathbf{Z}i}^{\mathbf{P},\mathbf{B}}$ is the solution to the following systems of linear equations with dimension $|\mathcal{X}|$:

$$\begin{aligned}\mathbf{W}_{\mathbf{Z}i}^{\mathbf{P},\mathbf{B}} &= \beta \mathbf{F}_{\mathbf{X}}^{\mathbf{P},\mathbf{B}}\ \mathbf{W}_{\mathbf{Z}i}^{\mathbf{P},\mathbf{B}} + (1-\mathbf{P}_i) * (1-\mathbf{B}_i) * \mathbf{z}_i(0,0) + (1-\mathbf{P}_i) * \mathbf{B}_i * \mathbf{z}_i(0,1) \\ &+ \mathbf{P}_i * (1-\mathbf{B}_i) * \mathbf{z}_i(1,0) + \mathbf{P}_i * \mathbf{B}_i * \mathbf{z}_i(1,1)\end{aligned} \tag{B.3}$$

where $\mathbf{P}_i$ is a $|\mathcal{X}| \times 1$ vector with the stacked CCPs of player i for every possible value of $\mathbf{X}_t$; $\mathbf{z}_i(Y_i, Y_j)$ is a matrix with $|\mathcal{X}|$ rows and the same number of columns as $z_{it}(Y_i, Y_j)$ such that a row of $\mathbf{z}_i(Y_i, Y_j)$ is equal to the vector $z_{it}(Y_i, Y_j)$ associated with a given value of $\mathbf{X}_t$; $*$ represents the Hadamard or element-by-element product; and $\mathbf{F}_{\mathbf{X}}^{\mathbf{P},\mathbf{B}}$ is the transition matrix of $\{\mathbf{X}_t\}$ induced by the vector of CCPs $\mathbf{P}_i$ and $\mathbf{B}_i$ such that the elements of this matrix are $(1-P_i(\mathbf{X}_t))f_i^{\mathbf{B}}(\mathbf{X}_{t+1}|0,\mathbf{X}_t) + P_i(\mathbf{X}_t))f_i^{\mathbf{B}}(\mathbf{X}_{t+1}|1,\mathbf{X}_t)$. It is clear that this system of equations has the following closed-form analytical expression: $\mathbf{W}_{\mathbf{Z}i}^{\mathbf{P},\mathbf{B}} = (\mathbf{I} - \beta \mathbf{F}_{\mathbf{X}}^{\mathbf{P},\mathbf{B}})^{-1}\ [(1-\mathbf{P}_i) * (1-\mathbf{B}_i) * \mathbf{z}_i(0,0)\ + (1-\mathbf{P}_i) * \mathbf{B}_i * \mathbf{z}_i(0,1)$ $+ \mathbf{P}_i * (1-\mathbf{B}_i) * \mathbf{z}_i(1,0)\ + \mathbf{P}_i * \mathbf{B}_i * \mathbf{z}_i(1,1)]$. Similarly, the vector $\mathbf{W}_{ei}^{\mathbf{P},\mathbf{B}}$ is the solution to the following systems of linear equations with dimension $|\mathcal{X}|$:

$$\mathbf{W}_{ei}^{\mathbf{P},\mathbf{B}} = \beta \mathbf{F}_{\mathbf{X}}^{\mathbf{P},\mathbf{B}}\ \mathbf{W}_{ei}^{\mathbf{P},\mathbf{B}} + \mathbf{e}_i^{\mathbf{P}} \tag{B.4}$$

where $\mathbf{e}_i^{\mathbf{P}}$ is a vector that contains the expected values $E(\varepsilon_{it} Y_{it}|\mathbf{X}_t, Y_{it}$ is optimal) for every value of $\mathbf{X}_t$. These conditional expectations only depend on the probability distribution of ε_{it} and on the choice probability $P_i(\mathbf{X}_t)$. For the logit and probit models we have the following closed expressions. When ε_{it} is extreme value distributed (logit): $E(\varepsilon_{it} Y_{it}|\mathbf{X}_t, Y_{it}$ optimal$) = Euler -$

$(1-P_i(\mathbf{X}_t))\ln(1-P_i(\mathbf{X}_t))$ $-P_i(\mathbf{X}_t)\ln(P_i(\mathbf{X}_t))$, where *Euler* represents Euler's constant. When ε_{it} has a standard normal distribution (probit): $E(\varepsilon_{it}Y_{it}|\mathbf{X}_t, Y_{it}\text{ optimal}) = \phi(\Phi^{-1}(P_i(\mathbf{X}_t)))$, where $\phi(.)$ and $\Phi^{-1}(.)$ are the PDF and the inverse-CDF of the standard normal.

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Table 1
Descriptive Statistics for the Evolution of the Number of Stores
Data: 422 markets, 2 firms, 5 years = 4,220 observations

	Burger King						McDonalds					
	1990	1991	1992	1993	1994	1995	1990	1991	1992	1993	1994	1995
# Markets with stores	71	98	104	118	131	150	206	213	220	237	248	254
Change in # markets with stores		17	6	14	13	19		7	7	17	11	6
# of stores	79	115	128	153	181	222	281	316	344	382	421	447
Change in # of stores		36	13	25	28	41		35	28	38	39	26
Mean # stores per market	1.11	1.17	1.23	1.30	1.38	1.48	1.36	1.49	1.56	1.61	1.70	1.76

Table 2
Reduced Form Probits for Decision to Open a Store
Data: 422 markets, 5 years = 2,110 observations per firm

	Burger King		**McDonalds**	
Explanatory Variable	Estimate (s.e.)	Marg. Effect (s.e.)	Estimate (s.e.)	Marg. Effect (s.e.)
1{own stores[t-1] >0}	-1.106 (0.249)*	-0.042 (0.010)*	-0.499 (0.254)*	-0.059 (0.032)*
# own stores[t-1]	-0.197 (0.101)*	-0.011 (0.005)*	-0.178 (0.062)*	-0.019 (0.006)*
1{other stores[t-1] >0}	0.317 (0.190)	0.017 (0.010)	0.460 (0.291)	0.061 (0.040)
# other stores[t-1]	0.091 (0.062)	0.005 (0.004)	-0.188 (0.099)*	-0.021 (0.011)*
log Population	-1.695 (0.815)*	-0.092 (0.046)*	-0.149 (0.673)	-0.016 (0.075)
log Population 15-29	2.962 (0.795)*	0.162 (0.047)*	1.310 (0.663)*	0.146 (0.074)*
log Population Density	0.203 (0.078)*	0.011 (0.004)*	0.108 (0.069)	0.012 (0.008)
log Distance Headquarters	0.238 (0.293)	0.013 (0.016)	-0.187 (0.345)	-0.021 (0.038)
(#own stores/Pop) in 1990	0.295 (0.071)*	0.016 (0.004)*	0.118 (0.049)*	0.013 (0.005)*
Prob. entry at mean x	0.023		0.055	
Time dummies (4)	YES		YES	
County dummies (61)	YES		YES	
log likelihood	-332.22		-455.23	
Pseudo R-square	0.270		0.150	

Table 3
Myopic Game of Entry for McDonalds and Burger King
Under the Assumption that Players' Beliefs are in Equilibrium
Data: 422 markets, 2 firms, 5 years = 4,220 observations

	$\beta = 0.00$ (not estimated)			
	Two Step Estimates		**NPL Estimates**	
	Burger King	**McDonalds**	**Burger King**	**McDonalds**
Variable Profits:				
θ_0^{VP}	4.904 (1.070)*	7.909 (2.289)*	4.864 (1.081)*	7.898 (2.287)*
θ_1^{VP} cannibalization	2.005 (0.869)*	3.510 (0.659)*	2.035 (0.831)*	3.466 (0.647)*
θ_2^{VP} competition	0.014 (0.046)	0.032 (0.051)	0.016 (0.044)	0.037 (0.053)
Fixed Costs:				
θ_0^{FC} fixed	0.378 (0.212)*	0.806 (0.248)*	0.374 (0.212)*	0.808 (0.247)*
θ_1^{FC} linear	3.099 (0.436)*	2.662 (0.405)*	3.103 (0.436)*	2.659 (0.405)*
θ_2^{FC} quadratic	-0.054 (0.064)	0.085 (0.041)	-0.052 (0.063)	0.087 (0.041)
Pseudo R-square	0.154		0.154	
Log-Likelihood	-895.5		-895.4	
Distance $\|\|P^K - P^{K-}\|\|$			0.00	
# NPL iterations	1		5	

Table 4
Dynamic Game of Entry for McDonalds and Burger King
Under the Assumption that Players' Beliefs are in Equilibrium
Data: 422 markets, 2 firms, 5 years = 4,220 observations

	$\beta = 0.95$ (not estimated)			
	Two Step Estimates		**NPL Estimates**	
	Burger King	**McDonalds**	**Burger King**	**McDonalds**
Variable Profits:				
θ_0^{VP}	0.5849 (0.1077)*	0.8303 (0.2968)*	1.098 (0.2169)*	0.9737 (0.3091)*
θ_1^{VP} cannibalization	-0.2096 (0.0552)*	-0.0024 (0.0392)	-0.0765 (0.0725)	0.2874 (0.0986)*
θ_2^{VP} competition	-0.0110 (0.0029)*	0.0008 (0.0027)	-0.0129 (0.0065)*	-0.0074 (0.0073)
Fixed Costs:				
θ_0^{FC} fixed	0.0784 (0.0213)*	0.0822 (0.0332)*	0.0788 (0.0307)*	0.0773 (0.0261)*
θ_1^{FC} linear	0.0790 (0.0420)*	0.1076 (0.0400)*	0.1509 (0.0282)*	0.1302 (0.0185)*
θ_2^{FC} quadratic	-0.0078 (0.0059)	-0.0034 (0.0023)	-0.0054 (0.0026)*	0.0001 (0.016)
Pseudo R-square	0.323		0.146	
Log-Likelihood	-655.7		-893.4	
Distance $\|\|P^K - P^{K-}\|\|$	4831.26		0.00	
# NPL iterations	1		31	

Table 5
Dynamic Game of Entry for McDonalds and Burger King
Under Assumption of Level-2 Rational Players
Data: 422 markets, 2 firms, 5 years = 4,220 observations

	$\beta = 0.95$ (not estimated)			
	95% Confidence Intervals		**NPL Estimates**	
	Burger King	**McDonalds**	**Burger King**	**McDonalds**
Variable Profits:				
θ_0^{VP}	[0.3228 , 0.7980]	[0.7296 , 1.0274]	1.098 (OUT)	0.9737 (IN)
θ_1^{VP} cannibalization	[-0.2381 , -0.0688]	[0.0648 , 0.2360]	-0.0765 (IN)	0.2874 (OUT)
θ_2^{VP} competition	[-0.0276 , 0.0070]	[-0.0115 , 0.0062]	-0.0129 (IN)	-0.0074 (IN)
Fixed Costs:				
θ_0^{FC} fixed	[0.0723 , 0.0846]	[0.0698 , 0.0799]	0.0788 (IN)	0.0773 (IN)
θ_1^{FC} linear	[0.0763 , 0.1271]	[0.1009 , 0.1461]	0.1509 (OUT)	0.1302 (IN)
θ_2^{FC} quadratic	[-0.0037 , 0.0002]	[-0.0010 , 0.0007]	-0.0054 (IN)	0.0001 (IN)