One Person, Many Votes: Divided Majority and Information Aggregation*

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Abstract

In elections, majority divisions pave the way to focal manipulations and coordination failures, which can lead to the victory of the wrong candidate. This paper shows how this flaw can be addressed if voter preferences over candidates are sensitive to information. We consider two potential sources of divisions: majority voters may have similar preferences but opposite information about the candidates, or opposite preferences. We show that when information is the source of majority divisions, Approval Voting features a unique equilibrium with *full information and coordination equivalence*. That is, it produces the same outcome as if both information and coordination problems could be resolved. Other electoral systems, such as Plurality and Two-Round elections, do not satisfy this equivalence. The second source of division is opposite preferences. Whenever the fraction of voters with such preferences is not too large, Approval Voting still satisfies full information and coordination equivalence.

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1 Introduction

In most electoral systems, even small divisions within the majority can have a dramatic impact on the election outcome. The history of US "first-past-the-post" elections offers many examples; two recent ones being the 1992 and 2000 presidential elections, in which the third candidate, R. Perot in 1992 and R. Nader in 2000, is regularly claimed to have deprived the majority of its victory. The impact of such divisions is almost as important in two-round systems. In 2002 in France, a vote split within the left led the socialist candidate, Lionel Jospin, to lose the first round by a hair's breadth to J.-M. Le Pen, an extreme-right candidate with no chance of winning the second round. Another case is Nicaragua, where the ex-Sandinista D. Ortega won the 2006 election despite being supported only by a minority. He owed his victory primarily to internal divisions among the right-wing majority.[1]

The issue raised by these examples dates back at least to Borda (1781). As shown repeatedly (see e.g. Myerson and Weber 1993, Cox 1997, Myerson 2002, Dewan and Myatt 2007), when a divided majority is facing a unified minority block, electoral systems produce (i) bad equilibria, in which the minority candidate gets elected and (ii) equilibrium multiplicity, which makes elections open to focal manipulations and to coordination failures. Designing an electoral system exempt from such problems proved impossible.

In this paper, we take a step back and reconsider the nature of majority divisions. Traditional formalizations assume that voters have a fixed preference ordering over candidates. We instead propose a model in which the preference ordering can be affected by information: within a voting block, voters agree about the ends but may be divided about the means. That is, in the spirit of Condorcet (1785), they have common policy goals but different information about each candidate's capacity to achieve them.

This introduces state-contingent preferences in the comparison of electoral systems: which of the two majority candidates is best depends on some state of nature. Majority divisions result from voters having opposite beliefs about which state prevails. We compare voting equilibria in three voting systems and find that Approval Voting strictly dominates Plurality and Runoff (or: *two-round*) elections. Contrarily to the latter systems, Approval Voting is not sensitive to majority divisions and produces a unique equilibrium, which saves voters from the risk of focal manipulations and coordination failures. In this unique equilibrium, the winning candidate is necessarily the one preferred by the majority under full information. Thus, Approval Voting satisfies what we call *full information and*

[1] Nicaragua's system is a runoff where a candidate wins in the first round if he obtains more than 35% and a 5-point lead over the nearest competitor. D. Ortega (left-wing) won with 38% because the right-wing majority divided their votes between E. Montealegre (28.3%) and J. Rizo (27.1%).

coordination equivalence.

This result extends directly to the presence of a fraction of voters with *stalwart* preferences. These are voters whose preference ordering over candidates can never be modified. No piece of information, however convincing, can influence their ranking of the candidates. We show that full information and coordination equivalence holds as long as the fraction of these voters is sufficiently small. Evidence and stylized facts provide support for the presence of a substantial fraction of swing voters in the electorate. First, poll patterns are far from flat: many elements of information about candidates or their policy produce massive swings in voter support. Second, presidential ratings are typically much higher after the election than before (Norpoth, 1996; Fox and Phillips, 2003) –our results actually provide a rationale for this initial jump. Finally, several historical incidents, such as the Watergate scandal or Al Qaeda's attack on Madrid demonstrate that the voters' response to information shocks can be largely sufficient to tip the election outcome.

The analysis of Approval Voting began with the works of Weber (1977, 1995) and Brams and Fishburn (1978, 1983) – see also Myerson (2002) and Laslier (2006) for more recent advances. Approval Voting allows voters to "approve of" (or vote for) as many candidates as they wish. Each approval counts as one vote, and the candidate who attracts the largest number of approvals wins the election. This is actually a very natural mechanism. We use it spontaneously when we organize appointments with several people, precisely when we need to aggregate information about their availabilities. The Arbitration Committee of Wikipedia also relies on such a mechanism to resolve disputes, and Approval polls are used to select the Committee itself. Furthermore, Approval Voting is used by many academic societies and by the United Nations, to elect the Secretary-General. Yet, it has never been adopted for head-of-state elections.[2]

Our results contradict two prejudices against Approval Voting, and may also help explain why it has not been implemented in large-scale elections. First, according to traditional analyses, Approval Voting would also display a multiplicity of equilibria (see e.g. Myerson and Weber 1993), and may produce inferior equilibria in which the Condorcet winner fails to be elected (see De Sinopoli *et al.* 2006 and Nuñez 2007). We show that these conclusions are no longer valid when we relax the assumption of perfect information about the value of each electoral outcome. Second, Approval Voting is at times accused of inducing "excessive closeness" among the candidates' results. Nagel (2007) calls this the "Burr dilemma": voters may end up voting indiscriminately for all the candidates in the majority. Similarly, Myerson and Weber (1993, p106) present an example in which all candidates obtain the same vote share in equilibrium. According to our results, the

[2] See also Brams (2007), as well as Laslier and Van der Straeten (2007) who ran a large-scale experiment during the 2002 presidential election in France.

evidence produced by Nagel cannot extend to large electorates: since voters have opposing *ex ante* preferences, they always have an incentive to deviate from a strategy of all voting for the same candidates. In contrast to Myerson and Weber, we introduce information uncertainty: some voters *believe* that the best candidate is A; others *believe* it is B. Yet, all realize that they may be wrong. Hence, each voter has an incentive to also rely on the information held by the other voters. In equilibrium, this incentive will imply that candidate A has the highest expected vote share when she is the best, and conversely when B is actually the best. Third, our results show that the incumbency advantage no longer exists under Approval Voting: leading politicians and parties cannot foreclose entry on the political marketplace (see also Dewan and Myatt 2007). This is because Approval Voting makes experimentation easier for the voters, which stiffens competition and reduces the rents of the main politicians and parties. In our view, this in itself helps explain why Approval Voting did not pervade to head-of-state elections.

Our modeling of large-scale elections draws on the Condorcet Jury Theorem (CJT) literature.[3] We rely on extended Poisson games to model a three-candidate election, and compare voting equilibria across electoral systems in the spirit of Myerson and Weber (1993).[4] Extended Poisson games were introduced by Myerson (1998a), who also shows that they simplify the analysis of the CJT. As in Austen-Smith and Banks (1996), the goal of the electorate in Myerson (1998a) is to select the "best" alternative. Depending on the state of nature, either A or B can be the best, but voters have different prior opinions about these alternatives. One of the main results of the CJT literature is that, in a two-candidate setting, there exists an equilibrium in which the best alternative wins almost certainly, despite the lack of information. This result is robust to changes in the information structure or in the size of the majority required to win – with the notable exception of the unanimity rule (Feddersen and Pesendorfer 1997, 1998).[5]

In our model, the majority always prefers both A and B to a third alternative, C. However, majority-block voters hold opposing convictions as to which of A and B is the best alternative: in the absence of additional information, some prefer A and the others prefer B. They also face opposition by the minority who staunchly supports C. Hence, the majority may be forced to avoid dividing their votes to prevent C from winning the election. In this setup, we analyze the equilibrium properties of Approval Voting, Plurality and Runoff elections. Only Approval Voting produces a unique equilibrium, in which the

[3] For 2-candidate elections, see Austen-Smith and Banks (1996), Feddersen and Pesendorfer (1996 and 1997), Myerson (1998a). For multicandidate elections, see Piketty (2000), Martinelli (2002) and Castanheira (2003).

[4] Though based on Poisson games, the nature of our results extends to multinomial distributions and to the setup of Myerson and Weber (1993).

[5] See Kim and Fey (2007) and Bhattacharya (2007) for precise necessary conditions on voter preferences.

best alternative is the sole likely winner.

The intuition is two-pronged. First, by its very design, Approval Voting allows voters to kill two birds with one ballot: they can vote for their most preferred alternative *and* lend support to their second choice if they perceive C as a threat – this is the classical argument in favor of Approval Voting. Second, we show that the trade-off between splitting majority ballots and eliciting information is drastically different under Approval Voting. This is the rationale for equilibrium uniqueness: when voters know that with some (even tiny) probability, their alternative might be "bad", they want to prevent any of the majority alternatives from being too much ahead of the other.[6] Hence, whenever there is an imbalance between the two alternatives, majority-group voters prefer to vote for both A and B. This reduces the imbalance *and* ensures that C remains weak. Only when vote shares are balanced and there is enough double-voting to drag C behind, will majority voters start to single-vote for their most preferred alternative. This is the channel through which voter preferences generate the information necessary to select the best alternative.

The paper is organized as follows: Section 2 lays out the model. Section 3 identifies actions that are strictly dominated under Approval Voting and identifies pivot probabilities for the remaining actions. Section 4 analyzes equilibrium behavior under Approval Voting. Section 5 and 6 analyze equilibria under Plurality Voting and Runoff respectively. Section 7 shows how our results extend to a population with a continuum of types and to the presence of "stalwart" voters, with opposite preferences. Section 8 concludes.

2 The Model

There are three alternatives, indexed by $P \in \{A, B, C\}$, two states of nature, $\omega \in \{a, b\}$, and three types of voters, $t \in \{t_A, t_B, t_C\}$. Conditional on the state of nature, types t_A and t_B hold identical preferences: they always want to elect the best alternative, which is A in state a and B in state b:

$$\begin{aligned} U(P, t_A, \omega) = U(P, t_B, \omega) &= 1 \text{ if } (P, \omega) = (A, a) \text{ or } (B, b) \\ &= 0 \text{ if } (P, \omega) = (A, b) \text{ or } (B, a) \\ &= -1 \text{ if } P = C, \end{aligned} \tag{1}$$

where $U(P, t, \omega)$ denotes the utility of a voter with type t when alternative P is elected and the true state is ω.

[6] If instead voters assign a zero-probability on their candidate being "bad", equilibrium multiplicity can be an issue (see Myerson and Weber 1993, Nuñez 2007 and Section 7.2 in this paper).

Yet, from an *ex ante* vantage point, types t_A and t_B have opposite convictions regarding alternatives A and B: they hold different beliefs as to which state is most likely. As detailed below, a voter with type t believes that the true state is ω with a probability $q(\omega|t)$. We impose that:

$$\infty > \frac{q(a|t_A)}{q(b|t_A)} > 1 > \frac{q(a|t_B)}{q(b|t_B)} > 0. \tag{2}$$

That is, information is imperfect and divides types t_A and t_B. The former believe that A is most likely to be the best alternative, whereas the latter believe it is alternative B. Additional information on the true state of nature could nevertheless affect these convictions (more on this below).

Types t_C are pure partisans: independently of the true state of nature, they always prefer alternative C. For the sake of tractability, they are also assumed indifferent between the other two alternatives:

$$\begin{aligned} U(P, t_C, \omega) &= 1 \text{ if } P = C \\ &= 0 \text{ if } P \in \{A, B\}. \end{aligned}$$

Timing. At the beginning of the game (**time 0**), nature chooses the state ω, which remains unobserved until after the election. The probabilities of states a and b are respectively $q(a)$ and $q(b)$, with $q(a) + q(b) = 1$. At **time 1**, nature selects a random number of voters from a Poisson distribution of mean n and, conditional on the state, assigns them a type t by iid draws.[7] The conditional probability of being assigned type t is $r(t|\omega)$, with $\sum_t r(t|\omega) = 1$, $\forall\omega$. These probabilities correlate with the true state of nature:

$$\begin{aligned} r(t_A|a) &> r(t_A|b), \\ r(t_B|a) &< r(t_B|b), \\ r(t_C|a) &= r(t_C|b), \end{aligned}$$

and, to ensure that our results cannot hinge on any type of symmetry across types t_A and t_B, we allow types t_A to be potentially more "abundant" than t_B:

$$r(t_A|a) + r(t_A|b) \geq r(t_B|a) + r(t_B|b).$$

Of course, the distribution of voters determines which type has the majority. We focus on the case:

$$r(t_C|\omega) < 1/2,$$

[7] The main properties of extended Poisson games are summarized in Appendix A1 and in the next section, where we also explain why our results extend to multinomial distributions.

which implies that types t_C are a strict minority.[8] Hence, types t_A and t_B compose the *majority block*, whereas types t_C form the *minority block*. The majority's preferred alternative, A or B, thus depends on the state of nature, a or b, which is unknown at the time of the election.

The election is held at **time 2**. The probabilities $q(\omega)$ and $r(t|\omega)$ are common knowledge. In contrast, neither the actual state of nature nor the actual number of voters of each type is observed: voters only know their own type, t. Through Bayesian updating, a voter with type t infers that the probability of state ω is $q(\omega|t)$:

$$q(\omega|t) = \frac{q(\omega)\, r(t|\omega)}{q(a)\, r(t|a) + q(b)\, r(t|b)}. \tag{3}$$

Clearly, condition (2) imposes restrictions on $q(\omega)$ and $r(t|\omega)$. As already explained, (2) implies that t_A and t_B voters are divided. However, these divisions are based only on the voter's perception of the candidates, formally represented by her type. Through Bayesian updating, additional elements of information will affect the voter's beliefs and may therefore modify her preferences over candidates. In particular, the information revealed by the election can have a major impact, by eliciting information about the distribution of preferences in the entire electorate (in Section 7.1, we consider a continuum of types and/or preference intensities).

Payoffs are realized at **time 3**: the winning alternative $W \in \{A, B, C\}$ is selected and each voter's utility $U(W, t, \omega)$ then realizes. In sections 5 and 6, we analyze Plurality and Runoff elections. Here, we only introduce Approval Voting.

Action set under Approval Voting. Under *Approval Voting*, each voter can cast a ballot on as many (or as few) alternatives as she wishes. Each approval counts as one vote: when a voter only approves of A, then only alternative A is credited with one vote. If the voter approves of both A and B, then both A and B are credited with one vote, and so on. Hence, the voters' action set is:

$$\Psi = \{A, B, C, AB, AC, BC, ABC, \varnothing\},$$

where, by an abuse of notation, action A denotes a ballot in favor of A only, action BC denotes a joint approval of B <u>and</u> C, etc., and $\varnothing$ denotes abstention. Thus, the difference between approval voting and other, more common, electoral rules is that a voter can cast a single, a double or a triple approval. Single approvals ($\psi = A$, B and C) act as positive

[8] For $r(t_C|\omega) > 1/2$, a majority of the electorate prefers to have C elected, independently of ω. This case is trivial to investigate: since types t_C are a majority, they can elect C with a probability that converges to 1 when population size increases to infinity.

votes: for instance, an A-vote can only be pivotal in favor of A, against B or against C. In a three-candidate setup, double approvals ($\psi = AB$, BC and AC) act as negative votes. For instance, if the voter plays AC, she is acting against B: her ballot can only be pivotal against that alternative, either in favor of A or of C. Finally, a triple approval (ABC) can never be pivotal: it is strategically equivalent to abstention.

Letting $x(\psi)$ denote the number of voters who played action $\psi \in \Psi$ at time 2, the *total number of approvals* received by alternatives A, B, and C are respectively:

$$\begin{aligned} X(A) &= x(A) + x(AB) + x(AC) + x(ABC), \\ X(B) &= x(B) + x(AB) + x(BC) + x(ABC), \\ X(C) &= x(C) + x(AC) + x(BC) + x(ABC). \end{aligned} \tag{4}$$

The alternative with the largest total number of approvals wins the election. Ties are resolved by the toss of a fair coin. We will see below that, given a Poisson-distributed total size of the population, each random variable $x(\psi)$ itself follows a Poisson distribution. This will imply that each voter has a strictly positive probability of being pivotal.

Strategy space and equilibrium. A type t's *strategy function* is any mapping $\sigma(t) : t \to \psi$ that specifies a probability distribution over the set of actions Ψ for each type t. $\sigma(\psi|t)$ denotes the probability that a randomly sampled voter of type t plays action ψ, and the usual constraints apply: $\sigma(\psi|t) \geq 0$ and $\sum_{\psi} \sigma(\psi|t) = 1$, $\forall t$. This strategy function $\sigma(t)$ reflects the fact that a voter can only condition her strategy on her type t.

Given the strategy function $\sigma(t)$ of each type t, a fraction:

$$\tau(\psi|\omega) = \sum\nolimits_{t} r(t|\omega) \; \sigma(\psi|t) \tag{5}$$

of the electorate is expected to play action ψ in state ω. We call $\tau(\psi|\omega)$ the *expected share of voters* who choose action ψ in state ω. Importantly, if types t_A and t_B play the same strategy $\sigma(t)$, then vote shares $\tau(\psi|\omega)$ are identical in the two states of nature. If instead they play different strategies, then expected shares vary with the state of nature.

We analyze symmetric Bayesian Nash equilibria of this voting game for an expected population size n that becomes infinitely large.[9] We shall say that:

[9] Note that the equilibrium mapping $\sigma(\psi|t)$ *must* be identical for all voters of a same type t, by the very nature of population uncertainty (see Myerson 1998b, p377, for more detail). Therefore, symmetry is necessarily part of the equilibrium. Section 7.1 extends the model to a continuum of types, in which case the equilibrium is in cutoff strategies.

Definition 1 *An equilibrium produces an* ***informational trap*** *if the expected result of the election is independent of the state of nature:*

$$\mathsf{E}_\sigma\left(X\left(P\right)|a\right) = \mathsf{E}_\sigma\left(X\left(P\right)|b\right), \ \forall P \in \{A, B, C\}.$$

In the presence of an informational trap, the outcome of the election does not reveal anything about the actual state of nature: the voter's prior preferences are then unaffected by the election outcome. We will see that this can only happen if $\sigma\left(t_A\right) = \sigma\left(t_B\right)$.

3 Approval Voting: Elimination of Dominated Strategies

The action set contains eight elements. Identifying strictly dominated strategies allows us to focus on only three of them.

Denoting by $\Pr\left(W|\omega\right)$ the probability that alternative $W \in \{A, B, C\}$ wins the election in state ω, the expected utility of a majority-block voter is:

$$\mathsf{E}U\left(t\right) = q\left(a|t\right) \ \left[\Pr\left(A|a\right) - \Pr\left(C|a\right)\right] + q\left(b|t\right) \ \left[\Pr\left(B|b\right) - \Pr\left(C|b\right)\right], \ t \in \{t_A, t_B\}. \quad (6)$$

This reads as follows: having observed her type t, the voter anticipates that the true state of nature is a with probability $q\left(a|t\right)$. In that case, by (1), her utility is 1 if A wins, 0 if B wins, and -1 if C wins. With probability $q\left(b|t\right) \equiv \left[1 - q\left(a|t\right)\right]$ the true state is b. In that case, her payoff is 0 if A wins, 1 if B wins, and -1 if C wins. The expected utility of a minority-block voter is:

$$\mathsf{E}U\left(t_C\right) = \Pr\left(C\right).$$

The value of each action depends on its probability of affecting the outcome of the election, *i.e.* on its probability of being *pivotal*. A ballot can be pivotal in two cases: when an alternative trails behind the leader by **exactly one** vote or when the leading alternatives have **the same** number of votes. It immediately follows that:

Lemma 1 *For a majority-block voter $t \in \{t_A, t_B\}$, in equilibrium:*

$$\sigma\left(A|t\right) + \sigma\left(B|t\right) + \sigma\left(AB|t\right) = 1. \quad (7)$$

For a minority-block voter, action $\psi = C$ is a strictly dominant strategy:

$$\sigma\left(C|t_C\right) = 1. \quad (8)$$

The proof is straightforward: consider a majority-block voter and compare actions AB and ABC. While the latter can never be pivotal, an AB-ballot can be pivotal against

C, either in favor of A or in favor of B. Both events increase a majority-type's expected utility. Hence, AB strictly dominates ABC. All other strict dominance relationships are obtained by performing similar two-by-two comparisons: AB strictly dominates ABC, $\varnothing$ and C; A strictly dominates AC; and B strictly dominates BC.

Lemma 1 tells us that we must focus on three undominated actions. Let $G(\psi|t)$ denote the *expected gain* of these actions, $\psi = A, B, AB$. This gain depends on the voter's type, summarized by $q(\omega|t)$, and on the strategy function $\sigma(t) \equiv \{\sigma(A|t), \sigma(B|t), \sigma(AB|t)\}$ of the other majority-block voters. These strategies determine the expected number of votes received by each alternative, and thereby the pivot probabilities $\Pr(piv_{PQ}|\omega)$:

$$\begin{aligned} G(A|t) &= q(a|t)\ \left[\Pr(piv_{AB}|a) + 2\Pr(piv_{AC}|a)\right] \\ &\quad + q(b|t)\ \left[\Pr(piv_{AC}|b) - \Pr(piv_{AB}|b)\right], \end{aligned} \tag{9}$$

$$\begin{aligned} G(B|t) &= q(a|t)\ \left[\Pr(piv_{BC}|a) - \Pr(piv_{BA}|a)\right] \\ &\quad + q(b|t)\ \left[\Pr(piv_{BA}|b) + 2\Pr(piv_{BC}|b)\right], \end{aligned} \tag{10}$$

$$\begin{aligned} \text{and } G(AB|t) &= q(a|t)\ \left[\Pr(piv_{BC}|a) + 2\Pr(piv_{AC}|a)\right] \\ &\quad + q(b|t)\ \left[\Pr(piv_{AC}|b) + 2\Pr(piv_{BC}|b)\right]. \end{aligned} \tag{11}$$

These pivot probabilities depend on the distribution of the number $x(\psi)$ of voters who play each action ψ. As shown by Myerson (1998a, 1998b, 2000), since the total number of voters follows a Poisson distribution of mean n, the realizations $x(\psi)$ follow mutually independent Poisson distributions: $x(\psi) \sim \mathcal{P}(n \cdot \tau(\psi|\omega))$, where $\tau(\psi|\omega)$ is the expected fraction of voters playing action ψ in state ω (see (5) above).

Under Approval Voting, the number of votes received by alternative A or B is the sum of two independent Poisson random variables: $X(A) = x(A) + x(AB)$ and $X(B) = x(B) + x(AB)$. A pivot probability is therefore the joint probability of two events, each one involving a different pair of candidates:

$$\begin{aligned} \Pr(piv_{PQ}|\omega) &= \frac{1}{2}\Pr(\overbrace{X(Q) - X(P) \in \{0,1\}|\omega}^{Q \text{ is ahead of } P \text{ by 0 or 1 vote}}) \times ... \\ &\qquad ...\Pr(\underbrace{X(R) < X(Q)\,|X(Q) - X(P) \in \{0,1\}, \omega}_{3^d \text{ alternative, } R\text{, trails behind}}) \\ &\quad + \underbrace{\tfrac{2\Pr(X(P)=X(Q)=X(R)|\omega)\ +\ \Pr(X(P)+1=X(Q)=X(R)|\omega)}{3}}_{\text{near three-way tie}} \end{aligned} \tag{12}$$

Property 1 below summarizes some of the properties proven by Myerson (1998a, 1998b, 2000) and extends them to Approval Voting (the proofs are in Appendix A1). Denoting

P_1, P_2 and P_3 the alternatives with respectively the largest, second largest, and lowest expected vote totals, we have:

Property 1 *For an increasingly large electorate size n, the probability that two alternatives P and $Q \in \{A, B, C\}$ have (almost) the same number of votes converges to zero at an exponential rate called the* magnitude *of the probability. We denote it $mag\,(PQ|\omega)$:*

$$mag\,(PQ|\omega) \equiv \lim_{n\to\infty} \frac{\log\left[\Pr\left(|X\,(P) - X\,(Q)| \leq 1|\omega\right)\right]}{n}. \tag{13}$$

The exact form of the different magnitudes $mag\,(PQ|\omega)$ are given in Property 2 in Appendix A1. It follows that:

a) *if two events have a different magnitude, then (Property 3 in Appendix A1):*

$$\lim_{n\to\infty} \tfrac{\Pr(X(P)=X(Q)|\omega)}{\Pr(X(P)=X(R)|\omega')} = 0 \text{ if and only if } mag\,(PQ|\omega) < mag\left(PR|\omega'\right), \tag{14}$$

with $P, Q, R \in \{A, B, C\}, P \neq Q \neq R$ and $\omega, \omega' \in \{a, b\}$.

b) *The magnitude of a pivot probability $\Pr\left(piv_{PQ}\right)$ is such that:*

$$\begin{aligned} mag\,(piv_{PQ}|\omega) &= mag\,(PQ|\omega) \text{ if } \Pr\left(X\,(R) \leq X\,(Q)\,|X\,(Q) - X\,(P) \in \{0,1\}\,,\omega\right) \underset{n\to\infty}{\to} 1 \\ &< mag\,(PQ|\omega) \text{ if } \Pr\left(X\,(R) \leq X\,(Q)\,|X\,(Q) - X\,(P) \in \{0,1\}\,,\omega\right) \underset{n\to\infty}{\to} 0. \end{aligned}$$

c) *Under Approval Voting, the pivot probability with the largest magnitude need not be the one between the top two alternatives. Yet, a sufficient condition for $mag\,(piv_{P_1P_2}|\omega) > mag\,(piv_{P_1P_3}|\omega) \geq mag\,(piv_{P_2P_3}|\omega)$ is that C is one of the top-two contenders in state ω (Property 4 in Appendix A1).*

The result summarized by equations $(13-14)$ has been called the *magnitude theorem* by Myerson (2000). The intuition is that pivot probabilities do not converge towards zero at the same speed. Hence, unless two events have the same magnitude, their likelihood ratio converges either to zero or to infinity. For instance, if two events have a magnitude *differential* of 0.01, their probability *ratio* is of an order of 10^{-44} with 10 000 voters and 10^{-435} with 100 000 voters.

Remark 1 *Note that this magnitude result is far from being specific to Poisson distributions. For instance, Myerson (2000, Section 4) shows that pivot probabilities under multinomial distributions are simply a monotone transformation of their Poisson equivalent.*[10] *Myerson and Weber (1993) rank pivot probabilities in a similar way. Since our*

[10] Myerson (2000) shows that limits of pivot probabilities under Poisson games are such that $\lim_{n\to\infty} \log\left(\Pr\left(piv_{PQ}\right)\right)/n = \mu$. In his Section 4, Myerson (2000) shows that, if the distribution is Multinomial instead of Poisson, then $\lim_{n\to\infty} \log\left(\Pr\left(piv_{PQ}\right)\right)/n = \log(\mu + 1)$, where μ is the limit under the Poisson distribution. Therefore, the limit likelihood ratio (14) is the same under both distributions.

results primarily depends on the magnitude of pivot probabilities, they do not hinge upon the assumption of Poisson games.

In addition to these classical results, Property *1c* tells us that the pivot probability ranking need not correspond to the expected ranking of vote shares. This is because the voters who double-vote for A and B introduce a correlation between $X(A)$ and $X(B)$, which reduces $mag(piv_{AB}|\omega)$. This correlation is taken care of by computing pivot probabilities on the mutually independent variables $x(\psi)$.

Remark 2 *The correlation between the number of votes obtained by candidates A and B implies that the ranking of pivot probabilities need not mirror the ranking of expected vote shares. This contrasts with the simplifying assumptions in Myerson and Weber (1993), who assume such a bijectional relation.*

4 Approval Voting: Equilibrium Analysis

Classically, elections with three or more alternatives suffer from information and coordination problems: which is the best alternative is unclear, and one voter's best response depends on the action profile of the rest of the electorate. In the present setup, under full information, alternative A should win in state a and alternative B should win in state b. Yet, the voters' lack of information means that they cannot make their ballot contingent on the state of nature. What is more, perfect information is not even sufficient to ensure that the best candidate wins. Indeed, voters could experience a coordination failure: as shown in Sections 5 and 6, all majority-block voters may be induced to vote for the same alternative in common electoral systems. We shall say that:

Definition 2 *Elections satisfy* ***full information and coordination equivalence*** *if equilibrium vote shares are such that:*

$$\begin{aligned} &\tau(A|a) + \tau(AB|a) > \max\{\tau(B|a) + \tau(AB|a), \tau(C)\} \text{ in state } a, \text{ and} \\ &\tau(B|b) + \tau(AB|b) > \max\{\tau(A|b) + \tau(AB|b), \tau(C)\} \text{ in state } b. \end{aligned} \tag{15}$$

That is, alternative A's expected vote share must be the largest one in state a and conversely for alternative B in state b. Asymptotically, the winning alternative is then the one preferred by a majority of the population under full information.[11]

[11] This concept of *full information and coordination equivalence* is the natural extension to multicandidate elections of Feddersen and Pesendorfer's (1997) concept of *full information equivalence.*

Typically, satisfying this constraint is not trivial in a three-candidate setting: first, C may win the election if the majority split their votes. Second, even if C is only supported by a small minority, there may be multiple equilibria, and hence a coordination problem. Third, the outcome cannot vary with the state of nature if the majority coordinate on exactly one alternative. Our main contribution is to show that these problems vanish under Approval Voting:

Theorem 1 *Under Approval Voting, the equilibrium is unique and satisfies* full information and coordination equivalence*: the equilibrium strategies are such that* (15) *holds.*

In other words, the possibility of double-voting, which is built into Approval Voting, profoundly modifies the trade-off that is present in other systems. When majority voters can use double-voting to avoid C's victory, coordinating behind only one alternative is both unnecessary and undesired. Indeed, if A's victory is threatened in state a, then even types t_B will be willing to lend support to A by double-voting, i.e. by playing AB. Importantly, this is not only valid when A is threatened by C, but also true when B threatens the victory of A in state a: types t_B understand that the true state might be a. Similarly, when B is threatened in state b, then types t_A will be willing to play AB. Only when A and B's vote shares are sufficiently high compared to C's and balanced with one another, majority voters are willing to divide their votes to aggregate information. As we show below, the simple fact that majority-group voters represent more than half of the electorate is sufficient to ensure that information aggregation takes place in equilibrium.

The purpose of this section is to prove this Theorem. Each of the next two subsections focuses on one aspect of the proof: first, we prove in Propositions 1 and 2 that there cannot be an informational trap under Approval Voting. Second, we derive the equilibrium strategies: Proposition 3 identifies them and shows that they are unique and induce full information and coordination equivalence. Section 7.1 shows why this result is robust to heterogeneity in priors and/or preference intensities and Section 7.2 shows that the result builds on information-sensitive preferences.

4.1 Absence of Informational Traps

In this subsection, we prove that there cannot be informational traps in equilibrium, either in pure or in mixed strategies (remember that informational traps arise if all majority types, t_A and t_B, adopt the same strategy profile in equilibrium). We underline the main trade-off in Proposition 1. Proposition 2 then shows that types t_A and t_B necessarily specialize in playing A and B respectively.

Proposition 1 *There cannot be an informational trap in which all majority-block voters play the same pure strategy. That is, none of the three corner strategies:*

$$\{\sigma(A|t), \sigma(B|t), \sigma(AB|t)\} \in \{\{1,0,0\}, \{0,1,0\}, \{0,0,1\}\}$$

in which all majority block voters $t = t_A, t_B$ play the same action with probability 1 can be an equilibrium.

Proof. See Appendix A2. ■

The intuition is as follows. Imagine that all majority-block voters are expected to play A. This would generate an informational trap, in which case the election result cannot influence voter preferences. In particular, a type t_B still wants to vote for B: this is her *preference motive*. Yet, she knows that, with a vote share of 0, B has virtually no chance of winning. So, her *strategic motive* induces her to support A. Under Approval Voting, t_B-voters can kill these two birds with one ballot: they can combine their strategic motive (vote A) together with their preference motive (vote B) through a joint AB approval; this deviation is always profitable.[12] Hence, approval voting frees voters from the trap of "having to" single-vote for a less attractive candidate.

The balance between these two motives is reversed when all majority voters are expected to double-vote. If they all play AB, alternatives A and B top the polls with the same expected vote share. In this case again, the election outcome cannot reveal any information about the state of nature. This means that types t_A and t_B maintain their prior preferences and that any of them would deviate and single-vote for her preferred alternative. Indeed, a single ballot has a very high probability of making the difference between A and B, whereas the other majority-block voters are already taking care of the strategic motive (trailing behind, C has virtually no chance of winning).

Hence, Proposition 1 eliminates three problematic equilibria. The first two candidate equilibria are the game theoretic materialization of Duverger's Law. In such equilibria, majority-block voters feel compelled to coordinate all their votes on only one alternative (we will see in Sections 5 and 6 that these equilibria exist under Plurality and Runoff elections). Such Duvergerian outcomes pose two problems. They prevent information aggregation and, most importantly, they erect barriers to entry: without sufficient initial

[12] This feature is specific to Approval Voting. Consider any other voting rule, in which the voter must withdraw some "voting points" from A if she wants to also vote for B. In that case, there is a conflict between the preference and strategic motives: the probability that a ballot is pivotal in favor of B is infinitesimal compared to the pivot probability in favor of A. Hence, any "voting points" withdrawn from A has a cost that is infinitely larger than the benefit of the point(s) given to B. As in a prisoner's dilemma, no voter can afford to express her preferences.

support, challengers are bound to lose the election even when a large fringe of the population perceives them as better than incumbent alternatives. Thus, Proposition 1 also shows that the incumbency advantage vanishes under Approval Voting.

The third candidate equilibrium has been termed the *Burr dilemma* by Nagel (2007). He documents the "[approval] *experiment* [that] *ended disastrously in 1800 with the infamous Electoral College tie between Jefferson and Burr*". Proposition 1 shows why such a "disaster" cannot happen in large-scale elections – the Electoral College involved few voters, whose behavior was dictated by party discipline.

Even though Proposition 1 eliminates these three candidate equilibria, it does not ensure that equilibrium vote shares are necessarily different in the two states of nature. Myerson and Weber (1993), for instance, present an example in which all candidates have the same vote share in equilibrium. This is another version of the Burr dilemma: A and B indeed end up in a tie. Our second proposition shows that this cannot happen in our setup: since majority-block voters t_A and t_B "specialize" into playing A and B respectively, there can never be an informational trap:

Proposition 2 *In equilibrium, we must have:* $\sigma(A|t_A) + \sigma(AB|t_A) = 1$ *and* $\sigma(B|t_B) + \sigma(AB|t_B) = 1$ *with* $\sigma(A|t_A) > 0$ *and* $\sigma(B|t_B) > 0$. *Hence, majority-block voters mix between their 'preferred alternative' and the joint AB approval.*

Proof. See Appendix A2. ■

The intuition for the proof is as follows: first, we show that a voter never wants to mix between actions A and B. Such a mixed strategy would imply that she is indifferent between the two alternatives. Expressed differently, the voter does not want to choose between them. However, a safer option is then to play action AB: this action has a higher probability of being pivotal against C, and can never be mistakenly pivotal, *e.g.* in favor of A against B when the true state of nature is b.

This intuition also relates to the "swing voter's curse": in Feddersen and Pesendorfer (1996), voters with imperfect information abstain, to avoid "noising" the election result. Proposition 2 shows why this incentive to abstain is absent under Approval Voting: double-voting is more effective than abstaining when there are more than two candidates.

It remains to see why types t_A and t_B necessarily play A and B with strictly positive probability. To understand this, imagine for a moment that no voter plays $\psi = A$. Even if we constrain $\sigma(A|t)$ to be equal to 0, the vote share of A will be larger in state a than in state b, because types t_A must play AB with a strictly higher probability than types t_B in equilibrium. This difference in vote shares implies that even types t_B do not want to be pivotal against A: they prefer to play AB in pure strategy. This leads to a contradiction: by Proposition 1, it cannot be an equilibrium that all majority types play AB with probability 1. Hence, the action $\psi = A$ must be played with strictly positive probability in equilibrium. Given the preference motive, types t_A can be identified as the ones playing A with strictly positive probability (they never play B), and conversely for types t_B.[13]

4.2 Equilibrium Uniqueness

From Proposition 2, we know that all majority-type voters include their *a priori* preferred alternative in their ballot: since they mix between A and AB, types t_A necessarily approve of A. Types t_B mix between B and AB, which always includes B. Hence, the strategy of a type t_A does not influence the vote count of alternative A. It only influences that of B: the more types t_A double-vote, the higher the expected vote share of B. Likewise, the strategy of a type t_B influences the expected vote count of alternative A.

The vote share of either alternative will thus increase when the incentives of types t_A and t_B become more aligned, *i.e.* when either type feels it must support the other group. Their incentives align, first, when there is a "major imbalance" between the expected vote shares of A and B or, second, when they need to fight alternative C.

A "major imbalance" occurs when either alternative A or B is too much ahead of the other. Imagine for instance that A is expected to receive many more votes than B. In that case, t_A-voters are quite certain that A wins in state a, given its lead. Instead, they are not quite certain that B wins in state b. They thus realize that they have to lend support to B as well: this does not threaten A in state a, but does give B a chance in state b. Hence, they prefer to play AB if they expect a major imbalance in favor of A.

The fight against C aligns incentives in the same way. Imagine that a vote for B is much more likely to be pivotal against C than against A (this happens when A and B's vote shares are not sufficiently above C's). In that case as well, a type t_A prefers to cast a double ballot: it provides additional insurance against the election of C.

These two cases lead to the same conclusion: if B's vote share is too low, either compared to A's or to C's, the incentives of types t_A become aligned with that of types t_B – this is again the strategic motive at work – which induces them to double-vote with a higher probability. By symmetry, if A's vote share is too low, then it is types t_B who

[13] In a setup with fixed preferences, Brams and Fishburn (2007, Theorem 2.1) show that a voter will always include her most preferred alternative in her ballot. One aspect of Proposition 2 is to show how their Theorem extends to voters whose preference ordering is state-dependent.

must lend support to A, and double-vote.

Previous propositions showed that the preference motive dominates when sufficiently many majority voters double-vote. In what follows, we show that there is a unique combination of strategies for types t_A and t_B that can prevent major imbalances between A and B, and a unique "aggregate level" of double-voting that balances the preference and strategic motives. This is why the equilibrium is unique.

Formally, using the expected gain functions $(9) - (11)$, we have:

$$\begin{aligned} G(A|t_A) - G(AB|t_A) \quad &= q(a|t_A)\ \left[\Pr(piv_{AB}|a) - \Pr(piv_{BC}|a)\right] \\ &\quad - q(b|t_A)\ \left[2\Pr(piv_{BC}|b) + \Pr(piv_{AB}|b)\right], \end{aligned} \tag{16}$$

$$\begin{aligned} G(B|t_B) - G(AB|t_B) \quad &= q(b|t_B)\ \left[\Pr(piv_{BA}|b) - \Pr(piv_{AC}|b)\right] \\ &\quad - q(a|t_B)\ \left[2\Pr(piv_{AC}|a) + \Pr(piv_{BA}|a)\right]. \end{aligned} \tag{17}$$

From Proposition 2, types t_A and t_B must single-vote with positive probability in equilibrium. A necessary condition to have $G(A|t_A) - G(AB|t_A) \geq 0$ is that $\Pr(piv_{AB}|a)$ be sufficiently large compared to the other three pivot probabilities in (16). Similarly, a necessary condition to have $G(B|t_B) - G(AB|t_B) \geq 0$ is that $\Pr(piv_{BA}|b)$ be sufficiently large compared to the other three pivot probabilities in (17). From Property 1, this requires:

$$\begin{aligned} mag(piv_{AB}|a) &\geq \max\{mag(piv_{AB}|b), mag(piv_{BC}|a), mag(piv_{BC}|b)\}, \\ mag(piv_{BA}|b) &\geq \max\{mag(piv_{BA}|a), mag(piv_{AC}|a), mag(piv_{AC}|b)\}. \end{aligned} \tag{18}$$

Let us first focus on the constraint that appears between the vote shares of alternatives A and B. The combination of the two inequalities in (18) imposes that the magnitudes $mag(piv_{AB}|a)$ and $mag(piv_{BA}|b)$ be equal. Since they must also be larger than all the magnitudes against C, we have by Property 4 in Appendix A1:

$$\begin{aligned} &\left(\sqrt{r(t_A|a)\cdot\sigma(A|t_A)} - \sqrt{r(t_B|a)\cdot\sigma(B|t_B)}\right)^2 = \\ &\qquad\qquad \left(\sqrt{r(t_A|b)\cdot\sigma(A|t_A)} - \sqrt{r(t_B|b)\cdot\sigma(B|t_B)}\right)^2. \end{aligned} \tag{19}$$

This condition depends on the two strategy profiles, $\sigma(A|t_A)$ and $\sigma(B|t_B)$. Yet, defining:

$$\rho \equiv \sigma(A|t_A)/\sigma(B|t_B),$$

one readily sees that condition (19) is satisfied iff:

$$\left|\sqrt{r(t_A|a)\cdot\rho} - \sqrt{r(t_B|a)}\right| = \left|\sqrt{r(t_B|b)} - \sqrt{r(t_A|b)\cdot\rho}\right|,$$

which has a unique solution in $\mathbb{R}^+$:

$$\rho^* = \left(\frac{\sqrt{r(t_B|a)} + \sqrt{r(t_B|b)}}{\sqrt{r(t_A|a)} + \sqrt{r(t_A|b)}}\right)^2. \tag{20}$$

This solution in turn implies: $\tau(A|a) > \tau(B|a)$ and $\tau(A|b) < \tau(B|b)$.

Hence, we are now left with one unknown variable: if we find the equilibrium probability $\sigma(B|t_B)$ with which types t_B single-vote in equilibrium, the value of $\sigma(A|t_A)$ follows immediately. The following proposition shows that there is a unique solution to $\sigma(B|t_B)$. This equilibrium value of $\sigma(B|t_B)$ is the highest one that allows (18) to be satisfied:

Proposition 3 *The equilibrium is unique and such that:*
i) $\sigma(B|t_B) = 1$, $\sigma(A|t_A) = \rho^*$ *iff, for this strategy profile,*

$$mag(piv_{AB}|a) = mag(piv_{AB}|b) \geq \max_{\omega} \{mag(piv_{AC}|\omega), mag(piv_{BC}|\omega)\}.$$

ii) *Otherwise,* $\sigma(B|t_B) = \bar{\sigma}$, $\sigma(A|t_A) = \rho^*\bar{\sigma}$ *with* $\bar{\sigma} \in (0,1)$ *such that:*

$$mag(piv_{AB}|a) = mag(piv_{AB}|b) = \max_{\omega} \{mag(piv_{AC}|\omega), mag(piv_{BC}|\omega)\}. \tag{21}$$

Proof. See Appendix A2. ■

Proposition 3 shows that there is a unique equilibrium value for $\sigma(A|t_A)$ and $\sigma(B|t_B)$. The reason is as follows: whenever C's vote share is sufficiently below that of A and B, the preference motive dominates: types t_B strictly prefer to single-vote for B, and so do types t_A, who want to single-vote for A. This increases the gap between A and B in both states of nature. The only obstacle to furthering this gap is the threat posed by C: if there exists a strategy profile for which (21) binds, then the strategic motive starts dominating again, and both types t_A and t_B prefer to double-vote with a sufficiently high probability. The equilibrium is reached when this strategic motive to beat C balances the preference motive, unless a corner solution is reached. The solution is unique because the perceived threat posed by C decreases monotonically with the fraction of voters who double-vote.

4.3 Numerical Examples

The examples focus on symmetric priors: $q(a) = \frac{1}{2} = q(b)$ and a symmetric distribution of types: $r(t_A|a) = r(t_B|b)$. This is meant to simplify exposition: from (20) and Proposition 3, symmetry imposes that $\sigma^*(A|t_A) = \sigma^*(B|t_B)$. We illustrate the effect of a variation in $r(t_C)$, the size of the minority group in the population, and of a variation in the ratio $r(t_A|a)/r(t_A|b)$, which proxies the quality of the information available to the voters.

Let $r(t_C) = 0.4$, $r(t_A|a) = 0.36$ and $r(t_A|b) = 0.24$. With these parameter values, as for the actual cases discussed in the introduction, the Condorcet loser, C, would asymptotically be sure to win the election if the majority divide their votes. Vote shares would indeed be: $\tau(C) = 0.4 > \tau(A|a) = \tau(B|b) = 0.36 > \tau(A|b) = \tau(B|a) = 0.24$. This implies that we are in case (*ii*) of Proposition 3, and that there must be some double-voting in equilibrium. The equilibrium strategy profile is $\sigma(AB|t_A) = 0.57 = \sigma(AB|t_B)$, which leads to the vote shares and magnitudes illustrated in Table 1.

Table 1: equilibrium vote shares (left) and magnitudes (right).[14]

Vote shares	state a	state b
A	0.497 (first)	0.445 (second)
B	0.445 (second)	0.497 (first)
C	0.4 (third)	0.4 (third)
Total	1.342	1.342

and

Magnitudes	state a	state b
$mag(piv_{AC}\|\omega)$	-0.0052	(small)
$mag(piv_{BC}\|\omega)$	(small)	-0.0052
$mag(piv_{AB}\|\omega)$	-0.0052	-0.0052

This example illustrates the effect of the double-vote: it allows the majority to "inflate" the expected vote shares of both A and B above the share of C. This is why the sum of the three vote shares exceeds 100% of the population: majority-block voters double-vote up to the point at which the magnitude of the pivot probability between A and B is equal to the largest magnitudes against C.

The equilibrium propensity to double-vote is directly related to the size of the minority. If the fraction of types t_C is sufficiently low, majority-group voters do not actually need to double-vote: let $r(t_C) = 0.25$, $r(t_A|a) = 0.45$ and $r(t_A|b) = 0.30$. With these parameter values, the quality of information is the same as in the previous example ($r(t_A|a)/r(t_A|b) = 1.5$) but full information and coordination equivalence obtains even if majority-group voters divide their votes. Indeed, with $\sigma(AB|t_A) = 0 = \sigma(AB|t_B)$, we have: $\tau(A|a) = \tau(B|b) = 0.45 > \tau(A|b) = \tau(B|a) = 0.30 > \tau(C) = 0.25$, and $mag(piv_{AB}|\omega)$ is strictly larger than the other magnitudes. We are therefore in case (*i*) of Proposition 3. More generally, in such a symmetric setup, majority-block voters double-vote in equilibrium if and only if $r(t_C) > r(t_A|b) = r(t_B|a)$ and, the higher is $r(t_C)$, the higher is the majority's propensity to double-vote (holding $r(t_A|a)/r(t_A|b)$ constant).

This shows that double-voting may vanish when $r(t_C)$ falls, and be valuable again when $r(t_C)$ increases. This observation directly links to Brams and Fishburn's (2005) case study of the Institute of Electrical and Electronics Engineers (IEEE). In 1986, because of a split among the majority, the minority-backed candidate almost won the election for the

[14] The pivot probability between the second and third candidates is infinitely lower than the pivot probability between the first and second candidate. In the absence of a closed-form solution for these magnitudes, we cannot compute their exact value.

presidency. This triggered the adoption of Approval Voting by the Institute. Subsequently, both majority divisions and minority size decreased, which induced the IEEE to revert to Plurality Voting. Arguably, the latter decision overlooks the option value of a double-vote:

> *According to the IEEE executive director* [...] *'few of our members were using* [multiple voting...]'. *Brams responded in an e-mail exchange (June 2, 2002) that since "candidates now can get on the ballot with 'relative ease'* [...] *the problem of multiple candidates* [...] *might actually be exacerbated ... and come back to haunt you [IEEE] some day"* (Brams and Fishburn 2005, p16).

Returning to the numerical examples, we now analyze the effect of an improvement in information. Surprisingly, better information induces *more* double-voting in equilibrium. The rationale is as follows: increasing $r(t_A|a)$ and decreasing $r(t_A|b)$ while holding $r(t_C)$ constant implies that, *ceteris paribus*, the gap between the first and the second alternative's vote shares increases. For a given strategy profile, the probability of being pivotal between A and B decreases in magnitude. In comparison, the gap between the first alternative and C does not increase as fast. Hence, the balance between the strategic and preference motives tilts in favor of the former: the relative value of a double vote increases. To illustrate this, set $r(t_A|a) = 0.48$ and $r(t_A|b) = 0.12$ and keep $r(t_C) = 0.4$ as in the first example. We find that $\sigma(AB|t_A) = 0.8580 = \sigma(AB|t_B)$ in equilibrium, and hence:

Table 2: equilibrium vote shares (left) and magnitudes (right).

Vote shares	state a	state b
A	0.583 (first)	0.532 (second)
B	0.532 (second)	0.583 (first)
C	0.4 (third)	0.4 (third)
Total	1.5144	1.5144

and

Magnitudes	state a	state b
$mag(piv_{AC})$	-0.0172	(small)
$mag(piv_{BC})$	(small)	-0.0172
$mag(piv_{AB})$	-0.0172	-0.0172

Compared to the first example, the equilibrium ranking remains the same but there is more double-voting and pivot magnitudes are lower, which means that the probability of a mistake, *i.e.* that A wins in state b or B wins in state a, decreases substantially.

5 Plurality Elections

Now that we have analyzed the properties of Approval Voting, we can compare them with those of other, commonly used, electoral systems. We analyze two such systems: *plurality elections* in this section, and *runoff elections* in the next one.

Under plurality, as under Approval Voting, the alternative receiving the most votes wins the election. The only difference is that voters can only cast a single ballot or abstain. That is, their action set is restricted to: $\Psi_{Plurality} = \{\varnothing, A, B, C\}$. Otherwise, all pivot probabilities remain the same as in Property 1, with the only difference that, by the definition of $\Psi_{Plurality}$, we have: $\sigma(AB|t) = 0$, $\forall t$ and hence $X(P) = x(P)$, $\forall P \in \{A, B, C\}$.

Theorem 2 shows that this single difference between the two electoral procedures is sufficient to induce multiplicity of equilibria. Moreover, as already highlighted by Piketty (2000), many such equilibria fail to produce full information and coordination equivalence:

Theorem 2 *Under plurality elections, there are at least three equilibria. The first and second are self-fulfilling equilibria in which all majority types vote for A (resp. B), because they expect the other alternative, B (resp. A) to receive no vote. These equilibria produce an informational trap.*

In the third equilibrium, majority types adopt different strategies, hence there is no informational trap. Yet, for $\tau(C) > 1/[2 + r(t_A|b)/r(t_A|a)]$, equilibrium vote shares are such that:

$$\tau(C) > \tau(A|a) \simeq \tau(B|b) > \tau(A|b) \simeq \tau(B|a) > 0.$$

In this equilibrium, candidate C wins with a probability that converges to 1 as $n \to \infty$.

Proof. See Appendix A3. ■

6 Runoff Elections

This section analyzes the properties of another commonly used electoral system: *Plurality Runoff elections*, also known as *two-round elections*. In this electoral system, a candidate wins outright if she collects more than 50% of the votes in the first round. If no candidate reaches this 50%-threshold, then a Runoff is organized between the two candidates with the most votes.[15] This Runoff procedure is often proposed as a solution to the coordination failures that lead to informational traps. Piketty (2000) for instance professes that Runoff elections should be able to separate the "communication stage", in which voters learn which of A and B is best, from the "election stage". This intuition finds support in Martinelli (2002) who analyses the equilibrium properties of Plurality Runoff elections

[15] Note that there exists other types of two-round elections in which the threshold for first-round victory is below 50% (for instance in Argentina, Nicaragua, Costa Rica and North Carolina). For an analysis of such two-round elections in Poisson games, see Bouton (2007).

with privately informed voters. However, in his analysis, Martinelli (2002) assumes away the risks that are present in the second round: the majority-backed candidate wins with probability 1. In contrast, we let, in each round, the population follow the same Poisson distribution as under the other electoral systems, which means that the probability of winning is only *asymptotically* equal to 1. As we show here, this implies that, unless types t_C represent a very small part of the electorate, Runoff elections suffer from the same informational traps as Plurality elections.

To show this, we need to check whether the first-period strategies $(\sigma(A,t), \sigma(B,t)) \in \{(1,0),(0,1)\}$ for $t = t_A, t_B$ can be an equilibrium. Solving the game backwards, we are therefore only interested in the subgames in which C reaches the second round. Let us focus on the subgame where A opposes C: in that case, all majority-block voters play $\psi = A$, and all minority-block voters play $\psi = C$. The expected utility of a majority type $t \in \{t_A, t_B\}$ negatively depends on the probability that C wins the election, $\Pr(C)$:

$$\begin{aligned}
\mathsf{E}U(t|A \text{ vs.} C \text{ in 2d round}) &= q(a|t) - \Pr(C) \\
&= q(a|t) - \left(\frac{\Pr\left[\tilde{X}(C)=\tilde{X}(A)\right]}{2} + \Pr\left[\tilde{X}(C) > \tilde{X}(A)\right]\right) \\
&< q(a|t) - \frac{\Pr\left[\tilde{X}(C)=\tilde{X}(A)\right]}{2} = q(a|t) - \Pr\left(piv_{AC}^2\right),
\end{aligned}$$

where $\Pr\left(piv_{AC}^2\right)$ denotes the second-round pivot probability between A and C. By Property 2, $\Pr\left(piv_{AC}^2\right)$ is proportional to:

$$\Pr\left[\tilde{X}(C) = \tilde{X}(A)\right] \propto \exp\left[-\left(\sqrt{1-\tau(C)} - \sqrt{\tau(C)}\right)^2 n\right].$$

This second-round risk influences the incentives of a majority block voter in the first round: consider the first-round strategy profile $\sigma(B|t_B) \to 0$ and $\sigma(B|t_A) = 0$, for which alternative B's expected vote share is vanishingly small. What is a given t_B-voter's best response? If she plays $\psi = A$ and is pivotal to elect A in the first round, she saves herself from the second-round risk. In comparison, action $\psi = B$ is valuable if a second round is organized and if her ballot is pivotal in bringing B to that round.

Comparing the probabilities of each of these events shows that:

Theorem 3 *Under Runoff elections, unless the fraction of types t_C is sufficiently small, there exist two self-fulfilling equilibria in which all majority types play $\psi = A$ (resp. B). These equilibria produce an informational trap.*

Proof. See Appendix A4. ■

The trade-off is self-explanatory. A majority voter has an incentive to abandon a trailing candidate (B in the above case) if the second-round risk is too high compared to the first-round chance of bringing the trailing candidate to the second round. Typically, the larger C's vote share, the higher the second-round risk, and the lower the probability that one vote may bring B to the second round. Surprisingly, even though we only focus on a lower bound of that risk (we only compute the probability that the two candidates tie in the second round), we find that a vote share of C as low as 6.7% is sufficient to generate such informational traps.[16]

Note however that Theorem 3 does *not* claim that there is no equilibrium with full information and coordination equivalence. Runoff elections actually feature many equilibria, and some of them do satisfy this equivalence. This is however immaterial to the analysis, for two reasons. First, the equilibrium under Approval Voting is unique. Approval Voting therefore Pareto-dominates Runoff elections. Second, organizing elections is extremely costly. Runoff elections may therefore cost about twice as much as Approval Voting elections, despite its less desirable properties.

7 Robustness

7.1 Heterogeneous types and cutoff strategies

Throughout, we worked under the assumption of two types of majority voters, who have identical preferences and information. What would happen if they had more heterogeneous priors or preference intensities? We show that this would only slightly affect the shape of the equilibrium under Approval Voting: instead of adopting a symmetric mixed strategy, voters specialize. Those most in favor of A (resp: B) single-vote for A (resp: B) and the moderate double-vote. This is related to Piketty (2000)'s idea of "labour division" between voters. This is also close to the equilibrium in Feddersen and Pesendorfer (1997).

If we compare this equilibrium with our result in symmetric mixed strategies, the proportion of voters who single and double-vote must remain the same. Formally, with probability $r(t_C)$ the voter is a minority type, t_C. With probability $1 - r(t_C)$ the voter is a majority type. In that case, his pre-election priors are $q(a|t) = t$, with $t \in [0, 1]$. Thus, a type t close to 0 strongly believes that B is the best candidate, whereas a type t close to 1 strongly believes that A must win.[17] $F(t|\omega)$ denotes the cumulated distribution

[16] As emphasized in Section 4.1, all our results directly extend to multinomial distributions. In the case of runoff elections, results would even be stronger with such a multinomial distribution. Indeed, the share of C sufficient to generate an informational trap converges to zero as population size increases.

[17] We could also consider different preference intensities (e.g. $U(A|a) = t$): it would yield the same results. The advantage of differentiating beliefs is that the support is then $[0, 1]$ instead of $[0, \infty]$, which makes the analysis more tractable.

of majority types t, with $F(t|\omega) = 0$, $\forall t \leq 0$, and $F(t|\omega) = 1$, $\forall t \geq 1$. In the previous sections, we assumed that $F(.)$ was discontinuous in $q(a|t_B)$ and $q(a|t_A)$. That is, only two types were present, with beliefs $q(a|t_A)$ and $q(a|t_B)$.

Our next proposition shows how our results extend to a continuous distribution of types, when there is first-order stochastic dominance in the distribution of types (this is related to the result in Bhattacharya 2007):

Proposition 4 *For $F(t|a) < F(t|b)$, $\forall t \in (0,1)$ and $F(t|\omega)$ continuous everywhere, there exists a unique equilibrium defined by two cutoffs, θ_A and θ_B such that: $\sigma(B|t) = 1$, $\forall t < \theta_B$, $\sigma(A|t) = 1$, $\forall t > \theta_A$ and $\sigma(AB|t) = 1$, $\forall \theta_B < t < \theta_A$. In this equilibrium, alternative A wins in state a and alternative B wins in state b.*

Proof. First, note that, by (16) and (17), if there exists a type θ_A such that $G(A|\theta_A) = G(AB|\theta_A)$, then $G(A|t') \gtreqless G(AB|t')$ iff $t' \gtreqless \theta_A$, and similarly for $G(B|\theta_B) = G(AB|\theta_B)$.

The rest of the proof follows the same steps as in Propositions 1-3, that lead to Theorem 1: following Proposition 1, $\theta_A = \theta_B = 1$ cannot be an equilibrium: with these cutoffs, all majority voters vote B. But then, all types $t > 1/2$ strictly prefer to deviate and play AB. By symmetry, $\theta_A = \theta_B = 0$ cannot be an equilibrium. Finally, $\theta_A = 1 > \theta_B = 0$ cannot either be an equilibrium: with these cutoffs, all majority voters vote AB. In this case, types $t < 1/2$ prefer to play B and those above $1/2$ prefer A. This shows that there does not exist any equilibrium with an informational trap. Next, following (16) and (17), a necessary condition for indifference is that (18) holds, which requires: $mag(piv_{AB}|a) = mag(piv_{AB}|b) \geq \max_\omega \{mag(piv_{AC}|\omega), mag(piv_{BC}|\omega)\}$. Proposition 3 showed that there exists only one set of vote shares that satisfy these conditions, and that it requires that $\tau(A|a) + \tau(AB|a) > \tau(B|a) + \tau(AB|a)$ and conversely in state b. These vote shares pinpoint the unique equilibrium cutoffs. ■

This proposition demonstrates that our results do not hinge on symmetric strategies nor on the voters' indifference in equilibrium. As shown below, the truly important assumption is that the distribution function is continuous in 0 and in 1: any voter who has pre-election priors strictly between zero and one know that, with some probability –possibly arbitrarily small–, their pre-election ranking of the candidates can be wrong. This is sufficient to make them want to extract information from the rest of the majority block, and Approval Voting gives them the possibility to do so efficiently.

In contrast, the literature traditionally focuses on the assumption that voters assign a zero-probability on the other voters in their group being right. In our setup, a prior preference ordering that can *never* be changed is akin to imposing that all voters have pre-election priors that are exactly $t = 0$ or $t = 1$, and not ε away from these values. Theorem 1, instead imposes that these priors are different from 0 and 1 with probability

1. Let us develop this point in more detail.

7.2 Approval Voting with Purely Partisan Voters

Up to now, we focused on majority voters who have identical, state-contingent, preferences but different information about the candidates. This is polar to the classical, Arrovian specification, in which voters are divided because they rank the candidates differently. As explained in the introduction, equilibrium multiplicity prevails in that case.

A natural question is which result extends to a world in which a fraction of the divisions are due to information, and the complementary fraction is caused by opposite preferences. To analyze this problem, we introduce a positive fraction of stalwart voters, who never change their preference ordering. We will show that full information and coordination equivalence requires that the fraction of such stalwart voters is not too high.

Using the notation for majority types introduced in Section 7.1, all majority voters are insensitive to information when:

$$r(0|\omega) + r(1|\omega) = 1,$$

where $r(0|\omega) = \lim_{t \underset{>}{\to} 0} F(t|\omega)$ and $r(1|\omega) = 1 - \lim_{t \underset{<}{\to} 1} F(t|\omega)$. That is, the distribution of types is entirely concentrated on the "stalwart" types $t = 0$ and $t = 1$.[18] These types are stalwart in the sense that their pre-election prior, $q(a|t)$, is equal to either 0 or 1. Such extreme priors preclude any updating: since they assign a 0-weight to one of the two states of nature, their Bayesian posterior must also assign a probability 0 to that state. Therefore, types $t = 0$ think that B is the best alternative with probability 1 and no piece of information, however convincing, can influence that belief.

Let us show why full information and coordination equivalence does not prevail in such a setup: assume that types $t = 1$ form 55% of the electorate in state a and 51% in state b.[19] In turn, set $r(t_C) = 0.4$ and hence $r(0|a) = 0.05$ and $r(0|b) = 0.09$, which implies that types $t = 0$, who support B, are the smallest group in the population.

Even though types $t = 0$ are the smallest group, there exists an equilibrium in which B wins the election in both states of nature. In this equilibrium, we have: $\sigma(AB|t_A) = 1$, $\sigma(B|t_B) = 1$ and $\sigma(C|t_C) = 1$: 60% of the electorate approve of B and 40% approve of C, whereas alternative A lies in between, with 51% or 55%. To show that this strategy profile is an equilibrium, we apply Properties 2 and 4 in Appendix A1. They reveal that

[18] In contrast, we assumed througout that: $r(0|\omega) + r(1|\omega) = 0$.

[19] The results of this example do not depend on the presence of two states of nature. We maintain them only to remain as close as possible to the initial setup.

the largest magnitude is always the one between B and C:

Magnitudes	state a	state b
$mag(piv_{AC})$	-0.062	-0.097
$mag(piv_{BC})$	-0.02	
$mag(piv_{AB})$	-0.05	-0.09

Given these magnitudes, types $t = 1$ strictly prefer to play $\psi = AB$ since $G(A|t=1) < G(AB|t=1)$, by (16). In contrast, types $t = 0$ strictly prefer to play $\psi = B$: their preferences being insensitive to information, they prefer to take advantage of their lead, and impose the election of B. In this equilibrium indeed, B's total number of approvals can never be inferior to A's, since $\tau(A) = 0$. Hence, alternative A can only win when it ties with B, i.e. when also B can win, which implies $G(B|t=0) > G(AB|t=0)$ by (17).

The contrast between this result and Theorem 1 shows that "swing" or "independent" voters profoundly affect the equilibrium properties of Approval Voting. It also suggests that a sufficiently large fraction of stalwart voters is necessary for inferior equilibria to exist in Approval Voting.

To determine this fraction, consider a setup in which swing and stalwart voters coexist, i.e. $0 < r(0|\omega) + r(1|\omega) < 1$. Proposition 5 identifies the critical fraction of stalwart voters:

Proposition 5 *If:*

$$r(0|\omega) > \frac{1 - 2\sqrt{(1 - r(t_C))\, r(t_C)}}{1 - r(t_C)},$$

i.e. if the fraction of stalwart voters is too large, Approval Voting displays multiple equilibria, and hence does not satisfy full information and coordination equivalence. The threshold is identical for $r(1|\omega)$*.*

Proof. See Appendix A5. ■

Theorem 1 and Proposition 4 showed that Approval Voting produces a unique equilibrium as long as there exists at least one element of information that can modify a voter's preference ordering. This makes the majority willing to share information and learn about the alternatives. Proposition 5 instead shows that equilibrium multiplicity is a concern when there are too many stalwart voters for whom no piece of evidence, as convincing as it might be, can affect their beliefs.

This result also sheds light on the relationship between the fraction of stalwart voters and the nature of the minority. If the minority is very extreme, $r(t_C)$ will be small and the majority be divided between, say, left and right. Proposition 5 shows that full information aggregation will then prevail even if the majority is composed of a relatively large fraction of stalwart voters. If instead the minority is one of the two main parties in the country, $r(t_C)$ is bound to be large. In such a case, the majority block is the other party, with two candidates. Most majority voters must then double-vote to prevent the minority from winning. Hence, a small fraction of stalwart voters may be sufficient to prevent information aggregation. Yet, the majority group should also be expected to be relatively homogeneous, since their choice is between close candidates. The fraction of such stalwart voters should thus, by nature, be extremely small.

8 Conclusion

We argued that one must take account of the voters' sensitiveness to information when studying the properties of electoral systems. Under imperfect information, the voters' preference ranking is bound to depend, among other things, on fresh information about candidate competence, probity or political preferences.

We proposed a model of elections that captures this information imperfection. Voters in the majority are divided about the candidate they prefer but know that they only have a fraction of the information needed to make a fully informed decision. A third candidate, backed by another part of the electorate, runs against the majority. Hence, voters in the majority run the risk of losing the election altogether if they divide their votes.

In this setup, we studied the asymptotic equilibrium properties of three electoral systems and showed that Approval Voting is ideally suited to information aggregation: it produces a unique equilibrium, in which the candidate who wins the election is actually the one preferred by a majority of the electorate under full information. The other two systems, Plurality and Runoff, produce multiple equilibria. This gives rise to coordination problems and allows a bad candidate to win almost certainly.

The reason why Approval Voting dominates the other systems is that majority divisions need not translate into divided votes. Voters can double-vote (that is: approve of their two candidates) both to fight the minority-backed candidate and to balance the support in favor of either majority alternative. Since majority voters can always outnumber the minority, there will always be room for single-voting, which assures that information is elicited in equilibrium and that the best candidate is assured to win.

Arguably, the model focused on a simplified baseline case, but the trade-offs and strategies that emerged are quite general. First, the equilibrium strategy proves extremely intuitive: voters only need to understand that a multiple ballot is valuable whenever a potentially good candidate is too weak or when a disliked candidate gets too strong. Generalizing the setup to a continuum of types shows that the pattern of specialization that emerges is even more intuitive: voters who are more or less indifferent between the majority candidates double-vote, and those most in favor of either candidate single-vote.

Second, these trade-offs should also be robust to several extensions not considered here. Think for instance of a world with more candidates. If there are k candidates in the majority and l candidates in the minority, the trade-off remains identical. As long as their primary objective is to fight one another, both majority-block and minority-block voters will "multiple-vote" for their own candidates. Within the majority, voters will also maintain the balance between all their potentially good candidates, to make sure that the best can win. Indeed, our results have shown that, whenever a candidate trails behind, all voters in the majority want to support her with a multiple ballot. Hence, although the analysis would become much more cumbersome given the number of strategies to consider, the main insights remain.

Finally, we have made the assumption that *all* voters in the majority value information aggregation. In contrast, the literature focuses on the polar case in which no voter updates beliefs. We saw in Section 7.2 that, if the fraction of such stalwart voters is too large, then multiple equilibria may also arise under Approval Voting. The nature of our contribution is thus to emphasize the role of "independent" or "swing" voters in real-world elections. Those have typically been overlooked by voting theory and we showed that they profoundly modify the properties of Approval Voting. A natural question for future research is thus to see how these voters affect the properties of other electoral systems, such as Instant Runoff, the Borda Count or Storable Votes.

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Appendices

Appendix A1 summarizes and extends to Approval Voting some properties of Poisson Games proven by Myerson (1998a, 1998b, 2000). Appendices A2, A3 and A4 demonstrate the claims made in Sections 4, 5 and 6 respectively.

Appendix A1: Some Properties of Poisson Voting Games

Property 2 *(Myerson 2000, Theorem 1 and extension to Approval Voting)*
Subject to $\sum_{\psi\in\{A,B,AB,C\}}\tau(\psi|\omega)=1$*, and for* $\omega\in\{a,b\}$*, given the expected numbers of votes* $n\tau(\omega)$*, the probability that the realized number of votes are* $x=\{x(A),x(B),x(AB),x(C)\}$ *is:*

$$\Pr(x|\tau(\omega)) \underset{n\longrightarrow\infty}{\longrightarrow} \max_{x} \frac{\exp[mag\,[x]]}{\prod\limits_{\psi\in\Psi}\sqrt{2\pi x(\psi)+\frac{\pi}{3}}},$$

$$\text{where: } mag\,[x]=\sum_{\psi}\frac{x(\psi)}{n}\left(1-\log(\frac{x(\psi)}{n\tau(\psi|\omega)})\right)-1 \quad (\leq 0) \tag{22}$$

For a large electorate (n large), the probability that two alternatives P *and* $Q\in\{A,B,C\}$ *have (almost) the same number of votes converges to zero at an exponential rate called the* magnitude *of the probability:*

$$mag\,(PQ|\omega)\equiv\lim_{n\to\infty}\frac{\log\left[\Pr\left(|x(P)-x(Q)|\leq 1|\omega\right)\right]}{n},$$

where the magnitudes $mag\,(PQ|\omega)$ *are given by:*

$$mag\,(AB|\omega) = -\left(\sqrt{\tau(A|\omega)}-\sqrt{\tau(B|\omega)}\right)^2, \tag{23}$$

$$mag\,(AC|\omega) = -\left(\sqrt{\tau(A|\omega)+\tau(AB|\omega)}-\sqrt{\tau(C|\omega)}\right)^2, \tag{24}$$

$$mag\,(BC|\omega) = -\left(\sqrt{\tau(B|\omega)+\tau(AB|\omega)}-\sqrt{\tau(C|\omega)}\right)^2. \tag{25}$$

Proof. (22) is the application of Theorem 1 in Myerson (2000), and (23) , (24) and (25) extend this theorem to Approval Voting. From Theorem 1 in Myerson (2000), the magnitude of the probability that alternatives A and C have (almost) the same number of votes is:

$$\lim_{n\to\infty}\frac{\log[\Pr(|X(C)-X(A)|\leq 1|\omega)]}{n} = \max_{x}\sum_{\psi}\frac{x(\psi)}{n}\left(1-\log\frac{x(\psi)}{n\tau(\psi|\omega)}\right)-1 \tag{26}$$

$$s.t.\ x(A)+x(AB)=x(C)$$

If we denote $x(A)+x(AB)=x$, $x(A)=\alpha x$, and $x(AB)=(1-\alpha)x$, we find that this is maximized in:

$$\begin{aligned} \alpha^*_{AC} &= \frac{\tau(A|\omega)}{\tau(A|\omega)+\tau(AB|\omega)}, \\ x^*_{AC} &= n\sqrt{\tau(C)\left[\tau(A|\omega)+\tau(AB|\omega)\right]}, \\ x(B)^*_{AC} &= n\tau(B|\omega). \end{aligned} \tag{27}$$

Substituting for α^*_{AC}, x^*_{AC}, and $x(B)^*_{AC}$ in (26) thus yields:

$$\lim_{n\to\infty} \frac{\log[\Pr(|X(C)-X(A)|\leq 1|\omega)]}{n} = -\left(\sqrt{\tau(A|\omega)+\tau(AB|\omega)}-\sqrt{\tau(C|\omega)}\right)^2.$$

By analogy:

$$\begin{aligned} \lim_{n\to\infty} \frac{\log[\Pr(|X(C)-X(B)|\leq 1|\omega)]}{n} &= -\left(\sqrt{\tau(B|\omega)+\tau(AB|\omega)}-\sqrt{\tau(C|\omega)}\right)^2, \text{ and} \\ \lim_{n\to\infty} \frac{\log[\Pr(|x(B)-x(A)|\leq 1|\omega)]}{n} &= -\left(\sqrt{\tau(A|\omega)}-\sqrt{\tau(B|\omega)}\right)^2. \end{aligned}$$

Note the symmetry between $mag(PQ)$ and $mag(QP)$:

$$\lim_{n\to\infty} \frac{\log[\Pr(|X(P)-X(Q)|\leq 1|\omega)]}{n} = \lim_{n\to\infty} \frac{\log[\Pr(|X(Q)-X(P)|\leq 1|\omega)]}{n}.$$

■

Property 3 *(Myerson 2000, Corollary 1) The relative probability of two events x and x' converges to ∞ as population size increases to infinity when the magnitude of x is larger than that of x', and conversely:*

$$\begin{aligned} \frac{\Pr(x|\tau(\omega))}{\Pr(x'|\tau(\omega))} & \underset{n\longrightarrow\infty}{\longrightarrow} \infty \text{ if } mag\,[x] > mag\,[x'] \\ & \underset{n\longrightarrow\infty}{\longrightarrow} 0 \text{ if } mag\,[x] < mag\,[x']\,. \end{aligned}$$

Property 4 *If C is expected to rank first in state ω, then, for $\tau(A|\omega)>\tau(B|\omega)$, we have:*

$$mag(piv_{AC}|\omega) > mag(piv_{BC}|\omega) \geq mag(piv_{AB}|\omega).$$

Conversely, for $\tau(A|\omega)<\tau(B|\omega)$, we have $mag(piv_{BC}|\omega) > mag(piv_{AC}|\omega) \geq mag(piv_{AB}|\omega)$. If C is expected to rank second in state ω, then, for $\tau(A|\omega)>\tau(B|\omega)$, we have:

$$mag(piv_{AC}|\omega) > mag(piv_{AB}|\omega) \geq mag(piv_{BC}|\omega).$$

Conversely, for $\tau(A|\omega)<\tau(B|\omega)$, we have $mag(piv_{BC}|\omega) > mag(piv_{AB}|\omega) \geq mag(piv_{AC}|\omega)$. That is, whenever C is expected to rank first or second, the pivot probability between the expected top (resp. bottom) two alternatives has the largest (resp. smallest) magnitude.

Proof. As formally stated in (12), the pivot probability between P and Q is the joint probability of two events. These two events can in fact be viewed as two constraints imposed on the number of votes to make a ballot pivotal: (i) Q is ahead of P by 0 or 1 vote and (ii) the 3^d alternative, R, trails behind. To compute the magnitude of the different pivot probabilities, we use Theorem 1 in Myerson (2000) and impose these constraints. Applying this Theorem to compute the magnitude of the pivot probability between A and C gives:

$$\begin{aligned} mag(piv_{AC}|\omega) &= \max_x \sum_\psi x(\psi)\left(1-\log\frac{x(\psi)}{n\tau(\psi|\omega)}\right)-1 \\ & \quad s.t.\ x(A)+x(AB)=x(C) \text{ and } x(C)\geq x(B)+x(AB) \end{aligned} \tag{28}$$

If we abstract from the constraint $x(C)\geq x(B)+x(AB)$, or if this constraint is not binding, from Property 2, (28) is maximized for α^*_{AC}, x^*_{AC} and $x(B)^*_{AC}$ as defined in (27). Substituting for α^*_{AC}, x^*_{AC}, and $x(B)^*_{AC}$ in (28) yields:

$$mag(piv^*_{AC}|\omega)=\lim_{n\to\infty}\tfrac{\log[\Pr(|X(C)-X(A)|\leq 1|\omega)]}{n}=-\left(\sqrt{\tau(A|\omega)+\tau(AB|\omega)}-\sqrt{\tau(C|\omega)}\right)^2.$$

We refer to this as the *unrestricted* magnitude (denoted by *).

If the constraint is binding, i.e. if $\alpha^*_{AC}x^*_{AC}\leq x(B)^*_{AC}$, the joint probability also depends on another event that has a strictly negative magnitude. Taking this constraint into account implies:

$$mag(piv_{AC}|\omega)\leq mag(piv^*_{AC}|\omega)=-\left(\sqrt{\tau(A|\omega)+\tau(AB|\omega)}-\sqrt{\tau(C|\omega)}\right)^2.$$

By analogy, it is immediate to check that:

$$\begin{aligned} mag(piv_{BC}|\omega) &\leq mag(piv^*_{BC}|\omega)=-\left(\sqrt{\tau(B|\omega)+\tau(AB|\omega)}-\sqrt{\tau(C|\omega)}\right)^2, \\ \text{and } mag(piv_{AB}|\omega) &\leq mag(piv^*_{AB}|\omega)=-\left(\sqrt{\tau(A|\omega)}-\sqrt{\tau(B|\omega)}\right)^2. \end{aligned}$$

Now, note that the three events piv_{AB}, piv_{AC} and piv_{BC} are identical if their respective constraints are binding. Indeed, whatever the event, a binding constraint implies: $x(A)+x(AB)=x(C)=x(B)+x(AB)$. We refer to the magnitude of this binding events as the *restricted* magnitudes (denoted by **):

$$mag(piv^{**}_{AC}|\omega)=mag(piv^{**}_{BC}|\omega)=mag(piv^{**}_{AB}|\omega),$$

which, by definition, are smaller than the lowest unrestricted magnitude:

$$mag(piv^{**}_{AC}|\omega)\leq\min_{P,Q\in\{A,B,C\}} mag\left(piv^*_{PQ}|\omega\right).$$

Having observed this, we are now in a position to prove that, if the expected ranking is $A>C>B$ in state ω, then:

$$mag(piv_{AC}|\omega)>mag(piv_{AB}|\omega)\geq mag(piv_{BC}|\omega). \tag{29}$$

The proof is in 3 steps: first, we compare the *unrestricted* magnitudes and show that:

$$mag\left(piv_{AC}^{*}|\omega\right) > mag\left(piv_{AB}^{*}|\omega\right). \tag{30}$$

This amounts to showing that:

$$\tau(A|\omega) + \tau\left(AB|\omega\right) > \tau(C|\omega) > \tau(B|\omega) + \tau\left(AB|\omega\right) \tag{31}$$

implies:

$$-\left(\sqrt{\tau\left(A|\omega\right) + \tau\left(AB|\omega\right)} - \sqrt{\tau\left(C|\omega\right)}\right)^2 > -\left(\sqrt{\tau\left(A|\omega\right)} - \sqrt{\tau\left(B|\omega\right)}\right)^2. \tag{32}$$

Rearranging terms, we find that (32) holds iff:

$$\sqrt{\tau\left(A|\omega\right)} - \sqrt{\tau\left(B|\omega\right)} > \sqrt{\tau\left(A|\omega\right) + \tau\left(AB|\omega\right)} - \sqrt{\tau\left(C|\omega\right)},$$

which is necessarily true. Hence, the ranking (31) indeed implies (30).

Second, we show that $mag(piv_{AC}|\omega)$ is always equal to the unrestricted magnitude. For this, we need to prove that: $x\left(A\right) + x\left(AB\right) = x\left(C\right)$ implies $x\left(C\right) > x\left(B\right) + x\left(AB\right)$ at the optimum, that is:

$$\alpha_{AC}^{**}x_{AC}^{**} > x\left(B\right)_{AC}^{**}.$$

Using (27) and performing some manipulations, we see that the latter inequality holds iff:

$$\sqrt{\frac{\tau\left(C\right)}{\tau\left(A|\omega\right) + \tau\left(AB|\omega\right)}} > \frac{\tau\left(B|\omega\right)}{\tau\left(A|\omega\right)}, \tag{33}$$

in which both sides are smaller than one. Hence: $\frac{\tau(B|\omega)}{\tau(A|\omega)} \leq \frac{\tau(B|\omega)+\tau(AB|\omega)}{\tau(A|\omega)+\tau(AB|\omega)} \leq \sqrt{\frac{\tau(B|\omega)+\tau(AB|\omega)}{\tau(A|\omega)+\tau(AB|\omega)}}$, and by (31), the last member of this inequality is smaller than $\sqrt{\frac{\tau(C)}{\tau(A|\omega)+\tau(AB|\omega)}}$, which proves that $mag(piv_{AC}|\omega)$ is always unrestricted. Hence $mag(piv_{AC}|\omega)$ is always larger than $mag(piv_{AB}|\omega)$, whether the latter be restricted or not.

Third, to complete the proof that (29) always holds under the expected ranking (31), we prove that $mag(piv_{BC}|\omega)$ is always the *restricted* magnitude $mag(piv_{BC}^{**}|\omega)$, which implies: $mag(piv_{AB}|\omega) \geq mag(piv_{BC}|\omega)$.

Mutatis mutandis, the derivation of the critical values α_{BC}^{**}, x_{BC}^{**}, and $x(A)_{BC}^{**}$ is identical to that of α_{AC}^{**}, x_{AC}^{**}, and $x\left(B\right)_{AC}^{**}$ in (27):

$$\begin{aligned}
\alpha_{BC}^{**} &= \frac{\tau\left(B|\omega\right)}{\tau\left(B|\omega\right) + \tau\left(AB|\omega\right)} \\
x_{BC}^{**} &= n\sqrt{\tau\left(C\right)\left[\tau\left(B|\omega\right) + \tau\left(AB|\omega\right)\right]} \\
x\left(A\right)_{BC}^{**} &= n\tau(A|\omega)
\end{aligned}$$

and the magnitude $mag(piv_{BC}|\omega)$ would be unrestricted iff:

$$\alpha_{BC}^{**}x_{BC}^{**} > x\left(A\right)_{BC}^{**} \tag{34}$$

To show that the latter inequality can never hold, we proceed as with (33) and show that:

$$\sqrt{\frac{\tau\left(C\right)}{\tau\left(B|\omega\right)+\tau\left(AB|\omega\right)}}<\frac{\tau\left(A|\omega\right)}{\tau\left(B|\omega\right)},$$

in which both fractions are larger than one. This implies: $\frac{\tau(A|\omega)}{\tau(B|\omega)} \geq \frac{\tau(A|\omega)+\tau(AB|\omega)}{\tau(B|\omega)+\tau(AB|\omega)} \geq \sqrt{\frac{\tau(A|\omega)+\tau(AB|\omega)}{\tau(B|\omega)+\tau(AB|\omega)}}$ and, by (31), the last member of this inequality is always larger than $\sqrt{\frac{\tau(C)}{\tau(B|\omega)+\tau(AB|\omega)}}$, which proves that $mag(piv_{BC}|\omega)$ is always restricted and completes the proof of (29).

The proof is identical for all the other possible rankings: $C > B > A$, $C > A > B$ and $B > C > A$, which proves the property. ■

Property 5 *(Myerson 2000, Theorem 2)) The probability that two alternatives, $P, Q \in \{A, B, C\}$, receive a number of votes that differs by a constant c ($c << n$) in state of the nature $\omega \in \{a, b\}$, is:*

$$\lim_{n\to\infty}\Pr\left(x(P)=x(Q)+c|\omega,\tau(P|\omega),\tau(Q|\omega)\right)=\left(\frac{\tau(P|\omega)}{\tau(Q|\omega)}\right)^{c/2}\frac{\exp[-\left(\sqrt{\tau(P|\omega)}-\sqrt{\tau(Q|\omega)}\right)^{2}n]}{2\sqrt{\pi n}\left(\tau(P|\omega)\tau(Q|\omega)\right)^{1/4}}.$$

Appendix A2: Proofs for Section 4

Lemma 2

$$G(A|t)\geq G\left(AB|t\right)\Longleftrightarrow\quad\frac{q(b|t)}{q(a|t)}\leq\frac{1}{M_1}\equiv\frac{\Pr(piv_{AB}|a)-\Pr(piv_{BC}|a)}{\Pr(piv_{AB}|b)+2\Pr(piv_{BC}|b)}\tag{35}$$

$$G(B|t)\geq G\left(AB|t\right)\Longleftrightarrow\quad\frac{q(a|t)}{q(b|t)}\leq M_2\equiv\frac{\Pr(piv_{BA}|b)-\Pr(piv_{AC}|b)}{\Pr(piv_{BA}|a)+2\Pr(piv_{AC}|a)}\tag{36}$$

Proof. Immediate from $(9)-(11)$. ■

Proof of Proposition 1.

Conjecture the following strategy functions: $\sigma\left(t_A\right)=\sigma\left(t_B\right)=\{1,0,0\}$. These strategies imply that $\tau\left(\psi|a\right)=\tau\left(\psi|b\right)$, $\forall\psi$. Therefore: $\Pr\left(piv_{PQ}\right)\equiv\Pr\left(piv_{PQ}|a\right)=\Pr\left(piv_{PQ}|b\right)$. Now, we show that playing $\psi=AB$ is a best response to $\sigma\left(t\right)$ for a type t_B:

$$\begin{aligned}G\left(AB|t\right)-G\left(A|t\right)\quad&=q\left(a|t\right)\ \{\Pr\left(piv_{BC}\right)-\Pr\left(piv_{AB}\right)\}\\&\quad+q\left(b|t\right)\ \{2\Pr\left(piv_{BC}\right)+\Pr\left(piv_{AB}\right)\}\\&=\left(1+q\left(b|t\right)\right)\Pr\left(piv_{BC}\right)+\left(q\left(b|t\right)-q\left(a|t\right)\right)\Pr\left(piv_{AB}\right).\end{aligned}\tag{37}$$

Since $q\left(b|t_B\right)>q\left(a|t_B\right)$, all terms in (37) are strictly positive, which proves that a type t_B always wants to deviate from $\sigma\left(t_A\right)=\sigma\left(t_B\right)=\{1,0,0\}$. By symmetry, $\sigma\left(t_A\right)=\sigma\left(t_B\right)=\{0,1,0\}$ cannot be an equilibrium either.

It remains to show that $\sigma\left(t_A\right)=\sigma\left(t_B\right)=\{0,0,1\}$ cannot be an equilibrium. That is, all majority types will never play $\psi=AB$ with probability 1. To see this, note that, by Properties 1 and 2:

$$\lim_{n\to\infty}\frac{\Pr\left(piv_{BC}\right)}{\Pr\left(piv_{AB}\right)}=\lim_{n\to\infty}\frac{\Pr\left(piv_{AC}\right)}{\Pr\left(piv_{BA}\right)}=0,$$

since alternatives A and B are expected to lead the election, with the same vote share.[20] Hence:

$$\begin{aligned} \lim_{n\to\infty} \tfrac{G(AB|t)-G(A|t)}{\Pr(piv_{AB})} &= q\left(b|t\right) - q\left(a|t\right), \\ \lim_{n\to\infty} \tfrac{G(AB|t)-G(B|t)}{\Pr(piv_{BA})} &= q\left(a|t\right) - q\left(b|t\right). \end{aligned}$$

The former value is strictly positive for types t_A and the latter is strictly positive for types t_B. Hence, both types strictly prefer to deviate from a pure AB vote, and single-vote for their preferred alternative. ■

Proof of Proposition 2.

From Proposition 1, we know that majority-block voters never play the same action in pure strategy. It thus remains to show that majority block voters never play the same mixed strategy in equilibrium. We begin by showing that $\sigma\left(A|t\right) > 0$ implies $\sigma\left(B|t\right) = 0$ and conversely, for any $t \in \{t_A, t_B\}$. We use a proof by contradiction.

We know that equilibrium strategies lie on the simplex $\{\sigma\left(A|t\right), \sigma\left(B|t\right), \sigma\left(AB|t\right)\}$. A necessary condition for A and B to be played with positive probability in equilibrium is that, for some $t \in \{t_A, t_B\}$:

$$G(A|t) = G(B|t) \geq G\left(AB|t\right), \tag{38}$$

and, from Lemma 2 (in this Appendix), $G(A|t), G(B|t) \geq G\left(AB|t\right)$ require $\Pr\left(piv_{AB}|a\right) > \Pr\left(piv_{BC}|a\right)$ and $\Pr\left(piv_{BA}|b\right) > \Pr\left(piv_{AC}|b\right)$.

Using (9) and (10), a necessary condition for $G(A|t) = G(B|t)$ is:

$$\frac{q\left(a|t\right)}{q\left(b|t\right)} = \frac{\Pr\left(piv_{BA}|b\right) - \Pr\left(piv_{AC}|b\right) + \Pr\left(piv_{AB}|b\right) + 2\Pr\left(piv_{BC}|b\right)}{\Pr\left(piv_{AB}|a\right) - \Pr\left(piv_{BC}|a\right) + \Pr\left(piv_{BA}|a\right) + 2\Pr\left(piv_{AC}|a\right)}. \tag{39}$$

Now, we prove that (38) can never hold: using Lemma 2, we identify a lower bound for M_1 and an upper bound for M_2. Then, we show that this lower bound for M_1 is strictly larger than the upper bound for M_2, whereas condition (38) requires:

$$M_1 \leq M_2, \tag{40}$$

hence the contradiction.

$M_1 = \frac{\Pr(piv_{AB}|b)+2\Pr(piv_{BC}|b)}{\Pr(piv_{AB}|a)-\Pr(piv_{BC}|a)}$ is strictly increasing in $\Pr\left(piv_{BC}|a\right)$ and $\Pr\left(piv_{BC}|b\right)$. A lower bound to M_1 is thus found by setting these two pivot probabilities equal to 0. Similarly, an upper bound to M_2 is found by setting $\Pr\left(piv_{AC}|a\right)$ and $\Pr\left(piv_{AC}|b\right)$ equal to zero. This establishes that:

$$\frac{\Pr\left(piv_{AB}|b\right)}{\Pr\left(piv_{AB}|a\right)} < M_1 \text{ and } M_2 < \frac{\Pr\left(piv_{BA}|b\right)}{\Pr\left(piv_{BA}|a\right)}, \tag{41}$$

and hence that a necessary condition for (40) is that:

$$\frac{\Pr\left(piv_{AB}|b\right)}{\Pr\left(piv_{BA}|b\right)} \frac{\Pr\left(piv_{BA}|a\right)}{\Pr\left(piv_{AB}|a\right)} < 1.$$

[20] We have two strategies being played: minority types play C and majority types play AB. Hence: $\tau\left(AB|\omega\right) = \left(1 - r\left(t_C\right)\right) > \tau\left(C\right) = r\left(t_C\right)$ and $\tau\left(A|\omega\right) = \tau\left(B|\omega\right) = 0$. Applying Property 1 yields the result.

Using Property 5 (in Appendix A1), the left-hand side of this expression is equal to:

$$\sqrt{\frac{\tau\left(A|a\right)}{\tau\left(A|b\right)}\frac{\tau\left(B|b\right)}{\tau\left(B|a\right)}},$$

which cannot be smaller than 1. Indeed, by (39), types t_A must vote for A with a higher probability than types t_B, since $\frac{q(a|t_A)}{q(b|t_A)} > \frac{q(a|t_B)}{q(b|t_B)}$. Hence, in equilibrium:

$$\frac{\tau\left(A|a\right)}{\tau\left(A|b\right)} \geq 1 \text{ and } \frac{\tau\left(B|b\right)}{\tau\left(B|a\right)} \geq 1. \tag{42}$$

It follows that $G(A|t) = G(B|t)$ implies $G\left(AB|t\right) > G(A|t)$, and therefore that a strict mixture between A and B is a strictly dominated strategy: $\sigma\left(A|t\right) > 0$ implies $\sigma\left(B|t\right) = 0$ and conversely.

It remains to prove that $\sigma\left(A|t_A\right)$ and $\sigma\left(B|t_B\right)$ are strictly positive in equilibrium. To this end, we show that:

$$\sigma\left(B|t_B\right) > 0 \text{ and } \sigma\left(A|t_A\right) = 0 \tag{43}$$

leads to a contradiction. Indeed, (43) implies $\tau\left(A|\omega\right) = 0$ in both states. Hence, by Property 2:

$$mag\left(piv_{BA}|\omega\right) = -\tau\left(B|\omega\right).$$

By (42), we have: $\tau\left(B|a\right) < \tau\left(B|b\right)$, which implies that $\lim_{n\to\infty} \Pr\left(piv_{BA}|b\right) / \Pr\left(piv_{BA}|a\right) = 0$ and therefore that $\lim_{n\to\infty} M_2 \leq 0$ in Lemma 2. Instead, $\sigma\left(B|t_B\right) > 0$ imposes that M_2 be strictly positive. This shows that $\sigma\left(A|t_A\right) = 0$ contradicts the possibility that $\sigma\left(B|t_B\right) > 0$. By symmetry, we cannot either have: $\sigma\left(A|t_A\right) > 0$ and $\sigma\left(B|t_B\right) = 0$.

Together with Proposition 1 and (42), this proves that, in equilibrium, we must have $\sigma\left(A|t_A\right) > 0$ and $\sigma\left(B|t_B\right) > 0$. From the first part of this proof, this also implies that: $\sigma\left(B|t_A\right) = 0 = \sigma\left(A|t_B\right)$. ■

Proof of Proposition 3.
To prove that there is an unique equilibrium, we proceed in two steps. First, we show that $\sigma\left(A|t_A\right) = \rho^*\sigma\left(B|t_B\right)$ is the unique best response of types t_A given the strategy of types t_B. Second, we prove that there is a unique equilibrium strategy $\sigma^*\left(B|t_B\right)$.

From (18) and (20), we must have in equilibrium:

$$\begin{aligned} mag\left(piv_{AB}|a\right) = mag\left(piv_{AB}|b\right) \geq \quad & \max\{mag\left(piv_{BC}|a\right), mag\left(piv_{BC}|b\right), \\ & mag\left(piv_{AC}|a\right), mag\left(piv_{AC}|b\right)\}. \end{aligned} \tag{44}$$

We can check that types t_A never want to deviate from $\sigma\left(A|t_A\right) = \rho^*\sigma\left(B|t_B\right)$: for any $\sigma\left(A|t_A\right) < \rho^*\sigma\left(B|t_B\right)$, we have $\sigma\left(AB|t_A\right) > 1 - \rho^*\sigma\left(B|t_B\right)$. This implies that the expected share of alternative B increases in both states and hence that: $mag\left(piv_{AB}|a\right)$ increases above $mag\left(piv_{AB}|b\right)$, whereas $mag\left(piv_{BC}|a\right)$ and $mag\left(piv_{BC}|b\right)$ decrease.

Using Lemma 2 and (44), this implies:

$$\frac{q(b|t_A)}{q(a|t_A)} < \lim_{n\to\infty} \frac{1}{M_1} \equiv \frac{\Pr(piv_{AB}|a)-\Pr(piv_{BC}|a)}{\Pr(piv_{AB}|b)+2\Pr(piv_{BC}|b)} = \infty,$$

and hence: $G(A|t_A) > G(AB|t_A)$. Therefore, $\sigma(A|t_A) < \rho^*\sigma(B|t_B)$ cannot be true in equilibrium.

For any $\rho^*\sigma(B|t_B) < 1$, we also have to check that $\sigma(A|t_A) > \rho^*\sigma(B|t_B)$ cannot be an equilibrium either. Following the same procedure as above, one can check that $\sigma(A|t_A) > \rho^*\sigma(B|t_B)$ implies:

$$\frac{q(b|t_A)}{q(a|t_A)} > \lim_{n\to\infty} \frac{1}{M_1} \equiv \frac{\Pr(piv_{AB}|a)-\Pr(piv_{BC}|a)}{\Pr(piv_{AB}|b)+2\Pr(piv_{BC}|b)} \leq 0,$$

which in turn implies $G(A|t) < G(AB|t)$. Hence, $\sigma(A|t_A) > \rho^*\sigma(B|t_B)$ cannot be true in equilibrium. Therefore, when (44) holds, $\sigma^*(A|t_A) = \rho^*\sigma(B|t_B)$ is the unique best response of types t_A to $\sigma(B|t_B)$.

It remains to prove that there is a unique equilibrium strategy $\sigma^*(B|t_B)$, which will always imply (44). Two cases must be considered:

Case 1: $G(B|t_B) - G(AB|t_B) \geq 0$ in $\sigma(B|t_B) = 1$, $\sigma(A|t_A) = \rho^*$.

In that case, $\sigma(B|t_B) = 1$ is the only possible best response for types t_B. Indeed, $\sigma(B|t_B) < 1$ would imply $\sigma(AB|t_B) > 0$. This induces an increase in the expected vote share of alternative A in both states of nature and hence that: $mag(piv_{BA}|b)$ increases above $mag(piv_{BA}|a)$, whereas $mag(piv_{AC}|a)$ and $mag(piv_{AC}|b)$ decrease. Using Lemma 2 and (44), this implies:

$$\frac{q(a|t_B)}{q(b|t_B)} < \lim_{n\to\infty} M_2 \equiv \frac{\Pr(piv_{BA}|b)-\Pr(piv_{AC}|b)}{\Pr(piv_{BA}|a)+2\Pr(piv_{AC}|a)} = \infty,$$

and hence $G(B|t_B) > G(AB|t_B)$. Therefore, $\sigma(B|t_B) = 1$ is the unique best response to $\sigma(A|t_A) = \rho^*$.

It remains to show that types t_B would deviate from any $\{\sigma(A|t_A), \sigma(B|t_B)\} = \{\rho^*\sigma, \sigma\}$ if $\sigma < 1$. To this end, we need to show that

$$\lim_{n\to\infty} \frac{G(B|t_B) - G(AB|t_B)}{\Pr(piv_{AB}|a)} = q(b|t_B)\frac{\Pr(piv_{BA}|b)}{\Pr(piv_{AB}|a)} - q(a|t_B)\frac{\Pr(piv_{BA}|a)}{\Pr(piv_{AB}|a)} > 0, \tag{45}$$

for any $\{\sigma(A|t_A), \sigma(B|t_B)\} = \{\rho^*\sigma, \sigma\}$, $\sigma < 1$.

The strategy of the types t_A implies:

$$\begin{aligned} \lim_{n\to\infty} \frac{G(A|t_A) - G(AB|t_A)}{\Pr(piv_{AB}|a)} &= q(a|t_A) - q(b|t_A)\frac{\Pr(piv_{AB}|b)}{\Pr(piv_{AB}|a)} = 0 \\ &\Longrightarrow \frac{\Pr(piv_{AB}|b)}{\Pr(piv_{AB}|a)} = \frac{q(a|t_A)}{q(b|t_A)}. \end{aligned}$$

By Myerson's offset theorem: $\Pr(piv_{BA}|\omega) = \Pr(piv_{AB}|\omega)\sqrt{\frac{\tau(A|\omega)}{\tau(B|\omega)}}$. Hence, (45) can be rewritten as:

$$\frac{q(b|t_B)}{q(a|t_B)}\frac{q(a|t_A)}{q(b|t_A)} > \sqrt{\frac{\tau(A|a)\,\tau(B|b)}{\tau(B|a)\,\tau(A|b)}}.$$

By (3), the left-hand side of this inequality is equal to: $\frac{\tau(A|a)\tau(B|b)}{\tau(B|a)\tau(A|b)} > 1$, which proves that (45) holds.

Case 2: $G(B|t_B) - G(AB|t_B) < 0$ in $\sigma(B|t_B) = 1$, $\sigma(A|t_A) = \rho^*$.

In this case, there must exist a $\bar{\sigma} \in (0,1)$ such that, for $\{\sigma(A|t_A), \sigma(B|t_B)\} = \{\rho^*\bar{\sigma}, \bar{\sigma}\}$, we have: $G(B|t_B) - G(AB|t_B) = 0$. Indeed, by Proposition 1, $G(B|t_B) - G(AB|t_B) > 0$ for $\{\sigma(A|t_A), \sigma(B|t_B)\} = \{0, 0\}$. The existence of $\bar{\sigma}$ immediately follows from the continuity of the G function.

This value of $\bar{\sigma}$ is unique and such that:

$$\begin{aligned} mag(piv_{AB}|a) = mag(piv_{AB}|b) = \quad & \max\{mag(piv_{BC}|a), mag(piv_{BC}|b), \\ & mag(piv_{AC}|a), mag(piv_{AC}|b)\}. \end{aligned} \tag{46}$$

Indeed, any $\sigma < \bar{\sigma}$ implies that the total expected vote shares of alternatives A and B increase. Since (46) implies that C is third in both states, the magnitudes $mag(piv_{PC}|\omega)$ must decrease, for any $P \in \{A, B\}$ and $\omega \in \{a, b\}$. In contrast, the magnitudes $mag(piv_{AB}|\omega)$ must increase, since:

$$\begin{aligned} mag(piv_{AB}|a) = mag(piv_{AB}|b) &= \left(\sqrt{r(t_A|a)\cdot\rho^*\sigma} - \sqrt{r(t_B|a)\cdot\sigma}\right)^2 \\ &= \left(\sqrt{r(t_A|a)\cdot\rho^*} - \sqrt{r(t_B|a)}\right)^2 \sigma \end{aligned}$$

is strictly increasing in σ. Hence (44) holds with a strict inequality for any $\sigma < \bar{\sigma}$. This implies that (45) holds, and hence that $G(B|t_B) - G(AB|t_B) > 0$ for any $\{\sigma(A|t_A), \sigma(B|t_B)\} = \{\rho^*\sigma, \sigma\}$, $\sigma < \bar{\sigma}$.

Similarly, one can check that (44) is violated for any $\sigma > \bar{\sigma}$ which implies $G(B|t_B) - G(AB|t_B) < 0$ for any $\{\sigma(A|t_A), \sigma(B|t_B)\} = \{\rho^*\sigma, \sigma\}$, $\sigma > \bar{\sigma}$. This proves that (46) must hold in $\{\sigma(A|t_A), \sigma(B|t_B)\} = \{\rho^*\bar{\sigma}, \bar{\sigma}\}$, and that the solution to $\bar{\sigma}$ is unique. ■

Appendix A3: Proof for Section 5

Proof of Theorem 2.

1) First, we prove that, for all majority types $t \in \{t_A, t_B\}$, $G(A|t) - G(B|t)$ is strictly positive if $\tau(B|\omega) \to 0$. This proves that, if B is expected to receive too few votes, all majority types strictly prefer to vote for A. By symmetry, it also proves that all majority types vote for B if they expect A to receive too few votes.

For any strategy profile, we have:

$$\begin{aligned} G(A|t) - G(B|t) = \quad & q(a|t) \quad \{2\Pr(piv_{AC}|a) + \Pr(piv_{AB}|a) + \Pr(piv_{BA}|a) - \Pr(piv_{BC}|a)\} \\ & + q(b|t) \quad \{\Pr(piv_{AC}|b) - \Pr(piv_{AB}|b) - \Pr(piv_{BA}|b) - 2\Pr(piv_{BC}|b)\}. \end{aligned} \tag{47}$$

By (5), for $\tau(B|\omega) \to 0$ we have: $\tau(A|\omega) \to 1 - r(t_C)$. Hence, by Properties 1 and 2, for any given $\omega = a, b$ we have:

$$\lim_{n\to\infty} \frac{\Pr(piv_{BC}|\omega)}{\Pr(piv_{AC}|\omega)} = \lim_{n\to\infty} \frac{\Pr(piv_{AB}|\omega)}{\Pr(piv_{AC}|\omega)} = \lim_{n\to\infty} \frac{\Pr(piv_{BA}|\omega)}{\Pr(piv_{AC}|\omega)} = 0.$$

Hence:

$$\lim_{\substack{n\to\infty \\ \tau(B|\omega)\to 0}} \frac{G(A|t) - G(B|t)}{\Pr(piv_{AC}|a)} = 2q(a|t) + q(b|t)\frac{\Pr(piv_{AC}|b)}{\Pr(piv_{AC}|a)},$$

which is strictly positive. This proves the existence of the two "sunspot" equilibria.

2) Second, we show the existence of the third equilibrium. Following Theorem 2 of Myerson (1998a), if a type $t \in \{t_A, t_B\}$ adopts a strictly mixed strategy, then the other type $t' \neq t$, $t' \in \{t_A, t_B\}$ votes for "his" candidate with probability 1. The reason is that $q(a|t_A) > q(a|t_B)$, which implies $G(A|t_A) - G(B|t_A) > G(A|t_B) - G(B|t_B)$ for any expected voting profile.

Having noted this, we know that a necessary condition for majority-types voters to adopt a different strategy is that:

$$\begin{aligned} &G(A|t_A) - G(B|t_A) \geq 0, \text{ and} \\ &G(A|t_B) - G(B|t_B) \leq 0. \end{aligned} \tag{48}$$

Next, remark that: *a)* pivot probabilities are continuous in the voters' propensity to cast their ballot on A and on B, and *b)* payoffs are bounded. Therefore, the difference $G(A|t) - G(B|t)$ is continuous in the voters' propensity to vote for A, and we can apply Kakutani's fixed point theorem.

Now, consider a strategy profile $\bar{\sigma}$ such that: $\tau(A|a) = \tau(B|b) \equiv \bar{\tau}$. If voters marginally increase their propensity to vote A above $\bar{\sigma}$, we have: $\tau(A|a) > \tau(B|b) > \tau(A|b) > \tau(B|a)$. By Property 1, for any such strategy profile, we have:

$$\begin{aligned} G(A|t) - G(B|t) &> 0 \text{ for both } t \in \{t_A, t_B\}, \text{ if } \tau(C) < \bar{\tau}, \\ G(A|t) - G(B|t) &< 0 \text{ for both } t \in \{t_A, t_B\}, \text{ if } \tau(C) > \bar{\tau}, \end{aligned}$$

and the inequalities are reversed if the voters' propensity to vote for A decreases below $\bar{\tau}$. By the continuity of the payoff functions, (48) must hold in a neighborhood of $\bar{\sigma}$.

Now, we show that, for $\tau(C) > 1/[2 + r(t_A|b)/r(t_A|a)]$, the following strategy profile is an equilibrium:

$$\begin{aligned} \sigma(\varnothing|t_A) &= 0 = \sigma(\varnothing|t_B), \\ \sigma(B|t_B) &= 1, \\ \sigma(A|t_A) &\simeq \frac{r(t_B|b) + r(t_A|b)}{r(t_A|a) + r(t_A|b)}, \text{ and } \sigma(B|t_A) = 1 - \sigma(A|t_A). \end{aligned} \tag{49}$$

For that strategy profile, we have $\tau(A|a) \simeq \tau(B|b) \equiv \bar{\tau}$ and: $\tau(C) > \bar{\tau} > \tau(A|b) \simeq \tau(B|a)$. By Property 1, this implies:

$$\lim_{n\to\infty} \frac{\Pr(piv_{BC}|a)}{\Pr(piv_{AC}|a)} = \lim_{n\to\infty} \frac{\Pr(piv_{AC}|b)}{\Pr(piv_{BC}|b)} = 0.$$

Finally, since alternative C's vote share is the largest of the three in both states of nature, we have by Property 4 in Appendix A1:

$$\lim_{n\to\infty}\frac{\max\left\{\Pr\left(piv_{AB}|a\right),\Pr\left(piv_{BA}|a\right)\right\}}{\Pr\left(piv_{AC}|a\right)}=\lim_{n\to\infty}\frac{\max\left\{\Pr\left(piv_{AB}|b\right),\Pr\left(piv_{BA}|b\right)\right\}}{\Pr\left(piv_{BC}|b\right)}=0.$$

It results that, in $\bar{\sigma}$:

$$\lim_{n\to\infty}\frac{G\left(A|t\right)-G\left(B|t\right)}{\Pr\left(piv_{AC}|a\right)}=2\left[q\left(a|t\right)-q\left(b|t\right)\frac{\Pr\left(piv_{BC}|b\right)}{\Pr\left(piv_{AC}|a\right)}\right],$$

and, by Kakutani's fixed point theorem, there must exist a strategy profile $\sigma(A|t_A)$ in the neighborhood of $\frac{r(t_B|b)+r(t_A|b)}{r(t_A|a)+r(t_A|b)}$ such that: $\lim\limits_{n\to\infty}\frac{G(A|t_A)-G(B|t_A)}{\Pr(piv_{AC}|a)}=0$. It remains to prove that abstention is strictly dominated. To this end, it can be verified that: $G\left(A|t_A\right)>0$ and $G\left(B|t_B\right)>0$, which can be compared to the value of abstention: zero. ■

Appendix A4: Proof for Section 6

Proof of Theorem 3. The probability that A is elected from the first round, with a majority of the votes is:

$$\Pr\left[X\left(A\right)\geq X\left(B\right)+X\left(C\right)+1\right].$$

For $\sigma(A|t_A)=1$ and $\sigma(A|t_B)\to 1$, we have $\tau(A|\omega)\to 1-r(t_C)$ and $\tau(B|\omega)\to 0$. The magnitude of this probability is therefore:

$$\lim_{\tau(B|\omega)\to 0}mag(piv_{AC}^{1}|\omega)=-\left(\sqrt{1-r(t_C)}-\sqrt{r(t_C)}\right)^2,\forall\omega\in\{a,b\},$$

where piv_{AC}^{1} denotes the event that a ballot is pivotal in electing A in the first round. In contrast, the probability that a B ballot is pivotal in bringing B to a second round is given by:

$$\tfrac{1}{2}\Pr\left[\max\left\{X\left(A\right),X\left(B\right),X\left(C\right)\right\}\leq\tfrac{X(A)+X(B)+X(C)}{2}\cap\min\left[X\left(A\right),X\left(C\right)\right]-X\left(B\right)\in\{0,1\}\right].$$

When alternative B's vote share approaches zero, the magnitude of this joint event converges to -1.

However, if $X\left(A\right)=X\left(B\right)+X\left(C\right)$, a ballot for A would be pivotal to elect A in the first round. Similarly, if $X\left(A\right)=X\left(B\right)+X\left(C\right)+1$, a B-ballot would be pivotal in forcing the organization of a second round. Hence, when a voter compares the two options, she values the A-ballot only in proportion to the second-round risk:

$$G\left(A|t\right)>\tfrac{1}{2}\Pr\left(piv_{AC}^{1}\right)\,\Pr\left(piv_{AC}^{2}\right),$$

where $\Pr\left(piv_{AC}^{2}\right)$ denotes the second-round pivot probability. Yet, the two probabilities, $\Pr\left(piv_{AC}^{1}\right)$ and $\Pr\left(piv_{AC}^{2}\right)$ are identical. Hence:

$$G\left(A|t\right)>\tfrac{1}{2}\Pr\left(piv_{AC}^{1}\right)^2.$$

Taking logarithms and dividing by n:

$$\frac{\log\left[\Pr\left(piv_{AC}^{1}\right)^{2}\right]}{n} \to -2\left(\sqrt{1-r(t_C)}-\sqrt{r(t_C)}\right)^{2},$$

which must be compared to the magnitude of the probability that a B ballot is pivotal in bringing B to a second round. That magnitude is equal to -1. Hence:

$$-2\left(\sqrt{1-r(t_C)}-\sqrt{r(t_C)}\right)^{2} \geq -1$$

is a sufficient condition for $G(A|t) > G(B|t)$. Solving it in $r(t_C)$ yields: $r(t_C) \geq 0.06699$. Hence, for any $r(t_C) \geq 0.06699$, there exists an informational trap equilibrium with $\sigma(A|t) = 1$, $t \in \{t_A, t_B\}$. By symmetry, there exists another equilibrium with $\sigma(B|t) = 1$, $t \in \{t_A, t_B\}$. ■

Appendix A5: Proof of Proposition 5

Proof. We show that, for $r(0|\omega)$ large enough, there exists an equilibrium in which B always wins in Approval Voting. This equilibrium is defined by the cutoffs $\theta_A = 1$ and $\theta_B = 0$ such that: $\sigma(B|0) = 1$ and $\sigma(AB|t) = 1$ $\forall t > \theta_B$. Obviously, $\sigma(C|t_C) = 1$.

For these strategies, expected vote shares are: $\tau(A|\omega) = 0$, $\tau(B|\omega) = (1-r(t_C))\,r(0|\omega)$, $\tau(AB|\omega) = (1-r(t_C))\,(1-r(0|\omega))$ and $\tau(C) = r(t_C)$. Using Properties 2 and 4 in Appendix A1, we have:

$$\begin{aligned}
mag(piv_{AB}|\omega) &\leq -(1-r(t_C))\,r(0|\omega)\\
mag(piv_{BC}|\omega) &= -\left(\sqrt{1-r(t_C)}-\sqrt{r(t_C)}\right)^{2}, \text{ and}\\
mag(piv_{AC}|\omega) &= 2\sqrt{r(t_C)\,(1-r(t_C))\,[1-r(0|\omega)]}-1.
\end{aligned}$$

The above strategy profile defines an equilibrium if:

$$G(B|t) > G(AB|t) \text{ for } t = 0, \tag{50}$$

$$G(A|t) < G(AB|t),\ \forall t > 0. \tag{51}$$

Types $t = 0$ prefer to play B except if $mag(piv_{AC}|b) > \max\{mag(piv_{AB}|b), mag(piv_{BC}|b)\}$. Since $mag(piv_{AC}|\omega)$ is necessarily smaller than $mag(piv_{BC}|\omega)$ for this strategy profile, we have that (50) is always satisfied and thus $\sigma(B|t) = 1$ is an equilibrium strategy for types $t = 0$. Condition (51) holds if:

$$\begin{aligned}
-(1-r(t_C))\,r(0|\omega) &< -\left(\sqrt{1-r(t_C)}-\sqrt{r(t_C)}\right)^{2}\\
r(0|\omega) &> \frac{1-2\sqrt{(1-r(t_C))\,r(t_C)}}{1-r(t_C)}.
\end{aligned}$$

The proof for $r(1|\omega)$ is similar. ■