

# Engel's Law and Commodity Differentiation\*

Vincenzo Merella<sup>†</sup>

Birkbeck College

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## Abstract

I propose a consumer theory that provides an endogenous explanation of the Engel's Law based on the qualitative differentiation of the commodities available for purchase. By introducing a new definition of consumption technology into a standard form of power utility, I develop a framework able to characterize the different commodities either as normal or inferior goods and, within these categories, as luxury or necessary goods. The model establishes a steady relationship between the optimal consumption bundles and the cost of quality upgrading, and may help in increasing the robustness of the results obtained by some recent models based on hierarchic preferences, the literature this work is mainly related to.

**Keywords:** Consumption, Quality, Comparative Statics, Nonhomothetic Preferences

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<sup>†</sup>Birkbeck College, University of London, United Kingdom. E-mail: [vmerella@ems.bbk.ac.uk](mailto:vmerella@ems.bbk.ac.uk)

# 1 Introduction

*Of all the empirical regularities observed in economic data, Engel's Law is probably the best established. (H.S. Houthakker)*

This paper develops a framework, based on commodity differentiation, that offers a rationale for the economic facts resulting from the income comparative statics on the household's optimal choice. Empirical evidence – from Engel (1895) to many subsequent studies (citations to be added) – suggests that households facing a diversified commodity set choose an optimal consumption structure that strongly depends on the level of available resources. Income generally differs across households in a given economy, across representative households of different countries, and even in the same household at different dates. Any nondegenerate income distribution implies a certain degree of discrimination among real expenditure possibilities, and this is expected to cause ever different optimal demand structures.

These issues are neglected by a great deal of the traditional economic literature. Most models completely abstract from income-related studies, and only a small number of contributions analyse the economic effects of commodity differentiation. Among them, some authors such as Romer (1990) and Xie (1998) propose a horizontal commodity diversification, while others like Aghion and Howitt (1992) and Stokey (1988) suggest a vertical differentiation. What all these contributions have in common is the adoption of homothetic utility functions to formalize the household's preferences, whose main implication is precisely that the optimal consumption structure is invariant to income changes.

In order to achieve a diversified demand structure, I propose a nonhomothetic specification of household's preferences on a differentiated set of commodities, directly derivable from a traditional power utility of a Dixit and Stiglitz (1977) type. I also argue that the properties of this formalization are better represented by considering both vertical and horizontal aspects of commodity differentiation. Following Grossman and Helpman (1991), close-substitutes commodities are regarded as vertically differentiated grades belonging to a common variety, in turn defined as the set of commodities satisfying a common need. Conversely, commodities of different varieties are considered as horizontally differentiated near-independent goods meeting different needs.<sup>1</sup>

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<sup>1</sup>Close complements are neglected by considering them as parts of a single commodity bundle.

Households thus face two choices: 1) the identification, within each variety, of which grades to consume; 2) the definition of the magnitude of the expenditure shares devoted to the different varieties.

A model with a homothetic utility ordering typically predicts that households optimally choose grades and expenditure shares with no regard to the level of available resources. This makes any exercise of income comparative statics redundant. On the contrary, this paper is able to characterize the different commodities as normal or inferior goods and, within these categories, as luxury or necessary goods. I find that the optimal expenditure shares depend on the distribution of quality, and that the choice of the optimal grades relies on the distribution of the relative cost of quality across the different varieties. Consistently with empirical evidence, the resulting consumption structure is strongly income-contingent. Homothetic properties hold as a specific case, requiring identical relative cost of upgrading consumption to higher-quality commodities for all varieties.

In some circumstances, the approach adopted here differs from standard textbooks comparative statics in the object of analysis.<sup>2</sup> The demand expansion paths may relate to the different varieties rather than to single commodities. In other words the Engel Curves, typically defined as the relationship between the amounts of a commodity that a household is willing to purchase and its income, are unable to distinguish between luxury and necessary goods. This is due to the process of substitution to higher-quality commodities, which implies that variety-specific prices do not hold constant with rising income, a prerequisite to the traditional wealth-effect analysis.

This paper relates to a recent literature that obtains nonhomotheticity from a hierarchic specification of household's preferences. First formalized by Jackson (1984) on the basis of the tree-structured utility functions proposed by Strotz (1957), this preference specification has been recently applied to the analysis of several macroeconomic issues, such as the role of income distribution in the adoption of new technologies (Murphy, Shleifer, and Vishny, 1989; Zweimuller, 2000), in the determination of economic growth performance (Falkinger, 1994; Mat-

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<sup>2</sup>See Reny (2000) and Varian (1992) among many others. Mas-Colell et al. (1995) opt for a more general approach. They appropriately point out that “[the] assumption of normal demand makes sense if commodities are large aggregates [in this paper named *varieties*] (e.g., food, shelter). But if they are very disaggregated (e.g., particular kinds of shoes), then because of substitution to higher-quality goods as wealth increases, goods that become inferior [commodities for which wealth effect on demand is negative] at some level of wealth may be the rule rather than the exception”. This argument inevitably extends to the related notions of luxury and necessary goods.

suyama, 2002; Foellmi and Zweimüller, 2006), and in the formation of the international trade structure (Matsuyama, 2000; Mitra and Trindade, 2005).<sup>3</sup> The preference structure presented here differ from a hierarchic specification because the consumption orderings are endogenously determined through the qualitative differentiation of the commodity set.

The rest of the paper is organized as follows. The next section proposes a formal definition of quality based on the principles of the characteristics theory. Section 3 describes the household's preferences, with a particular focus on the definition of consumption technology, and the budget constraint. Section 4 offers a heuristic solving of the household's problem. Section 5 concludes. The appendix contains all relevant analytical derivations.

## 2 Commodity Space and Definition of Quality

This section proposes one of the virtually infinite possible definitions of quality. Alternative definitions of quality can be found, among many others, in Stokey (1988) and Merella (2006). Although not essential (several contributions, including those of Aghion and Howitt (1992) and Grossman and Helpman (1991) mentioned above, provide no formalization of the quality index), I opt for stating an explicit definition of quality as it may be useful for the economic interpretation of some of my findings. The reader may, however, skip the discussion below and jump to the next section without losing the thread.

When qualitative differentiation is introduced by assigning each commodity a value in a quality ladder, it is essential to perform a strict separation between quantitative and qualitative aspects. For this reason, in stating my definition of quality I abstract from any quantitative issue, referring to each commodity as a (distinctly measured) unit object, and to its components as proportions of this object. Following the Lancasterian (1966) tradition, I assume a commodity is outlined by its underlying characteristics, and is so depicted by the allocations of characteristics it contains. I depart, however, from Lancaster's theory in assuming that the characteristic allocations lie in a unit interval, and that a commodity is identified properly only if the allocations of the different characteristics it contains sum up to one.

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<sup>3</sup>Other contributions where preferences have a nonhomothetic structure include Aoki and Yoshikawa (2002), who investigate economic growth issues constructing a model based on logistic Engel curves; Galor and Moav (2004), who analyse household's bequest behaviour; and Voigtländer and Voth (2005), who apply nonhomothetic preferences to a historical context.

Formally, I assume we observe a characteristic set  $\mathbb{J} \subset \mathbb{N} : \{j \leq J\}$ , where  $j$  stands for a generic characteristic and  $J$  is the number of elements in  $\mathbb{J}$ . Each  $j$  identifies one dimension in the characteristic allocation space  $\mathbb{Y} \subset \mathbb{R}_+^J : \{y_j \in [0, 1], \forall j \in \mathbb{J}\}$ , where  $y_j$  measures the  $j$ th characteristic allocation. A commodity is defined as a  $(J \times 1)$  vector  $\mathbf{y} \equiv [y_1, y_2, \dots, y_J]'$  in  $\mathbb{Y}$ , such that  $\sum_{j=1}^J y_j = 1$ . The resulting commodity set, denoted by  $\mathbb{S} \subset \mathbb{Y}$ , can be geometrically represented by a  $(J - 1)$  *dimensional simplex*, an object often used in studies involving probabilities. In figure 1, the simplex is portrayed by the gray wired triangle for a  $J = 3$  case. Each point in the simplex represents a commodity (in figure 1, one example is given by point  $\mathbf{y}$ ), qualitatively different from all others.

The commodity set so obtained can be ordered by defining a proper datum point. Given the nature of the problem, a sensible benchmark is provided by human needs. These can be defined as predetermined and objectively identifiable ideal objects, whose set is denoted by  $\mathbb{V} \subset \mathbb{N} : \{v \leq V\}$ , where  $v$  stands for a generic need and  $V$  is the number of elements in  $\mathbb{V}$ . I assume each need  $v$  is fully satisfied by a commodity equivalent (hereafter called *bliss*) in  $\mathbb{S}$ , denoted by  $\mathbf{y}_v$ . By comparing a generic commodity to the bliss, we obtain a need-specific measure of qualitative differentiation. I name this measure *quality*, and denote it by  $q_v(\mathbf{y}) \in \mathbb{R}_+$  for some commodity  $\mathbf{y}$  with reference to need  $v$ .

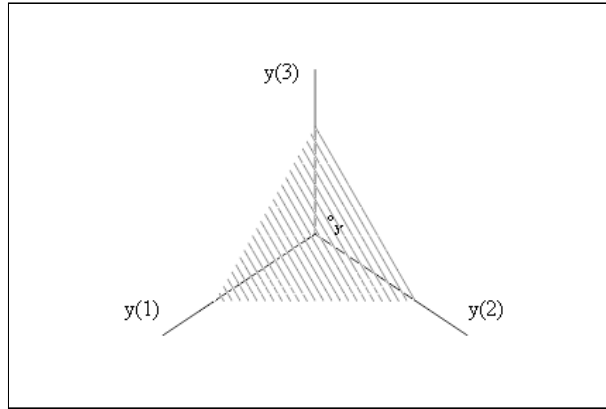


Figure 1 – A Commodity in the Characteristics Space

In this framework, it is straightforward to imagine an inverse relation between quality and the Euclidean distance from the reference commodity and the bliss, denoted by  $d_v(\mathbf{y}) \equiv |\mathbf{y} - \mathbf{y}_v|$ . Quality can thus be formally represented by the function  $q_v(\mathbf{y}) = f[d_v(\mathbf{y})]$ , with  $f'(\cdot) < 0$ .

Function  $f$  can be given a number of possible formalisations. A convenient way to define it is to allow quality to range in all positive real numbers as stated above. As we shall see below, other domains may, however, be more appropriate in specific contexts.

According to the framework developed so far, each commodity is able to satisfy, at least to some extent, all human needs. Although it possibly reflects reality in greater details, such a model is extremely complex, and providing the household's optimisation with a proper economic interpretation becomes virtually impossible. Without heavy loss of relevant information, the analysis can be eased by assuming that one commodity can only satisfy one need. This univocal relation greatly simplifies the computation of marginal utility of consumption, so the resulting demand functions take a more convenient form. A qualitative priority condition is a suitable way to link each commodity to a single need: that is, a commodity satisfies the need associated to the highest level of quality.

Denoting by  $\mathbb{S}_v \subset \mathbb{S}$  the subset of commodities satisfying the  $v$ th need, I formally express the above condition as  $\mathbf{y} \in \mathbb{S}_v \Leftrightarrow q_v(\mathbf{y}) = \max_{u \in \mathbb{V}} \{q_u(\mathbf{y})\}$ . If quality of a commodity is the same with regard to two or more needs, then without loss of generality I assume that commodity is in the subset with the lowest  $v$ . Under these assumptions, the subsets of commodities satisfying different needs are such that  $\mathbb{S}_v \cap \mathbb{S}_z = \emptyset$ . As a result, we may sensibly suppose that marginal utility of a commodity in one subset to be influenced only by consumption of commodities in the same subset. Accordingly, I assume that two commodities in the same subset exhibit a unit marginal rate of substitution. Commodities in different subsets are conversely said to be *independent*. I shall formalise these assumptions in section 3.

The commodity subsets defined above form different groups of commodities, each devoted to the satisfaction of a different need. I shall name these groups *varieties*, and denote them by the same index  $v \in \mathbb{V}$  as needs. Using the definition of quality, the elements in  $\mathbb{S}_v$  can then be mapped into a unidimensional commodity space, denoted by  $\mathbb{G}_v \subset \mathbb{R}_+ : \{g_v \equiv q_v(\mathbf{y})\}$ , where  $g_v$  is the generic element of  $\mathbb{G}_v$  representing a commodity in variety  $v$ . I shall name the commodities so sorted as *grades* belonging to the same variety. Notice that several grades may display the same level of quality. Among them, however, household's demands will be set according to minimum-price criteria.

In a commodity space so defined, the different grades belonging to the same variety are

directly indexed by quality. As it may generate confusion, I keep the two measure formally distinct, although they are identical measures: the grade index is depicted by  $g_v$ ; quality is denoted by  $q(g_v)$ . For future reference, it is useful to set  $q(g_v) = 1$  as the qualitative benchmark. I shall thus refer to commodity  $g_v$  as a *low-quality* grade if  $q(g_v) < 1$ ; as a *high-quality* grade if the opposite holds.

### 3 Preferences and Budget Constraint

The preference specification of a generic household  $i$  is defined over aggregate consumption  $C^i$ , according to the traditional CIES form of power utility:

$$U^i = \frac{(C^i)^{1-\sigma} - 1}{1-\sigma}. \quad (1)$$

where  $\sigma$  is a parameter measuring the elasticity of marginal utility to aggregate consumption. Using aggregators of the Dixit-Stiglitz (1977) type, I in turn define aggregate consumption as:

$$C^i = \left[ \sum_{v=1}^V (C_v^i)^\alpha \right]^{1/\alpha}. \quad (2)$$

where  $v$  indexes the different varieties,  $\alpha \in [0, 1]$  is a parameter measuring the elasticity of substitution between consumption of any pair of varieties, and  $C_v^i$  denotes another aggregate, which I shall name *variety-specific consumption*, formally given by:

$$C_v^i = \int_0^\infty c^i(g_v) dg_v, \quad (3)$$

where  $g_v$  indexes the grades belonging to variety  $v$ , and  $c^i(g_v)$  represents the *consumption technology* of commodity  $g_v$ .<sup>4</sup> The twofold aggregation, across varieties and across grades, required to define aggregate consumption is due to the double (horizontal and vertical) nature of commodity differentiation.

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<sup>4</sup>The reader can interpret the different varieties as *horizontally* differentiated commodities, and the different grades as *vertically* differentiated commodities. For a more precise definition, I refer to section 2. The elasticity of substitution parameter in (3) is implicitly set to one, to reflect the unit marginal rate of substitution between consumption of two grades belonging to the same variety. The term consumption technology is borrowed from Lancaster .

### 3.1 The Consumption Technology

The preference specification, as so far developed, is pretty standard. The novelty of my approach to the consumer theory lies in the formalization of the consumption technology. In its more atomistic meaning, the consumption technology can be defined as the primary measure of satisfaction of a given need an individual derives by consuming one type of commodity, and its output as the value of the relevant consumption activity. In most economic models, the consumption technology of a generic commodity  $g_v$  is simply given by the amount consumed, or *quantity*, denoted by  $x^i(g_v)$ , and formally expressed by  $c^i(g_v) = x^i(g_v)$ . I shall name this function *traditional* consumption technology. The marginal contribution of an increase in quantity to the value of the consumption activity, given by  $D_{x^i(g_v)}c^i(g_v) = 1$ , is constant and identical for all commodities.<sup>5</sup>

Along with the quantitative aspect, in the studies adopting a vertical commodity differentiation quality is regarded as a fundamental element of the consumption activity. The consumption technology is accordingly expressed by a function of both quantity and quality, and is typically given the multiplicative form  $c^i(g_v) = q(g_v)x^i(g_v)$ . I shall name this function *linear consumption technology*. The marginal contribution of an increase in the amount consumed to the value of the consumption activity, given by  $D_{x^i(g_v)}c^i(g_v) = q(g_v)$ , is invariant to quantity and proportional to quality. The resulting relations between the consumption technology (on the  $y$ -axis, labelled  $c(g,v)$ ) and quantity (on the  $x$ -axis, labelled  $x(g,v)$ ) are portrayed in figure 2. Line B is drawn for a unit level of quality, so it also depicts the functional form of the traditional consumption technology. Lines A and C are respectively drawn for a low-quality and a high-quality commodity.

Although this qualitatively differentiated approach adds important information concerning the household's choice, for instance by delivering an endogenous commodity consumption ordering, it has serious limits when considering the income comparative statics. The linear consumption technologies are homogeneous across the different commodities, and when they are nested into a monotonic utility function, they generate a homothetic formalisation of preferences. Given prices, the income expansion paths are straight lines, so the different commodities will be con-

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<sup>5</sup>Hereafter, the compact differentiation notation  $D_z^i f(z) \equiv \partial^i f(z) / \partial z^i$  is systematically used. The notation  $f(a)$  indicates the value of the function  $f(z)$  evaluated at  $z = a$ .



sumed in the same proportions at any level of income. This rules out the analysis of economic phenomena like those summarised by the Engel's law.

In order to account for such phenomena, I propose to reinforce qualitative differentiation by formalising consumption as a power function, in which quantity appears at its argument and quality at its exponent:

$$c^i(g_v) = [x^i(g_v)]^{q(g_v)}. \quad (4)$$

I shall name this function *power consumption technology*. The marginal contribution of an increase in the amount consumed to the value of the consumption activity, given by  $D_{x^i(g_v)} c^i(g_v) = q(g_v) [x^i(g_v)]^{q(g_v)-1}$ , depends positively on quality, while the role of quantity, which settles  $c^i(g_v)$  concavity, is uncertain, as the sign of  $D_{x^i(g_v)}^2 c^i(g_v) = [q(g_v) - 1] [D_{x^i(g_v)} c^i(g_v)] / x^i(g_v)$  changes from negative to positive at  $q(g_v) = 1$ . The consumption technology is thus concave for low-quality commodities and convex for high-quality commodities.

The resulting relations between the consumption technology (on the  $y$ -axis, labelled  $c(g,v)$ ) and quantity (on the  $x$ -axis, labelled  $x(g,v)$ ) are portrayed in figure 3, for the same levels of quality as in figure 2. Quality differentiation is reinforced in that, for high-quality commodities, the consumption technologies diverge much faster. More importantly, they are non-homogeneous in quantity across the different commodities. With the adoption of power consumption technologies, utility is a non-homothetic function, and the resulting income expansion paths are non-linear. It is therefore possible to develop a theory, based on this framework, explaining the Engel's law through commodity differentiation.

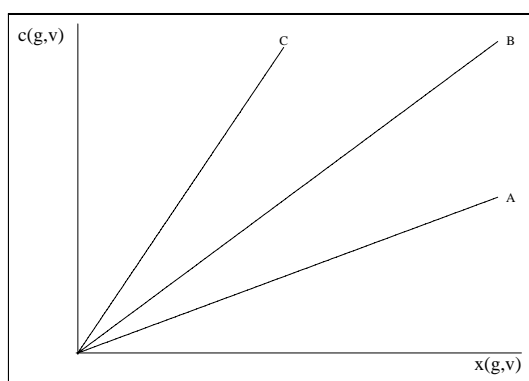


Figure 2 – Linear Consumption Technologies

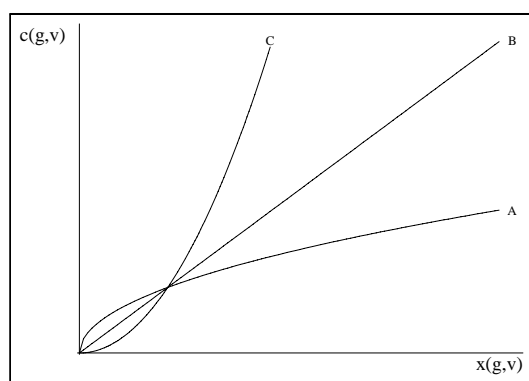


Figure 3 – Power Consumption Technologies

The functional form of this technology implies that, for a given and sufficiently small amount of consumption (for  $x^i(g_v) < 1$ ), higher values of the consumption activity may come from commodities of lower quality. This fact, which can be economically interpreted as undesirability of quality when consumption is quantitatively scarce, follows from the different marginal contributions provided by high-quality and low-quality commodities to the value of the consumption activity. The contribution of high-quality commodities is tiny for small amounts of consumption, and increases exponentially with quantity. The opposite holds for low-quality commodities, whose contribution is huge for small amounts of consumption and decreases exponentially with quantity. As I shall show in section 4, households accordingly consume low-quality commodities when they get low incomes, and switch to high-quality commodities when they reach higher standards of living.

Following from the above interpretation of quality desirability, and given that the “take over” (the quantity at which high-quality commodities begin yielding higher values of consumption activity than low-quality commodities) happens at  $x^i(g_v) = 1$ , we may interpret a unit amount of consumption as the minimum quantity of a commodity that properly satisfies one need. On the one hand, for amounts of consumption smaller than unit, households feel quantitatively dissatisfied and are willing to replace quality with quantity: in this sense, quality is undesirable because of scarcity of quantitative consumption. On the other, for amounts of consumption greater than unit, households feel quantitatively satisfied and are willing to replace quantity with quality. The latter thus becomes increasingly desirable with rising income.

### 3.2 The Utility Function

Using the aggregates (2) and (3), the power consumption technology (4) is nested into (1) to get an explicit form of utility function, given by:

$$U^i = \frac{\left[ \sum_{v=1}^V \left( \int_0^\infty [x^i(g_v)]^{q(g_v)} dg_v \right)^\alpha \right]^{\frac{1-\sigma}{\alpha}} - 1}{1 - \sigma} \quad (5)$$

I assume that commodities in different varieties are *independent* in the sense that there exist no cross-effect of consumption of a commodity belonging to one variety on marginal utility of

commodities of other varieties.<sup>6</sup> The relevant effect resulting from (5) is easily eliminated by imposing  $\alpha = 1 - \sigma$ . Utility (5) can be further simplified by assuming a unit elasticity of marginal utility to consumption, which implies that  $\sigma = 1$ . To avoid indefiniteness of the utility function, the above assumption would require the variety set to be normalized to a unit interval. This can be, however, avoided by adopting a direct definition of log-utility, and redefining aggregate consumption in logarithmic terms, to get  $U^i = \log C^i \equiv \sum_{v=1}^V \log C_v^i$  as in Grossman and Helpman (1991). Using (3) and (4), we finally obtain the form:<sup>7</sup>

$$U^i = \sum_{v=1}^V \log \left( \int_0^\infty [x^i(g_v)]^{q(g_v)} dg_v \right). \quad (6)$$

The utility function (6) takes the familiar logarithmic form widely used in the economic literature. The only difference lies in the change from linear to power consumption technologies.

A first hint of the consequences of using the new preference formalisation is well captured by computing marginal utility of consumption. About this, it is worth stressing that the decision problem faced by the households in a market economy is to choose quantitative levels of consumption of the different commodities available for purchase, in order to satisfy a given set of needs. The household's object of choice is solely the amount of consumption. Characterisation of matters such as marginal utility of consumption and concavity must be taken, for each commodity  $g_v$ , with respect to the variable  $x^i(g_v)$ . According to the chain rule, marginal utility of consumption can be expressed as  $D_{x^i(g_v)} U^i = D_{c^i(g_v)} U^i \cdot D_{x^i(g_v)} c^i(g_v)$ , where  $D_{c^i(g_v)} U^i = \left( \int_0^\infty c^i(g_v) dg_v \right)^{-1}$  takes the same value for all commodities belonging to the same variety. In addition, since in the absence of qualitative differentiation it holds that  $D_{x^i(g_v)} c^i(g_v) = 1$ , in this case  $D_{c^i(g_v)} U^i$  also corresponds to marginal utility of consumption.

Under the assumption of linear consumption technologies,  $D_{x^i(g_v)} c^i(g_v) = q(g_v)$ , so marginal utility of consumption is proportional to  $D_{c^i(g_v)} U^i$ . In other words,  $D_{x^i(g_v)} U^i$  is a linear function of marginal utility of consumption in the absence of qualitative differentiation. The resulting demand functions are simply rescaled, and the equilibrium conditions are qualitatively analogous to those delivered by the traditional framework. The utility function is in fact homothetic, so the

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<sup>6</sup>I refer to section 2 for a more precise definition of independency among the different commodities.

<sup>7</sup>For future reference, it is useful to define each addend of the sum on the right-hand side of (7) as *variety-specific utility*, and denote it  $U_v^i \equiv \log \left( \int_0^\infty [x^i(g_v)]^{q(g_v)} dg_v \right)$ .

income comparative statics is unable to predict phenomena like those summarised by the Engel's Law. Given prices, the income expansion paths are straight lines, and the different commodities are thus consumed in the same proportions at any level of income.

Under the assumption of power consumption technologies,  $D_{x^i(g_v)}c^i(g_v) = g_v [x^i(g_v)]^{g_v-1}$ , so marginal utility of consumption takes a more complex form. As discussed in section 3, the marginal contribution of quantity to the value of the consumption activity varies with the amount of commodity consumed, and quality plays a crucial role in shaping the functional form of this contribution. In other words, the equilibrium conditions are dramatically different from those delivered by the traditional framework. The utility function is non-homothetic, the income expansion paths are curvilinear, so the different commodities are consumed in different proportions at each level of income. In short, the resulting model is able to explain the fundamentals of the Engel's Law through commodity differentiation.

Another aspect is important to notice. Although consumption technologies are convex for high-quality commodities, the utility function is concave in  $x^i(g_v)$  with regard to any commodity. This can be easily seen by considering a single-good economy. Utility (6) simplifies to:

$$U^i = q(g_v) \ln x^i(g_v). \quad (7)$$

As required by standard microeconomic axioms, marginal utility of consumption, given by  $D_{x^i(g_v)}U^i = q(g_v) / x^i(g_v)$ , is positive and decreasing ( $D_{x^i(g_v)}^2U^i = -q(g_v) / [x^i(g_v)]^2$ ). Utility is thus concave with regard to consumption of commodities of all quality levels.

Finally, it is worth pointing out that the effects of quality set another difference with frameworks based on linear consumption technologies. Marginal utility of quality, given by  $D_{q(g_v)}U^i = \ln x^i(g_v)$ , is positive, invariant to the level of quality, and log-proportional to quantity. The effect of quality on marginal utility of quantity, measured by  $D_{x^i(g_v),q(g_v)}^2U^i = [x^i(g_v)]^{-1}$ , is also positive. While the results on concavity and marginal utility of consumption are in line with the traditional specification of preferences, the effects of quality conversely set a distinctive feature: in the traditional models, marginal utility of quality is decreasing and unaffected by quantity; in addition, quality has no effect on marginal utility of consumption. In my framework, marginal utility of quality is non-decreasing and depending on quantity. More importantly, quality posi-

tively affects marginal utility of consumption. These interactions between quality and quantity represent the break with the existing literature of commodity differentiation, and constitute the first step towards an endogenous explanation of the Engel's Law.

### 3.3 The Production Side

As this paper is only concerned with the analysis of the household's behaviour, the production side of the economy is kept as simple as possible. I assume manufacturing has no fixed cost, and each commodity  $g_v$  has a specific constant marginal cost  $w(g_v)$ , positively related to its quality, that is  $D_{g_v} w(g_v) > 0$ . Markets are perfectly competitive, and firms inelastically offer each commodity at a price, denoted by  $p(g_v)$ , determined by the minimum cost requirement, i.e.  $p(g_v) \equiv w(g_v)$ .

The household's resources are therefore constrained by the following linear budget constraint:

$$\sum_{v=1}^V \left( \int_0^{\infty} w(g_v) x^i(g_v) dg_v \right) = W, \quad (8)$$

where  $x^i(g_v)$  measures the quantity consumed of commodity  $g_v$ , and  $W$  represents the household's income.

## 4 A Heuristic Approach to Household's Optimisation

The household's problem consists of maximising the utility function (6) by choosing quantitative levels of consumption of the different commodities available for purchase, subject to the budget constraint (8). The ingredients households observe to solve this problem are, for each pair of commodities  $g_v$  and  $h_z$  (where it must hold that either  $g \neq h$  or  $v \neq z$ , but not necessarily together): 1) the marginal rate of substitution, denoted by  $MRS(g_v, h_z) = - [D_{x^i(g_v)} U^i] / [D_{x^i(h_z)} U^i]$ ; 2) the price ratio  $w(g_v) / w(h_z)$ ; 3) concavity of the indifference curves. These ingredients may vary depending on the pair of commodity considered. In particular, for commodities belonging to the same variety (i.e.  $v = z$ ), concavity of the indifference curve is tightly linked to those of the consumption technologies.

In the economic models of households behaviour making use of linear consumption tech-

nologies, the indifference curves associated to grades of the same variety are invariably linear. The marginal rate of substitution, given by  $MRS(g_v, h_v) = -g_v/h_v$  (with  $g_v = h_v = 1$  if the traditional consumption technology is adopted), is constant. Unless  $w(g_v)/w(h_v) = g_v/h_v$  (in which case households are indifferent to consume any mixed bundle of the relevant commodities), households purchase either grade, depending on the comparison of the price ratio to the marginal rate of substitution. They consume only commodities  $g_v$  if  $w(g_v)/w(h_v) < g_v/h_v$ ; only commodities  $h_v$  if the opposite holds. In addition, the price ratio holding constant, this choice is conclusive in that it does not change with varying income.

A number of fundamental features are introduced by using power consumption technologies. First, households specialise in consumption only when quality of the commodities is sufficiently high. Conversely, they opt for a mixed consumption bundle when quality is sufficiently low. Finally, in a context of varied qualities (i.e. when high-quality and low-quality grades are jointly considered), both the cases of a mixed bundle and of consumption specialisation may arise. As we shall see, all equilibria are contingent to the level of income.

#### 4.1 High-Quality Grades of the Same Variety

Suppose first that quality of all commodities is sufficiently high. In this case, the indifference curves are concave, since the marginal rate of substitution between each pair of grades  $g_v, h_v \in \mathbb{G}_v$ , given by

$$MRS(g_v, h_v) = -\frac{g_v [x^i(g_v)]^{g_v-1}}{h_v [x^i(h_v)]^{h_v-1}}, \quad (9)$$

is negative and monotonically decreasing from 0 to  $-\infty$ . The equilibrium always requires a corner solution.

A graphical illustration is portrayed by figure 4 for a a generic pair of grades such that  $q(h_v) < q(g_v)$ . At the level  $W$  of income, the budget line is depicted by BC. The equilibrium is identified by the intersection point E between BC and the indifference curve IC associated to the highest possible utility. Households consume only grade  $h_v$ . If income rises to  $W'$ , then the budget line shift upwards to the position BC'. The indifference curve IC' intersects BC' in two points, both denoted by E'. The new solution implies that households are indifferent between consuming either grade  $g_v$  or  $h_v$ . Any mixed bundle of the two commodities is, however, excluded

from the set of potential solutions, as it would belong to an indifference curve associated to a lower utility. A further income rise to  $W''$ , taking the budget line to the position  $BC''$ , identifies a new equilibrium in  $E''$ : households specialise in consumption of grade  $g_v$ .

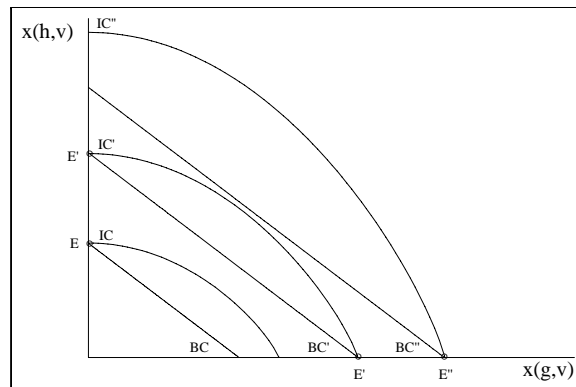


Figure 4 – Optimal Choice: The Case of High-Quality Commodities

By repeating the same argument for all possible levels of income, we obtain a *locus* of equilibrium points known as the *income expansion path*, denoted by IEP. This curve is used to characterize the demand of a commodity, according to its reaction to income variations. In figure 4, IEP follows the segment  $OE'$  along the  $y$ -axis for low incomes, and the half-line on the  $x$ -axis from  $E'$  to infinity for high incomes. While no direct characterization follows from the analysis of *IEP* as obtained in figure 4, it is apparent the households turn to consuming commodities of higher quality with rising income.<sup>8</sup>

The rationale of the above result lies in the definition of the consumption technologies. As income rises, households are willing to increase the amount of consumption. From figure 3, we know that increasing consumption implies, for high-quality grades, rising marginal contribution to the value of the consumption activity. To maximise this contribution, households devote all resources to purchase a single grade, demanding the one producing the highest output (in consumption technology terms). If income is low, then the quantity is low, the marginal contribution is modest for all grades and, considering the same quality for all commodities, higher for grade of lower quality. By setting different affordable quantities, the price differentials greatly influence

<sup>8</sup>Mas-Colell, et al. (1995) consider the grades replaced in the upgrading mechanism due to rising income as *inferior goods*. This mechanism is in turn regarded as an ordinary phenomena when commodities are very disaggregated. See footnote 2 for further details.

the marginal contribution of each grade. With rising income, the latter increases exponentially. As this occurs at different speed for different grades, the effect of price differentials is gradually offset, and households accordingly switch from consuming one grade to another, whose marginal contribution rising speed is higher. Since the former is by construction the grade of lower quality, the optimal choice turns in favour of commodities of higher quality.

## 4.2 Low-Quality Grades of the Same Variety

Suppose now that quality of all commodities is sufficiently low. In this case, the indifference curves are convex, since the marginal rate of substitution between each pair of grades  $g_v, h_v \in \mathbb{G}_v$ , given by (9), is negative and monotonically increasing from  $-\infty$  to 0. The equilibrium always requires an interior solution.

Again, I turn to a graphical illustration to assist the analysis. This is portrayed by figure 5 for a generic pair of grades such that  $q(h_v) < q(g_v)$ . At the level  $W$  of income, the budget line is depicted by BC. The equilibrium is identified by the tangent point E between BC and the indifference curve IC associated to the highest possible utility. Households consume a mixed bundle of the two grades. If income rises to  $W'$ , then the budget line shift upwards to the position BC'. The indifference curve IC' is tangent to BC' in the new equilibrium point E'. The new solution also implies that households consume a mixed bundle of the two grades. So it does a further income rise to  $W''$ , taking the budget line to the position BC'' and identifying, by tangency to the indifference curve IC'', another mixed consumption bundle in E''.

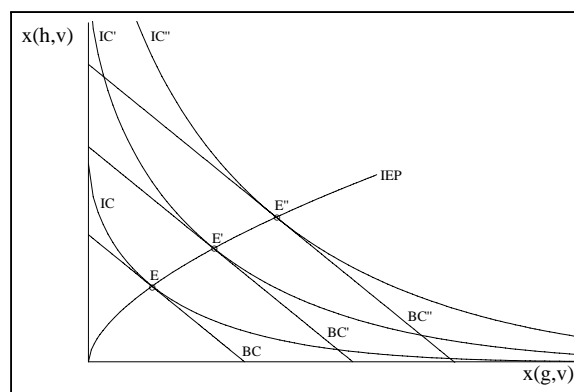


Figure 5 – Optimal Choice: The Case of Low-Quality Commodities



By repeating the same argument for all possible levels of income, we obtain the income expansion path denoted by IEP. In figure 5,  $IEP$  bends towards the  $x$ -axis. According to the standard microeconomic definition, commodity  $g_v$  can be characterized as a *luxury good* when compared to commodity  $h_v$ , which is in turn regarded as a *necessary good*. This example clearly shows that, being able to characterize a commodity as luxury or necessity according to its quality level, a framework with neoclassical preferences and power consumption technology is able to give an endogenous explanation of the Engel's Law based on commodity differentiation.

The rationale of the above result still lies in the definition of the consumption technologies, although it is hardly the same as that discussed for high-quality commodities. As mentioned above, rising income induces households to increase the amount of consumption. From figure 3, we know that increasing consumption implies, for any low-quality grade, a drop in the marginal contribution to the value of the consumption activity. To maximise this contribution, households raise the demand of each commodity in such a way that the marginal contribution remains proportional among the different grades. Rising income implies an increase in the amount of consumption and a fall in the marginal contribution of each commodity. This fall happens at different speed for different grades. For those grades whose marginal contribution falls faster, a smaller increase in demand is required than for those grades whose marginal contribution is more stable. Since the former are, by construction, the grades of lower quality, the proportions on income spent on the different commodities change in favour of those of higher quality.

### 4.3 Low-Quality versus High-Quality Grades

Suppose finally that the existence of commodities of all quality levels. In this case, the indifference curves are partly concave and partly convex. The marginal rate of substitution between each pair of grades  $g_v, h_v \in \mathbb{G}_v$ , given by (9), is negative, starts from 0, first decreases and then rises back to 0. The equilibrium may deliver either a corner or an interior solution.

Once more, a graphical illustration, portrayed by figure 6 for a generic pair of grades such that  $q(h_v) < 1 < q(g_v)$ , may help in describing my findings. At the income level  $W$ , the budget line is depicted by BC. Households consume only grade  $h_v$ , as illustrated by the equilibrium point  $E$ , identified by the intersection of the indifference curve IC associated to the highest possible

utility to BC. If income rises to  $W'$ , then the budget line shifts upwards to the position BC'. The indifference curve IC' is tangent to BC' and also intersects it, as the two equilibrium points labelled  $E'$  show. Households are indifferent between consuming only grade  $h_v$  and purchasing a mixed bundle of the two grades. A further income rise to  $W''$ , taking the budget line to the position BC'', identifies a new equilibrium in  $E''$ , in which the indifference curve IC'' is tangent to the budget line. Households opt for consuming a mixed bundle of the two grades.

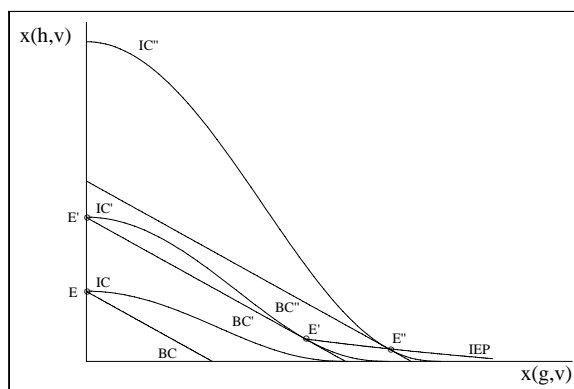


Figure 6 – Optimal Choice: The case of Commodities of Mixed Qualities

By repeating the same argument for all possible levels of income, we obtain the income expansion path, depicted by IEP. In figure 6, IEP bends *backwards* towards the  $x$ -axis. This implies that, with rising income, the increase in the demand for grade  $g_v$  is accompanied by a fall in the demand for grade  $h_v$ . According to microeconomic textbooks, a commodity (in this case,  $h_v$ ) whose demand falls with rising income is to be interpreted as an *inferior good*, as opposed to those commodities (like  $g_v$ ), named *normal goods*, whose demand increases.<sup>9</sup>

The rationale of the above result lies once more in the definition of the consumption technologies and, as it may be expected, combines elements of those holding for low-quality and high-quality commodities. At a sufficiently low level of income, households specialize in consumption of low-quality grades. Rising income induce households to increase the amount of consumption. As figure 3 shows, it causes the marginal contribution of low-quality grades to the value of the consumption activity to fall. When income is sufficiently high, the marginal contribution of high-quality grades, which increases with rising consumption, is such that households

<sup>9</sup>Once again, commodities become inferior goods only at a certain level of income. As a result, the Mas-Colell et al. (1995) argument about this issue apply also here. For further details, see footnote 2.

can afford a mixed bundle of low-quality and high-quality commodities delivering equal marginal contributions. Once the high-quality commodity is introduced in the bundle, further rises in the income levels induce households to increase its demand and decrease that for low-quality grades, to maintain the same (rising) marginal contribution from each commodity.

Notice that there exists no finite level of income such that the low-quality grades come out of the equilibrium bundle. This is due to the fact that their marginal contribution to the consumption technology increases as the amount consumed falls, and tends to infinity as the amount of consumption approaches zero. Setting a null quantity of low-quality grades would then require choosing an infinite amount of consumption of the high-quality commodity, an impossible task under strictly positive price and limited income.

#### 4.4 Commodities of Different Varieties

Let now study another aspect of the household's problem, the choice of the optimal expenditure shares devoted to satisfying the different needs. This analysis concerns commodities of different varieties. In this case, the indifference curves are convex and perfectly well-behaved. The marginal rate of substitution between any pair of commodities  $g_v \in \mathbb{G}_v$  and  $h_z \in \mathbb{G}_z$ ,  $v \neq z$ , given by

$$MRS(g_v, h_z) = -\frac{g_v [x^i(g_v)]^{g_v-1} \int_0^\infty [x^i(h_z)]^{h_z-1} dh_z}{h_z [x^i(h_z)]^{h_z-1} \int_0^\infty [x^i(g_v)]^{g_v-1} dg_v}, \quad (10)$$

is negative and increases from  $-\infty$  to zero. The equilibrium always requires an interior solution.

A graphical illustration is portrayed by figure 7 for a generic pair of commodities belonging to different varieties. For an income level  $W$  such that the budget line is identified by BC, the tangent indifference curve IC depicts the equilibrium point E. Households consume a mixed bundle of commodities of the different varieties, whose proportions are set by equating the marginal rates of substitution to the relevant price ratio. If a rise in the income to level  $W'$  occurs, the budget constraint shifts upwards to the position BC'. A new equilibrium point E' is thus depicted by the tangent indifference curve IC'. Demands for both commodities increase, to such an extent that the expenditure shares devoted to each variety remain constant. A further rise in the level of income to  $W''$  identify, through the same mechanism, a new equilibrium in the point E'', according to which the same expenditure proportions hold.

By repeating the same argument for all possible levels of income, we identify a linear income expansion path, depicted by *IEP*. Following microeconomic textbooks, both commodities are to be considered normal goods, and nothing can be said about the luxury or necessary nature of the commodities.

The rationale of the above result lies in the definition of the utility function. Variety-specific consumption is additively mapped into utility by using the same (logarithmic) functional form. As a result, there is no cross-effect between consumption of one commodity and marginal utility of consumption of a commodity of a different variety and, with rising income and all other conditions holding fixed, each variety displays the same speed in the fall of marginal utility of consumption. Households are therefore induced to keep the same expenditure proportions at each level of income. The above argument does not account, however, for the changes in the optimal grades consumed for each variety, as pointed out by the previous subsections. Since the expenditure proportions do depend on the quality of the different grades, rising income may imply different quality-upgrading dynamics for each variety. Looking at the marginal rate of substitution given above, it is easy to conjecture that the expenditure proportions will change in favour of the variety whose quality evolves faster.

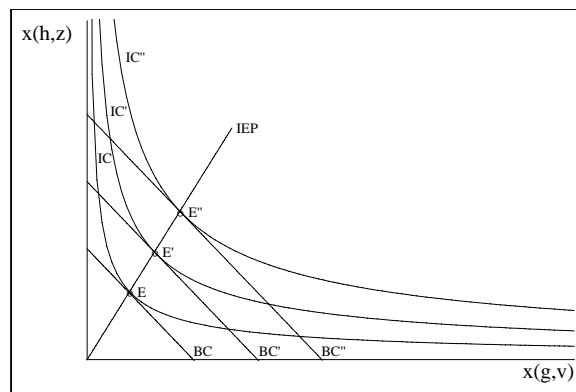


Figure 7 – Optimal Choice: The Case of Commodities of Different Varieties

All things considered, constant expenditure proportions are a specific case when income varies, which happens just if households consume either the same optimal grades for all levels of income, or these bundles change in the same fashion for all varieties. In all other cases, the income expansion path is not linear. The different varieties are to be considered normal goods,

and may be characterized as luxury or necessary goods depending on the speed of the quality turn-over.

## 4.5 Remarks

The joint consideration of section 4.4 together with each of sections 4.1-4.3 completes the study of household's optimization for the three cases analysed in the previous section. By pairing off sections 4.1 and 4.4 we obtain a solution of the consumer's problem in a multiple-variety setting with high-quality commodities. In the same way, sections 4.2. and 4.4 offer the same solution for low-quality grades. The generalised solution, which consider commodities of all possible qualities, is provided by pairing off sections 4.3 and 4.4.

The results obtained in section 4 can thus be appropriately merged to conjecture about the predictions following from the more general approaches just mentioned. With regard to high-quality commodities, it is easy to notice that since households specialise in consumption of a single grade, the integrals in the last term of (10) degenerate to the value of the consumption activity concerning the optimal grade. This term cancels out with the second term of the same equation, and the marginal rate of substitution between two generic varieties therefore reduces to the first term of (10). The income expansion paths are linear only if all optimal grades remain fixed at each level of income, or the quality-upgrading dynamics due to rising income are the same for all varieties. If conversely quality-upgrading of the different varieties happens at different speed, then the value of the first term of (10) varies, and the income expansion paths bend towards the varieties exhibiting faster qualitative evolutions.

The matters are more complicated when considering low-quality commodities, as households consume a mixed bundle of grades at each level of income. This clashes with the analysis in section 4.4, which conversely concerns single grades belonging to different varieties. A different approach to the study should be therefore adopted. We may, however, conjecture that a similar mechanism to that discussed in section 4.2 applies also in this case, even tough it would probably depend on the evolution of the consumption bundle composition within each variety. Given the presence of high-quality commodities (and the consumption specialisation households opt for, according to section 4.1), we may also conjecture that the solution for the more general approach derived

by considering jointly sections 4.3 and 4.4, would possibly be more similar to that regarding high-quality than that about low-quality commodities.

All in all, I can conclude that income rises produce a qualitative evolution in the optimal grades consumed, and this in turn generates a change in the proportions of income spent in satisfying the different needs. By means of this mechanism, my theory is able to explain, through commodity differentiation, the traditional characterization of the different goods due to the reaction of the relevant demand to changes in the level of available resources. A more specific approach, including a formalised solution of the model and the analytical study of its implications, will be the subject of a following article.

## 5 Conclusions

This paper has illustrated a theory that explains the Engel's Law, along with other issues related to the income comparative statics on consumer's demand, through commodity differentiation. The resulting framework, based on a standard specification of preferences, has the main novelty in the introduction of a power consumption technology to value the activity of consuming a given commodity. The income expansion paths resulting from the solution of the household's problem are non-linear, so the model accounts for other dynamics than those following from optimization under homothetic preferences.

I find that commodities of sufficiently low quality must be regarded as inferior goods when compared to commodities of higher quality. A similar definition applies to the grades of lower quality when they are replaced by those of better standards in the consumption specialisation households optimally choose when they face a commodity set made up exclusively of high-quality grades. When facing a set made up of low-quality commodities, on the contrary, households optimally consume a mixed bundle of grades, whose expenditure proportions evolve with rising income in favour of the commodities of higher quality. An analogous mechanism operates for the expenditure shares devoted to satisfying the different needs. In these two cases, the commodities whose expenditure shares increase are regarded as luxury goods, while they are considered necessary goods if the opposite holds.

I conclude by pointing out the an important limit of this approach. Households optimally

specialise in consumption of a single grade when comparing two high-quality commodities. They conversely consume a mixed bundle of grades when comparing a high-quality commodity with a low-quality one. This bizarre prediction, implying that low-quality grades are preferred to all high-quality grades but one with no regard to the relevant relative prices, is due to unboundedness of the marginal contribution of low-quality commodities to the value of the consumption activity. To deliver more likely predictions, a bound to this contribution should be introduced, and possibly be linked to the commodity level of quality, on the basis that an inferior good can hardly satisfy the need as a normal good does, so its consumption could never provide an infinite contribution to the value of the consumption activity. The study of these issues will be part of future research.

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