WELFARE LOSSES UNDER COURNOT COMPETITION*

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Abstract

We find that when the good is homogeneous, firms are identical, compete in quantities and produce under constant returns, the percentage of welfare losses (PWL) is small with as few as five competitors for a class of demand functions which includes linear and isoelastic cases. However, PWL under fixed costs and asymmetric firms can come up to large numbers. We provide exact formulae of PWL and robust constructions of markets were PWL is close to one in these two cases. We show that the market structure that maximizes PWL is either monopoly or dominant firm, depending on demand. Finally we prove that PWL are minimized when all firms are identical, a clear indication that the assumption of identical firms biases downwards the estimation of PWL.

*This paper was presented in the universitis of Murcia and Pablo de Olavide, Sevilla. I am grateful to the audiences there and to Carmen Beviá, Miguel González-Maestre, Javier López-Cuñat and Felix Marcos for helpful comments and to the Spanish Ministry of Education for financial support under grant BEC2002-02194.
1. Introduction

In his classical contribution Cournot (1837, Chapter 8) established that when the number of firms in a market tends to infinity, oligopolistic equilibrium tends to perfect competition. As a corollary, Welfare Losses (WL), measured as the difference between social welfare in the optimal and the equilibrium allocation, tend to zero. But, what happens when the number of firms is finite? Is perfect competition a good approximation or, on the contrary WL are significant? (see Hotelling (1938) and Yarrow (1985) for an early treatment of this problem).

As a first cut to the problem, assume that all firms are identical and costs and demand are linear. It is easily calculated that the percentage of WL under Cournot competition, denoted by $PW_L$, is $1/(1+n)^2$ where $n$ is the number of firms. Thus a market composed by 7 identical firms ("the seven sisters") produces a $PW_L$ of 1.56%, not a big number.\(^1\) This poses a serious question: were WL systematically small there would be little to be gained by considering oligopolistic behavior: A simple equilibrium concept like perfect competition may be preferable. Moreover, the motivation for public policies is weakened under small WL. Then, the dilemma is, either we find environments in which oligopoly produces WL much greater than those found in the linear model or, we abandon the oligopoly model as a leading model describing markets.

Let us first comment on papers that relevant to our problem. Hardy (2000) studies a model with quadratic demand and presents numerical calculations. He finds that WL can be up 30\% larger than those in the linear model, which is encouraging but still does not solve the problem. Anderson and Renault (2003) calculate $PW_L$ under the assumptions made above except that they assume an inverse demand function of the form $p = A - bx^n$, ($x$ is aggregate output and $p$ market price).\(^2\) They do not study if these $PW_L$ differ substantially from those in the linear model. Johari and Tsitsiklis (2005) show that if firms are identical, average costs are not increasing and the inverse

\(^1\)This formula shows that once linearity is assumed, as done implicitly by Harberger (1954), WL seldom comes up to big numbers except if the number of firms is very small. A list of other empirical papers measuring WL in oligopoly can be found in Tullock (2003) p. 2.

\(^2\)This form of demand generalizes both linear ($\alpha = 1$) and isoelastic (with elasticity of demand $1/\alpha$) forms and allows for computation of equilibria. See González-Maestre (2000) for an early application.
demand function is concave, $PWL$ is bounded above by $1/(2n + 1)$, which still not very large because a market with seven firms achieves more than 93% of maximum welfare.

Our paper is a quest for markets where oligopoly produces large WL and, thus, it is a relevant model of market competition. More specifically, the purpose of our paper is twofold.

1: To provide workable formulae for $PWL$ which depend, as far as possible, on magnitudes that are observable.\(^3\) We regard these formulae as the main contribution of the paper from the point of view of applied economics because they show which variables to look at when dealing with WL in an actual market.

2: To use these formulae to construct markets where the Cournot equilibria yields large $PWL$, sometimes close to one.\(^4\) These constructions are the main contribution of the paper from the theoretical point of view because they show that oligopoly theory is valid as a general description of markets with the perfectly competitive case as a limit.

In Section 2 we consider the baseline model, which is that of Anderson and Renault. We might expect that for suitable values of $\alpha$, WL were much higher than those in the linear case. However, by using numerical methods we find that the maximum $PWL$ obtained in this case is not very different from the one obtained in the linear case. Moreover, for some values of $\alpha$, $PWL$ are arbitrarily small. Thus, the consideration of a more general class of demand functions does not bring significant WL associated with oligopoly, but on the contrary it adds to the suspicion that WL may be small. We then turn our attention to fixed costs and heterogeneous firms.\(^5\).

In Section 3 we consider free entry with a fixed cost. We provide formulae for the maximal and the minimal $PWL$ where this magnitude depends on the number of firms and $\alpha$. We show that $PWL$ could be arbitrarily close to one for any exogenously given observation about market price, output and number of firms (Proposition 1). Here WL are due to the combination of the form of the demand function -we show that $PWL$

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\(^3\)The parameter $\alpha$, which can be estimated but not observed, enters in the formula of $PWL$ calculated by Anderson-Renault (2003), so it is unavoidable in the more general set ups considered in this paper.

\(^4\)Johari and Tsitsiklis (2005) offer an example of a market where $PWL$ is arbitrarily close to one but in which the inverse demand function is not differentiable.

\(^5\)Other attempts to find higher WL focus on issues outside market competition like "X-Inefficiency" (Leibenstein (1966) and Rent-Seeking (Tullock (1967)).
with linear demand are large but far from zero- and overentry -because the optimal number of firms is one.

In Section 4 we consider heterogeneous firms. We provide a formula for \( \text{PWL} \) where this magnitude depends (positively) on the share of the largest firm, (negatively) on the Hirschman-Herfindahl concentration index, denoted by \( H \), and on \( \alpha \). We find that there are combination of market shares, number of firms and \( \alpha \) that yield \( \text{PWL} \) close to one whereas \( H \) is close to zero (Proposition 2). We check the robustness of this construction by considering the effects on \( \text{PWL} \) when one of the above magnitudes is held fixed. In all cases, \( \text{PWL} \) are large -not necessarily close to one- and negatively correlated with \( H \).\(^6\) This points out that \( H \) is not a reliable measure of WL.\(^7\) Next, we prove that the market structure that maximizes \( \text{PWL} \) is a dominant firm when \( \alpha > 0 \) and monopoly for \( \alpha < 0 \) (Proposition 3). Finally we prove that \( \text{PWL} \) are minimized when firms are identical (Proposition 4). This shows that proper care of the heterogeneity of firms is essential to obtain estimates of \( \text{PWL} \) that are not biased towards small \( \text{PWL} \).

Finally, in Section 5 we offer some thoughts about our results. Our main conclusion is twofold. On the one hand, the search for WL should focus on economies of scale and asymmetric firms, two facts that are seldom considered in the applied literature. On the other hand the oligopoly model still alive and well as a leading model in the study of markets. Moreover, in some cases we turn tables instead: the classical vision of markets as places where surplus is created may be too optimistic and markets may create little or no surplus at all, at least in several relevant cases.

It goes without saying that important causes of WL are not considered here, i.e. product differentiation, investment, R&D, quality, location, etc. The analysis of the impact of these variables on WL requires the consideration of games that are more complicated than those considered here and, consequently, they are left for future research.

\(^6\)In the linear case considered above, \( H \) is positively related to \( \text{PWL} \). However, for \( n = 5 \), \( \text{PWL} = 2.7\% \), still pretty small but \( H = .2 \), a value that is considered a threshold for some anti-trust authorities.

\(^7\)That social welfare is increasing in the marginal cost of small firms was first pointed out by Lahiri and Ono (1988). For a criticism of the idea that concentration is generally bad for social welfare see Daughety (1990) and Farrell and Shapiro (1990).
2. The Baseline Model

There is a representative consumer with a utility function \( U = Ax^{\frac{b\alpha+1}{\alpha+1}} - px \) where \( x \) is aggregate output, \( p \) is the market price, \( A \geq 0, b\alpha > 0 \) and \( \alpha > -1 \). The maximization of utility generates an inverse demand function \( p = A - bx^\alpha \). Notice that if \( \alpha < 0, b < 0 \), and \( A = 0 \) we have an isoelastic function \( p = -bx^\alpha \). The linear case occurs if \( \alpha = 1 \).

There are \( n \) identical …rms each producing a single output denoted by \( x_i \). Thus \( x = \sum_{i=1}^{n} x_i \). Marginal cost is constant and denoted by \( c \). Profits for firm \( i \) are \( \pi_i \equiv (p-c)x_i \). Defining \( a \equiv A - c \) we have that \( \pi_i \equiv (a - bx^\alpha)x_i \). Assume \( ab > 0 \). If firms compete à la Cournot, first order condition of profit maximization yields \( a - bx^\alpha - b\alpha x^{\alpha-1}x_i = 0 \). It is easy to check that the second order condition holds and that equilibrium is symmetric. Thus Cournot equilibrium output and market price are

\[
x^* = \left( \frac{an}{b(n+\alpha)} \right)^{\frac{1}{\alpha}} \quad \text{and} \quad p^* = \frac{A\alpha + cn}{n + \alpha}.
\]

(2.1)

Social welfare, denoted by \( W \), is the sum of industry profits and the utility of the representative consumer, i.e. \( W = ax - \frac{bx^{\alpha+1}}{\alpha+1} \). The optimal aggregate output is found by maximizing \( W \), namely

\[
x^o = \left( \frac{a}{b} \right)^{\frac{1}{\alpha}}.
\]

(2.2)

Social welfare in equilibrium and in the optimal allocation, are, respectively

\[
W^* = \frac{a^{\frac{\alpha+1}{\alpha}} n^{\frac{1}{\alpha}} \alpha(n + \alpha + 1)}{b^{\frac{\alpha}{\alpha}} (n + \alpha)\left(\frac{1}{\alpha}(n + \alpha)(\alpha + 1)\right)} \quad \text{and} \quad W^o = \frac{a^{\frac{\alpha+1}{\alpha}} \alpha}{b^{\frac{1}{\alpha}} (\alpha + 1)}.
\]

(2.3)

From (2.3), the percentage of WL (Anderson-Renault ([2003], p. 262), denoted by \( PWL \) is

\[
PWL \equiv \frac{W^o - W^*}{W^o} = 1 - \frac{n^{\frac{1}{\alpha}} (n + \alpha + 1)}{(n + \alpha)\left(\frac{1}{\alpha}(n + \alpha)(\alpha + 1)\right)} \equiv L(\alpha, n).
\]

(2.4)

The following properties of \( L(\cdot, \cdot) \) are easily established:

i) \( \lim_{n \to \infty} L(\alpha, n) = 0 \).

ii) \( \lim_{\alpha \to -1} L(\alpha, n) = 0 \).

iii) \( \lim_{\alpha \to \infty} L(\alpha, n) = 0 \).

iv) \( L(\alpha, \cdot) \) decreases with \( n \).
v) $L(\cdot, n)$ is quasiconcave in $\alpha$.

i) is the usual property of large economies, as noticed in the Introduction. The explanation of ii) is that when $\alpha \to -1$, the market produces in the limit an infinite amount of surplus, so the loss caused by oligopoly tends to zero. iii) is caused by the fact that when $\alpha \to \infty$, inverse demand is flat so firms can not influence price and optimal and equilibrium output are identical. ii) and/or iii) imply that there are markets where, for a given $n$, $PWL$ is as small as we wish, something impossible in the case of quadratic utility functions. iv) shows that, when there are no technological issues at stake, the more competition, the better. Finally v) follows from the fact that Anderson and Renault (2003) proved that $W^o/W^*$ is quasi-concave on $\alpha$. So $W^*/W^o$ is quasi-convex and $-W^*/W^o$ is quasi-concave, so it is $1 - W^*/W^o$.

We now study $PWL$ as a function of $\alpha$, see Figure 1. Notice that v) guarantees that the local maximum found there is a global maximum.

![Figure 1: $PWL$ for $n = 1$ (black), 2 (red), 3 (light red), 4 (green) and 5 (brown).](image)

Table 1 below shows the maximum $PWL$, denoted by $PWL$, and the corresponding values in the linear model, denoted by $PWLL$, for selected values of $n$. Notice that iv) above guarantees that for $n$ larger than 10, $PWL$ will be smaller than 2.2%.
TABLE 1

<table>
<thead>
<tr>
<th>n</th>
<th>PWL</th>
<th>PWLL</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>.27</td>
<td>.25</td>
</tr>
<tr>
<td>2</td>
<td>.18</td>
<td>.11</td>
</tr>
<tr>
<td>3</td>
<td>.076</td>
<td>.0625</td>
</tr>
<tr>
<td>4</td>
<td>.058</td>
<td>.04</td>
</tr>
<tr>
<td>5</td>
<td>.044</td>
<td>.027</td>
</tr>
<tr>
<td>6</td>
<td>.0357</td>
<td>.02</td>
</tr>
<tr>
<td>7</td>
<td>.032</td>
<td>.0156</td>
</tr>
<tr>
<td>8</td>
<td>.027</td>
<td>.012</td>
</tr>
<tr>
<td>9</td>
<td>.024</td>
<td>.01</td>
</tr>
<tr>
<td>10</td>
<td>.022</td>
<td>.008</td>
</tr>
</tbody>
</table>

Notice that the relative difference between PWL and PWLL increases with n (from 5.2% for n = 1 to 250% for n = 10). However this effect occurs for small values of PWLL and is not strong enough to obtain significant WL in the cases in which the linear model yields small WL. Given this and that PWL can be much smaller than PWLL, we have to conclude that the consideration of a more general class of utility functions alone is not helpful to finding significant WL in Cournot competition.

3. Fixed Costs and Free Entry

In this section we assume that in order to produce, firms must incur in a fixed cost, denoted by k, and that there is an infinite number of potential firms. The number of active firms in equilibrium is denoted by n. Given n, output is determined as in the previous section. We assume that the decision of entry is prior to the decision on output.\(^8\) Thus, equilibrium under free entry implies that if n firms are in the market, firm n has non negative profits but firm (n + 1) has non positive profits, formally

\[
\frac{aa^{\frac{1+\alpha}{\alpha}}n^{\frac{1-\alpha}{\alpha}}}{b^{\frac{1}{\alpha}}(n+\alpha)^{\frac{1+\alpha}{\alpha}}} \geq k \geq \frac{aa^{\frac{1+\alpha}{\alpha}}(n+1)^{\frac{1-\alpha}{\alpha}}}{b^{\frac{1}{\alpha}}(n+\alpha+1)^{\frac{1+\alpha}{\alpha}}}.
\] (3.1)

Welfare in a Cournot equilibrium with free entry is

\[W^* = \frac{a^{\frac{\alpha+1}{\alpha}}n^{\frac{1}{\alpha}}\alpha(n+\alpha+1)}{b^{\frac{1}{\alpha}}(n+\alpha)^{\frac{1}{\alpha}}(n+\alpha)(\alpha+1)} - nk.
\] (3.2)

In an optimal allocation, aggregate output equals the one in (2.2) and the number of active firms is one because the existence of economies of scale. Thus, social welfare in

\(^7\)Lopez-Cuñat (1999) has shown that, under conditions that are met here, the equilibrium considered in this paper is a subset of equilibrium when both decisions are simultaneous. For the latter, see Novshek [1980] and Ushio [1983]).
the optimal allocation is
\[ W^* = \frac{a^{\frac{\alpha+1}{\alpha}}}{b^\frac{\alpha}{\alpha}} - k. \]  
(3.3)

Thus, the percentage of WL can be written as
\[ PWL = \frac{\frac{a^{\frac{\alpha+1}{\alpha}}}{b^\frac{\alpha}{\alpha}} - \frac{a^{\frac{\alpha+1}{\alpha}}}{b^\frac{\alpha}{\alpha}} n^{\frac{\alpha}{\alpha}} (n+\alpha+1)}{\frac{a^{\frac{\alpha+1}{\alpha}}}{b^\frac{\alpha}{\alpha}} - k} + (n - 1)k. \]  
(3.4)

In order to have a workable formula, in which \( PWL \) depends on variables that are observable, we substitute \( k \) for the upper and lower bounds for this variable in (3.1). It is clear that \( PWL \) is increasing on \( k \). Thus, the maximum value of \( PWL \), denoted by \( MA(\alpha, n) \), occurs for the maximum value of \( k \), namely
\[ MA(\alpha, n) = \frac{(n + \alpha)^{\frac{1+\alpha}{\alpha}} - n^{\frac{1}{\alpha}} (n + \alpha + 1) + (n - 1)n^{\frac{1-\alpha}{\alpha}} (\alpha + 1)}{(n + \alpha)^{\frac{1+\alpha}{\alpha}} - n^{\frac{1-\alpha}{\alpha}} (\alpha + 1)}. \]  
(3.5)

The minimum value of \( PWL \), denoted by \( MI(\alpha, n) \), occurs for the minimum value of \( k \), namely
\[ MI(\alpha, n) = \frac{(n + \alpha + 1)^{\frac{1+\alpha}{\alpha}} - n^{\frac{1}{\alpha}} (n + \alpha + 1 + 1)^{\frac{1+2\alpha}{\alpha}} + (n - 1)(n + 1)^{\frac{1-\alpha}{\alpha}} (\alpha + 1)}{(n + \alpha + 1)^{\frac{1+\alpha}{\alpha}} - (n + 1)^{\frac{1-\alpha}{\alpha}} (\alpha + 1)}. \]  
(3.6)

A first approach to the understanding of (3.5) and (3.6) is to picture \( MA(\cdot, n) \) and \( MI(\cdot, n) \) for selected values of \( n \). This is done in Figures 2 and 3 below.
FIGURE 2: $MA(\cdot, 1)$ and $MI(\cdot, 1)$ (black) and $MA(\cdot, 10)$ and $MI(\cdot, 10)$ (red).

FIGURE 3: $MA(\cdot, 2)$ and $MI(\cdot, 2)$ (black) and $MA(\cdot, 20)$ and $MI(\cdot, 20)$ (green).
We now state the properties of $MA(\cdot, \cdot)$ and $MI(\cdot, \cdot)$ that correspond to i)-iv) in the previous section.

\begin{itemize}
    \item[i')] $\lim_{n \to \infty} MI(\alpha, n) = \lim_{n \to \infty} MA(\alpha, n) = 0$.
    \item[ii')] $\lim_{\alpha \to 1} MI(\alpha, n) = \lim_{\alpha \to 1} MA(\alpha, n) = 0$.
    \item[iii')] $\lim_{\alpha \to \infty} MI(\alpha, n) = \frac{n-1}{n}, \lim_{\alpha \to \infty} MA(\alpha, n) = 1$.
    \item[iv')] Neither $MI(\alpha, \cdot)$ nor $MA(\alpha, \cdot)$ are monotonic on $n$.
\end{itemize}

i') implies that $\lim_{k \to 0} PWL = 0$, since (3.1) implies that when $k \to 0$, $n \to \infty$. Variations of this result have been obtained by Dasgupta and Ushio (1981), Fraysse and Moreaux (1981) and Guesnerie and Hart (1985). i') and ii') are identical to i) and ii) in the previous section. However iii') is very different from iii) because it says that markets with very large $\alpha$'s could be very inefficient. For large values of $\alpha$, it is striking the contrast between monopoly and markets with a large number of firms: In the former it is possible to construct examples where $PWL$ is arbitrarily small and in the latter such examples are not possible. This is due to the fact that when $n$ is very large, there are large WL due to the discrepancy between $n$ and the optimal number of firms, namely one. Finally iv') is proved in Figure 1. The reason for this -apparently paradoxical- result is that $k$ changes in order to maintain the free entry condition (3.1).\footnote{It is also noteworthy that $\lim_{\alpha \to \infty}(MA(\alpha, n) - MI(\alpha, n)) = \frac{1}{n}$, i.e. the upper and the lower bounds of $PWL$ are closer in economies with a large number of firms.}

We now use the second part of iii') to show that there are markets with arbitrary large $PWL$ even if these markets are required to yield values of price, output and number of firms that are exogenously given. To formalize this, we say that a Market is a collection of real numbers $(A, c, b, \alpha, k)$ such that $A, k \geq 0$, $(A-c)\alpha > 0$ and $ab > 0$. An Observation is a triple $(p_i, x_i, n)$ where $p$ is market price, $x_i$ is output of firm $i$ and $n$ is the number of active firms. The last variable is a positive integer and the others are positive real numbers. Now we have the following:

\textbf{Proposition 1.} Given an observation $(p, x_i, n)$, there is a market $(\hat{A}, \hat{c}, \hat{b}, \hat{\alpha}, \hat{k})$ such that $(p, x_i, n)$ is a Cournot equilibrium with free entry for this market (i.e. they fulfill (2.1) and (3.1)), and $PWL$ are arbitrarily close to one.
Proof: For $k$ equal to the maximum value in (3.1), the percentage of WL is

$$MA(\alpha, n) = \frac{(n + \alpha) \frac{1+\alpha}{\alpha} - n \frac{1}{\alpha} (n + \alpha + 1) + (n - 1)n \frac{1-\alpha}{\alpha} (\alpha + 1)}{(n + \alpha) \frac{1+\alpha}{\alpha} - n \frac{1-\alpha}{\alpha} (\alpha + 1)}, \quad (3.7)$$

Let $\hat{\alpha}$ be such that $MA(\hat{\alpha}, n)$ is arbitrarily close to one. By iii') above this is possible. Now set

$$\hat{A} = \frac{p(n + \hat{\alpha}) - \hat{c}n}{\hat{\alpha}}, \quad \hat{k} = \frac{\hat{\alpha}(\hat{A} - \hat{c}) \frac{1+\alpha}{\alpha} n \frac{1-\alpha}{\alpha}}{\hat{b} \frac{1}{\alpha} (n + \hat{\alpha}) \frac{1+\alpha}{\alpha}}, \quad \hat{b} = \frac{(\hat{A} - \hat{c})n^{1-\hat{\alpha}}}{\hat{r}^{\hat{b}}(n + \hat{\alpha})}$$

If we set $\hat{c} = 0$, the previous system can be solved easily, i.e.

$$\hat{A} = \frac{p(n + \hat{\alpha})}{\hat{\alpha}}, \quad \hat{k} = pr_i, \quad \hat{b} = \frac{pn^{1-\hat{\alpha}}}{\hat{\alpha} \hat{r}^{\hat{b}}}$$

Plugging the values of $\hat{A}$ and $\hat{b}$ defined above into (2.1) we obtain

$$x^* = \left( \frac{\hat{A}n}{\hat{b}(n + \hat{\alpha})} \right)^{\frac{1}{\hat{\alpha}}} = n r_i \quad \text{and} \quad p^* = \frac{\hat{A} \hat{\alpha}}{n + \hat{\alpha}} = p.$$  

And from (3.1) (with equality) and the definition of $\hat{k}$ it follows that

$$\frac{\hat{\alpha} \hat{A} \frac{1+\alpha}{\alpha} n \frac{1-\alpha}{\alpha}}{\hat{b} \frac{1}{\alpha} (n + \hat{\alpha}) \frac{1+\alpha}{\alpha}} = \frac{\hat{\alpha}(\hat{A} - \hat{c}) \frac{1+\alpha}{\alpha} n \frac{1-\alpha}{\alpha}}{\hat{b} \frac{1}{\alpha} (n + \hat{\alpha}) \frac{1+\alpha}{\alpha}} \iff \frac{n \frac{1-\alpha}{\alpha}}{(n + \hat{\alpha}) \frac{1+\alpha}{\alpha}} = \frac{n \frac{1-\alpha}{\alpha}}{(n + \hat{\alpha}) \frac{1+\alpha}{\alpha}},$$

which for $\hat{\alpha} > 1$, has only one solution namely $n = n$ so the proof is complete. ■

The explanation of this result is that we have constructed a market in which, in equilibrium, profits are zero and, in the limit, consumer surplus is also zero since from (2.1) we have that

$$U = \frac{\alpha}{(\alpha + 1)b^{\frac{1}{\alpha}} \left(n \frac{a}{\alpha + 1}\right)^{\frac{1+\alpha}{\alpha}}}.$$

The intuition of the latter equation is that large values of $\alpha$ make inverse demand flatter and flatter so consumer surplus goes to zero when $\alpha$ goes to infinite. The difference with the previous section -where $\lim_{\alpha \to \infty} L(\alpha, n) = 0$- arises from the fact that in the latter industry profits are not zero, but when $\alpha$ tends to infinite they tend to a.
Comparing Proposition 1 with the results obtained in the previous section we see that the consideration of fixed costs allows the possibility of finding large \( PWL \). This is because in this case, we add to the misallocation due to the wrong output, the misallocation due to the wrong number of firms.\(^{10}\) The latter comes up to very large numbers because in our model the optimal number of firms is one.\(^{11}\) But the role of preferences should also be emphasized. In the linear case, the corresponding expressions to (3.5) and (3.6) are (see Figure 4).

\[
MA(1, n) = \frac{2n - 1}{n^2 + 2n - 1} \quad \text{and} \quad MI(1, n) = \frac{2n^3 + 3n^2 + 2n + 2}{(n + 1)^2 (n^2 + 4n + 2)}.
\]

Even though for large values of \( n \) PWL are substantial (i.e. for \( n = 15 \), the minimum \( PWL \) is 10.14\% which is just below the PWL in the case of no free entry with two firms),

\(^{10}\)Very similar conclusions are achieved if the cost function is \( cx_i + x_i^2d/2 \) with \( d < 0 \), i.e. under increasing returns to scale. Assuming that \( 2c(d + b) > -da \) costs are positive in the optimum and in equilibrium. In this case \( PWL \) can be very large too: For instance for large \( n \) and \( 2c = A \), \( PWL \approx 2/3 \). Again it can be shown that WL are essentially due to the fact that oligopoly entails too many firms.

\(^{11}\)Overentry may also occur even if the marginal cost is increasing, see von Weizsäcker (1980).
both $MA(1,n)$ and $MI(1,n)$ tend to zero as $n \to \infty$, which was not the case when $\alpha$ was allowed to vary. Moreover, in this case, values of $PWL$ arbitrarily close to one can not be obtained for a given $n$. The reason is that the utility of the representative consumer when $\alpha = 1$ is always positive.

4. Non Identical Firms

Suppose now that firms have different productivities. Let $c_i$ be the marginal cost of firm $i$. Without loss of generality let $c_1 \leq c_i$ for all $i$. Let $a_i \equiv A - c_i$. We will assume that for all $i$, $(n + \alpha - 1)a_i > \sum_{j \neq i} a_j$. This assumption guarantees that all firms produce a positive output in equilibrium (see Equation (4.1) below). Cournot equilibrium is easily shown to be unique and given by

$$x_i^* = \left( \frac{\sum_{j=1}^{n} a_j}{b(n + 1)} \right)^{\frac{1}{\alpha}} \left( \frac{a_i(n + \alpha)}{\sum_{j=1}^{n} a_j} - 1 \right) \quad \text{and} \quad x_i^* = \left( \frac{\sum_{j=1}^{n} a_j}{b(n + 1)} \right)^{\frac{1}{\alpha}}. \tag{4.1}$$

Social welfare is now

$$W^* = \sum_{i=1}^{n} a_i \left( \frac{\sum_{j=1}^{n} a_j}{b(n + 1)} \right)^{\frac{1}{\alpha}} \left( \frac{a_i(n + \alpha)}{\sum_{j=1}^{n} a_j} - 1 \right) - \frac{b}{\alpha + 1} \left( \frac{\sum_{j=1}^{n} a_j}{b(n + 1)} \right)^{\frac{1}{\alpha + 1}}. \tag{4.2}$$

which when all $a_i$’s are identical reduces to (2.3). In the optimal allocation only the technology in the hands of Firm 1 is used and accordingly

$$x^o = \left( \frac{a_1}{b} \right)^{\frac{\alpha + 1}{\alpha}} \quad \text{and} \quad W^o = \frac{\alpha a_1^{\frac{\alpha + 1}{\alpha}}}{(\alpha + 1) b^\frac{\alpha + 1}{\alpha}}. \tag{4.3}$$

In order to have a workable expression for $PWL$ that depends on observable variables alone, let us define $s_i$ as the market share of firm $i$. Clearly, $\sum_{i=1}^{n} s_i = 1$ and $s_1 \geq s_i$, $i = 2, \ldots, n$. Then, from (4.1),

$$s_i \equiv \frac{x_i}{x} = \frac{a_i(n + \alpha) - \sum_{j=1}^{n} a_j}{\alpha \sum_{j=1}^{n} a_j} \quad \Rightarrow \quad a_i = \frac{(\alpha s_i + 1) \sum_{j=1}^{n} a_j}{n + \alpha}. \tag{4.4}$$

For future reference, we will say that a list of market shares $(s_1, s_2, \ldots, s_n)$ is a Market Structure. It is clear from (4.4) that any vector $(a_1, a_2, \ldots, a_n)$ yields a unique market
structure compatible with Cournot equilibrium and that given a market structure we can construct a vector \((a_1, a_2, \ldots, a_n)\) (in fact an infinite number of vectors) whose Cournot equilibrium yields this market structure. Given this equivalence, we will focus in this section on market structure that has the advantage of being observable.

Plugging the last part of (4.4) into (4.2) and after lengthy calculations we obtain \(PWL\) as a function of \(\alpha\) and the market structure, namely

\[
PWL = \frac{(1 + \alpha s_1)^{\frac{\alpha+1}{\alpha}} - (\alpha + 1) \sum_{i=1}^{n} s_i^2 - 1}{(1 + \alpha s_1)^{\frac{\alpha+1}{\alpha}}} \equiv P(s_1, \sum_{i=1}^{n} s_i^2, \alpha). \tag{4.5}
\]

When all firms are identical, (4.5) reduces to (2.4). It is noteworthy that \(PWL\) here depends only on three variables:

- \(\alpha\).
- The market share of the largest firm \(s_1\).
- The Hirschman-Herfindahl index of concentration denoted by \(H \equiv \sum_{i=1}^{n} s_i^2\).

Equation (4.5) allows computation of \(PWL\) from \(s_1\) and \(H\) assuming that demand is linear or isoelastic (where \(\alpha\) is the inverse elasticity of demand). It also allows to plot \(PWL\) as a function of \(\alpha\) for actual market structures and see how this function looks like. For instance, the numbers below represent shares of different firms in the Spanish gasoline market. Our data do not include operators with less than .013 of market share, but the consideration of these operators hardly would make any difference in \(H\).

<table>
<thead>
<tr>
<th>.14</th>
<th>.178</th>
<th>.074</th>
<th>.048</th>
<th>.034</th>
<th>.026</th>
<th>.023</th>
<th>.014</th>
</tr>
</thead>
</table>

**TABLE 2**

This market has been voiced repeatedly as not very competitive. In this case, (4.5) reads like follows:

\[
PWL = \frac{(1 + \alpha s_1)^{\frac{\alpha+1}{\alpha}} - (\alpha + 1) \cdot 2093 - 1}{(1 + \alpha \cdot 41)^{\frac{\alpha+1}{\alpha}}}.
\]

\(^{12}\)In fact, \(s_1\) and \(H\) are not independent but we prefer to write (4.5) in this way to highlight the role of \(H\) in the formula.
Looking at Figure 5 is clear that, except for very special values of $\alpha$, $PWL$ are large. Indeed for values of $\alpha$ larger than $-0.6$, $PWL$ is larger than $0.10\%$. When demand is concave ($\alpha \geq 1$), $PWL$ is always larger than $0.28\%$.

![Graph showing PWL vs. Alpha](image)

**FIGURE 5**

Notice the following properties of $P(\cdot)$ as defined by (4.5):

- $i^\prime\prime\prime$) $\lim_{\alpha \to -1} P(s_1, H, \alpha) = 0$.
- $i^\prime\prime\prime\prime$) $\lim_{\alpha \to 0} P(s_1, H, \alpha) = \frac{1}{s_1} \left( s_1 - \sum_{i=1}^{n} s_i^2 \right)$.
- $i^\prime\prime\prime\prime\prime$) $P(\cdot, H, \alpha)$ is increasing on $s_1$.
- $i^\prime\prime\prime\prime\prime\prime$) $P(s_1, \cdot, \alpha)$ is decreasing on $H$.

$i^\prime\prime\prime$) is identical to i). When firms are identical $i^\prime\prime\prime\prime$) reduces to ii).$^{15}$ Point $iii')$ agrees with the received wisdom: the larger the dominant firm, the closer to monopoly, and hence the larger $PWL$. However, $iv^\prime\prime\prime$) is counterintuitive because it says the larger the

---

$^{13}$ As we mentioned before we take $s_1$ and $H$ as independent when in fact they are not.

$^{14}$ Also, $\lim_{\alpha \to 0} PWL(s_1, H, \alpha) = \frac{e^{s_1 - 1} - H}{e^{s_1} - 1}$.

$^{15}$ If $PWL$ is written as a function of $a_i(s, i^\prime)$ hold and $i^\prime\prime\prime$) reads $\lim_{\alpha \to -\infty} PWL = 1 - \sum_{i=1}^{n} a_i / \sum_{j=1}^{n} a_j$. 

15
concentration, the less WL. The reason is that when \( H \) increases, production is shifted to the less efficient firms which causes social welfare to fall.

Our first result is that when \( \alpha, n \) and the market structure can be chosen simultaneously, \( PWL \) can be arbitrarily close to one and at the same time the concentration index \( H \) arbitrarily low.

**Proposition 2.** There exists \((\alpha, n, s_1, \ldots, s_n)\) for which \( PWL \) is arbitrarily close to one and \( H \) is arbitrarily close to zero.

**Proof:** From iv”) the maximal \( PWL \) occurs when \( s_2 = s_3 = \ldots = s_n \). Denoting these shares by \( y \), we have that \( s_1 + (n - 1)y = 1 \). Plugging this in (4.5) we have that

\[
P(s_1, n, \alpha) = \frac{(1 + \alpha s_1) \frac{\alpha + 1}{\alpha} - (\alpha + 1)(s_2^2 + \frac{(1-s_1)^2}{n-1}) - 1}{(1 + \alpha s_1) \frac{\alpha + 1}{\alpha}}.
\]

(4.6)

\( PWL \) is increasing on \( n \) so the maximum \( PWL \) obtains when \( n \) is arbitrarily large, i.e.

\[
\lim_{n \to \infty} P(s_1, n, \alpha) = \frac{(1 + \alpha s_1) \frac{\alpha + 1}{\alpha} - (\alpha + 1)s_2^2 - 1}{(1 + \alpha s_1) \frac{\alpha + 1}{\alpha}}.
\]

(4.7)

We easily compute

\[
\lim_{\alpha \to \infty} \lim_{n \to \infty} P(s_1, n, \alpha) = \lim_{n \to \infty} \lim_{\alpha \to \infty} P(s_1, n, \alpha) = 1 - s_1.
\]

It follows from above that when \( \alpha \) and \( n \) are very large and \( s_1 \) very small, \( PWL \) is arbitrarily close to one (since limits are interchangeable our procedure is robust). The restriction \( s_1 \geq s_i \), \( i = 2, \ldots, n \) when firms \( 2, \ldots, n \) are identical, is equivalent to \( ns_1 \geq 1 \). This inequality holds when the order of magnitude at which \( n \) tends to \( \infty \) is larger than the order of magnitude at which \( s_1 \) tends to 0.

Finally, it can be easily shown that when firms 2 to \( n \) are identical,

\[
H = \frac{ns_1^2 + 1 - 2s_1}{n-1} = \frac{s_1^2 + \frac{1}{n} - 2s_1}{1 - \frac{1}{n}},
\]

which when \( n \to \infty \) and \( s_1 \to 0 \) tend to zero. \( \blacksquare \)
We now perform a robustness test on the previous result by checking that would happen to the \( PWL \) constructed above and to \( H \) if one of the variables in this construction is held fixed.

First is clear that for given \( s_1, PWL = 1 - s_1 \) which for sensible values of \( s_1 \) are very large but not close to one. Also, \( H = s_1^2 \). Thus in this case \( PWL = 1 - \sqrt{H} \) so high concentration is good for welfare and viceversa.

If \( n \) were given,

\[
\lim_{\alpha \to \infty} \frac{(1 + \alpha s_1)^{\alpha + 1} - (\alpha + 1)(s_1^2 + \frac{(1-s_1)^2}{n-1}) - 1}{(1 + \alpha s_1)^{\alpha + 1}} = \frac{s_1 n - s_1^2 n - 1 + s_1}{s_1 n - s_1}.
\]

This expression achieves a maximum when \( s_1 = \frac{1}{\sqrt{n}} \). For this value of \( s_1 \) \( PWL \) amount to \( \frac{\sqrt{n-1}}{\sqrt{n}+1} \). In this case \( PWL \) are large -the minimum value of \( PWL \) is 0.17- but not close to one. Also \( H = \frac{2(\sqrt{n}-1)}{\sqrt{n}(n-1)} \) which is decreasing in \( n \), see Figure 6. So in this case \( H \) and \( PWL \) go in opposite directions when \( n \) varies.

\[
\begin{align*}
\text{FIGURE 6: } &PWL \text{ (black) and } H \text{ (red) for given values of } n. \\
\end{align*}
\]
Finally, if \( \alpha \) were given, for \( n = \infty \),

\[
PWL = 1 - \frac{(\alpha + 1)s_1^2 + 1}{(1 + \alpha s_1)^{\frac{\alpha + 1}{\alpha}}}.\]

First order condition of \( PWL \) maximization implies \( s_1^2(1 - \alpha^2) - 2s_1(1 + \alpha) + 1 + \alpha = 0 \).

If \( \alpha = 1 \), the maximum is achieved at \( s_1 = \frac{1}{2} \) and \( PWL = \frac{1}{3} \). If \( \alpha \neq 1 \) we have two solutions, \( s_1 = \frac{-1 \pm \sqrt{\alpha}}{\alpha - 1} \). The root \( \frac{-1 - \sqrt{\alpha}}{\alpha - 1} \) can be discarded because if \( \alpha > 1 \) it is negative, if \( \alpha < 1 \) it is larger than one and if \( \alpha \in (-1, 0) \) it is not defined. If \( \alpha \in (0, \infty) \), \( \frac{-1 + \sqrt{\alpha}}{\alpha - 1} \in [0, 1] \) and since it can be easily shown that the maximum is interior, \( s_1 = \frac{-1 + \sqrt{\alpha}}{\alpha - 1} \) maximizes \( PWL \). The latter can be written as

\[
PWL = 1 - \frac{1}{(\alpha - 1)^2} \frac{(\alpha + 1)(\sqrt{\alpha} - 1)^2 + 1}{\frac{\alpha}{\alpha - 1} (\sqrt{\alpha} - 1) + 1} \left( \frac{1}{\alpha + 1} \right)^{\frac{1}{\alpha + 1}} \quad \text{for} \quad \alpha \in (0, \infty),
\]

which increases with \( \alpha \), see Figure 7.\(^{16}\) Again, \( PWL \) are large but not close to one. In this case \( H = (\frac{-1 + \sqrt{\alpha}}{\alpha - 1})^2 \) so \( H \) and \( PWL \) go in opposite directions with respect to \( \alpha \). For \( \alpha \in (0, -1) \) the maximum obtains when \( s_1 = 1 \), i.e. monopoly.

\(^{16}\)Notice that \( \lim_{\alpha \to 1} PWL = \frac{1}{3} \) and \( \lim_{\alpha \to -1} \frac{-1 + \sqrt{\alpha}}{\alpha - 1} = \frac{1}{2} \) which equal the values obtained when \( \alpha = 1 \).
FIGURE 7: PW L (black) and H (red) for given values of $\alpha$

Summing up, in the three cases considered, PW L are easily made large, but not close to one and H is far from being a reliable measure of PW L.

We now perform a more demanding exercise where PW L is studied by varying only either the market structure or $\alpha$.

We first concentrate on how market shares affect PW L. A market structure such that $s_1 > s_2 = \ldots = s_n > 0$ will be called a Dominant Firm. A limit case of a dominant firm is Monopoly where only $s_1$ is positive.

**Proposition 3.** For $\alpha \geq 1$, PW L is maximized when the market structure is a dominant firm with $s_1 = \frac{n+3}{2n+2}$ if $\alpha = 1$ and $s_1 = \frac{-n-1+\sqrt{1+\alpha n+n^2}}{\alpha n-n}$ if $\alpha > 0$. For $\alpha < 0$ the market structure that maximizes PW L is monopoly.

**Proof:** The maximum of PW L in (4.5) over $\sum_{i=1}^{n} s_i = 1$ exists (by Weierstrass’ theorem). As mentioned before, it occurs when $s_2 = s_3 = \ldots = s_n$. So, let us consider PW L as given by (4.6). The extrema of this expression with respect to $s_1$ can be located, either when \[
\frac{\partial P(s_1,n,\alpha)}{\partial s_1} = 0 \] or in the bounds of the interval in which $s_1$ must lie, namely $s_j \leq s_1 \leq 1$ for all $j > 1$. Since $(n-1)s_j \leq s_1$ the previous inequality can be written as $\frac{1}{n} \leq s_1 \leq 1$. Now, rewrite (4.6) as follows:

\[
P(s_1,n,\alpha) = 1 - \frac{(\alpha + 1)(ns_1^2 - 2s_1 + 1) + n - 1}{(n-1)(1 + \alpha s_1)^{\frac{n+1}{2}}}.
\]

\[
\frac{\partial P(s_1,n,\alpha)}{\partial s_1} = s_1^2(n - n\alpha^2) - s_1(2 + 2n + 2\alpha + 2n\alpha) + 2 + \alpha(3 + n + \alpha) + n
\]
\[
(n-1)(\alpha s_1 + 1)^{\frac{n+2}{2}}
\]

\[
\frac{\partial P(s_1,n,\alpha)}{\partial s_1} = 0 \iff s_1^2(n - n\alpha^2) = 2s_1(1 + n + \alpha + n\alpha) - 2 - \alpha(3 + n + \alpha) - (*)
\]

We have three possible cases: If $\alpha = 1$, the solution to (4.9) is $s_1^* = \frac{n+3}{2n+2} \in [\frac{1}{n}, n]$. Then, the maximum must be located either at $s_1 = \frac{1}{n}$, at $s_1 = 1$ or at $s_1 = \frac{n+3}{2n+2}$. We easily compute,

\[
P(1,n,1) = \frac{1}{4}, \quad P\left(\frac{1}{n},n,1\right) = \frac{1}{(n+1)^2}, \quad P\left(\frac{n+3}{2n+2},n,1\right) = \frac{n+1}{3n+5}.
\]
From these expressions we obtain the desired result.

If $\alpha > 1$ from the first order condition we obtain two solutions,

$$s_1^* = \frac{-n - 1 \pm \sqrt{1 + \alpha n + \alpha^2 n + \alpha n^2}}{\alpha n - n}.$$  \hspace{1cm} (4.10)

Clearly only the solution with a plus sign in front of the square root is feasible. We will show that for this solution, $s_1^* = \frac{1}{n}$ we would have $\alpha^2(n - 1) + n^2(\alpha - 1) - \alpha n + 1 < 0$ which is impossible because the left hand side achieves a minimum when $n = 2$ and $\alpha = 1$. Similarly, if $s_1^* > 1$, $\alpha n - \alpha - n + 1 < 0$, which again is impossible.  

Finally, notice that since there is only one value of $s_1$ for which $\frac{\partial P(s_1, n, \alpha)}{\partial s_1} = 0$ the shape of $P(\cdot, n, \alpha)$ is determined by the sign of $\frac{\partial P(s_1, n, \alpha)}{\partial s_1}$ at $s_1 = \frac{1}{n}$ and $s_1 = 1$. From (4.8),

$$\text{sign} \frac{\partial P(\frac{1}{n}, n, \alpha)}{\partial s_1} = \text{sign}(n + \alpha + na - \frac{1}{n} + \alpha^2 - 2\frac{1}{n} \alpha - \frac{1}{n} \alpha^2) > 0 \hspace{1cm} (4.11)$$

because the expression on the right hand side is increasing in $\alpha$ and for $\alpha = -1$ equals to zero. Also from (4.8) we obtain that

$$\text{sign} \frac{\partial P(1, n, \alpha)}{\partial s_1} = \text{sign}(\alpha - na + \alpha^2 - n\alpha^2) = \text{sign}(\alpha(1 + \alpha)(1 - n)) < 0. \hspace{1cm} (4.12)$$

So the interior solution is indeed a maximum.

Finally let us consider the case $\alpha < 1$. Suppose that the negative root in (4.10) is less than one. Then

$$-n - 1 - \sqrt{1 + \alpha n + \alpha^2 n + \alpha n^2} < 1 \quad \Leftrightarrow \quad -\sqrt{1 + \alpha n + \alpha^2 n + \alpha n^2} > \alpha n + 1,$$

which is impossible. So there is, at most, one interior solution. Suppose first that $\alpha > 0$. From (4.11-12) we get that $\text{sign} \frac{\partial P(\frac{1}{n}, n, \alpha)}{\partial s_1} > 0$ and $\text{sign} \frac{\partial P(1, n, \alpha)}{\partial s_1} < 0$ which implies that maximum PWL are achieved at the interior solution. If $\alpha = 0$, the positive root in (4.10) equals one. Finally, if $\alpha < 0$, from (4.11-12), we have that $\text{sign} \frac{\partial P(\frac{1}{n}, n, \alpha)}{\partial s_1} > 0$ and $\text{sign} \frac{\partial P(1, n, \alpha)}{\partial s_1} > 0$, which given that there is, at most one value of $s_1$ for which $\text{sign} \frac{\partial P(\cdot, n, \alpha)}{\partial s_1}$ switches from positive to negative means that $P(\cdot, n, \alpha)$ is increasing, so it achieves the maximum when $s_1 = 1$.  

\hspace{1cm} \[17\text{Notice that when } \alpha \rightarrow 1 \text{ both the numerator and the denominator in the definition of } s_1^* \text{ go to zero.}\]
Proposition 3 implies that the most deleterious market structure is not always monopoly, the target of the wrath of economists since Adam Smith. In many cases a dominant firm structure is even worse because firms other than 1 do not add much competition to the market and they are technological inefficient. We notice that under maximal $PWL$,

$$H = \frac{ns_1^2 + 1 - 2s_1}{n - 1} \quad \text{and} \quad PWL = \frac{(1 + \alpha s_1)^{\frac{\alpha + 1}{\alpha}} - (\alpha + 1)(ns_1^2 + \frac{(1-s_1)^2}{n-1}) - 1}{(1 + \alpha s_1)^{\frac{\alpha + 1}{\alpha}}},$$

so $H$ is decreasing on $n$ but $PWL$ is increasing on $n$. Also, $H$ is increasing on $s_1$ but $P(\cdot, n, \alpha)$ is not necessarily so. Thus, in this case, again, the concentration index $H$ is a poor measure of WL.

We now study the market structure that minimize $PWL$.

**Lemma 1.** Suppose that $(\hat{s}_1, \hat{s}_2, \ldots, \hat{s}_n)$ minimizes $P(s_1, \sum_{i=1}^n s_i^2, \alpha)$. Then $\frac{\partial}{\partial s_i} \hat{s}_i, \hat{s}_j, j > 1$ such that $\hat{s}_1 > \hat{s}_i \geq \hat{s}_j > 0$.

**Proof:** Increasing $\hat{s}_i$ by an small amount, say $dx$, and decreasing $\hat{s}_j$ by $dx$ too is feasible -i.e. $\hat{s}_i + dx$ and $\hat{s}_j - dx \in [0, s_1]$, increases $H$ and so decreases $PWL$ which contradicts that these are minimized. $\blacksquare$

Lemma 1 implies that there are only three kind of market structures that could be candidates to minimize $PWL$.

1: All firms produce the same output. Market structure reads $(\frac{1}{n}, \frac{1}{n}, \ldots, \frac{1}{n})$.

2: All firms minus one, say $n$, produce the same output. Market structure reads $(x, x, \ldots, y)$ with $x > y$.

3: A number of firms, say $1, \ldots, m$ with $m < n$ produce the same output, and the remaining firms produce zero output. Market structure reads $(\frac{1}{m}, \ldots, \frac{1}{m}, 0, \ldots, 0)$.

But option 3) can not minimize $PWL$ since it was established that when all firms are identical, $PWL$ decreases with the number of (active) firms (Property iv) in Section 2). So we are left with options 1 and 2.

**Proposition 4.** The market structure that minimizes $PWL$ is when all firms produce the same output.
Proof: Notice that market structures 1 and 2 can be written as $(x, x, ..., 1 - (n-1)x)$ with $x \in \left[\frac{1}{n-1}; \frac{1}{n}\right]$, where the lower bound of this interval comes from $1 \geq (n-1)x$. In this case $H = (n-1)x^2 + (1 - (n-1)x)^2$. Plugging this into the definition of $PWL$ we obtain

$$PWL = 1 - \frac{(\alpha + 1) \left( (n-1)x^2 + (1 - (n-1)x)^2 \right) + 1}{(1 + \alpha x)^{\frac{\alpha + 1}{\alpha}}} \equiv PW(\alpha, x, n).$$

Now, computing $\frac{\partial PW(\alpha, x, n)}{\partial x}$ this expression is found to be equal to

$$\frac{-(\alpha + 1)}{(1 + \alpha x)^{\frac{\alpha + 1}{\alpha}}} \left[ 2n^2x - 2nx - 2n + 2 - \frac{(1 + \alpha) \left( (n-1)x^2 + (1 - (n-1)x)^2 \right) + 1}{1 + x\alpha} \right].$$

Solving for $\frac{\partial PW(\alpha, x, n)}{\partial x} = 0$ we obtain the following. If $\alpha = 1$, 

$$\frac{\partial PW(\alpha, x, n)}{\partial x} = 0 \iff 4n + 4x + 2 - 4n^2x = 0 \iff x = \frac{2n + 1}{2n^2 - 2} < \frac{1}{n-1}.$$

So only boundary solutions are feasible and $PWL$ is minimized when $x = \frac{1}{n}$. If $\alpha \neq 1$, 

$$\frac{\partial PW(\alpha, x, n)}{\partial x} = 0 \iff x = \frac{-n^2 + 1 \pm \sqrt{n^4 + 1 + 2\alpha n^3 + \alpha^2 n^2 - 3\alpha n^2 - \alpha^2 n - 2n^3 + \alpha n}}{(\alpha - 1)(n^2 - n)}.$$

Suppose that $\alpha > 1$. Clearly, the negative root is not feasible, so consider the positive root, say $x^*$. If $x^* \leq \frac{1}{n}$, it must be that $(n-1)(\alpha^2 + \alpha n - 1 - n) \leq 0$ which for $n > 2$ and $\alpha > 1$ is impossible.

Suppose that $\alpha < 1$. If the negative root is less or equal than $\frac{1}{n}$, we have that 

$$-\sqrt{n^4 + 1 + 2\alpha n^3 + \alpha^2 n^2 - 3\alpha n^2 - \alpha^2 n - 2n^3 + \alpha n} \geq (n + \alpha)(n - 1)$$

which is impossible. Take the positive root. For this root to be larger or equal than $\frac{1}{n-1}$ we need that $n(1 - \alpha) \leq \alpha^2 - 3\alpha + 2$ or $n \leq \frac{\alpha^2 - 3\alpha + 2}{1 - \alpha}$. The right hand side of this inequality has a maximum at 3 when $\alpha \to -1$. Since this value of $\alpha$ is never actually achieved, this inequality only may hold when $n = 2$. But $\frac{\partial PW(\alpha, 0.5, 2)}{\partial x} = \frac{0.5\alpha + 1.5}{0.5\alpha + 1} > 0$ which means that the minimum is achieved at the boundaries of $x$. Since in this case these bounds imply monopoly and duopoly, by iv) in Section 2 we achieve the desired result. ■
The implication of this result is that disregarding firms heterogeneity stacks the deck in favour of small welfare losses because the assumption that all firms are identical implies that PWL are minimal among all market structures.

Finally, we consider the effect of \( \alpha \) alone on PWL. We have little to say about the value of \( \alpha \) that maximizes PWL because first order conditions of maximization with respect to \( \alpha \) are not very informative.\(^{18} \) Given this we follow an indirect route. We look for the maximum PWL that can be achieved for given \( n \). This is obtained by plugging the value of \( s_1 \) that maximizes PWL, denoted by \( s(\alpha, n) \), into \( P(s_1, n, \alpha) \), i.e. \( P(s(\alpha, n), n, \alpha) \equiv F(\alpha, n) \), say. This function reads

\[
F(\alpha, n) = 1 - \frac{(\alpha + 1)(ns(\alpha, n)^2 - 2s(\alpha, n) + 1) + n - 1}{(n - 1)(1 + \alpha s(\alpha, n))^\frac{n+1}{n}}, \text{ where}
\]

\[
s_1(\alpha, n) = \frac{-n - 1 + \sqrt{1 + \alpha n^2 + \alpha^2 n^2 + \alpha n^2}}{\alpha n} \text{ if } \alpha > 0 \text{ and } s_1(\alpha, n) = 1 \text{ otherwise.}
\]

Figure 8 shows \( F(\cdot, n) \) for several values of \( n \) and for \( \alpha \in (0, 50] \). As it was mentioned before for \( \alpha \in (-1, 0) \) maximal PWL are obtained under monopoly. This is why all the curves in the figure tend to the same value when \( \alpha \to 0 \), namely to PWL under monopoly, see Table 1 above. Now we state some properties of \( F(\cdot, \cdot) \).\(^{19} \)

a) \( F(\alpha, \cdot) \) is increasing in \( n \).

This is proved by extending the functions \( P(s_1, \cdot, \alpha) \) and \( F(\alpha, \cdot) \) to have real values in the domain. It is clear that such functions are differentiable in \( n \). Now compute,

\[
\frac{\partial F(\alpha, n)}{\partial n} = \frac{\partial P(s(\alpha, n), n, \alpha)}{\partial n} = \frac{\partial P(s_1, n, \alpha)}{\partial s_1} \frac{\partial s_1}{\partial n} + \frac{\partial P(s_1, n, \alpha)}{\partial n} = \frac{\partial P(s_1, n, \alpha)}{\partial n},
\]

where the last equality comes from the fact that \( s(\alpha, n) \) maximizes PWL with respect to \( s_1 \) (this is the envelope theorem). Finally, it was established in Proposition 2 that \( P(s_1, \cdot, \alpha) \) is increasing in \( n \) so the desired result follows. This result implies that, for any number of firms, it is possible to find PWL of, at least, the magnitude of \( F(\alpha, 2) \) which for values of \( \alpha \in (0, 50] \) never goes below 0.2097.

\(^{18}\) \( \frac{\partial P(s_1, n, \alpha)}{\partial s_1} > 0 \iff (n - 2s_1 + \alpha - 2\alpha s_1 + ns_1^2 + ns_1^2 \alpha) \frac{1}{\alpha} \ln (s_1\alpha + 1) = n - 2s_1 + ns_1^2. \)

\(^{19}\) Properties of \( s(\alpha, n) \) when \( \alpha > 0 \) are: a) \( \lim_{n \to \infty} s(\alpha, n) = \frac{1}{\sqrt{n+1}}. \) b) \( \lim_{\alpha \to \infty} s(\alpha, n) = \frac{1}{\sqrt{n}}. \) c) \( \frac{\partial s(\alpha, n)}{\partial \alpha} > 0 \iff 3an + n + \alpha n^2 + n^2 - 2 < 2\sqrt{(1 + \alpha n + \alpha^2 n + \alpha n^2)(n + 1)} \) and \( d) \frac{\partial s(\alpha, n)}{\partial n} > 0 \iff (an + \alpha^2 n - 2\sqrt{(1 + \alpha n + \alpha^2 n + \alpha n^2) + 2})(1 - \alpha) > 0. \)
Proposition 4 says that minimal $PWL$ obtain when all firms are identical and thus, minimal $PWL$ are given by the function $L(\cdot, \cdot)$ in (2.4). Notice that since $L(\alpha, \cdot)$ is decreasing in $n$ and $F(\alpha, \cdot)$ is increasing in $n$, the difference between maximal and minimal $PWL$ increases with $n$ for a given $\alpha$, see Figure 9. Thus, the more firms the larger the impact of heterogeneity on $PWL$. Figure 9 also suggests that for a given $n \geq 5$, the distance between these two magnitudes increases with $\alpha$. Finally, since $P(\cdot, n, \alpha)$ is continuous in $s_1$, for a given $n$ and $\alpha$, any $PWL$ between minimal and maximal $PWL$ are reachable by the choice of $s_1$. 

![Figure 8:PW L for n = 2 (red), 3 (light red), 4 (green), 5 (brown) and 10 (black).](image)
FIGURE 9: Maximal and Minimal PWL for $n = 2$ (black), 5 (red) and 10 (green).

We end this section by stating the limiting properties of $F(\cdot, \cdot)$.

\[ \lim_{\alpha \to \infty} F(\alpha, n) = \left( \frac{\sqrt{n}}{\alpha} \right)^3 + \frac{\sqrt{n} - 2n}{\sqrt{n}}. \]

\[ \lim_{n \to \infty} F(\alpha, n) = \frac{\alpha + 1}{\alpha - 1} - \frac{(1 + \sqrt{n})^2}{(\alpha - 1)^2} - \frac{(1 + \sqrt{n})^2 - 1}{(\alpha - 1)^2} - 1. \]

Notice that in both cases PWL are high even for small values of $\alpha$ and $n$, see Figure 8.
5. Conclusions

When one observes public policies on oligopolies one sees some concern about the number and the relative size of firms. But the question of the output set by oligopolists is cause of little or no concern at all. This paper provides some justification to this attitude. In two different settings, namely free entry and heterogeneous firms, we found that welfare losses due to the wrong number of firms can be quite substantive as found in Sections 3 and 4. On the contrary welfare losses due to the divergence between equilibrium and optimal output are small, even with as few as four firms in the market as shown in Section 2. This conclusion, though, is likely to be exaggerated by the fact that under our assumptions the optimal number of firms is one. It is possible that if an U-shaped average cost curve was assumed this conclusion would be weakened. Other factor likely to bring down our estimates of welfare losses is the consideration of other solution concepts (e.g. Bertrand). Thus our results are just a first cut to the problem.

Our results have a number of implications for the empirical literature that measures
welfare losses. Typically, this literature has attempted to measure welfare losses arising from oligopolistic output setting. Our results suggest that this attempt is misguided because welfare losses arising from this source are likely to be small. However welfare losses due to overentry or to asymmetric firms can be quite substantial. Lack of consideration of these points biases downwards our estimates of welfare losses.

References


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