# Assessing risky social decisions*

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May 2006

**Abstract**

This paper analyzes social decisions under risk. A new argument is proposed in order to defend the ex-post approach against Diamond's (1967) famous critique of Harsanyi's (1955) utilitarian theorem. It contributes to characterizing the criterion consisting in computing the expected value of the "equal-equivalent". Characterizations of the ex-post maximin and leximin criteria, as well as a variant of Harsanyi's theorem, are also obtained. It is examined how to take account of concerns for ex-ante fairness within this ex-post approach. Related issues are also addressed, such as the rescue problem or preference for catastrophe avoidance.

Keywords: risk, social welfare, ex ante, ex post, fairness, Harsanyi theorem.

JEL Classification: D63, D71, D81.

*This paper has benefitted from many conversations with A. Williams, from discussions with S. Anand, R. Bradley, J. Broome, P. Casal, T. Christiano, M. Clayton, G. Cohen, D. Franklin, L. Gevers, P. Hammond, R. Hockett, C. List, F. Maniquet, P. Mongin, C. Murray, S. Orr, M. Otsuka, J. Roemer, D. Wikler, J. Wolff, as well as from reactions from the audience of a seminar at the LSE.

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# 1 Introduction

This paper defends a family of criteria for the evaluation of risky social decisions, namely, those which rely on the expected value of the equal-equivalent utility. The equal-equivalent utility of a given distribution of utilities is the utility level which, if equally enjoyed by the population, yields a distribution that is judged as good as the examined distribution. If the equal-equivalent utility is lower than the average utility, this criterion displays aversion to ex-post inequalities. The ex-post maximin criterion, which computes the expected value of the smallest utility, is one example.

The literature on risky social decisions has been strongly influenced by Harsanyi's (1955) defense of the utilitarian criterion. Harsanyi showed that this criterion has the unique virtue of being at the same time based on expected social utility, i.e. corresponding to rationality at the social level, and based on ex-ante individual preferences, i.e. respecting the Pareto principle. Hammond (1983) and Broome (1991) have offered additional arguments in favor of this criterion, and made a careful scrutiny of the scope of this result.[1] A well known drawback of the utilitarian criterion is its indifference to ex-ante and ex-post inequalities, while intuition suggests that ex-ante inequalities matter, as noted by Diamond (1967) and Keeney (1980a), and ex-post inequality aversion appears also rather pervasive (Meyer and Mookherjee 1987, Harel et al. 2005). In the next section, a brief survey of the literature is proposed, with a description of the main difficulties which have been identified.

This paper proposes a defense of the ex-post approach based on the argument that one should try to mimic an omniscient evaluator whenever this is possible. It is further argued that this argument is fully compatible with ex-post inequality aversion, and that

[1] See also Weymark (1991) for a careful examination of Harsanyi's arguments in favor of utilitarianism, and Mongin (1994) for an analysis of the connection with social choice theory.

the reference to an omniscient evaluator actually questions the Pareto principle applied to individual expected utilities. Section 3 introduces the "omniscience principle", and Section 4 formalizes it and derives its consequences over the definition of an ordering of risky social alternatives. In particular, it offers characterizations of the ex-post equal-equivalent, maximin and leximin criteria. Section 5 provides a variant of Harsanyi's (1955) theorem based on the omniscience principle instead of the expected utility assumption, and discusses the conflict between aversion to ex-post inequality and the Pareto principle. Section 6 comes back to the problems described in Section 2 and in particular examines how concerns for ex-ante fairness can be accommodated by the ex-post criteria proposed here. Section 7 concludes.

## 2 Three problems

Social decisions under risk have raised many puzzles and controversies. We list three of them here.

**Ex ante versus ex post.** Harsanyi's (1955) aggregation theorem derives a linear (or affine) social welfare function from the combined requirements that the social objective should be computed as the expected value of social welfare over ex post distributions and should also respect individuals' ex-ante preferences. This can be viewed as an impossibility theorem for egalitarians: It is impossible to have the slightest degree of inequality aversion over the distribution of von Neumann-Morgenstern (VNM) utilities, under these two requirements. And it is almost an impossibility theorem for utilitarians as well since they must measure utilities in terms of VNM functions if they want to satisfy both requirements.[2]

[2]This is no problem for utilitarians like Harsanyi or Mirrlees who are inclined to adopt VNM utilities anyway as the good measure.

Diamond (1967) criticized the principle of maximizing the expected value of social welfare and suggested that inequality aversion over individual expected utilities made a lot of sense in the context of fair lotteries. It is better to toss a coin than give a prize directly to one individual. Hammond (1983), however, provided a defense of the expected utility approach for the social planner in terms of dynamic consistency. Diamond's argument, indeed, seems likely to produce dynamically inconsistent decisions. After a coin is tossed, one would still be in the same situation when considering another coin tossing: It would always be "more fair" to toss an additional coin, and that seems absurd.[3] Hammond defended the ex-post approach by combining the requirement of dynamic consistency with an independence axiom which requires decisions to depend only on future possibilities. Epstein and Segal (1992), however, proposed to reconcile ex-ante egalitarianism and dynamic consistency by letting decisions depend on the past, i.e. by dropping Hammond's independence axiom. For instance, whether flipping a coin appears necessary or not may depend on whether a previous coin flipping has already taken place.[4]

At any rate, there is a conflict between the *ex-post approach*, based on the expected value of social welfare, and the *ex-ante approach*, which computes social welfare from the distribution of expected utilities. The problem has been nicely summarized in Broome (1991) and Ben Porath et al. (1997) as follows.[5] Consider two individuals, Ann and Bob, and two equally probable states of natures, $s_1$ and $s_2$. There are three lotteries to compare, $L_1$, $L_2$ and $L_3$, which are described in the following tables by their utility consequences

[3] See also Myerson (1981), Broome (1984b, 1991) and Hammond (1988, 1989).

[4] The independence axiom can be saved by enriching the description of consequences, incorporating information about the history from which they emerge. But when this is done the axiom loses its bite, a classical dilemma for independence axioms in decision theory.

[5] See also Fishburn (1984) for an early axiomatic analysis of these issues.

for Ann and Bob.

| $L_1$ | $s_1$ | $s_2$ |
|---|---|---|
| Ann | 1 | 1 |
| Bob | 0 | 0 |

| $L_2$ | $s_1$ | $s_2$ |
|---|---|---|
| Ann | 1 | 0 |
| Bob | 0 | 1 |

| $L_3$ | $s_1$ | $s_2$ |
|---|---|---|
| Ann | 1 | 0 |
| Bob | 1 | 0 |

Harsanyi's additive criterion is indifferent between the three lotteries. An ex-post approach is indifferent between $L_1$ and $L_2$ (provided it is impartial between Ann and Bob), because the ex-post distribution is $(0, 1)$ for both lotteries, while an ex-ante approach is indifferent between $L_2$ and $L_3$, because the individuals face the same individual lotteries (0 or 1 with equal probability) in the two lotteries.

But, intuitively, $L_1$ is worse than $L_2$ which is worse than $L_3$. Indeed, with $L_1$ Ann is advantaged for sure whereas $L_2$ randomizes over the two individuals, as with a coin tossing. This is Diamond's (1967) observation. With $L_3$, there is the same ex-ante prospect for each individual as with $L_2$, but with less inequality ex post, as observed in Broome (1991).

Facing this problem, Ben Porath et al. (1997) suggest taking a weighted sum of an ex-ante and an ex-post criterion.[6] Broome (1991) proposes to introduce a measure of ex-ante fairness in the measurement of individual utilities. In a similar vein, Hammond (1983) alludes to the possibility of recording intermediate consequences jointly with terminal consequences.

[6] A similar proposal is made in Gajdos and Maurin (2004).

**Risk equity versus catastrophe avoidance.** A related problem was raised by Keeney (1980a), who argued that for public decisions which submit individuals $1, ..., n$ to independent risks of death $(p_1, ..., p_n)$, "risk equity" requires a preference for a more equal over a less equal distribution of probabilities (for a fixed sum of probabilities $p_1 + ... + p_n$). He also noted that many people express a preference for "catastrophe avoidance", meaning that for a given expected number of fatalities, one prefers a smaller number of actual fatalities (even if they must be, logically, more probable).[7]

Now, in this simple context, an impartial ex-post criterion simply has to define preferences over lotteries $(\pi_f)_{f=1,...,n}$ where $\pi_f$ is the probability of having $f$ fatalities. Keeney (1980a) then shows that, if the social criterion is an expected utility $\sum_{f=1}^{n} \pi_f u(f)$, the utility function must be strictly convex (risk prone) under the equity requirement, but strictly concave (risk averse) under catastrophe avoidance.[8]

Broome (1982), invoking Diamond's argument, questioned the expected utility assumption as suitable for discussing ex-ante fairness, and Fishburn (1984)[9] examined the compatibility of various requirements of this kind with or without the expected utility hypothesis.

We can relate this problem to the previous one by examining the application of these notions to a two-agent population. Assuming four equiprobable states of nature, consider the four following lotteries, together with the corresponding distribution of probabilities for fatalities (death is represented by a 0 utility).

[7] This notion is used by Bommier and Zuber (2006) in the context of intergenerational equity.

[8] Strict convexity for a function taking only $n$ values is understood here as

$$u(f) < \frac{1}{2} [u(f-1) + u(f+1)]$$

for all $f = 2, ..., n-1$. The reverse inequality defines strict concavity.

[9] See also Keeney (1980b), Keeney and Winkler (1985) and Fishburn and Straffin (1989).

| $L'_1$ | $s_1$ | $s_2$ | $s_3$ | $s_4$ |
|---|---|---|---|---|
| Ann | 1 | 1 | 1 | 1 |
| Bob | 0 | 0 | 0 | 0 |

$\longrightarrow$

| $f$ | 0 | 1 | 2 |
|---|---|---|---|
| $\pi_f$ | 0 | 1 | 0 |

| $L'_2$ | $s_1$ | $s_2$ | $s_3$ | $s_4$ |
|---|---|---|---|---|
| Ann | 1 | 1 | 0 | 0 |
| Bob | 0 | 0 | 1 | 1 |

$\longrightarrow$

| $f$ | 0 | 1 | 2 |
|---|---|---|---|
| $\pi_f$ | 0 | 1 | 0 |

| $L'_3$ | $s_1$ | $s_2$ | $s_3$ | $s_4$ |
|---|---|---|---|---|
| Ann | 1 | 1 | 0 | 0 |
| Bob | 1 | 1 | 0 | 0 |

$\longrightarrow$

| $f$ | 0 | 1 | 2 |
|---|---|---|---|
| $\pi_f$ | .5 | 0 | .5 |

| $L'_4$ | $s_1$ | $s_2$ | $s_3$ | $s_4$ |
|---|---|---|---|---|
| Ann | 1 | 1 | 0 | 0 |
| Bob | 1 | 0 | 1 | 0 |

$\longrightarrow$

| $f$ | 0 | 1 | 2 |
|---|---|---|---|
| $\pi_f$ | .25 | .5 | .25 |

Lotteries $L'_1, L'_2, L'_3$ are respectively equivalent to $L_1, L_2, L_3$. Risk independence between individuals is obtained for $L'_1$ and $L'_4$, and risk equity requires to prefer an equal chance of death for both individuals to a sure fatality, therefore ranking $L'_4$ above $L'_1$. It seems intuitive to extend this argument and rank $L'_2$ over $L'_1$, even if risk is not independent across agents in $L'_2$. But an ex-post criterion is unable to distinguish between $L'_1$ and $L'_2$ (the right-hand tables are identical), this is the above problem of accommodating ex-ante fairness in a consistent criterion.

Catastrophe avoidance requires preferring one death for sure to a 50% risk of two deaths, therefore ranking $L'_1$ above $L'_3$. The conflict between risk equity and catastrophe avoidance, for an ex-post criterion, is apparent on the right-hand tables, where one sees that from $L'_1$ to $L'_4$ and from $L'_4$ to $L'_3$, the same operation on probabilities of fatalities is performed:

| $f$ | 0 | 1 | 2 |
|---|---|---|---|
| $\triangle\pi_f$ | +.25 | -.5 | +.25 |

A criterion such as expected utility, which is linear with respect to probabilities, must therefore prefer $L'_3$ to $L'_4$ if $L'_4$ is preferred to $L'_1$. This clashes with catastrophe avoidance, i.e. ranking $L'_1$ above $L'_3$.

Moreover, since $L'_1$ is equivalent to $L_1$ and $L'_3$ to $L_3$, catastrophe avoidance goes directly against the aversion to ex-post inequalities which, in the previous example, motivated a preference for $L_3$ above $L_1$. We therefore have a second problem, namely, a conflict between preference for catastrophe avoidance on one side and preference for ex-post equality on the other side.

**Rescue versus prevention.** Schelling (1968) has made the striking observation that people are generally much more willing to rescue identified people currently at great risk than to invest in the prevention of future uncertain fatalities, even if the expected number of lives saved is greater with the prevention policy. This apparently irrational preference has been analyzed in many ways, such as the difference between known victims and "statistical lives", the symbolic expression of our commitment to life by undertaking rescue operations, or preferences about how to die.[10]

Although all of these considerations are relevant in order to understand the full force of the appeal of rescue operations in concrete examples, it seems possible to undertand this as, more than anything else, another instance of the ex-ante vs. ex-post dilemma. An ex-post approach would minimize the number of fatalities and would favor the prevention policy. But ex-ante, the rescue operation looks preferable because it reduces the inequality of expected utility between the victims and the rest of the population.

[10] For a penetrating and synthetic analysis of this issue, see Fried (1969).

Formally, one can describe the problem in terms of independent[11] risks $(p_1, ..., p_n)$, with $p_i = \bar{p}$ for the $m$ victims and $p_i = \underline{p} < \bar{p}$ for the rest of the population. The expected number of fatalities is $p_1 + ... + p_n$ and can be reduced by $\triangle$. The rescue operation reduces $p_i$ to $\bar{p} - \triangle/m$ for the victims, while the prevention policy reduces $p_i$ to $\underline{p} - \triangle/(n-m)$. Let us assume that $\bar{p} - \triangle/m \geq \underline{p}$, so that the rescue operation does not provide extra protection against future risks to the saved victims. Under this assumption, one sees that Keeney's "risk equity" principle induces a strict preference for the rescue operation, since one can obtain the distribution

$$(\underbrace{\bar{p} - \triangle/m}_{m}, \underbrace{\underline{p}}_{n-m})$$

from the distribution

$$(\underbrace{\bar{p}}_{m}, \underbrace{\underline{p} - \triangle/(n-m)}_{n-m})$$

by a series of inequality reducing transfers of probability from the $m$ victims to the $n-m$ others. It has been assumed here that $\triangle$ is the same for the rescue operation and the prevention policy, but if there is a strong preference for rescue in this case, one will still have preference for rescue in cases when the expected number of saved lives is greater with prevention.

# 3 Diamond's critique revisited

Diamond's critique, which is a key reference for arguments against the ex-post approach, is not as powerful as usually considered. Consider the Ann-Bob example again. In $L_1$, Ann wins for sure. Suppose that one proposes a new lottery $L_1^*$ which uses a selection device. This selection device comes with a name on it, which indicates who will be selected

[11] Independence of risks is not essential to the rescue problem and is introduced only for convenience. For instance, it is also assumed in Pratt and Zeckhauser (1996).

(Ann or Bob). If the name is "Ann", there is no difference between $L_1$ and $L_1^*$. If the name is "Bob", by impartiality this should not make a difference either. One should therefore be indifferent between $L_1$ and $L_1^*$ since they offer the same perspective of one individual winning for sure.

Now consider another lottery $L_1^{**}$ which uses the same selection device except that the name is not readable. Since, for $L_1^*$, the same evaluation was made independently of the name written on the device when we could read it, nothing should change when we cannot read it and therefore we should again be indifferent between $L_1$ and $L_1^{**}$. Now, what is the difference between $L_2$, in which a coin is tossed, and $L_1^{**}$? Coin tossing is also a deterministic mechanism and, although it does not always select the same winner, it does select the same winner for given initial conditions. Therefore, for the given initial conditions in which the coin will be tossed, the coin is like the other device with a non-readable name on it. Coin tossing just brings the illusion that both agents can win, whereas, just as in $L_1$, $L_1^*$ or $L_1^{**}$, one is bound to lose and the other is bound to win. Now, suppose we replace the coin by a truly indeterministic mechanism. What difference does it make? We are still sure that one agent will win and the other will lose, and we do not know ex-ante who that is. There is no practical difference between coin tossing and a truly random device.

Now, one would still prefer $L_2$ to $L_1$ (and $L_1^*$, $L_1^{**}$) if, in $L_1$, Ann was selected because she is, say, the niece of the king's butler. We see that the example is actually insufficiently specified. Let us therefore add that, in $L_1$, Ann was actually selected by a previous coin tossing. This makes it quite plausible that $L_1$ and $L_2$ are actually equivalent. (The importance of impartial processes will be discussed further in Section 5.)

The ex-post approach is justified here by the argument that, for an impartial evaluator, it is equivalent to know that Ann will win or to know that one individual will win.

Suppose, in another variant of the example, that the evaluator knows that Ann will get the prize for sure whereas Ann and Bob wrongly believe that a coin will be tossed. Social evaluation in this case should be based on the evaluator's better knowledge and not on the agents' illusions. Now, in the case a coin is tossed, the situation is actually equivalent to the illusion case. The evaluator knows that one will win and one will lose, and therefore knows that both are wrong to believe that they have a 50% chance of winning. Again one of them has false hopes and the other wrong fears, exactly as in the previous situation. The only difference with the previous situation is that the evaluator does not know the name of the winner. But this is not relevant when he is impartial.

In a nutshell, adopting the ex-post viewpoint is equivalent to making use of superior information about the distribution of luck. The evaluator knows only the statistical distribution of luck and not the personal distribution, but this does not make any difference when he is impartial. An evaluator who adopts the ex-post approach is making the same judgments as an omniscient evaluator who would know the exact distribution of luck. Since the omniscient evaluator would obviously make the right judgments, there is no reason for a non-omniscient evaluator not to mimic him when this is possible.

Now, mimicking an omniscient evaluator is not always possible and this argument is correct only for micro risks as in $L_2$. In $L_3$ there is a macro risk and the evaluator does not know whether both will win or both will lose. But for macro risks there is no conflict between the ex-ante approach and the ex-post approach, when the evaluator has no better information than individual agents. The choice between the ex-ante and the ex-post approach is important only for micro risks, and the above argument applies precisely in that case.

# 4 Ex-post criteria

In this section we formalize the argument of the previous section and derive its consequences.

The framework is as simple as possible. The population is finite and fixed, $N = \{1, ..., n\}$.[12] The set of states of nature $S$ is finite and the evaluator has a fixed probability vector $\pi = (\pi^s)_{s\in S}$, with $\sum_{s\in S} \pi^s = 1$. It does not matter whether the probabilities are objective or subjective, but if they are subjective they belong to the evaluator.[13] Since what happens in zero-probability states can be disregarded, we simply assume that $\pi^s > 0$ for all $s \in S$.

The evaluator's problem is to rank lotteries $U$, where $U = (U_i^s)_{i\in N, s\in S} \in \mathbb{R}^{n|S|}$ describes the utility attained by every $i$ in every state $s$. Let $\mathcal{L}$ denote the set of such lotteries. Let $U_i$ denote $(U_i^s)_{s\in S}$ and $U^s$ denote $(U_i^s)_{i\in N}$. The social ordering over the set $\mathcal{L}$ is denoted $R$ (with strict preference $P$ and indifference $I$).

We now introduce requirements that $R$ should satisfy. The first, and the main one, is based on the argument of the previous section. It says that $R$ should be consistent with the choices of an omniscient evaluator, and formulates it in the following way. An omniscient evaluator would always face sure lotteries, i.e. lotteries with a given utility vector occurring for sure. Now, consider an ordinary lottery (with different utility vectors for different states). If, in a given state $s$, we replace $U^s$ by another vector $V^s$ such that, if such vectors were obtained for sure, $U^s$ would be judged equivalent to $V^s$, then we

[12] This means that we consider the possibility of individuals being killed but not risky decisions that may affect their mere existence. This is because issues of variable population trigger a whole set of different considerations that are better kept aside here. On variable population issues, see Hammond (1996), Blackorby et al. (2006) and Broome (2004).

[13] We ignore here how the evaluator's probability vector could be related to the agents' beliefs. There is an important literature on this issue, see e.g. Mongin (1995).

should be indifferent between the original lottery and the new one. The reason is that an omniscient evaluator would be indifferent. Indeed, if he knew that $s$ would occur, he would be indifferent by the assumption that $U^s$ is equivalent to $V^s$. If he knew that $s$ would not occur, he would also be indifferent to this irrelevant change of utilities in a non-occurring state.

Let $[U^s]$ denote the degenerate lottery in which vector $U^s$ occurs in all states of nature.

**Axiom 1 (Omniscience Principle)** *For all $U, U' \in \mathcal{L}$, one has $U \; I \; U'$ if $U'$ differs from $U$ only by replacing $U^s$, for some $s \in S$, by $V^s$ such that $[U^s] \; I \; [V^s]$.*

If we were dealing with individual decision theory, this principle would simply say that replacing an outcome in a given state of nature by another which is equivalent (as assessed by the ranking of sure outcomes) never affects the value of a lottery. This is a basic consequentialism principle that is already contained, for individuals, in our setting here since we deal with lotteries described in terms of utility and not in terms of underlying arguments. Replacing a decision by another yielding the same $U$ automatically yields individual and social indifference in our framework. The above axiom simply extends this basic principle to the social ordering.[14]

The second axiom expresses the idea that for pure macro risks, the evaluation should coincide with the ex-ante appreciation of individual situations, since the evaluator cannot

[14] In the formulation of Omniscience Principle, we have implicitly assumed that obtaining $U^s$ in all states of nature is equivalent to knowing that $s$ is the true state of nature. An alternative formulation of Omniscience Principle would say that $U^s$ can be replaced by $V^s$ when the evaluator would be indifferent between these two vectors if he attributed probability 1 to state $s$. This would slightly complicate the analysis because it would require considering different vectors $\pi$, and with this version of Omniscience Principle one could obtain ex-post criteria with state-dependent social utilities. Since state-dependent social utilities do not appear particularly attractive, we ignore this complication here.

take advantage of his knowledge of the statistical distribution of luck in this case.[15] A pure macro risk is described here as a situation in which in all states of nature, the distribution of utilities is equal.

**Axiom 2 (Pareto for Macro Risks)** *For all* $U, U' \in \mathcal{L}$ *such that for all* $i, j \in N$, $U_i = U_j$ *and* $U'_i = U'_j$, *one has* $U\, R\, U'$ *if for all* $i \in N$, $\sum_{s\in S} \pi^s U_i^s \geq \sum_{s\in S} \pi^s U_i'^s$.

The next two axioms are standard monotonicity and continuity conditions with respect to $U$. Vector inequalities are denoted $\geq, >, \gg$ .

**Axiom 3 (Weak Monotonicity)** *For all* $U, U' \in \mathcal{L}$, *one has* $U\, P\, U'$ *if* $U \gg U'$.

**Axiom 4 (Continuity)** *Let* $U, U' \in \mathcal{L}$ *and* $(U(t))_{t\in\mathbb{N}} \in \mathcal{L}^{\mathbb{N}}$ *be such that* $U(t) \to U$. *If* $U(t)\, R\, U'$ *for all* $t \in \mathbb{N}$, *then* $U\, R\, U'$. *If* $U'\, R\, U(t)$ *for all* $t \in \mathbb{N}$, *then* $U'\, R\, U$.

We can now state a first result, which describes how $R$ must be computed in order to satisfy these axioms.

**Theorem 1** $R$ *satisfies the four axioms if and only if, for all* $U, U' \in \mathcal{L}$,

$$U\, R\, U' \Leftrightarrow \sum_{s\in S} \pi^s e\left(U^s\right) \geq \sum_{s\in S} \pi^s e\left(U'^s\right),$$

*where* $e\left(U^s\right)$ *is defined by the condition*

$$\left[U^s\right]\ I\ \left[\left(e\left(U^s\right), ..., e\left(U^s\right)\right)\right].$$

[15] It is not assumed here that $\pi$, used by the evaluator, coincides with the agents' beliefs, since an evaluation simply has to be made with the best knowledge of the evaluator himself. Therefore the Pareto axioms in this paper do not necessarily reflect a respect for individual ex-ante preferences. More basically they describe a social evaluation that respects the evaluation of individual situations by the same evaluator: "If the evaluator thinks that nobody is ex-ante worse-off, he also thinks the lottery is not worse".

The function $e(U^s)$ is the "equal-equivalent" value of $U^s$, i.e. the utility level which, if enjoyed uniformly by all agents, would yield a distribution that is equivalent to $U^s$. The theorem says that the expected value of the equal-equivalent should be the criterion. The proofs of the results (and examination of the independence of axioms) are in the appendix, but the proof of this one is particularly simple. Weak Monotonicity and Continuity guarantee the existence of the function $e(.)$. Omniscience Principle implies that, for all lotteries $U$, one can replace each vector $U^s$ by $(e(U^s), ..., e(U^s))$, so as to obtain a lottery which involves a pure macro risk. One therefore simply has to know how to rank lotteries with pure macro risks, and the rest of the ordering is automatically derived from this. Pareto for Macro Risks then intervenes in order to require relying on expected utilities for such lotteries.

The result is compatible with $e$ being partial in favor of some agents, and it is easy to exclude biased criteria by imposing the following anonymity requirement.

**Axiom 5 (Anonymity)** *For all $U, U' \in \mathcal{L}$, one has $U\, I\, U'$ if $U'$ differs from $U$ only by permuting the vectors $U_i$.*

With this axiom added to the list, the function $e$ must be symmetrical in its arguments.

The classical utilitarian criterion is a special case of this, and corresponds to the situation in which $e(U^s)$ equals the average utility in $U^s$. But the above result is also compatible with incorporating inequality aversion into $R$. Let us introduce an adaptation of the Pigou-Dalton principle to this setting. The original Pigou-Dalton principle is about inequality reduction between two agents with unequal utilities (or incomes, in the standard framework of inequality studies). We apply the same idea when one agent has greater utility in all states of nature, and inequality reduction is performed in every state of nature.

**Axiom 6 (Pigou-Dalton Equity)** *For all $U, U' \in \mathcal{L}$ such that for some $i, j \in N$, some $\delta \gg 0$,*

$$U_i' - \delta = U_i \geq U_j = U_j' + \delta$$

*while $U_k = U_k'$ for all $k \neq i, j$, one has $U \, R \, U'$.*

Contrary to the classical Pigou-Dalton principle, it is not required here that the transfer always strictly improves the lottery. This weaker version is more reasonable.[16] Indeed, it is questionable that among very rich people, a Pigou-Dalton transfer strictly improves the situation. In particular, the maximin criterion which focuses on the worst-off only does not satisfy the strict version of the Pigou-Dalton principle because it is indifferent to transfers among the better-off, but it satisfies this weaker version.

**Corollary 1** *$R$ satisfies the six axioms if and only if, for all $U, U' \in \mathcal{L}$,*

$$U \, R \, U' \Leftrightarrow \sum_{s \in S} \pi^s e\left(U^s\right) \geq \sum_{s \in S} \pi^s e\left(U'^s\right),$$

*where $e\left(U^s\right)$ is defined as above and is Schur-concave.*[17]

This result, however, is compatible with an arbitrarily weak, even zero, aversion to inequality (with the strict version of the axiom one would obtain that $e$ must be *strictly* Schur-concave but that is still compatible with an arbitrarily weak positive aversion to inequality). One may want to require a greater concern for equality. In particular, one can adapt the axiom of Hammond equity (Hammond 1976) to this case.

[16] It is also more similar to the Hammond Equity condition which is introduced later, and which was formulated by Hammond (1976, 1979) in terms of weak preference.

[17] The function $e$ is Schur-concave if $e(AU^s) \geq e(U^s)$ for all $n \times n$ non-negative matrices $A$ such that $\sum_{i=1}^n A_{ij} = \sum_{j=1}^n A_{ij}$ for all $i, j$ (i.e. bistochastic matrices). It is strictly Schur-concave if $e(AU^s) > e(U^s)$ for all matrices $A$ satisfying the above conditions and additionally $A_{ij} \neq 0, 1$ for some $i, j$ (i.e. bistochastic matrices which are not permutation matrices).

**Axiom 7 (Hammond Equity)** *For all $U, U' \in \mathcal{L}$ such that for some $i, j \in N$,*

$$U_i' \gg U_i \geq U_j \gg U_j'$$

*while $U_k = U_k'$ for all $k \neq i, j$, one has $U \, R \, U'$.*

This axiom is logically stronger than Pigou-Dalton Equity and requires an infinite inequality aversion in two-person situations. With this axiom substituted to Pigou-Dalton Equity, the ex-post maximin criterion, which computes the expected value of the smallest utility, is singled out.

**Corollary 2** *$R$ satisfies the first four axioms and Hammond Equity if and only if, for all $U, U' \in \mathcal{L}$,*

$$U \, R \, U' \Leftrightarrow \sum_{s \in S} \pi^s \min_{i \in N} U_i^s \geq \sum_{s \in S} \pi^s \min_{i \in N} U_i'^s.$$

A drawback of the maximin criterion is that it does not satisfy the following stronger monotonicity condition.

**Axiom 8 (Strong Monotonicity)** *For all $U, U' \in \mathcal{L}$, one has $U \, P \, U'$ if $U \geq U'$ and $U_i \gg U_i'$ for some $i \in N$.*

This axiom is satisfied, along with the others except Continuity, by the ex-post leximin criterion, defined as follows. Let $U_{(i)}^s$ denote the utility of $i$th rank (by increasing order) in vector $U^s$. The symbol $\geq_{lex}$ denotes the ordinary leximin criterion, which compares two vectors by comparing the smallest component, and if they are equal it compares the second smallest component, and so on. The ex-post leximin criterion weakly prefers $U$ to $U'$ if

$$\left( \sum_{s \in S} \pi^s U_{(i)}^s \right)_{i \in N} \geq_{lex} \left( \sum_{s \in S} \pi^s U_{(i)}'^s \right)_{i \in N}$$

In other words, this criterion computes the expected value of the utility of $i$th rank in $U$, for all $i = 1, ..., n$, and applies the standard leximin criterion to such vectors.

The ex-post leximin criterion is not the only one which satisfies Omniscience Principle, Pareto for Macro Risks, Hammond Equity and Strong Monotonicity. This list is also satisfied, for instance, by the criterion which is defined by

$$\left(z_i(U)\right)_{i \in N} \geq_{lex} \left(z_i(U')\right)_{i \in N},$$

where

$$\begin{aligned} z_1(U) &= \sum_{s \in S} \pi^s U^s_{(1)}, \\ z_{i+1}(U) &= z_i(U) + \min_{s \in S} \left(U^s_{(i+1)} - U^s_{(i)}\right) \text{ for } i = 1, ..., n-1. \end{aligned}$$

One sees that this criterion does not rely on $\pi$ in order to evaluate the situation of the agents who are not the worst-off. This is rather questionable in situations where the evaluator has no better data than $\pi$, on the basis of the statistical distribution of utilities, in order to make his judgment. It would make sense to extend the application of the Pareto principle to situations in which the agents may have unequal utilities but are always ranked in the same way whatever the state of nature. Such situations may contain micro risks which are insurable, but they do not contain reversals of utility rankings in different states of nature. In absence of such reversals, one can never be sure of what an omniscient and impartial evaluator would say, and relying on the agents' expected utilities in these cases is fully compatible with the Omniscience Principle axiom.

Let us say that $U$ is comonotone if for all $s, s' \in S$ and all $i, j \in N$, $\left(U^s_i - U^s_j\right)\left(U^{s'}_i - U^{s'}_j\right) \geq 0$.

**Axiom 9 (Pareto for Comonotone Risks)** *For all $U, U' \in \mathcal{L}$ such that $U$ and $U'$ are comonotone, one has $U \, R \, U'$ if for all $i \in N$, $\sum_{s \in S} \pi^s U^s_i \geq \sum_{s \in S} \pi^s U'^s_i$.*

This axiom is satisfied by the ex-post maximin and leximin criteria, as well as by many other ex-post criteria. In combination with the other axioms, however, strenghtening Pareto for Macro Risks into Pareto for Comonotone Risks forces us to adopt the ex-post leximin criterion.

**Theorem 2** *R satisfies Omniscience Principle, Pareto for Comonotone Risks, Hammond Equity, Strong Monotonicity and Anonymity if and only if it is the ex-post leximin criterion.*

## 5 A variant of Harsanyi's theorem

In his famous aggregation theorem, Harsanyi (1955) directly assumed that the evaluator seeks to maximize the expected value of social welfare. On the basis of the expected utility theorem, this assumption can be replaced by an independence and a continuity requirement. Here we can obtain a variant of this result which relies on Omniscience Principle instead of independence and continuity conditions. Note that Omniscience Principle is not logically related to independence or the sure-thing principle. For instance, the doubly maximin criterion which relies on

$$\min_{\substack{s\in S\\ i\in N}} U_i^s$$

satisfies Omniscience Principle but certainly not independence or the sure-thing principle. Conversely, an ex-post criterion based on a state-dependent social welfare function, i.e.

$$\sum_{s\in S} \pi^s W^s(U^s)$$

obeys the sure-thing principle but violates Omniscience Principle when the $W^s$ functions are different, so that

$$\sum_{s\in S} \pi^s W^s(U^0) = \sum_{s\in S} \pi^s W^s(V^0)$$

does not imply $W^s(U^0) = W^s(V^0)$ for all $s \in S$.[18]

Let us extend the Pareto requirement further in order to capture Harsanyi's requirement of respecting agents' ex-ante preferences.

**Axiom 10 (Pareto)** *For all $U, U' \in \mathcal{L}$, one has $U\,R\,U'$ if for all $i \in N$, $\sum_{s\in S} \pi^s U_i^s \geq \sum_{s\in S} \pi^s U_i'^s$.*

We have the following result, which characterizes the classical utilitarian criterion.

**Theorem 3** *$R$ satisfies Omniscience Principle, Pareto, Weak Monotonicity and Anonymity if and only if, for all $U, U' \in \mathcal{L}$,*

$$U\,R\,U' \Leftrightarrow \sum_{\substack{s\in S \\ i\in N}} \pi^s U_i^s \geq \sum_{\substack{s\in S \\ i\in N}} \pi^s U_i'^s.$$

The proof of this result takes a very different route from Harsanyi's theorem since there is no continuity and one cannot even use the intermediate argument that the ordering must be an ex-post criterion (which, by Theorem 1, would be true under Continuity).

This theorem is problematic for those who would also like $R$ to display aversion to inequality. It rules out inequality aversion with respect to utilities (although it is of course compatible with inequality aversion with respect to the arguments of individual utility functions, if they can be concave) and, for instance, implies that $L_2$ is equivalent to $L_3$ in the Ann-Bob example. But the weak pillar of this result is the Pareto axiom. As explained above, the Pareto requirement is safe when applied to macro risks, possibly

[18] There is, however, a connection between the omniscience principle and the sure-thing principle because the latter means that preferences over subsets of states are independent of other states, while a variant of the former could be formulated as saying that preferences over *single* states are independent of other states. This shows that, in essence, the ommniscience principle is much weaker than the sure-thing principle.

comonotone risks, but much more questionable when applied across the board as in this last result.

It is worth emphasizing this point here, since the Pareto axiom has been firmly defended by many authors, most notably Hammond (1983, 1996) and Broome (1991). The attraction of the Pareto axiom comes from the apparently obvious observation that altering the correlation of utilities does not affect individual agents if the personal (marginal) distribution of utilities remains the same for each of them.

There is a questionable assumption in this observation, however. This assumption is that the good of each agent is fully represented by the distribution of his personal utility. This is far from obvious. Consider for instance an omniscient observer. For an observer who knows the true state of nature, the good of an agent is his utility in this state and not the whole distribution. It may happen that a lottery provides a more alluring distribution of utilities to the agent than another lottery, but an omniscient observer may know that it is actually worse because in the true state of nature it lowers utility. The good of an individual agent is not fully captured by the ex-ante distribution of utility. His good depends on the (unknown) true state of nature. It therefore appears exaggerate to say that respecting personal good requires judging on the sole basis of personal distributions of utility.

At this point, one may ask what the relevance of the true state of nature (or the omniscient observer) is, when nobody is omniscient in reality. As we have seen above, however, one need not always be omniscient in order to guess what an omniscient observer would think. And whether such guessing can be done or not depends precisely on the correlation of utilities across agents. Presumably, it would be reasonable for an individual agent to trust an omniscient evaluator who respects individual ex-post preferences. Indeed, when the individual would be tempted by a lottery, the omniscient evaluator would always tell him whether the eventual utility obtained would be high or low. And there is no reason to ask the omniscient evaluator to have no aversion to inequality. If this omniscient evaluator judges that a lottery is worse than another even though every agent sees that his personal distribution of utilities appears better, who should be trusted? The omniscient evaluator or the ignorant individuals? It seems that *individual agents can reasonably agree to put aside their isolated evaluation of their own distribution of utility when they have additional information about the omniscient evaluator thanks to the correlation between their utilities.*

In conclusion: Of the three basic principles of Omniscience, Pareto and inequality aversion which prove to be incompatible here, the one that must go is Pareto. For the record, we state this impossibility as a separate result here, since it is worth noting that it does not involve Weak Monotonicity if the following version of minimal egalitarianism is retained.

**Axiom 11 (Minimal Egalitarianism)** *For all* $U, U' \in \mathcal{L}$ *such that* $\sum_{i \in N} U_i = \sum_{i \in N} U'_i$ *and for all* $i, j \in N$, $U_i = U_j$ *and either* $U'_i \ll U'_j$ *or* $U'_i \gg U'_j$, *one has* $U \, P \, U'$.

**Theorem 4** *No R satisfies Omniscience Principle, Pareto, Minimal Egalitarianism and Anonymity.*

# 6 Back to the problems

Equipped with an ex-post criterion of the sort discussed in the previous section, how can we tackle the three problems listed in Section 2? In order to fix ideas, we will focus here on the ex-post maximin (or leximin) principle.

**Ex ante versus ex post.** By construction, an ex-post criterion cannot distinguish between $L_1$ and $L_2$ in the Ann-Bob example and this indifference was justified in Section

3. It seems, however, impossible to say that concerns for ex-ante fairness, which are so widespread, are based on pure illusion. They must be accommodated in some way.

One possibility is suggested by Deschamps and Gevers (1979). One can imagine that, in the example, there are two periods. It may then happen that the total value of a two-period "life" is a function of two arguments: the expected utility at $t = 1$ and the final utility at $t = 2$. Even if the evaluator considers the expected utility as measuring an illusion for both agents (an optimistic illusion for one, a pessimistic illusion for the other), this illusion has real welfare effects, and such effects may matter in the evaluation of a life.[19] The following table computes the relevant figures for the three lotteries and the two agents:

| | Ann | | | Bob | | |
|---|---|---|---|---|---|---|
| | $Eu$ | $u(s_1)$ | $u(s_2)$ | $Eu$ | $u(s_1)$ | $u(s_2)$ |
| $L_1$ | 1 | 1 | 1 | 0 | 0 | 0 |
| $L_2$ | .5 | 1 | 0 | .5 | 0 | 1 |
| $L_3$ | .5 | 1 | 0 | .5 | 1 | 0 |

The value of a life in state $s$ may be $U(Eu, u(s))$. One then computes the expected value at $t = 1$ of social welfare as:

| | $E\min_i U_i$ |
|---|---|
| $L_1$ | $U(0,0)$ |
| $L_2$ | $U(.5,0)$ |
| $L_3$ | $.5U(.5,1) + .5U(.5,0)$ |

One then easily obtains the desired ranking of lotteries. $L_2$ is better than $L_1$ because it offers better prospects at $t = 1$ to the worst-off. $L_3$ is better than $L_2$ because it offers the

[19] Deschamps and Gevers suggest that, if the prize was sent by mail without any explanation, welfare effects would be absent and tossing a coin would lose its value. However, the evaluator may want to take account of welfare effects that would be obtained if the agents were correctly informed of the process.

possibility for the worst-off to be better off than at $L_2$.[20]

The ranking of $L_1$ and $L_2$ is obtained in spite of retaining an ex-post approach. The idea is not exactly to incorporate a measure of ex-ante fairness into individual utilities (Broome 1991) or to adopt an approach that is intermediate between ex-ante and ex-post (Hammond 1983), but it is close to both, since it measures individual utility over the whole life, and this includes the prospects one faces in the beginning of one's life.

Diamond's argument was criticized above for implying that a new randomization would always be more fair. With the intertemporal welfare approach which has just been examined, one similarly obtains the conclusion that the later the randomization, the better, since it makes the ex-ante period longer and this contributes to reducing the inequality in the values of the two lives. In this case it is not absurd at all.

Another way to account for ex-ante fairness is to view random devices as mechanisms that guarantee impartiality of the allocation process.[21] Tossing a coin makes sure that, if Ann is the winner, she does not receive the prize because she is the niece of the king's butler. Interestingly, there are random devices which work in certain contexts but not in others, suggesting that the set of acceptable reasons for being selected depends on the

[20] With a more general equal-equivalent function $e$, one would obtain:

| | $Ee(U_1, U_2)$ |
|---|---|
| $L_1$ | $e(U(1,1), U(0,0))$ |
| $L_2$ | $e(U(.5,1), U(.5,0))$ |
| $L_3$ | $.5e(U(.5,1), U(.5,1)) + .5e(U(.5,0), U(.5,0))$ |

One then obtains $L_2$ better than $L_1$ if $e$ is sufficiently inequality averse, and $L_3$ better than $L_2$ if $e$ is supermodular (and symmetric).

[21] That the only value of flipping a coin may be to avoid prejudice and bias is considered as a plausible view in the concluding sentence of Broome (1984a): 'In the end we may be forced to the conclusion that the only merit of random selection is the political one of guarding against partiality and oppression' (p. 55).

context. For instance, for certain surveys the samples are selected on the basis of the birth date of individuals (e.g. those who are born between Oct. 4 and and Oct. 8). This is obviously a deterministic "random" device but it yields impartial decisions. Nonetheless, while this seems acceptable for certain statistical studies, it would hardly get approval for the distribution of valuable prizes (unless the dates themselves were randomly selected).

Impartiality of the allocation process is not directly examined by a consequentialist criterion that only looks at the lottery over final utilities and not at the characteristics of the generating process, but nevertheless, in applications, it can easily be justified by the ex-post maximin criterion. If the niece of the king's butler was systematically advantaged, this would show in an unequal distribution of utility, and an impartial process would quickly appear superior. It is also possible for a comprehensive consequentialist criterion to put intrinsic value on certain impersonal goods, such as the quality of procedures.

**Risk equity versus catastrophe avoidance.** Risk equity over independent risks can be satisfied by an ex-post inequality averse criterion, as noticed in Keeney (1980a) and illustrated by the fact that the ex-post maximin criterion ranks $L_4'$ above $L_1'$ (see Section 2 for the definition of these lotteries). This appears relevant for the discussion of the rescue problem pursued below.

What about preference for catastrophe avoidance? We have seen that it directly goes against aversion to ex-post inequalities, and therefore it cannot be accommodated by the maximin criterion proposed here. It may be that the apparently widespread preference for catastrophe avoidance relies on an illusion. Fearing large numbers of fatalities appears somewhat paradoxical if one would not similarly want to avoid joint losses of smaller magnitude, or joint gains. It appears obvious, by equity, that a $1/n$ chance that $n$ individuals all gain \$10 is preferable to the certainty of one individual alone gaining \$10. By symmetry, one should prefer a $1/n$ risk that $n$ individuals lose \$10 to the certainty of one individual losing $10. Now, dying prematurely is just a loss of bigger magnitude, so that one should presumably uphold the same judgment in the case of fatalities.

Nonetheless, there are two ways of making sense of catastrophe avoidance. In the case of a big loss, one may abandon inequality aversion and adopt "triage" preferences that seek to minimize the number of individuals suffering losses.[22] For instance, one might consider it less bad if one individual dies painfully than if two individuals die without pain. With such social preferences, it is indeed possible to combine preference for catastrophe avoidance in the case of large losses such as fatalities and aversion to ex-post inequality in the case of gains and small losses.

Another way of putting catastrophe avoidance in positive light is to take account of the fact that fatalities may not only consist in the premature death of some individuals but may also prevent the existence of other individuals (their descendants). Suppose that the death of one individual prevents the existence of $q$ other individuals, but that the death of $n$ individuals prevents many more than $qn$ individuals from coming into existence. For instance, if $n$ is the size of the whole population it may be that human life is wiped out entirely. In such a context it may be reasonable to prefer one fatality for sure to a $1/n$ risk of $n$ fatalities.

In summary, preference for catastrophe avoidance should not be adopted as an axiom, and can only appear as the result of special "triage" preferences in the case of big losses, or come up naturally as the result of taking account of exponential externalities in the case of certain hazards.

**Rescue versus prevention.** The rescue problem is connected to risk equity and concerns for ex-ante fairness, as explained above. One may think that the preference for

[22] An example of such social preferences is the poverty head count, which accepts worsening the situation of the remaining poor if this reduces the number of poor. This kind of social preferences is examined in Roemer (2004).

rescue is not related with issues of impartiality, since both the rescue operation and the prevention policy can be perfectly impartial. But it may still the case that one considers the lottery of accidents or diseases to be an improper way of allocating the uncertain time of death. In this way, rescue operations fight the unfairness of accident lottery.[23]

The consideration of time periods can also be relevant in some cases. Suppose that, before any victims are hit, one examines whether to devote resources to the logistics of rescue operations or to prevention activities. This suggests distinguishing three periods. The ex-ante period is before any harm is done, the ex-post period is after every uncertainty has been resolved, and in between there is an interim period when victims are there and it is not known whether they can be saved or not. If the interim situation is long enough or more generally has sufficient weight in the evaluation, it may be rational to prefer the rescue policy even if it is less efficient ex-ante and ex-post, because it provides better prospects for the victims in the interim period.

In order to illustrate this, consider two extreme examples of a health condition that hits at a certain age and kills after some time. In the first, people are hit around the age of fifty, and are saved or lost within one week. In the second example, the condition appears in the first year of life, but has no effect until the age of fifty, when people survive or die suddenly. It appears reasonable to prefer the prevention policy in the first case, since people enjoy a lower risk throughout their life, and the interim period is short. In contrast, a rescue policy may appear better in the second example, because even though more individuals are hit, they live a long part of their life with the hope of being saved.

But, even if one sticks to a pure ex-post approach and ignores the welfare effect

[23] One can object that prevention is a more efficient way of curbing the operation of the accident lottery. But if future accidents are not certain, it may be better to fight against the consequences of an accident that has occurred than to try to prevent accidents that may never occur. This argument is developed below.

of ex-ante or ex-interim utility, there is a way to make sense of the apparent duty of rescue that many people feel compelling. There is a difference between victims facing a sure prospect of death if no rescue is undertaken and other people facing a slightly greater risk in absence of prevention, and this difference can be described in terms of ex-post distributions. With the rescue operation there is still a positive probability that everybody will survive, whereas the prevention policy condemns the victims for sure. The ex-post maximin criterion, if it is applied by maximizing the probability of having no fatality,[24] therefore favors the rescue operation. This may also be true when the victims are not condemned to death in absence of rescue. Let us take up again the example of Section 2, with independent risks, and examine the choice between the rescue lottery

$$(\underbrace{\bar{p} - \triangle/m}_{m}, \underbrace{\underline{p}}_{n-m})$$

and the prevention lottery

$$(\underbrace{\bar{p}}_{m}, \underbrace{\underline{p} - \triangle/(n-m)}_{n-m}),$$

with $\bar{p} - \triangle/m \geq \underline{p}$. The probability of no fatality is, respectively,

$$\left(1 - \bar{p} + \frac{\triangle}{m}\right)^m (1 - \underline{p})^{n-m} \text{ and } (1 - \bar{p})^m \left(1 - \underline{p} + \frac{\triangle}{n-m}\right)^{n-m}.$$

One prefers the rescue lottery if $\bar{p} = 1$ (the victims are condemned in absence of rescue), or if $\bar{p} < 1$ and

$$\left(1 + \frac{\triangle}{(1 - \bar{p})\, m}\right)^m > \left(1 + \frac{\triangle}{(1 - \underline{p})\,(n - m)}\right)^{n-m}.$$

This is satisfied for all $n$ such that $n - m > m$ if

$$1 + \frac{\triangle}{(1 - \bar{p})\, m} > e^{\frac{\triangle}{(1-\underline{p})m}}.$$

[24] Applying the maximin criterion in this way is questionable since a fatality is just a reduction of lifetime and does not typically bring lifetime utility to a low number. There is an apparent irrationality in treating death as different from other utility losses.

To fix ideas, let us assume that victims are saved with the rescue operation, in the sense that $\bar{p} - \triangle/m = \underline{p}$ (after rescue, they face the same risks as the rest of the population). The above inequality then reads

$$1 + \frac{\bar{p} - \underline{p}}{(1 - \bar{p})} > e^{\frac{\bar{p}-\underline{p}}{1-\underline{p}}},$$

which holds true whenever $\bar{p} > \underline{p}$. This shows that even a pure ex-post approach, if it is egalitarian, can rationalize the preference for rescue. This is not, in essence, different from Keeney's observation that an ex-post egalitarian criterion can obey "risk equity" when the risks are independent, but it seems to find a relevant application here.

# 7 Conclusion

The expected value of the equal-equivalent utility yields a criterion which satisfies the omniscience principle and the Pareto principle restricted to macro risks. The omniscience principle excludes ex-ante egalitarian social welfare functions, but we have seen that concerns for ex-ante fairness can be accommodated to a substantial, perhaps sufficient, extent by an ex-post criterion of this sort. The utilitarian criterion does obey the omniscience principle, and combining this principle with a full-fledged version of the Pareto principle imposes embracing utilitarianism (under mild monotonicity and anonymity conditions). But we have seen that the omniscience principle actually weakens the case for the Pareto principle because ex-ante preferences at the individual level ignore correlations, a crucial information for anyone wanting to mimic an omniscient evaluator. Therefore we may conclude that ex-post egalitarianism emerges from this analysis as the most plausible approach to evaluating risky social alternatives.

There is, however, one line of defense of the ex-ante Pareto principle which has been ignored here. As mentioned in Hammond (1983) and Kolm (1998), one may associate the principle of respecting individual preferences over risk with the idea that individuals have the right to take risks and should be able to assume the consequences. Hammond concluded that such considerations may 'show that the usual consequentialist utilitarian approach to collective choice needs to be supplemented somewhat by non-utilitarian ideas such as liberty and equality of opportunity' (1983, p. 203). Now, a theory of responsibility and compensation for bad luck has been developed since then and provides tools for the analysis of the idea that individuals can sometimes be held responsible for the degree of risk to which they expose themselves.[25] A detailed analysis of this issue is the topic of a companion paper.

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[25] See e.g. Roemer (1998), and, for a survey of this literature, Fleurbaey and Maniquet (2006).

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# Appendix: Proofs

**Proof of Theorem 1**: By Weak Monotonicity and Continuity, there is always a solution $e$ to the equation

$$[U^s] \; I \; [(e, ..., e)],$$

and therefore the function $e(U^s)$ is well-defined. By Weak Monotonicity, $e(U^s) > e(U'^s)$ whenever $U^s \gg U'^s$. By Continuity, $e$ is continuous.

Take any lottery $U, U' \in \mathcal{L}$ such that

$$\sum_{s \in S} \pi^s e(U^s) = \sum_{s \in S} \pi^s e(U'^s).$$

By application of Omniscience Principle, one can replace each $U^s$ by $(e(U^s), ..., e(U^s))$. Let $\bar{U}$ denote the resulting lottery in which, by construction, $\bar{U}_i^s = \bar{U}_j^s$ for all $s \in S$ and all $i, j \in N$. One has $U \, I \, \bar{U}$. Similarly one can replace each $U'^s$ by $(e(U'^s), ..., e(U'^s))$ and construct $\bar{U}'$ such that $U' \, I \, \bar{U}'$. By Pareto for Macro Risks, $\bar{U} \, I \, \bar{U}'$. By transitivity, $U \, I \, U'$.

Consider now the case

$$\sum_{s\in S}\pi^s e\left(U^s\right) > \sum_{s\in S}\pi^s e\left(U'^s\right).$$

By the continuity and monotonicity properties of $e$ proved above, there is $V \in \mathcal{L}$ such that $V \gg U'$ and

$$\sum_{s\in S}\pi^s e\left(U^s\right) = \sum_{s\in S}\pi^s e\left(V^s\right).$$

By the previous step, $U\, I\, V$ and by Weak Monotonicity, $V\, P\, U'$. By transitivity, $U\, P\, U'$.

We now check that no axiom is redundant.

Drop Omniscience Principle. Take the maximin criterion applied to the vector of expected utilities.

Drop Pareto for Macro Risks. Take a criterion based on $\sum_{s\in S}\hat{\pi}^s e\left(U^s\right)$ for some arbitrary $\hat{\pi}$ that differs from $\pi$.

Drop Weak Monotonicity. Take universal indifference.

Drop Continuity. Take the ex-post leximin criterion.

**Proof of Corollary 1**: By Theorem 1, we know that for all $U, U' \in \mathcal{L}$,

$$U\, R\, U' \Leftrightarrow \sum_{s\in S}\pi^s e\left(U^s\right) \geq \sum_{s\in S}\pi^s e\left(U'^s\right).$$

Pigou-Dalton Equity and Anonymity imply that $e(.)$ must satisfy Pigou-Dalton Equity and Anonymity on $U^s$ vectors. This implies that it must be Schur-concave (see Marshall and Olkin 1979, pp. 22, 54). Conversely, if $e(.)$ is Schur-concave (and therefore symmetric), then $R$ satisfies Pigou-Dalton Equity and Anonymity.

Let us check that no axiom is redundant. For the first four axioms, the counter-examples are the same as for Theorem 1.

Drop Anonymity. Take a function $e(.)$ that satisfies Pigou-Dalton Equity but is not symmetric, such as:

$$e(u_1,...,u_n) = \frac{1}{|\{i \in N \mid u_i \leq u_1\}|} \sum_{\substack{i \in N \\ u_i \leq u_1}} u_i.$$

Drop Pigou-Dalton Equity. Take a strictly Schur-convex $e(.)$.

**Proof of Corollary 2**: By Theorem 1, we know that for all $U, U' \in \mathcal{L}$,

$$U \, R \, U' \Leftrightarrow \sum_{s \in S} \pi^s e\left(U^s\right) \geq \sum_{s \in S} \pi^s e\left(U'^s\right).$$

Hammond Equity implies that $e(.)$ must satisfy Hammond Equity on vectors $U^s$. Take two vectors $U^s$ and $V^s$ such that $\min_{i \in N} U^s < \min_{i \in N} V^s$. Construct $U'^s \gg U^s$ such that for some $i_0 \in N$ and for all $i \neq i_0$,

$$\begin{aligned} \min_{i \in N} U^s &< U'^s_{i_0} < U'^s_i, \\ U'^s_{i_0} &< \min_{i \in N} V^s, \end{aligned}$$

and $V'^s \ll V^s$ such that for all $i \in N$,

$$U'^s_{i_0} < V'^s_i < \min_{i \in N} V^s.$$

By repeated applications of Hammond Equity between $i_0$ and the other agents, one can construct $U''^s$ from $U'^s$, such that for some $i_0 \in N$ and for all $i \neq i_0$,

$$U'^s_{i_0} < U''^s_{i_0} < U''^s_i < V'^s_i.$$

By Hammond Equity, $e(U''^s) \geq e(U'^s)$. By Weak Monotonicity, $e(U^s) < e(U'^s)$, $e(U''^s) < e(V'^s) < e(V^s)$. Therefore $e(U^s) < e(V^s)$.

By Continuity, it then follows that for all $U^s, V^s$, $e(U^s) \leq e(V^s)$ if and only if $\min_{i \in N} U^s \leq \min_{i \in N} V^s$.

Take any $U^s$ and define $V^s = (\min_{i \in N} U^s, ..., \min_{i \in N} U^s)$. One has $e(U^s) = e(V^s)$, and since $V^s$ is egalitarian, $e(V^s) = V^s_1 = ... = V^s_n = \min_{i \in N} U^s$. Therefore $e(U^s) = \min_{i \in N} U^s$.

Let us check that no axiom is redundant. For the first four axioms, the counter-examples are the same as for Theorem 1.

Drop Hammond Equity. Take classical utilitarianism.

**Proof of Theorem 2**: Consider any lottery $U$. Let $V$ be defined by $V_i^s = U_{(i)}^s$ for all $s \in S$ and all $i \in N$. By Anonymity, for all $s \in S$, $[U^s]\ I\ [V^s]$. Therefore, by Omniscience Principle, $U\, I\, V$.

Notice that $V$ is comonotone. The above argument implies that we can restrict attention to comonotone lotteries and we do so hereafter. By Pareto for Comonotone Risks, there is an ordering $\tilde{R}$ over $\mathbb{R}^n$ such that $U\, R\, U'$ if and only if

$$\left(\sum_{s\in S}\pi^s U_i^s\right)_{i\in N}\ \tilde{R}\ \left(\sum_{s\in S}\pi^s U_i'^s\right)_{i\in N}.$$

By Hammond Equity, Anonymity and Strong Monotonicity, this ordering satisfies the following property: For all $u, v \in \mathbb{R}^n$, if there are $i, j \in N$ such that for all $k \neq i, j$, $u_k = v_k$, then $u\, \tilde{P}\, v$ if and only if

$$\begin{aligned} \min\{u_i, u_j\} &> \min\{v_i, v_j\} \text{ or:} \\ \min\{u_i, u_j\} &= \min\{v_i, v_j\} \text{ and } \max\{u_i, u_j\} > \max\{v_i, v_j\}. \end{aligned}$$

(See e.g. the proof of Th. 5 in Hammond 1979 for details about this step.) By Hammond (1979, Th. 4), $\tilde{R}$ is then the leximin ordering.

Let us check that no axiom is redundant.

Drop Omniscience Principle. Take the leximin criterion applied to the vector of expected utilities.

Drop Pareto for Comonotone Risks. Take the ex-post leximin criterion based on some arbitrary $\hat{\pi}$ that differs from $\pi$.

Drop Hammond Equity. Take classical utilitarianism.

Drop Strong Monotonicity. Take $R$ based on the computation of

$$\sum_{s\in S}\pi^s\left(\min_{i\in N}U_i^s-\max_{i\in N}U_i^s\right).$$

Drop Anonymity. Let $\geq_{lexico}$ denote the lexicographic criterion that compares the first component of vectors, and then the second component, and so on. Take $R$ defined by: $U\,R\,U'$ if

$$\left(\sum_{s\in S}\pi^s U_i^s\right)_{i\in N} >_{lex} \left(\sum_{s\in S}\pi^s U_i'^s\right)_{i\in N} \text{ or:}$$
$$\left(\sum_{s\in S}\pi^s U_i^s\right)_{i\in N} =_{lex} \left(\sum_{s\in S}\pi^s U_i'^s\right)_{i\in N} \text{ and } \left(\sum_{s\in S}\pi^s U_i^s\right)_{i\in N} \geq_{lexico} \left(\sum_{s\in S}\pi^s U_i'^s\right)_{i\in N}.$$

**Proof of Theorem 3**: By Pareto, there is an ordering $\tilde{R}$ over $\mathbb{R}^n$ such that $U\,R\,U'$ if and only if

$$\left(\sum_{s\in S}\pi^s U_i^s\right)_{i\in N}\tilde{R}\left(\sum_{s\in S}\pi^s U_i'^s\right)_{i\in N}.$$

By Omniscience Principle and Anonymity, one can permute $U_i^s$ and $U_j^s$ without changing the value of a lottery. Such a permutation has the effect of adding $\pi^s\left(U_j^s-U_i^s\right)$ to $\sum_{s\in S}\pi^s U_i^s$ and of subtracting the same quantity from $\sum_{s\in S}\pi^s U_j^s$. By the universal domain $\mathcal{L}$, this implies that for all $u,u'\in\mathbb{R}^n$ and for all $a\in\mathbb{R}$, $u\,\tilde{I}\,u'$ if $u'$ is obtained from $u$ by transferring $a$ from $i$ to $j$, for any $i,j\in N$.

By a finite number of such transfers (as proved in Hardy, Littlewood and Polya 1952, pp. 47-48), one can obtain the vector $(\bar{u},...,\bar{u})$, where $\bar{u}=\frac{1}{n}\sum_{i\in N}u_i$, from any $u\in\mathbb{R}^n$. Therefore, for every $u\in\mathbb{R}^n$, $u\ \tilde{I}\ (\bar{u},...,\bar{u})$.

By Weak Monotonicity, one then obtains that for all $u,v\in\mathbb{R}^n$,

$$u\,\tilde{R}\,v\Leftrightarrow\frac{1}{n}\sum_{i\in N}u_i\geq\frac{1}{n}\sum_{i\in N}v_i.$$

Let us check that no axiom is redundant.

Drop Omniscience Principle. Same as in Th. 1.

Drop Pareto. Take the ex-post maximin criterion.

Drop Weak Monotonicity. Take universal indifference.

Drop Anonymity. Take the criterion based on the expected utility of one particular agent.

**Proof of Theorem 4**: Consider the following two-agent, two-state lotteries (with equiprobable states):

$$U = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}, \; V = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, \; W = \begin{pmatrix} 2 & 2 \\ 0 & 0 \end{pmatrix}, \; X = \begin{pmatrix} 0 & 0 \\ 2 & 2 \end{pmatrix}.$$

By Pareto, $U$ is equivalent to $V$ and by Minimal Egalitarianism, $V$ is better than $W$. By transitivity, $U$ is better than $W$. But by Anonymity, $X$ is equivalent to $W$, so that by Omniscience Principle, $W$ is equivalent to $U$, a contradiction.

The tedious generalization of the argument to $n$ agents, more than two states of nature and an arbitrary $\pi$ is omitted here.

We now check that no axiom is redundant.

Drop Omniscience Principle. Take the leximin criterion applied to the vector of expected utilities.

Drop Pareto. Take the ex-post leximin criterion.

Drop Minimal Egalitarianism. Take classical utilitarianism.

Drop Anonymity. Same as in the proof of Th. 2.