Housing as a Measure for Long-Run Risk in Asset Pricing*

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Abstract

I evaluate the effects of long-run consumption growth risk and housing consumption risk on asset prices. Current asset values are affected by the risk-return tradeoff in the long-run. Housing plays an important role in the economy. As an asset, it is particularly sensitive to long-run risk-return trade off; as a consumption component, it accounts for one fifth of the total expenditures in non durable goods and services. The investment horizon for housing is usually distant in the future. Investors fear shocks that can affect the value of their house for a long period of time. Such shocks affect substantially the services obtained from the house and its price as an asset as well. I use a non-separable utility function with non-housing consumption and consumption of housing services, which generates an intertemporal composition risk, besides the traditional consumption growth risk. The composition risk has effects for the valuation of cash flow growth fluctuations far into the future due to the persistence of consumption growth. I provide a closed form solution for the valuation function despite the non-separability. This allows me to quantify the price of risk in the long-run with inputs from vector autoregressions. I evaluate the different exposure to long-run risk of a cross section of portfolios of securities, and characterize the price of risk for different investment horizons. The model also explains the spread of the returns to different portfolios sorted in book to market and housing returns, at different investment horizons.

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Introduction

Equilibrium prices and expected returns under the consumption based capital asset pricing model (CAPM) are determined by consumption growth risk. Recent evidence[1] suggests that the use of long-run aggregate consumption risk helps explain cross-sectional and aggregate stock returns.

The time variation of risk premia is an important element in asset pricing models. Expected returns are high in recessions, when people might be less willing to hold risky assets. It is an empirical fact that stock returns are predictable by instruments that are informative about the business cycle. Pakos (2006) and Yogo (2006) use the consumption of durable goods for the purpose of obtaining a factor of risk that implies time varying risk premia. In Piazzesi, Schneider, and Tuzel (2007) consumers fear recessions when consumption is low, but they specially fear severe recessions, when additionally housing expenditures are low relative to total consumption expenditures. This risk also implies a time varying risk premium, higher at business cycle troughs than at peaks. Another highly acclaimed family of models are those featuring habit formation[2], which work in the direction of obtaining countercyclical variation of the price of risk as well. Return dynamics and the cross-section have diverted attention from the equity premium, which remains unexplained, or explained by higher levels of risk aversion.

I propose a consumption based model that exploits the pricing implications of the risk in the long-run, and implies a time varying risk premium. I use recursive preferences over non-housing consumption and consumption of housing services. Risk premia depend on the exposure of assets' cash flows to risk, and change with the investment horizon considered. I find that housing is more exposed to risks that arise in the long-run, around 40 quarters, while the high and low book-to-market portfolios are more exposed —therefore highly rewarded— when horizons of 10 to 15 quarters are considered. The model captures the spread between portfolios sorted in book-to-market in the short run, as well as in the long-run.

There are two main features of the model. First, I propose an alternative constant elasticity of substitution aggregation between non-durable goods and housing services in the utility function.

[1] Bansal and Yaron (2004) and Hansen, Heaton, and Li (2006) are examples of empirical success of long-run risk models.

[2] Abel (1990), Constantinides (1990), and Campbell and Cochrane (1999) are examples featuring habit formation, and Chen and Ludvigson (2004) and Fillat and Garduño (2006) are examples of empirical estimation of the Campbell and Cochrane model.

Secondly, I consider recursive preferences as first introduced in Epstein and Zin (1991). Individuals are concerned about three types of risk or, in finance terms, there are three priced factors: consumption growth risk, composition risk —that arises from the non-separable utility function—, and long-run consumption growth risk, result of the recursive utility. I obtain an explicit solution for the pricing function (stochastic discount factor), identify the three factors of risk, and evaluate the response of the factors' pricing to shocks that have an effect in the long-run.

The representative consumer derives utility from consumption of non-housing goods and housing services, which are imperfect substitutes. Fluctuations in the expenditure shares of these goods have an effect on the expected returns, and also on the long-run valuation of an asset's risky cash flows. Piazzesi, Schneider, and Tuzel (2007) have shown that a model with housing services offers an explanation for the long-horizon predictability of excess stock returns. As in the standard model, investors value highly consumption when a recession occurs, so they sell claims on future consumption expecting it higher than today. That is the consumption growth risk, the factor that appears in the standard consumption based CAPM. Besides, investors are even more fearful to changes in expenditure shares on housing —changes in the *composition* of their consumption bundle—. Claims on future streams of consumption are sold desperately in periods where the relative quantity of housing services consumption is low due to the substitutability. So in very bad moments, caused by a low share of housing expenditures, the intra-temporal substitution causes even lower prices. A non-separable utility function allows to identify better the links between risky asset returns and macroeconomic factors. Lustig and Nieuwerburgh (2004) present a similar model, where the housing collateral plays the role of the variable that predicts expected returns, since constrained homeowners, whose collateral value declines, become more risk averse.

I use the class of "generalized expected utility" preferences proposed by Epstein and Zin (1991), or recursive preferences, to parameterize independently intertemporal elasticity of substitution and risk aversion. this representation of preferences has the major advantage of changing risk aversion without necessarily modifying the eagerness of consumers to smooth over time, therefore keeping the risk free rate close to observed values. The Euler equation obtained from the power utility states that differences in risk across portfolios are due to contemporaneous covariances with consumption. Within a recursive utility framework, the agent does not need to perfectly smooth expected marginal utility over time, and long-run consumption growth determines differences in risk. Mehra and Prescott (2003) acknowledge that this class of preferences could potentially solve the equity premium puzzle, with an important problem. The original empirical analysis in Epstein and Zin (1991) hinges on the return on all invested wealth which is not observable, and the market portfolio is used as a proxy. Instead of following the approximation, I solve explicitly for the value function, as a function of the underlying state of the economy. There is recent empirical success of models that investigate the effects of the long-run consumption growth risk, generated by the recursive preferences specification. Bansal and Yaron (2004), Bansal, Dittmar, and Lundblad (2005), and Hansen, Heaton, and Li (2006) present the theoretical background and the empirical findings. In particular, Hansen, Heaton, and Li (2006) finds a closed form solution for the special case of unitary elasticity of intertemporal substitution, that avoids the use of an approximation for the wealth portfolio[3]. They measure the long-run risk return tradeoff for the valuation of cash flows and solve a model where the price of risk is decomposed in one-period price of risk and long-run price of risk. Their model is successful in explaining the cross section of returns, in particular the spread between high and low book-to-market portfolio returns, in the long-run. It also offers an explanation for the particular dynamics of housing returns.

Summarizing, I obtain a three factor model, where the factors are aggregate sources of macroeconomic risk: consumption growth risk, composition risk, and long-run risk.

There are two important implications that arise from the use of a non-separable utility function and recursive preferences. First, the higher covariance of the *composition* risk with excess returns helps explaining the equity premium in the short run. Second, the housing expenditure shares have a non-negligible effect on the long-run price of risk, in addition to the price of long-run consumption growth risk. Different exposures of the assets to the long-run consumption risk help explain the cross-section of returns.

By finding a closed form solution for the model with non-separability across non-housing and housing consumption, I can identify the effect of the expenditure shares on the price of risk. I consider an endowment economy as in Lucas (1978), with two *trees*. One delivers non-housing consumption and the other delivers housing services. I propose a statistical model for the exogenously given consumption process and for the housing consumption. The latter is specified indirectly, through the expenditure

[3]Epstein and Zin (1991) relies on it, fact that is criticized in Mehra and Prescott (2003).

shares in non-housing. To ensure it is bounded between zero and one, expenditure shares follow a log-linear-quadratic process. This strategy implies solution forms germane to the risk-sensitive optimal control problems[4]. I consider corporate earnings and aggregate current net stock of private residential structures growth as the determinants of the state of the economy. The proposed model generates a heteroskedastic stochastic discount factor that implies countercyclical price of risk. When the state of the economy is low, either because corporate earnings are low or the growth in private residential stock is low, marginal utility is high, therefore risk prices are high. In some sense, this feature is related to the habits models, where the state variable determines the risk aversion and therefore, the price of risk. The state of the economy determines the magnitude of the response of asset prices to shocks to consumption growth. I estimate the parameters of the statistical model for the endowment of consumption and housing services, and evaluate prices and risk premia for different investment horizons with 5 portfolios sorted in book-to-market, housing, a claim to consumption, and a 3-month T-bill.

There are several studies in the housing literature that analyze the effects of adjustment costs in consumption and the fact that one cannot freely adjust the size of the house from period to period. Examples are found in Stokey (2007), Chetty and Szeidl (2005), and Flavin and Nakagawa (2004). Adjustment costs do not affect asset prices, only endogenous relative quantities, which are an outcome of intra-temporal first order condition. As long as relative expenditures are measured with aggregate data, we can safely compute prices from the Euler equation, imposing the equilibrium conditions.

In Section I, I present the model, the intra and inter-temporal first order conditions, and the solution of the model. Possible interpretation of the parameters is discussed. Section II is devoted to the description of the asset returns used in the estimation and the covariation with the relevant variables of the model. I estimate the dynamics of the system of variables in Section III and discuss the pricing results obtained in Section IV. Section V concludes.

[4]See Whittle (1990).

I Model

Consider an economy with a representative agent who derives utility from a consumption bundle, $\mathcal{C}_t$. The utility function is recursive as in Epstein and Zin (1989) and can be written as follows:

$$V_t = \left[(1-\beta)\mathcal{C}_t^{(1-\rho)} + \beta E_t\left[V_{t+1}^{1-\gamma}\right]^{\frac{1-\rho}{1-\gamma}}\right]^{\frac{1}{1-\rho}}. \tag{1}$$

Intertemporal elasticity of substitution is measured by $1/\rho$. A higher the value of ρ implies that agents are less willing to substitute consumption over time. Independently, a constant elasticity of substitution aggregation is used for the risk adjustment term. The parameter γ is the coefficient of risk aversion, and determines the curvature of the value function. Recursive preferences allow to modify the willingness of the agents to smooth consumption over time independently from their willingness to smooth consumption over different states of the world. The subjective discount factor is represented by β, and V_{t+1} is the continuation value of a consumption plan from $t+1$ on. The conditional expected value operator $E_t[\cdot]$ is defined as the expected value conditional on the set of information $\mathcal{F}_t$ that the agent has at time t.

The consumption bundle $\mathcal{C}_t$ is composed by two goods: consumption of housing services, S_t, and non-housing consumption, C_t, which represents consumption of non-durables.

$$\mathcal{C}_t = \left(C_t^{\frac{\varepsilon-1}{\varepsilon}} + w_t S_t^{\frac{\varepsilon-1}{\varepsilon}}\right)^{\frac{\varepsilon}{\varepsilon-1}}, \tag{2}$$

where ε is the elasticity of substitution between housing and non-housing consumption, and w_t represents a preference shift. It shifts the preferences towards housing services within a period and captures secular trends in the observed relative consumption of non-housing and housing.

There are three assets in the economy: a house, a stock, and a risk-free bond. I follow an endowment economy approach, introduced by Lucas (1978). In this economy, there are two *trees* with a positive supply: the house and the stock. The bond is in zero net supply. The house pays a stream of housing services and the stock pays a stream of non-housing consumption goods and services. The agent chooses consumption of non-housing goods and services, housing services, and asset holdings subject to

$$p_t^C C_t + p_t^S S_t + q_t^C \theta_t^C + q_t^S \theta_t^S = (q_t^C + p_t^C \bar{C}_t)\theta_{t-1}^C + (q_t^S + p_t^S \bar{S}_t)\theta_{t-1}^S, \tag{3}$$

where q_t^C and q_t^S are the prices at which the two assets trade, and θ_t^C and θ_t^S are holdings of the two assets.

A Intra-temporal first order condition

The static first order condition results in the marginal rate of substitution and relative prices relationship,

$$\frac{p_t^C}{p_t^S} = \frac{1}{w_t}\left(\frac{C_t}{S_t}\right)^{-\frac{1}{\varepsilon}}. \tag{4}$$

Multiplying both sides of (4) by relative quantities, C_t/S_t, we obtain:

$$\frac{p_t^C C_t}{p_t^S S_t} = \frac{1}{w_t}\left(\frac{C_t}{S_t}\right)^{\frac{\varepsilon-1}{\varepsilon}}. \tag{5}$$

The left hand side of (5) is the ratio of expenditures, which is well measured in the data. On the right hand side we have the relative quantities. I define the non-housing expenditure shares as the fraction of total expenditures, which will prove useful in the solution of the model:

$$\alpha_t \equiv \frac{p_t^C C_t}{p_t^C C_t + p_t^S S_t}; \qquad \frac{1}{\alpha} = 1 + w_t \left(\frac{S_t}{C_t}\right)^{\frac{\varepsilon-1}{\varepsilon}}. \tag{6}$$

If the two goods are substitutes, $\varepsilon \geq 1$, and an increase in housing consumption relative to non-housing causes a decrease in non-housing expenditure shares, α_t. Conversely, if they are complements, or $\varepsilon \leq 1$, and increase in housing consumption relative to non-housing implies a decrease in non-housing expenditure shares. Expenditure shares in non-housing consumption are stationary over time. Although the autocorrelation is 0.98 with quarterly data, assuming that the expenditure shares are not stationary would imply convergence to 1 or zero with probability one, which is not realistic. Furthermore, under that assumption, one of the two type of goods would vanish from the utility function, and that is hardly possible to infer from the data.

Figure 1 shows the evolution of the expenditure shares in the last 55 years and Figure 2 shows the prices and quantities of housing services relative to non-housing consumption.

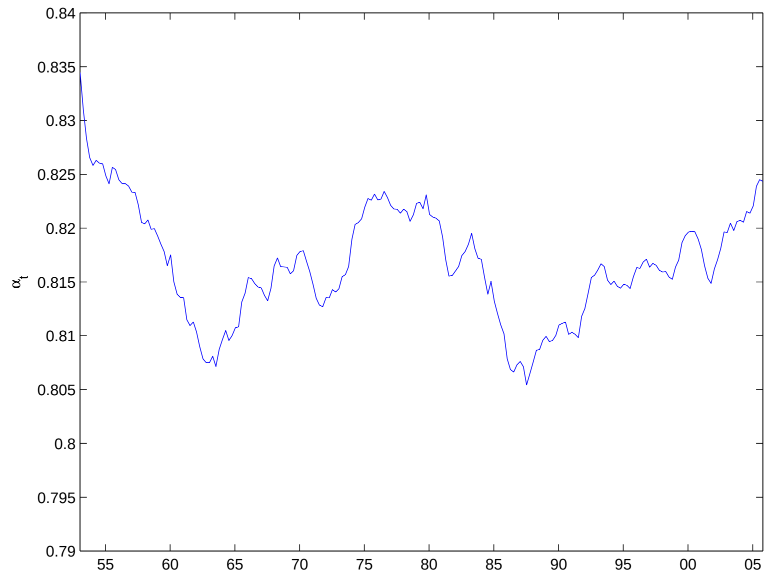

Figure 1: **Expenditure Shares.** Expenditure shares of non-housing consumption over total expenditure in non-durable goods and services, corresponding to α in the model. Source: NIPA Personal Income and Outlays, Table 2.3.5. Quarterly data, 1953-2006.

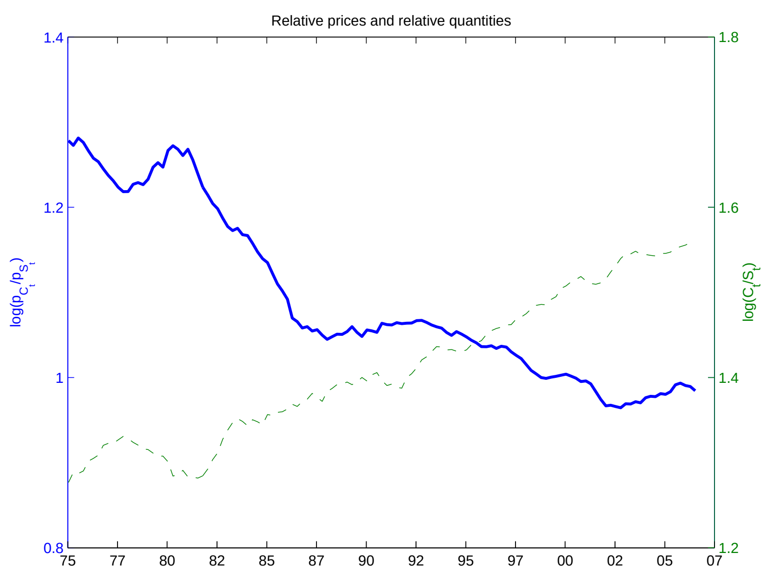

Figure 2: **Relative prices and relative quantities.** On the left axis, the bold line shows the evolution of relative prices in the last 30 years. On the right axis, the dashed line shows the ratio of the quantity indexes for housing services over non-housing consumption computed by the BEA. Source: NIPA Personal Income and Outlays, Tables 2.3.4 and 2.3.6. Quarterly data, 1975-2006.

Even though we do not observe that expenditure shares converge to 1 or zero, in the last 30 years measured prices of housing services relative to non-housing consumption have increased over time. The construction of the quantity index from NIPA implies that the relative quantities have decreased over the last 30 years as well. The reversed pattern is observed previous to 1975. There is a low frequency component driving measured relative prices and relative quantities, but relative expenditures remain stationary, although not constant. The process w_t captures this low frequency component driving relative prices and relative quantities, and keeps relative expenditures and expenditure shares stationary over time. This can be seen in (4) and (5). If w_t was to be kept constant, relative prices and relative quantities being integrated of order one, I(1), would imply a decrease in the left hand side of both (4) and (5). While the data show a decrease in relative prices, the effect on relative expenditures following (5) is not observed. The only case where a time trend in relative quantities could imply a decreasing trend in prices and not in expenditure shares corresponds to unitary elasticity of substitution between housing services and non-housing consumption. But that case implies a constant expenditure share, the special case when the CES specification coincides with the Cobb-Douglas. The data do not support the assumption of constant expenditure shares. The functional form for the aggregation between non-housing and housing consumption assumes homogeneity, which is supported by the fact that expenditure shares remained stable in a period were income increased substantially.

Identification of the elasticity of substitution between non-housing and housing consumption is not possible without a good measure for w_t. Equation (5) implies a cointegration relationship between w_t and S_t/C_t, with the cointegrating vector $[1, 1 - 1/\varepsilon]$, so that the right hand side is stationary, as the left hand side is. Since there is no good measure for either relative quantities or for w_t, I evaluate asset prices for a range of elasticities of substitution.

B Inter-temporal first order condition and Pricing

In this section I find a closed form solution for both the value function, expressed recursively in (1), and for the stochastic discount factor implied by the model.

Equilibrium in the endowment economy is characterized by stochastic processes for aggregate output of the two goods $\{\bar{C}_t, \bar{S}_t\}$, that in equilibrium must be equal to C_t and S_t respectively, a vector of prices $\{p_t^C, p_t^S\}$, and a vector of portfolio holdings $\theta_t^S = \theta_t^C = 1$, that maximize (1), subject to the budget constraint (3). I propose a model for the evolution of consumption and housing, and compute the prices that support them as equilibrium quantities.

B.1 Endowment

I assume that consumption follows a moving-average process, in particular, I express it as in Bansal and Yaron (2004), where consumption growth follows a random walk plus a state variable that causes persistent changes, therefore predictable.

$$c_{t+1} - c_t = \mu^c + \phi^c x_t + \sigma_0^c \nu_{t+1} \tag{7}$$

$$x_{t+1} = \delta x_t + \sigma_0^x \nu_{t+1}, \tag{8}$$

where ν_{t+1} is a multivariate normally distributed vector, with variance equal to the identity matrix. Lower-case letters represent natural logarithm of upper-case, so $\log C_t = c_t$. The unconditional long-run average of consumption growth is represented by μ^c.

The predictable component of consumption growth x_t follows a persistent autoregressive process. It is interpreted as the state of the economy, and it determines the long-run expected consumption growth. Changes in the state of the economy determine whether we are in an economy with a high or low long-run consumption growth, leveraged by ϕ^c. If consumption followed a random walk, there would be no persistent shocks to consumption growth, and the current shock would vanish in one period. Conversely, in this economy consumers fear shocks because they affect their conditional expectation of consumption growth for several periods through $\phi^c x_t$. There is no strong evidence against either of these possibilities. The purpose of my paper is to observe how returns behave when investors do fear shocks that may lower consumption growth of non-housing goods and also unbalance their optimal mix of housing and non-housing consumption for more than a period. Therefore the model relies on assuming that consumption has a small predictable component that imposes the persistence of the shocks. The conditional specification allows to identify the long-run response of consumption to innovations and to estimate it empirically —as in Hansen, Heaton, and Li (2006) and Malloy, Moskowitz, and Vissing-Jørgensen (2006)—, using VAR analysis.

I model the endowment of housing indirectly through expenditure shares, rather than quantity of housing services. Expenditure shares are an intra-period equilibrium result, from (4). Therefore, there is a one-to-one relationship between real quantities and expenditure shares for a given value of w_t. There exists an endowment process of housing services for which there exists a vector of prices that supports the process of expenditure shares. In particular, I model the evolution of the log expenditure shares as a linear quadratic function of the state of the economy x_t. As shares, they must lie in the unit interval, which is granted with a linear quadratic function of a normally distributed process:

$$-\log \alpha_t = \mu^{\alpha} + \phi^{\alpha} x_t + x_t' \Psi x_t. \tag{9}$$

This specification imposes conditions on ϕ^{α} and Ψ in order to keep the expenditure shares between 0 and 1. Equation (9) implies that expenditure shares are persistent if x_t is also persistent. The quarterly autocorrelation of non-housing expenditure shares is 0.98, supporting the persistent specification in (9).

B.2 Valuation

Define the stochastic discount factor as the marginal valuation of a stream of future value expressed in terms of non-housing consumption. The choice of numeraire is not innocuous. Empirically, it allows to use data on non-housing consumption and expenditure shares. It has been argued that these data are well measured[5], relative to the constructed quantity and price indexes for housing services. By homogeneity of degree one, I can express the value function as

$$V_t = \frac{\partial V_t}{\partial \mathcal{C}_t} \mathcal{C}_t + E_t \left[\frac{\partial V_t}{\partial V_{t+1}} V_{t+1} \right]. \tag{10}$$

Scaling the value function by the marginal valuation of non-housing consumption, I obtain the shadow valuation of a stream of future value expressed in terms of marginal value of non-housing consumption. Therefore it is a valid one-period stochastic discount factor (SDF_{t+1}), which is simplified

[5] A discussion on this follows in the description of the data.

to the following expression:

$$SDF_{t+1} = \beta \left(\frac{V_{t+1}}{E_t \left[V_{t+1}^{1-\gamma} \right]^{\frac{1}{1-\gamma}}} \right)^{\rho-\gamma} \left(\frac{C_{t+1}}{C_t} \right)^{-\rho} \left(\frac{1 + \left(w_{t+1} \frac{S_{t+1}}{C_{t+1}} \right)^{\frac{\varepsilon-1}{\varepsilon}}}{1 + \left(w_t \frac{S_t}{C_t} \right)^{\frac{\varepsilon-1}{\varepsilon}}} \right)^{\frac{1-\varepsilon\rho}{\varepsilon-1}} . \tag{11}$$

The discount factor is composed by three *factors*. The first term corresponds to the risk adjustment, originated by the recursive utility function. It can be also interpreted, following the literature on robustness, as the fear that consumers have to wrongly specify the model according to which they take decisions, as in Anderson, Hansen, and Sargent (2003)'s work on model misspecification. If agents are risk neutral, this term captures the innovation or disparity between realized value and expected value of a stream of future consumption. The second factor is the consumption growth risk, as in the consumption based CAPM. The third factor captures the composition risk. It reflects the fact that investors fear severe recessions that happen when the consumption of housing services falls relative to aggregate consumption. When consumption growth is low, agents' marginal utility increases. But if, besides a low consumption growth, there is a decline in housing services relative to non-housing consumption, the marginal utility also increases, when the two goods are substitutes ($\varepsilon \geq 1$). Therefore, if consumption of non-housing is declining *and* housing consumption also declines relative to non-housing, the substitutability between the two types of goods causes a greater increase in marginal utility. Shocks causing either of these increases of marginal utility are even more feared if they have a persistent effect, which is captured by the first factor.

From (4)-(6), we can express the third term —corresponding to the *composition* risk— only as a function of the non-housing consumption expenditures as a share of total expenditures, α. Therefore the composition risk is fully described by the change in expenditure shares. I relegate to the Appendix the detailed algebra to obtain it. The stochastic discount factor that results is

$$SDF_{t+1} = \beta \left(\frac{V_{t+1}}{E_t \left[V_{t+1}^{1-\gamma} \right]^{\frac{1}{1-\gamma}}} \right)^{\rho-\gamma} \left(\frac{C_{t+1}}{C_t} \right)^{-\rho} \left(\frac{\alpha_{t+1}}{\alpha_t} \right)^{\frac{1-\varepsilon\rho}{1-\varepsilon}} . \tag{12}$$

Once expenditure shares are substituted in, the role of the preference process w_t disappears. Whether the role of w_t is to correct for the measurement error in the quantity and price indexes, or to account for the quality improvements corresponding to housing services, once the discount factor is expressed as a function of the expenditure shares, the intra-temporal problems are overcome. Expenditure shares implicitly include improvements in quality, since they are an intra-temporal equilibrium result, and they are less subject to measurement error.

The risk adjustment is not directly observable, since the value function depends non-linearly on the two types of goods and the continuation value. Equivalently, the model could be solved as a dynamic asset allocation problem. Doing so entails the use of the return to the total wealth portfolio. Instead of following the portfolio choice approach approximating the returns on the wealth portfolio, I focus on the explicit solution of the value function. For tractability, I consider the case where the elasticity of intertemporal substitution ρ is fixed at one, for which I can obtain a closed form solution for the value function and for the stochastic discount factor. There is mixed evidence about the value of the intertemporal elasticity of substitution. Jones, Manuelli, and Siu (2000) examine a real business cycle model and conclude that it should be calibrated between 0.8 and 1. Conditional Euler equation estimations with stock returns in Hansen and Singleton (1983) and Hall (1988) find its value smaller and closer to zero. Heterogeneity of agents is also explored in the literature. Attanasio, Banks, and Tanner (2002) and Vissing-Jørgensen (2002) conclude that the elasticity of substitution is closer to 1 for stockholders and rather smaller for non-stockholders. I consider the case of $\rho = 1$, which gives the exact solution for the value function.

I consider the value function scaled by the consumption of non-housing goods and services, as I have done above for the stochastic discount factor. Non-housing consumption becomes the numeraire, for practical reasons, but also for empirical reasons regarding the price indexes[6]. Thus, the scaled value function is:

$$\frac{V_t}{C_t} = \left((1-\beta)\left(\frac{1}{\alpha_t}\right)^{\frac{\varepsilon(1-\rho)}{\varepsilon-1}} + \beta E_t \left[\left(\frac{V_{t+1}}{C_{t+1}}\right)^{1-\gamma} \left(\frac{C_{t+1}}{C_t}\right)^{1-\gamma} \right]^{\frac{1-\rho}{1-\gamma}} \right)^{\frac{1}{1-\rho}}. \tag{13}$$

[6]See Piazzesi, Schneider, and Tuzel (2007) for the analysis of aggregate consumption as a numeraire.

Define $v_t = \log\left(V_t/C_t\right)$. Taking logs and rewriting the term inside of the expectation,

$$v_t = \frac{1}{1-\rho}\log\left((1-\beta)\left(\frac{1}{\alpha_t}\right)^{\frac{\varepsilon(1-\rho)}{\varepsilon-1}} + \beta E_t\left[e^{(1-\gamma)(v_{t+1}+c_{t+1}-c_t)}\right]^{\frac{1-\rho}{1-\gamma}}\right). \tag{14}$$

As I mentioned above, I focus on the limiting case of $\rho = 1$. The value function becomes

$$\lim_{\rho\to 1} v_t = (1-\beta)\frac{\varepsilon}{\varepsilon-1}\log\frac{1}{\alpha_t} + \frac{\beta}{1-\gamma}\log E_t\left[e^{(1-\gamma)(v_{t+1}+c_{t+1}-c_t)}\right]. \tag{15}$$

The second term is as in Hansen, Heaton, and Li (2006). Additionally I obtain that the expenditure shares have a permanent effect on the recursive log-value function, which now I proceed to solve, assuming an equilibrium process for consumption growth and expenditure shares.

Proposition 1. *The value of a consumption plan of future housing and non-housing consumption at time t, expressed in terms of non-housing consumption, is a log-linear quadratic function of the state of the economy given by x_t. The value function depends linearly on the log of expenditure shares, and risk-adjusted linearly on the consumption growth. The solution of the value function as a function of the state x_t is*

$$v_t = D + Fx_t + x_t' H x_t \tag{16}$$

Proof. See Appendix A.2 for the verification of the functional form and the values of the coefficients.

□

The values of D, F, and H are functions of the underlying parameters. The matrix H in the quadratic term is the solution of a well defined Riccati equation, a result familiar in the risk-sensitive optimal control literature[7]. Naturally, H is the only set of parameters related to Ψ, the quadratic form for the logarithm of the shares that ensures the permanence of the level of the shares between 0 and 1. If the shares were approximated by a linear process, H would be equal to zero.

Using (16), the one-period stochastic discount factor at $t+1$ can be solved as well, in a linear-quadratic heteroskedastic function. I also leave the details of the derivation and the explicit solution

[7] See Whittle (1990) for an extensive review.

for each of the coefficients to Section A.3 in the Appendix. Taking logarithms of the expression in (12), we obtain the discount factor as function of the risk-adjusted expected continuation value, the consumption growth and the level of expenditure shares in non-housing consumption. Innovations in the stochastic discount factor are determined by innovations in consumption growth, in expenditure shares growth, and in the risk adjustment:

$$\begin{aligned} sdf_{t+1} = \log\beta + (1-\gamma)\left[v_{t+1} - \frac{1}{1-\gamma}\log E_t\left[e^{(1-\gamma)(v_{t+1}+c_{t+1}-c_t)}\right]\right] \\ - \gamma(c_{t+1}-c_t) + \log\alpha_{t+1} - \log\alpha_t. \end{aligned} \tag{17}$$

Proposition 2. *The solution of the logarithm of the discount factor is a log-linear quadratic function of the state and of the vector of shocks to the economy.*

$$sdf_{t+1} = J + Kx_t + x_t'Lx_t + M(x_t)\nu_{t+1} + \nu_{t+1}'N\nu_{t+1}. \tag{18}$$

Proof. See Appendix A.3. □

Equation (17) explicitly expresses the stochastic discount factor as a function of the consumption growth, the growth of the expenditure shares, and the long-run discounted impulse response of both processes, with a level term and a quadratic adjustment. Substituting in the processes for consumption growth and expenditure shares growth, the linear-quadratic heteroskedastic function is obtained as a function of the underlying parameters and the value of the state variable x_t. I leave the details for the Appendix, as well as the precise functional form for the coefficients. The time-varying coefficient corresponding to the first order effect of the shock in the pricing function, $M(x_t)$, is specially important.

C Risk Prices

I want to focus on the coefficient of the shock, $M(x_t)$, which determines —to a first order approximation— the one-period price of risk. When studying the pricing relations, what matters is the covariance of the returns with the innovations in the stochastic discount factor. The component that is not predetermined is the coefficient corresponding to the shock, and it is described by

$$- \sigma_0^c + (1-\gamma)(F\sigma_0^x - \sigma_0^c) - \phi^\alpha \sigma_0^x + 2x_t' \delta((1-\gamma)H - \Psi)\sigma_0^x \tag{19}$$

and the second order effect caused by the quadratic term, N. Focusing on the first order effects, there are some interesting insights. The first term, $-\sigma_0^c$ is the one-period exposure of consumption to risk. It captures the one-period response of consumption growth to a shock. The second term, $(1-\gamma)(F\sigma_0^x - \sigma_0^c)$, captures the long-run response of consumption and expenditure shares to a shock today. It is the component of the price implied by the recursive formulation. The third term, $2x_t'\delta((1-\gamma)H - \Psi)\sigma_0^x$, accounts for the consumption risk implied by the non-separability between housing services and non-housing consumption. Larger sensitivity of the expenditure shares to the state x_t is captured by ϕ^α, and reflected in higher prices. The intuition for this extra term comes from the effect of the shock in the composition of consumption. The higher the ϕ^α, the more negative is the effect that a shock causes in the expenditure shares, therefore, the counter-cyclicality of the non-housing expenditure shares becomes more acute. Expression (19) becomes the mean of the normally distributed shock ν_{t+1} under the risk-neutral probabilities. The last term captures the heteroskedasticity of the pricing function. It arises from the quadratic form, and introduces time variation of the price of risk, or equivalently time varying expected returns. If we eliminate the long-run risk by setting δ to a matrix of zeroes, the only effect that remains is the response of current consumption growth, which is basically the original consumption based CAPM, plus the contemporaneous composition risk.

The term $(F\sigma_0^x - \sigma_0^c)$ deserves special attention, since it determines the effects of long-run risk. It is a function of the discounted response of consumption and expenditure share to a contemporaneous shock to the economy. It appears in the level term of (18). It captures not only the intertemporal composition risk originated by the recursive utility formulation, but also the intra-temporal composition risk due to the non-separability between housing and non-housing consumption. It is important to remark that the intra-temporal composition risk does have an effect in the long-run, since its price does not vanish in one period. After substituting F for its value, and rearranging terms, $(F\sigma_0^x - \sigma_0^c)$ can be expressed as

$$F\sigma_0^x + \sigma_0^c = \sigma^c(\beta\zeta) + (1-\beta)\frac{\varepsilon}{\varepsilon-1}\sigma^\alpha(\beta\zeta)$$
$$- (1-\beta)\frac{\varepsilon}{\varepsilon-1}\sigma_0^{x\prime} S_{\beta\zeta}\sigma_0^x + 2\beta(1-\gamma)\sigma_0^c\zeta\sigma_0^{x\prime} H\delta(I-\beta\zeta\delta)^{-1}\sigma_0^x, \tag{20}$$

where $\sigma^c(\beta\zeta)$ and $\sigma^\alpha(\beta\zeta)$ represent the long-run discounted impulse responses of non-housing consumption and expenditure shares, respectively, and S_β solves $S_\beta - \beta^2\delta' S_\beta\delta = \Psi$. There are three shocks in the system that affect the state of the economy, x_{t+1}. I decompose them in one temporary and two permanent. The state variable is stationary but persistent, therefore, changes in the state cause a long-lasting review of the long term expected consumption growth. Consumers acknowledge this fear through prices, and therefore it is intuitive that cash-flows that are more exposed to the long-lasting shocks will be priced down, contrary to the cash-flows whose responses to shocks are less enduring.

To evaluate the stochastic discount factor with the data, is more convenient to focus on (17). As in the CCAPM investors care about the covariance between returns and consumption growth, investors in this economy care about covariance between (1) consumption growth, (2) expenditure shares growth, and (3) long-run consumption growth and expenditure shares responses to a shock.

If the NIPA aggregation of non-durable goods and services were correct, the term $-\sigma_0^c + (1-\gamma)(F\sigma_0^x - \sigma_0^c) - \phi^\alpha\sigma_0^x$ would be captured by the response of the aggregate consumption, $\mathcal{C}_t$, and the quadratic adjustment would also disappear from the picture[8]

D Valuation of Long-Run Cash Flows

After obtaining the stochastic discount factor, we are interested in the valuation of cash flows, or dividends, that are generated by the portfolios. Dividend growth is modeled as an exponential of a random walk with time trend and a persistent, predictable component that, like consumption, affects the long-run expected growth rate of dividends.

$$d_{t+1} - d_t = \mu^d + \phi^d x_t + \sigma_0^d \nu_{t+1}. \tag{21}$$

[8]That is the focus of Hansen, Heaton, and Li (2006).

The logarithm of the dividends can be expressed as an additive process, composed by a time trend, an additive martingale, and a transitory component, function of the stationary state variable x_t in differences. It can be expressed as

$$d_{t+1} - d_t = \mu^d + (\sigma_0^d + \phi^d (I-\delta)^{-1}\sigma_0^x)\nu_{t+1} - \phi^d (I-\delta)^{-1}(x_{t+1} - x_t), \tag{22}$$

where the coefficient of ν_{t+1} captures the exposure of cash flows to long-run risk, as it can be seen that $\sigma_0^d + \phi^d (I-\delta)^{-1}\sigma_0^x$ is the long-run impulse response to a shock to dividends, which I denote by π from now on. To determine the price of such additive process, we have to take into account the pricing of both the growth component and the permanent component. In the limit, the pricing of the transitory component goes to zero. The expected growth of a dividend process that follows (22) is the expectation of a log-normal variable with the variance correction, $\eta = \mu^d + \frac{\pi\pi'}{2}$.

Assets differ differ precisely in the long-run response to the shock, denoted by π. They may differ in the temporary part, but when computing the limiting prices and limiting returns, where cash flows are discounted far into the future, the temporary effect vanishes. Thus, the rate of return is described by the exposure to long-run risk and the growth rate component. Limiting results are invariant to the choice of the temporary component. The differences in exposure of portfolio cash flows to long-run risk are reflected in different prices. Two different concepts arise: on the one hand, there is the *long-run exposure* of the cash flows to the risk, which determines how cash flows evolve in the future, given current aggregate shocks. The other concept is the *price of long-run risk*, which corresponds to the value that agents assign to cash flows that offer a persistent exposure to risk. I have expressed the stochastic discount factor, or pricing function, and the dividends as functions of the underlying state process for x_t. Section III is devoted to estimate the consumption and expenditure shares dynamics (7), (8), and (9) in order to quantify how exposure to risk, as in (22), is priced.

The state follows a continuous-state Markov process. Therefore, the pricing function maps states at period t into valuations of cash flows at $t+j$. In a discrete state space, a Markov transition matrix would do the job of mapping functions of the state into functions of the state in the future. To obtain valuations of more than one period it suffices to raise the transition matrix that maps states into valuations to the power of the horizon. In the limit, the valuation of a perpetuity would be given by the solution to an eigenvalue problem. The analogy for the continuous state is the valuation operator, instead of the transition matrix, and the eigenfunction problem, instead of the eigenvalue problem. Following Hansen, Heaton, and Li (2006), define $\mathcal{P}\varphi(x_t)$ as the valuation operator that assigns value to the cash flow that is received at time $t+1$, as function of the state of the economy at time t. As explained above, the process for the dividends is decomposed in a growth component, a permanent component, and a transitory component. The valuation operator assigns a value to each of them and in the limit, the contribution from the transitory component vanishes. The one-period valuation operator can be written as

$$\mathcal{P}\varphi(x_t) = E\left[S_t^{t+1}\, e^{\mu^d + \pi\nu_{t+1}}\, \varphi(x_{t+1}) \middle| x_t = x \right]. \tag{23}$$

To obtain the value of a cash flow that occurs in the future, it suffices to apply the operator as many times as periods until the payment is realized. The functional form of the operator is maintained after being applied recursively period by period. In the limit, the function that solves recursively the problem is the eigenfunction. The eigenvalue that solves the functional equation is the rate at which the valuation decays over time due to two competing forces: on the one hand the growth rate of the cash flow, and on the other hand, the required discount rate, or limiting rate of return. The growth rate is given by the deterministic trend of the dividends process, so the asymptotic risk adjusted rate of return, r, can be obtained from the difference between the growth rate, η, and the decay rate, κ. Thus, in the limit, when $j \to \infty$, the decay rate is a fixed point of the equation

$$\mathcal{P}\varphi(x) = e^{-\kappa}\varphi(x) \tag{24}$$

that implies that the valuation of a temporary component is just the temporary component discounted at the limiting rate for one period. The solution for the above eigenfunction consist in finding the functional form of the transient component, $\varphi(x)$. It has been shown in Proposition 2 that the discount factor follows a linear quadratic process in the state and in the vector shocks, the eigenfunction is an exponential quadratic function of the state,

$$\varphi(x_t) = e^{-ax_t - \frac{1}{2}x_t' b x_t} \tag{25}$$

where b solves a Riccati equation that is detailed in the Appendix, and a is found by collecting terms that are interacting with x_t. The solution for a and b is given in the Appendix.

Combining (23), (24), and (25), the decay rate is obtained as a function of the underlying parameters. κ, the eigenvalue, is the asymptotic decay rate of the value of the cash flow.

The pricing operator is a conditional expectations operator. The price of an asset that pays a stochastic process of dividends forever has to be equal to the discounted sum of the same dividends, discounted by a proper stochastic discount factor. Dividing both sides by the dividends at time t we have an asset that pays the price dividend ratio every period, and its price dividend ratio at time t is given by

$$\frac{P_t}{D_t} = E\left[\sum_{j=1}^{\infty}\left(\prod_{s=1}^{j} S_{t+s}^{t+s+1}\right)\frac{D_{t+j+1}}{D_t}\middle| x_t = x_m\right]. \tag{26}$$

The contribution of the dividends corresponding to a certain period $t+j$ to the valuation at time t is given by each of the terms inside the infinite summation. Equation (26) is the summation of the valuations of each of the cash flows in the future. The contribution of each of them is given by the operator introduced in (23), applied as many times as periods in the future the cash flow occurs. The expected growth rate of the cash flows is $\eta = \mu^d + (\pi \cdot \pi')/2$, with μ^d and π different for each asset.

$$\lim_{j\to\infty} E\left[\left(\prod_{s=1}^{j} S_{t+s}^{t+s+1}\right) e^{\eta j + \sum_{s=0}^{j}\pi\nu_{t+1+s}}\ \varphi(x_{t+j+1})\middle| x_t = x_m\right] = e^{\eta j}\ \mathcal{P}^j\ v = e^{\eta}e^{-\kappa}v \tag{27}$$

The last equality corresponds to the eigenvalue problem, and κ is the dominant eigenvalue, which remains different than zero in the limit, when $j \to \infty$. The value of cash flows that are originated in several periods in the future corresponds to the addition of the values of the contribution of each of the periods where cash flows occur.

The asymptotic decay rate can be expressed as

$$\begin{aligned} -\kappa = &-J - \mu^d + \frac{1}{2}\log|I - 2\Sigma(N + \sigma_0^{x\prime} b \sigma_0^x)| \\ &+ (\pi - \pi^*(\pi))' \left(\Sigma^{-1} - 2(N + \sigma_0^{x\prime} b\sigma_0^x)\right)^{-1} (\pi - \pi^*(\pi)). \end{aligned} \tag{28}$$

This is an important result. π captures the long-run impulse response of the cash flows to current fluctuations, the exposure of cash flows to long-run risk. The price of risk, π^*, has to be determined implicitly, because at the same time it also depends on the exposure itself. A linear dependence between price of risk and exposure has been derived in Hansen, Heaton, and Li (2006), but for the quadratic case, we obtain an implicit price of risk, since π is part of a:

$$\pi^*(\pi) = -\sigma_0^c - \phi^\alpha \sigma_0^x + (1-\gamma)(F\sigma_0^x + \sigma_0^c) + a\sigma_0^x. \tag{29}$$

The price of risk is analogous to the coefficient of the shock in the solution to the stochastic discount factor, (19). The components are (i) the immediate response of consumption growth, (ii) the consumption composition, (iii) the long-run impulse response of consumption growth and expenditure shares, and (iv), the quadratic adjustment. The special case of $\pi = 0$, where the asset cash flows do not fluctuate in response to shocks, corresponds to the risk free asset which has no exposure to risk, and its cash flows do not grow over time either, $\mu^d = 0$, which is the long-run riskless return studied in Alvarez and Jermann (2001).

With this analysis I compute the one-period returns of an asset that has a given long-run exposure to risk and a given growth rate component. In the limit, the dividend-to-price ratio is equal to the inverse of the decay rate: the price of a stream of dividends is the sum of the valuation of each of the cash flows, which is a result of two forces: the decay rate of the valuation and the growth rate. If we scale by the initial dividend, to have the price dividend ratio on the left hand side, we eliminate the growth component from the right hand side, and only the decay rate is left:

$$R_{t,t+1} = \frac{P_{t+1} + D_{t+1}}{P_t} = \frac{P_{t+1}}{P_t} e^{\kappa}. \tag{30}$$

Substituting the one-period capital gain from (23), we have that

$$R_{t,t+1} = e^{\mu^d + \pi\nu_{t+1}} \frac{\varphi(x_{t+1})}{\varphi(x_t)} e^{\kappa} \tag{31}$$

The log return is therefore composed by the growth of the cash flow, the decay rate, and the temporary component:

$$r_{t,t+1} = \mu^d + \pi\nu_{t+1} - a(x_{t+1} - x_t) - (x_{t+1} - x_t)'b(x_{t+1} - x_t) + \kappa. \tag{32}$$

The log of the expected one-period return is equal to

$$\begin{aligned}\log E_t(R_{t,t+1}) &= \kappa + \mu^d + (a + a\delta)x_t + x_t'(b + b\delta)x_t - \frac{1}{2}\log|I - 2\Sigma\sigma_0^{x\prime}b\sigma_0^x| \\ &+ \frac{1}{2}\left(\pi + a\sigma_0^x + 2x_t'\delta b\sigma_0^x\right)\left(\Sigma^{-1} - 2\sigma_0^{x\prime}b\sigma_0^x\right)^{-1}\left(\pi + a\sigma_0^x + 2x_t'\delta.b\sigma_0^x\right)'\end{aligned} \tag{33}$$

Analogously, the k-periods returns take into account the deterministic trend, the accumulated response, and the transitory component for k periods. As k grows, the transitory component vanishes, and in the limit, the log of the expected value of (31) results in

$$r = \kappa + \mu^d + \frac{\pi \cdot \pi'}{2}.. \tag{34}$$

The one-period risk free rate can be obtained by computing the inverse of the conditional expectation of the stochastic discount factor:

$$\begin{aligned}r^f_{t,t+1} &= -J - Kx_t - x_t'Lx_t + \frac{1}{2}\log|I - 2\Sigma N| \\ &- \frac{1}{2}M(x_t\left(\Sigma^{-1} - 2N\right)^{-1}M(x_t)'.\end{aligned} \tag{35}$$

Setting the state variable x_t to its unconditional mean, the risk free reduces to

$$\begin{aligned}r^f_{t,t+1} &= -J + \frac{1}{2}|I - 2\Sigma N| - \frac{1}{2}(-\sigma_0^c - \phi^\alpha\sigma_0^x + (1-\gamma)(F\sigma_0^x + \sigma_0^c)) \\ &\times \left(\Sigma^{-1} - 2N\right)^{-1}(-\sigma_0^c - \phi^\alpha\sigma_0^x + (1-\gamma)(F\sigma_0^x + \sigma_0^c))'.\end{aligned} \tag{36}$$

Having derived the relevant variables like one-period and j-periods returns, limiting returns and decay rates, price-dividend ratio, and risk-free rates, I proceed to describe the data and the empirical strategy to estimate the underlying parameters of the model.

II Data

Data on onsumption is obtained from the Personal Consumption Expenditures of the National Income and Product Accounts (NIPA). I use quarterly post-war data from the tables of chapter 2, specifically from 1953 to 2005. Data are obtained from Table 2.3.5 of the NIPA, which presents total expenditures. For non-housing consumption goods I use expenditures on non-durable goods and services. For housing services, I use line 14, "Housing Services". It has been pointed out in Prescott (1997) that several components of consumption have been badly defined. Among these components is owner-occupied housing. Since there are no market prices for owner-occupied housing services, price indexes must be constructed. The approximation used by the Bureau of Economic Analysis is based on imputing rental prices of similar houses. The price of a commodity should account for what it costs to the household consuming it, which depends on many other factors like tax situation (for the deductions), size of mortgage, etc. which the current methodology does not take into account. An indication of this type of bias can be seen in the Consumer Expenditure Survey (CEX). In 2004, the shares of expenditure on shelter over total expenditures were computed to be 18.4%[9]. That share is 16.9% for homeowners and 23.8 for renters. Acknowledged this, NIPA statistics separate the dollar expenditures on housing services into a price index and a quantity index.

Besides the problem mentioned above, these series accumulate new problems, observed by the Boskin Report documents[10]. The quality of the houses today is considerably better than several years ago, and have several features that were not present when in the beginning of the series. There exists a reasonable measurement error in the CPI component p_t^s that affects inversely the quantity index s_t. There are no data on houses' quality improvement, therefore no data either on the quality improvement of housing services. The presence of w_t in the preferences specification is meant to overcome the bias, and capture the secular movement in relative quantities consumed of housing services, perhaps provided from housing units with an improved set of characteristics, rather than in raw measures of square meters and number of rooms.

According to (4), which must hold every period, the log of relative prices must be linear on the

[9]This share from the CEX includes some expenses like maintenance, repairs, insurance, and other expenses that do not enter in the definition of housing services used throughout the paper.

[10]Boskin et al. (1996) and more recently Gordon and van Goethem (2005) report that there has been a downward bias in the CPI for shelter since its computation began.

log of relative quantities. Looking at (5), relative expenditures must be linear on relative quantities. As it is observed in Figure 2, measured relative prices and relative quantities seem to share the same trend in opposite directions. If we take seriously those measurements, by (5), the expenditure ratio should also have the same trend as the quantities ratio. Therefore, expenditure shares should converge eventually to 1 or to 0, forcing one of the goods in the utility function to disappear, which has not been observed in the data.

While data about relative prices and relative quantities have been subject to criticism, data on expenditures seem less likely to be subject to those flaws. The observed total expenditures in housing services include the quality measure. But the quantity index constructed from the expenditures does not. Since the stochastic discount factor can be written as a function of non-housing consumption growth and expenditure shares solely, the problem of the quality improvement disappears for that matter. It becomes irrelevant for the purpose of pricing a set of assets. It remains crucial for estimating the elasticity of substitution between the two type of goods, but that is not the main focus of this paper. The sample starts in January 1953, which appears to be the period when the high re-stocking after the war slowed down. The aim of the model is not to explain the response of prices to large events, like the WWII, which triggered a period of restocking of durable goods.

Figure 1 shows that expenditure shares do not converge to either 1 or zero. Data on expenditure shares, more reliable than quantity and price indexes, show that the expenditure shares are stationary. Over the last 50 years, the average consumer has spent his money 80/20 in non-housing goods and services and housing services respectively.

Figure 3 shows the evolution of expenditure shares and consumption growth. Figure 4 shows the evolution of the expenditure shares with the long-run discounted consumption growth. It can be observed that a decline of the expenditure shares is accompanied by a decrease in the upcoming consumption growth of the next 16 quarters. The correlation of long-run discounted consumption growth and expenditure shares is higher for a longer horizon than for the contemporaneous. This shows potential improvements as a pricing factor, since it is added to the contemporaneous correlation studied by Piazzesi, Schneider, and Tuzel (2007), which does not exploit the long-run relationships.

The price index of the non-housing composite good has been computed using the expenditure

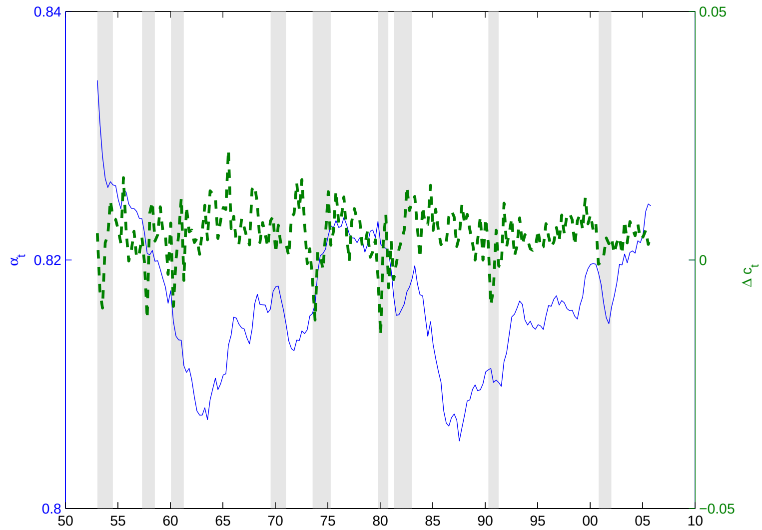

Figure 3: **Expenditure shares and consumption growth.** Non-housing expenditure share over total expenditure (left axis and solid line), and consumption growth series (right axis and dashed line). Shaded areas are NBER recessions.

shares of each of the categories that compose the aggregate as weights.

Homotheticity between housing and non-housing consumption is assumed throughout the paper. Pakŏs (2004) argues against its validity. Figure 1 shows that the expenditure shares did not increase with an increase of the income during the last decades. It is not evidence against non-homotheticity, but clearly shows that there is no steady decline in relative expenditures. If relative quantities are increasing over time, according to (5), relative expenditures should also trend over time. But that is not the case, since expenditure shares are fairly stable over the year, clearly not converging to one or zero. Therefore the preference tilt towards housing services acts so that the equations (4) and (5) are both consistent with the data.

A Returns

Table I reports descriptive statistics for the value weighted market from the CRSP, which contains the NYSE-AMEX-NASDAQ stocks, 5 portfolios sorted by book-to-market (book equity over market equity) from the data library of K. French, and returns on housing.

Dividends for the market portfolio and the book-to-market were created from data on returns with and without dividends. The difference between the two results in the dividend yield, while the

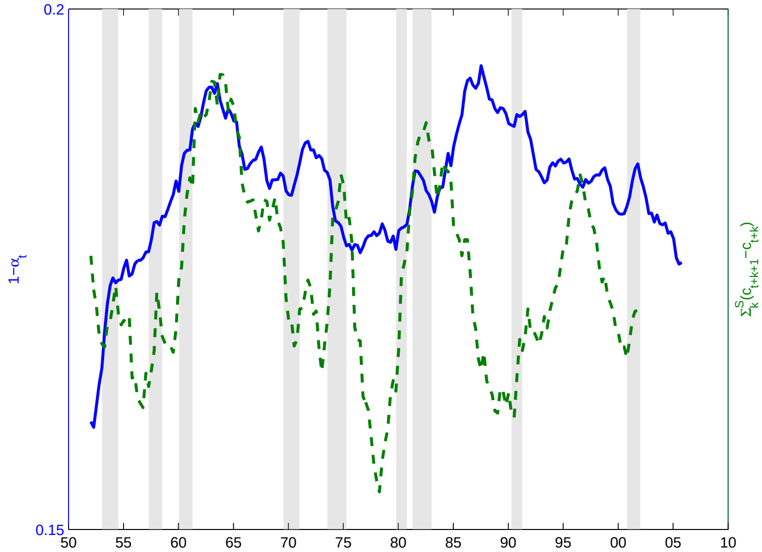

Figure 4: **Expenditure shares and long-run discounted consumption growth.** Non-housing expenditure share over total expenditure (left axis and solidline), and long-run discounted consumption growth series (right axis and dashed line) assuming $\beta = 0.95^{1/4}$ and a 4-year horizon. Shaded areas are NBER recessions.

composition of the returns ex-dividend results in the prices relative to the price at $t = 0$. It remains to set the dividends at the beginning of the sample to construct the entire series for dividend price ratio and dividends recursively. I initialize the price at time zero so that the dividends are equal to aggregate earnings.

I compose the returns quarterly and compute a 4-quarters moving-average of the dividend-price ratio to eliminate the strong seasonal component that this variable presents.

I use the 3-month T-Bill as the risk free rate, given that agents' decision horizon is one quarter. Data on consumption is quarterly. As we can observe in Table I, there is a significant correlation between housing returns and consumption growth, long-run consumption growth, and with expenditure shares. The correlation between returns on housing and future consumption growth is on average twice as much as the rest of the portfolios. This indicates that returns on housing are sensitive to shocks that affect consumption or the composition of consumption in a persistent manner.

Table I: **Descriptive Statistics.** Sample mean and standard deviation, annualized, for the real returns of the market, 5 portfolios sorted in book-to-market, housing and 3 month T-bill. Column Δc_{t+1} shows the correlation of the corresponding returns with consumption growth, column $\sum_{j=1}^{J} \Delta c_{t+j}$ shows the correlation of the returns with consumption growth during the subsequent 24 quarters, and the last column shows the correlation between the returns and the growth in the expenditure shares of non-housing consumption over the aggregate consumption.

	Mean	St.Dev.	Δc_{t+1}	$\sum_{j=1}^{J} \Delta c_{t+j}$	$\Delta\alpha_{t+1}$
$R^{\mathbf{mkt}}$	7.38	33.23	0.26	0.10	0.07
Q1	6.24	37.78	0.26	0.11	0.10
Q2	7.52	33.49	0.22	0.06	0.04
Q3	9.21	30.28	0.22	0.07	0.04
Q4	10.03	31.60	0.27	0.08	0.05
Q5	11.02	36.21	0.25	0.07	0.05
R^h	5.43	3.88	0.15	0.16	0.16
R^f	4.88	2.52	0.14	0.32	0.31

B Housing Returns

To compute housing returns, I impute the housing services as the dividend stream obtained from the value of the house, which basically are rents paid by tenants or imputed by owners. Aggregate data on value of the residential structures does include the value of the land, which is approximately 36% according to Census data. Following Flavin and Yamashita (2002), I subtract the depreciation and the property taxes, which are estimated to be 2.5% with an assumed 33% marginal tax rate. Depreciation is assumed to be equal to maintenance costs, necessary to keep constant the physical condition of the house,

$$R^H_{t+1} = \frac{p^H_{t+1} H_{t+1} + p^s_{t+1} S_{t+1}}{p^H_t H_t} \left(\frac{p^H_t H_t}{p^H_{t+1} H_{t+1}} \right) \left(\frac{p^H_{t+1}}{p^H_t} \right) - \delta - (1 - 0.33) \cdot 0.025. \tag{37}$$

The data that I use for $p^H_t H_t$ are from the aggregate net value of residential stock, in the Fixed Assets Tables of the BEA. There are several sources for the change in housing prices. Results are not sensitive to the use of either of them. The first price index is the widely used S&P Case-Shiller housing price index. It is the reference price index for the futures market on real estate that trades in the Chicago Mercantile Exchange Group. It is the benchmark for the OFHEO housing price index. The Case-Shiller index is an index based on repeated sales and for that reason it is claimed to keep

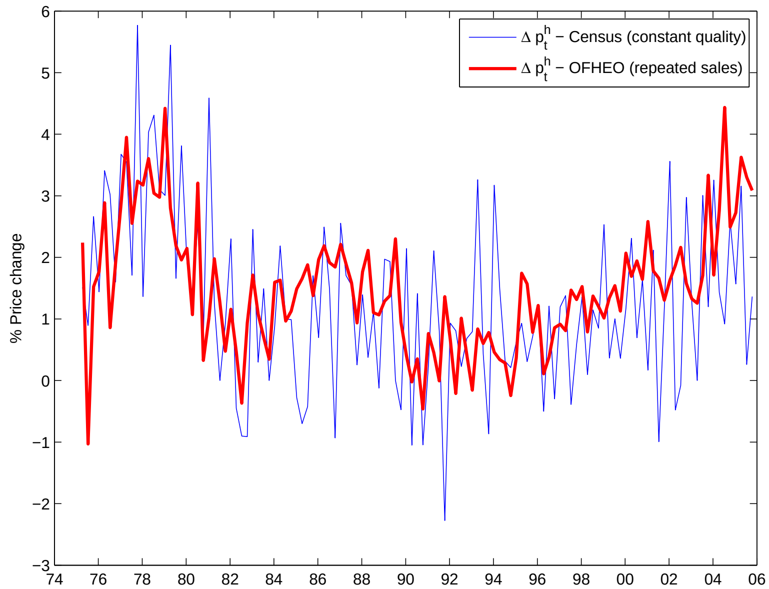

Figure 5: **Housing Price Indexes.** Evolution of gross growth of prices according to the OFHEO index of repeated sales, based on Case-Shiller index, and the Census Bureau price index of constant-quality housing.

quality constant. The second index is constructed by the Census Bureau. Quality is taken seriously, and the index is of a Laspeyres type. Hedonic regressions are used to compute the contribution of a set of characteristics to the price of the house, and the index is constructed keeping constant these characteristics. If the hedonic regressions are correctly specified, the index keeps quality constant by construction. The third index is obtained from the Table 5.3.4 in NIPA, corresponding to the price index for private investment in residential structures, and it provides the longest span of data, concordant with the rest of the data used in the paper. It mimics almost perfectly the index computed by the Census Bureau. Both the Case-Shiller index and the Census Bureau start in the 1975, when Freddie-Mac and Fannie Mae started collecting data on house prices, regionally and nationally. Figure 5 shows the evolution of the price changes in the two shorter indexes, the Case-Shiller and the Census Bureau. They follow the same trend, although the Census Bureau index is considerably more volatile. There is a discrepancy in the last few observations, where the Case-Shiller index exceeds the Census Bureau index. Eventually, the discrepancy could be a source of bias, but it has been deteced for a small fraction of the data —the last 2 years, specifically.

For the annual depreciation, Malpezzi, Ozanne, and Thibodeau (1987) obtained the value of 4.3%.

Figure 6 shows the net returns on aggregate housing, using the two types of price indexes, according to (37).

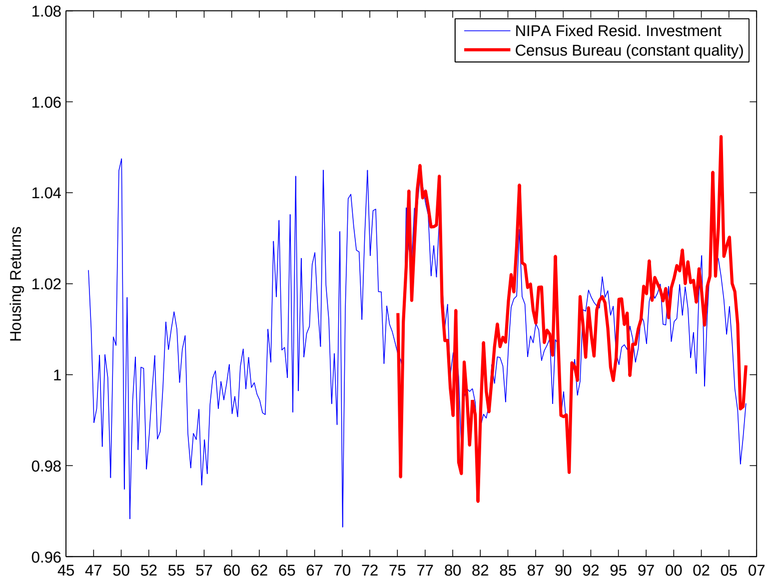

Figure 6: **Housing Returns.** Returns to aggregate housing computed with two different price indexes. The bold line represents returns computed using the Case-Shiller index, which is also the base to compute the OFHEO housing price index. The thinner line represents aggregate net housing returns using as price index the index for investments in residential structures, from NIPA.

III Estimating Long-Run Risk

The endowment of goods and services is governed by the system defined in (7), (6), and (8) which is rewritten here:

$$c_{t+1} - c_t = \mu^c + \phi^c x_t + \sigma_0^c \nu_{t+1}$$

$$-\log \alpha_t = \mu^\alpha + \phi^\alpha x_t + x_t' \Psi x_t$$

$$x_{t+1} = \delta x_t + \sigma_0^x \nu_{t+1}.$$

The state of the economy is determined by a vector of two variables: corporate earnings and aggregate housing stock growth. Corporate earnings are a predictor of consumption and a source of aggregate risk, as has been motivated in Hansen, Heaton, and Li (2006). Earnings are very persistent, more persistent than consumption growth, with an autocorrelation coefficient of 0.96. Therefore high earnings today predict high earnings tomorrow, and higher earnings in the future will be translated in higher consumption, which will catch up eventually. A second component of the state vector is the aggregate value of household residential stock. A similar rationale justifies the inclusion of the variable as state. It predicts the future stream of housing services, as a higher stock of housing delivers a higher level of housing services. Conversely, a slowdown in construction of new residential structures decreases the number of houses available to extract services from, lowering the expenditures in housing services, all else equal. Therefore, it is a well motivated predictor of future measured housing expenditure shares. The data vector is

$$z_t = \begin{bmatrix} c_{t+1} - c_t \\ \log\left(\frac{\alpha_{t+1}}{\alpha_t}\right) \\ e_t - \hat{c}_t \\ h_{t+1} - h_t \end{bmatrix}. \tag{38}$$

The estimation of the structural model follows a two-stage procedure. With the quadratic process for the expenditure shares, it is not feasible to write the likelihood function and estimate directly the parameters. I use the *indirect inference* method first introduced by Smith (1993), and Gourieroux, Monfort, and Renault (1993). Indirect inference is useful when the likelihood function of the problem is intractable. It consist of estimating the exact likelihood function of an approximated model, and the objective is to minimize the distance of the estimates of the approximated model with real data and with data simulated according to the structural model. The method yields consistent and asymptotically normal estimates of structural parameters.

More precisely, let β_0 be the set of structural parameters, and $y_t(\beta_0)$ the data generated by the model for a given β_0. Under the assumption that z_t and $y_t(\beta)$ are stationary and ergodic for all β, and that there exists a unique β_0 for which z_t and $y_t(\beta_0)$ have the same distribution, β_0 can be estimated by using an auxiliary model for which the likelihood can be obtained. Let me denote by $L_S(\{y_S(\beta)\};\theta)$ the likelihood function of the auxiliary model, with size S simulated data, for a given β. Maximizing the likelihood produces

$$\hat{\theta}_S \equiv \arg\max_{\theta\in\Theta} L_S(\{y_S(\beta)\};\theta). \tag{39}$$

Now let $L_T(\{z_T\};\theta)$ be the likelihood of the auxiliary model with the real data, of size T. We can obtain the parameter estimates of the auxiliary model with data, $\hat{\theta}_T$ from

$$\hat{\theta}_T \equiv \arg\max_{\theta\in\Theta} L_T(\{z_T\};\theta) = \sum_{s=\ell}^{T} f(z_t,\ldots,z_{t-\ell};\theta). \tag{40}$$

The first stage of the method requires to estimate an alternative –auxiliary– model. I use a linear vector-autoregressive model, with 2 lags. The vector $\hat{\theta}_T$ includes the autoregressive parameters and the error covariances. The second stage consists of simulating a size $S = T \times H$ path of artificial data, $\{y_S\}(\beta_T)$, for a given set of structural parameters, β_T, and a constant H. With the simulated data, the auxiliary model is estimated again to obtain $\hat{\theta}_S(\beta)$. The criterion consist of finding the $\hat{\beta}_T$ that minimizes the distance between the auxiliary estimates with the data, $\hat{\theta}_T$, and with the simulated data, $\hat{\theta}_S(\beta)$. The moments are weighted by the asymptotic covariance matrix of the data estimates $\hat{\theta}_T$. The asymptotic covariance matrix is estimated with the Newey-West methodology weighting up to 2 lags of autocovariances.

Hence, let the auxiliary model be a VAR with ℓ lags,

$$z_t = A_0 + A_1 z_{t-1} + \ldots + A\ell z_{t-\ell} + v_{t+1}. \tag{41}$$

The estimated parameters for θ_T, the auxiliary model with the real data, $\{z_t\}$, are estimated consistently by $\hat{\theta}_T$, which includes the coefficients of the VAR and the covariance matrix of the error term. The vector $\hat{\theta}_S(\beta)$ includes the parameter estimates with the simulated data $\{y_t(\beta)\}$, for a given β. The indirect based estimator for β_0 results from finding the set of structural parameters β from the minimization of the following quadratic form:

$$\hat{\beta}_T \equiv \arg\min_{\beta\in C}(\hat{\theta}_T - \hat{\theta}_S(\beta))' W_T(\hat{\theta}_T - \hat{\theta}_S(\beta)) \tag{42}$$

where W_T is a weighting matrix that can be computed with data only. The optimal weighting matrix, W_T^* corresponds to the asymptotic variance-covariance matrix of the estimates in the vector θ_T. To estimate the asymptotic variance covariance of the auxiliary parameters, I used the Newey-West estimator with 5 lags for the auto-covariances Γ_k:

$$\hat{W}_T^* = \left[\frac{1}{T}\nabla^2 \log(f(x_t, \ldots, x_{t-\ell}; \hat{\theta}_T))\right] \left[\Gamma_0(\hat{\theta}_T) + \sum_{k=1}^{5}\left(1 + \frac{k}{K+1}\right)(\Gamma_k(\hat{\theta}_T) + \Gamma_k(\hat{\theta}_T)')\right]^{-1}$$
$$\left[\frac{1}{T}\nabla^2 \log(f(x_t, \ldots, x_{t-\ell}; \hat{\theta}_T))\right] \tag{43}$$

Table II list the results for the structural parameters, which include the linear and the quadratic components of the model dynamics, as well as the variances and covariances of the innovations. The standard deviations are computed from the asymptotic variance of the structural parameters, developed in Smith (1993),

$$\sqrt{T}(\hat{\beta}_T - \beta_0)\left(1 + \frac{T}{S}\right)(J(\beta_0)' W_T^* J(\beta_0))^{-1} \tag{44}$$

where $J(\beta_0) = \nabla h(\beta)$ and $h(\beta)$ is the functional form that determines the dependence of the auxiliary parameters θ on the structural parameters β.

Once the parameters are obtained, the impulse responses of consumption and expenditure shares contain information about how consumption of both housing and non-housing responds to shocks to the economy. The theoretical impulse response functions are as follows:

$$\sigma_t^x = \delta^{t-1}\sigma_0^x \tag{45}$$

$$\sigma_t^c = \begin{cases} \sigma_0^c & t = 0, \\ \phi^c \delta^{t-1}\sigma_0^x & t > 0 \end{cases} \tag{46}$$

$$\sigma_t^\alpha = \phi^\alpha \delta^{t-1}\sigma_0^x + (\delta^{t-1}\sigma_0^x{}'(\delta^{t-1}\sigma_0^x). \tag{47}$$

The long-run impulse responses of the expenditure shares when the quadratic process is assumed involve the solution of a Lyapunov equation. It characterizes the discounted sum of the quadratic form in which the shock affects the future evolution of the shares. The long-run response of the state weighted by the sensitivity of the shares to the state, ϕ^α, captures the long-run impulse response of the shares.

$$\sigma^\alpha(1) = \phi^\alpha(I-\delta)^{-1}\sigma_0^x + \sigma_0^{x\prime} S\sigma_0^x, \quad \text{where } S - \delta' S\delta = \Psi \tag{48}$$

The long-run impulse response of consumption has two components. The contemporaneous response of consumption to a shock, σ_0^c, and a term that captures the long-run exposure of consumption growth to a shock: how the shock is transmitted to the future. If we were to assume that consumption growth is a random walk with no predictable component, ϕ^c would be zero, and this effect would not be present in the long-run, returning to the traditional consumption based CAPM, where only contemporaneous consumption growth matters.

$$\sigma^c(1) = \sigma_0^c + \phi^c(I-\delta)^{-1}\sigma_0^x. \tag{49}$$

Expenditure shares do not enter in the dynamics of consumption growth, independently of the assumption for the expenditure shares. Thus, consumption growth only captures first order effects, with no quadratic terms. This response of consumption characterizes the long-run exposure of consumption to risk, the long-run risk in consumption.

I leave the details for the Appendix, but the term $F\sigma_0^x + \sigma_0^c$, which appears repeatedly in the pricing formulas, summarizes the long-run discounted response of consumption and shares, risk adjusted:

$$F\sigma_0^x + \sigma_0^c = \sigma^c(\beta\zeta) + (1-\beta)\frac{\varepsilon}{\varepsilon - 1}\sigma^\alpha(\beta\zeta) + \chi \equiv \sigma^*(\beta\zeta), \tag{50}$$

where χ is a quadratic adjustment of the innovation in the value, and $\sigma^{c,\alpha}(\beta\zeta)$ is the long-run impulse response of consumption and shares respectively, discounted by the factor $\beta\zeta\delta$, and $\zeta \equiv (I - 2(1-\gamma)\sigma_0^{x\prime} H\sigma_0^x)^{-1}$. Hence, the coefficient corresponding to the shock ν_{t+1} in the discount factor becomes

$$-\sigma_0^c + (1-\gamma)\sigma^*(\beta\zeta) - \phi^\alpha\sigma_0^x + 2(1-\gamma)x_t'\delta' H\sigma_0^x - 2x_t'\delta'\Psi\sigma_0^x. \tag{51}$$

The first two terms are the one-period price of risk associated to consumption growth and the long-run price of risk, associated to the long-run consumption growth and housing expenditures share. It is the price of risk because it gives, in units of consumption, how much of it is lost, or gained, when a shock occurs. The long-run price of risk tells how much consumption is sacrificed in the long-run when a shock occurs. This decomposition is similar to the one in Hansen, Heaton, and Li (2006), and Campbell and Vuolteenaho (2004), but here the long-run price of risk takes into account the effects of housing in the long-run. Composition risk affects prices contemporaneously and also has a persistent effect, affecting both one period risk prices and long-run risk prices. The last two terms in (51) imply that the risk premium is time varying, since there is an interaction between the state x_t and the source of risk. A positive shock to the state decreases the price of risk due to the effect of the heteroskedastic terms. Therefore, countercyclical risk premium is obtained. In bad states, when non-housing consumption is low, marginal rate of substitution is high. When expenditure shares are also low, marginal rate of substitution is even higher. It has the same effect of the surplus consumption ratio variable in the Campbell and Cochrane (1999). The closer the agents get to the subsistence level, the more risk averse they become, and the higher is price of risk. Agents in this economy are not more risk averse, but the risk that they face when decreasing the expenditure shares is a sign for future consumption growth to be lower for a while, so they value much more the assets with good payoffs in the bad state. This result is promising insofar it allows to identify 3 sources of risk, parameterize them, and estimate them.

IV Results

The results of the indirect inference estimation of the endowment processes for consumption growth, expenditure shares in non-housing consumption, and the state of the economy given by corporate earnings and growth of aggregate housing value are summarized in Table II. The indirect inference estimator, as other simulation estimators, corrects the inconsistency caused by the estimation of the approximated model at the cost of an increased variance. This effect could be attenuated by increasing the length of the simulations[11]. It has been imposed that the innovations on the state do not affect contemporaneous consumption growth, although they affect future consumption growth through the persistent component. The persistence of the predictable component in consumption growth is given

[11] Or the number of simulations. In this case I opted to use the approach of simulating a long series of data, and maximize one likelihood function, instead of simulating several series, and compute the average of the estimators of each of them. For more about the possible empirical approaches of indirect inference, see Gourieroux, Monfort, and Renault (1993).

Table II: **Indirect Inference.** Estimates of the evolution of the state, $x_{t+1} = \delta x_t + \sigma_0^x \nu_{t+1}$, consumption growth, $c_{t+1} - c_t = \mu^c + \phi^c x_t + \sigma_0^c \nu_{t+1}$, and non-housing expenditure shares, $-\log \alpha_t = \mu^\alpha + \phi^\alpha x_t + x_t' \Psi x_t$.

x_{t+1}		Δc_{t+1}		$-\log \alpha_t$	
Param.	Value	Param.	Value	Param.	Value
δ_{11}	0.9164	ϕ^c_{11}	0.0158	μ^α	0.24
δ_{12}	-3.4048	ϕ^c_{12}	0.5267	ϕ^α_{11}	-0.034
δ_{21}	0.0044	$\sigma^c_{0\,11}$	0.012	ϕ^α_{12}	-0.1077
δ_{22}	0.8677	$\sigma^c_{0\,12}$	0	Ψ_{11}	-0.3704
$\sigma^x_{0\,11}$	0.0094	$\sigma^c_{0\,13}$	0	Ψ_{12}	0.4224
$\sigma^x_{0\,12}$	-0.036			Ψ_{22}	2.1759
$\sigma^x_{0\,13}$	0				
$\sigma^x_{0\,21}$	0.0141				
$\sigma^x_{0\,22}$	0.0001				
$\sigma^x_{0\,23}$	-0.0002				

Table III: **Dividends Equations.** Estimates of the sensitivity of dividend growth to changes in the state variable, given by ϕ^d. σ_0^d captures the immediate response of dividend growth to a shock.

	$R^{\mathbf{mkt}}$	R^1	R^2	R^3	R^4	R^5	R^h
ϕ^d_1	0.0059	0.0069	-0.1018	0.1061	0.0727	0.1366	-0.0138
ϕ^d_2	0.7843	1.2104	-0.4893	-0.1170	1.4841	2.1290	-0.5104
$\sigma^d_{0\,11}$	-0.0111	0.0573	-0.1405	0.1428	0.0754	0.1114	-0.0751
$\sigma^d_{0\,12}$	0.0011	0.0024	-0.0206	0.0161	0.0160	0.0270	-0.0093
$\sigma^d_{0\,13}$	0.0002	-0.0008	0.0023	-0.0024	-0.0012	-0.0019	0.0013
$\sigma^d_{0\,14}$	0.0196	0.0515	0.0446	0.0293	0.0332	0.0522	0.0094

by the matrix δ, in particular by the diagonal, close enough to one. The matrix $\sigma = [\sigma_0^c, \sigma_0^x, \sigma_0^d]'$ has been normalized to be lower triangular, for identification purposes. Thus the error term is normalized to have the identity as variance matrix.

To estimate the evolution of cash flows and the exposure of the cash flows to the shocks in the economy, I add to the system an extra equation, given by (21). It is necessary to add a fourth shock. The model with consumption growth, expenditure shares, and the state vector remains autonomous. The value of the parameters corresponding to dividends are summarized in Table III. These values determine the exposure of the cash flows in the long-run, by means of the impulse response functions, $\pi = \sigma_0^d + \phi^d (I - \delta)^{-1} \sigma_0^x$.

Figure 7 shows the impulse responses of non-housing consumption and expenditure shares to a

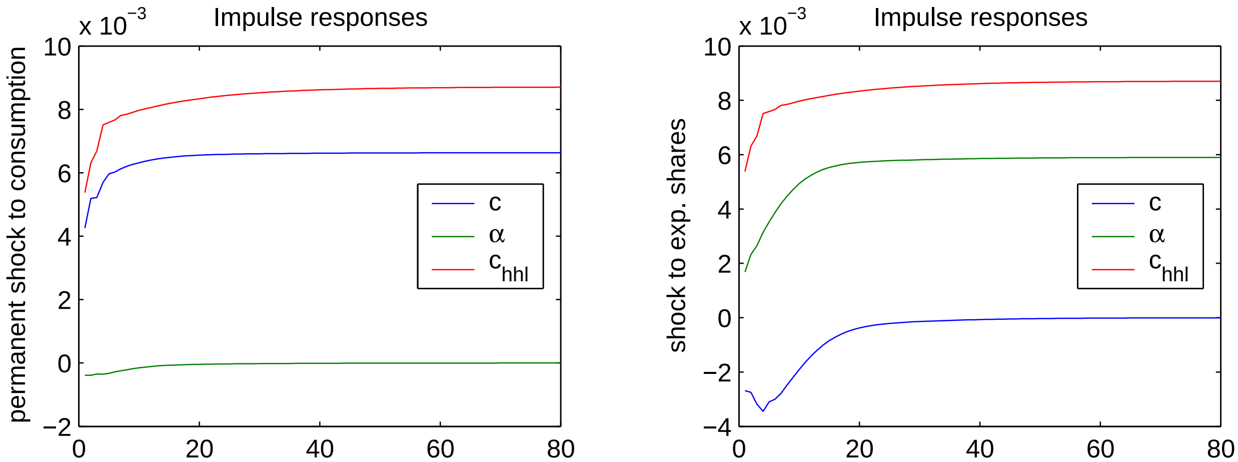

Figure 7: **Impulse response.** Responses of non-housing consumption and non-housing expenditure shares to a shock that has permanent effects on non-housing consumption on the left panel, and to a shock that has permanent effects on both non-housing consumption and expenditure shares on the right panel. The shocks are normalized to have a unit standard deviation. The comparison with aggregate consumption is done with a two-dimensional VAR including aggregate consumption growth and earnings, under the cointegration assumption between earnings and consumption.

shock which maximizes the permanent effect on consumption. For comparison, it is shown the effect of the same shock when aggregate consumption is used (both non-housing and housing), as in Hansen, Heaton, and Li (2006). The impact of a permanent shock is more than twice in the long-run than in the immediate period of the shock. Regarding the housing expenditure shares, there is no immediate response, but in the long-run it picks up and ends up being non-negligible, about one fifth of the consumption impact. The combination of both responses determines the price of the long-run risk, as it has been shown in the previous section, together with the appropriate discounting of the responses, as in (51).

As a first exploration of the model performance, Figure 8 displays the stochastic discount factor as in (11), and its tree components: risk adjustment, consumption growth, and expenditure shares growth. Most of the variation comes from the consumption growth and from the continuation value term. To obtain the closed form solution for the value function and the stochastic discount factor, the elasticity of intertemporal substitution has been set to one. For that special case, the factor corresponding to expenditure shares growth is raised to the power of one. For ρ different than one, intertemporal elasticity of substitution plays a crucial role: a value close to one amplifies the volatility of this factor. Few outlier observations might be the driving forces of the stochastic discount factor process, if raised to a very high power. With $\rho = 1$, this problem disappears. The only channel through which ε plays a role in this special case is the continuation value, as it can be seen in (15).

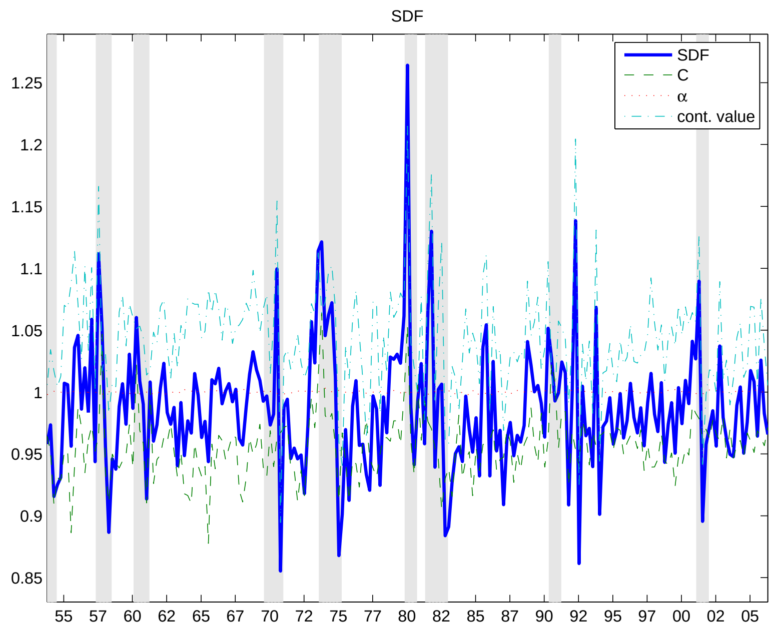

Figure 8: SDF with the data, for $\gamma = 2$.

Table IV shows the asymptotic returns for each portfolio. A 4% spread between the low book-to-market and the high book-to-market portfolios is captured for reasonable levels of risk aversion, in particular for $\gamma = 5$. The fundamental difference between the two extreme portfolios sorted in book-to-market is the growth of the cash-flows, rather than the decay rate in the valuation. The fact that the top quintile book-to-market portfolio has a higher asymptotic growth rate is reflected in a higher required rate of return for an infinite investment horizon. On the other hand, the growth rate of the cash-flows generated by housing is much smaller. That explains a lower rate of return in the limit. The risk free rate does not have any real cash flow growth, therefore the rate of return coincides with the valuation decay, which is at the same time the expected value of the inverse of the stochastic discount factor, as it has been shown in (36). The last row of the table shows the same results for a hypothetical claim on consumption growth.

Alternatively, Table V displays the one-period returns, for the same set of assets, for different levels of risk aversion. The model implies that the one-period returns required for the low and high book-to-market portfolios are higher than in the long-run. They are assets whose exposure is lower

Table IV: $\gamma = 5$, $\varepsilon = 1.5$. Asymptotic results for decay rate, growth rate, rate of return, and average price dividend ratio for the return on the market, the return on the low book-to-market portfolio, the high book-to-market portfolio, and the returns on housing, 3 month T-Bill, and consumption claim.

	Decay	Growth	Asy. Return	P/D
$R^{\mathbf{mkt}}$	2.00	5.63	7.64	49.75
R^1	2.21	5.87	8.07	45.17
R^5	1.90	10.15	12.06	52.38
R^h	6.24	1.04	7.28	15.91
R^f	7.40	0.00	7.40	12.87
R^c	4.08	3.67	7.75	24.37

in the short run. The exposure of the high book-to-market portfolio, R^5, is relatively larger than the one for R^1. That can be inferred from the fact that the spread is much higher in the short run than in the long-run, comparing Table VI with V. The housing asset is also more exposed to long-run risk since the required rate of return for a longer horizon is higher than the one-period return. The risk premia as a function of the investment horizon are plotted in Figure 9 for levels of risk aversion from 1.5 to 30. What can be observed more clearly in figure 10 is how differently the cash flows of R^1 and R^5 are exposed to the risk as a function of time. For the lowest level of risk aversion considered, 1.5, there is not a big difference in terms of slope. The other extreme, for risk aversion equal to 30, shows how they differ. The low book-to-market portfolio has a much flatter term structure than the high book-to-market. The exposure of the R^5 portfolio increases dramatically during the first 5 periods, and after that decreases slowly to the long-run level. The exposure of R^1 is much flatter, and once the required rate of return increases, it remains at that level. The term structure shows how cross-sectional spreads are different for different horizons of investment. So, even though the one-period returns for R^1 and R^5 are not strikingly different, at horizons of 10 quarters approximately, the higher exposure of the high book-to-market portfolio considerably increases its returns above the other portfolios.

Table V: **One Period Returns**. Annualized one-period returns, assuming the state of the economy x_t is at its steady state, $x_t = [0, 0]$, for different values of risk aversion and elasticity of substitution between housing and non-housing consumption, $\varepsilon = 1.1$.

	γ						
	1.5	5	10	15	20	25	30
$R^{\mathbf{mkt}}$	6.86	6.98	7.10	7.22	7.33	7.45	7.53
R^1	7.76	8.00	8.33	8.67	9.01	9.35	9.69
R^5	7.88	8.25	8.77	9.29	9.81	10.33	10.85
R^h	4.52	5.02	5.51	6.01	6.50	6.99	7.34
R^f	6.28	6.33	6.38	6.44	6.49	6.54	6.58
R^c	7.59	7.59	7.59	7.59	7.59	7.59	7.59

Table VI: **Asymptotic Returns**. Annualized asymptotic returns for different values of risk aversion, and elasticity of substitution between housing and non-housing consumption, $\varepsilon = 1.1$.

	γ						
	1.5	5	10	15	20	25	30
$R^{\mathbf{mkt}}$	6.99	7.10	7.22	7.34	7.45	7.57	7.65
R^1	8.85	9.09	9.43	9.76	10.10	10.44	10.78
R^5	10.06	10.43	10.95	11.47	11.99	12.51	13.03
R^h	5.58	6.07	6.57	7.06	7.56	8.05	8.39
R^f	6.28	6.33	6.38	6.44	6.49	6.54	6.58
R^c	7.60	7.60	7.60	7.60	7.60	7.60	7.60

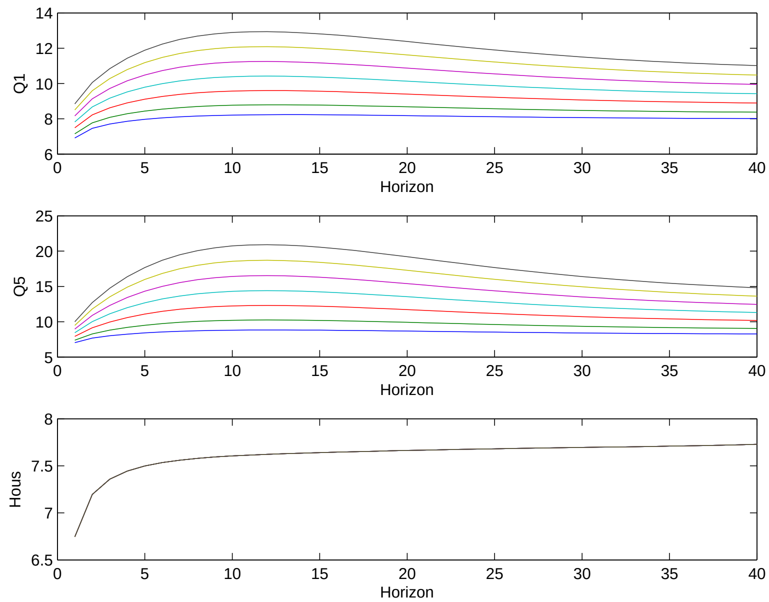

Figure 9: **Term structure of risk.** Risk premia for three of the portfolios: low book-to-market (Q1), high book-to-market (Q5) and housing. The different lines are for different levels of risk aversion, from 1.5 to 30, 1.5 corresponding to the flatter term structure.

I also calibrate the results for different values of the elasticity of substitution between housing and non-housing consumption. It has been discussed that for the particular case of intertemporal elasticity of substitution, $\rho = 1$, its effect is diminished. In Figure 11, bold lines and thin lines represent the term structure for the 4 portfolios considered, for the two extreme values of ε, 1.1 and 2, respectively. As it can be seen, there are no striking differences between the two values of intertemporal elasticity of substitution. Across assets, we observe the same pattern as before. Notice how the exposure to risk affects housing for this particular case. At a very short horizon, risk increases required return, but there is a region for which increasing the horizon decreases the return. After 10 quarters, the required risk increases asymptotically. Risk premia differences are widened at horizons of 7-10 quarters for the two portfolios sorted in book-to-market.

A general pattern observed in the results implied by the model is that ordering in risk premia is maintained for all investment horizons. The level and slope is substantially different for the different assets taken into consideration.

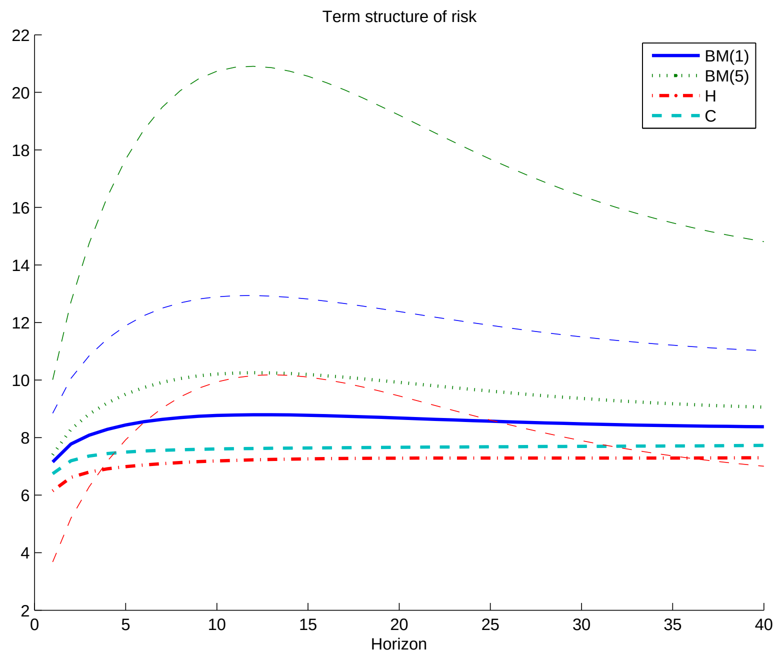

Figure 10: **Term structure of risk.** Bold lines $\theta = 5$, thin lines $\theta = 30$. Elasticity of substitution between non-housing and housing consumption $\varepsilon = 1.1$.

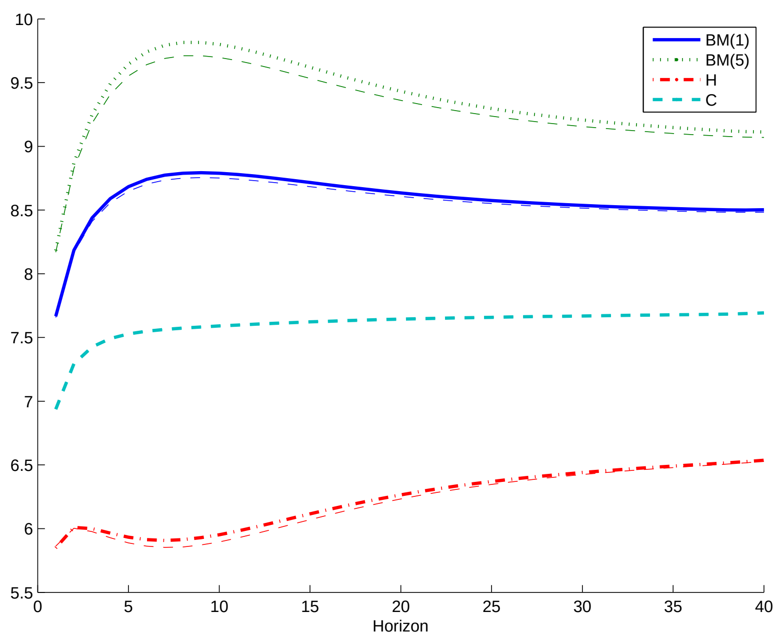

Figure 11: **Term structure of risk.** Bold lines, $\varepsilon = 1.1$, thin lines, $\varepsilon = 2$. $\theta = 5$.

V Conclusions

I have proposed a consumption based model that exploits the pricing implications of risk in the long-run, and implies a time varying risk premium. I use recursive preferences over non-housing consumption and consumption of housing services. Risk premia depend on the exposure of assets' cash flows to risk, and the premia change with the investment horizon considered. I find that housing is more exposed to risks that arise in the long-run, around 40 quarters, while the high and low book-to-market portfolios are more exposed —therefore highly rewarded— when horizons of 10 to 15 quarters are considered. The model captures the spread between portfolios sorted in book-to-market in the short run, as well as in the long-run.

In this paper I have accomplished a theoretical challenge. Solving the value function and the pricing function with non-separabilities allows to identify the sources of risk and consequently estimate the prices associated to them. There are three main driving forces, parameterized and identified: (1) contemporaneous consumption growth risk, as the regular consumption based model; (2) composition risk, that arises from the fact that consumers decide a basket of non-separable consumption goods and services; and (3) long-run consumption growth, which captures the inter-temporal composition risk. The long-run risk requires a highly structured but flexible and interpretable model. The solution of the complex model with non-separability opens a new line of research. In particular, I have solved the stochastic discount factor, where I have found that the price of risk depends on the response of consumption, the effect of the expenditure shares in it, and a heteroskedastic term. It is shown that there is information about future consumption growth in the continuation value that it is not present in C_{t+1}, but in the state x_t that determines future housing and non-housing consumption. The heteroskedasticity has another interesting implication. The risk premia are time varying. In states where housing stock and/or corporate earnings predict lower future housing services or consumption growth, prices of risk are higher.

I have also shown that risk premium is sensitive to the investment horizon that investors consider. There is a hump-shaped term structure of risk premia for assets that are more exposed to short-run cash flow variations. For housing, the risk premium follows an inverted U-shape. Investors with a short investment horizon face higher risks and require higher returns. This can be considered as a speculative type of real estate investment. After 4 quarters, risks decline, together with expected returns. In the long-run, houses are more exposed to risks and the longer the investment horizon, the more risk they carry. Thus, for longer horizons (30 to 40 quarters) the expected return on housing increases.

There is yet another implication of using Epstein-Zin preferences, stated by Uhlig (2006). We have to consider all components of consumption believed to contribute to the utility of the agent. Non-separability across states in the Epstein-Zin framework induces a non-separability across goods. In this paper, I have considered non-separabilities where the two goods are non-housing and housing consumption. My approach is promising for capturing the implications of non-separability between consumption and leisure and evaluate the consequences of labor market risks on asset prices. The methodology presented here allows to evaluate the exact model empirically, without the necessity of approximating the return on the wealth portfolio.

Finally, this methodology avoids the criticism of modifying conditional moments of consumption growth to match aggregate facts, besides assuming a non-random walk model. This fact, along with using utility functions with parameters changing with the economic environment is heavily criticized in Zin (2001). I obtain endogenously a time-varying price of risk, since the stochastic discount factor is heteroskedastic. If the random walk versus the small predictable component for consumption growth are statistically not distinguishable, the use of either is equally well founded.

Appendix

A.1 Marginal Valuation and Stochastic Discount Factor

A.2 Value Function

Substituting the guess and the processes of equilibrium defined in equations (7) and (8) into 15, we have

$$
\begin{aligned}
v_t =&(1-\beta)\frac{\varepsilon}{\varepsilon-1}\left(\mu^\alpha+\phi^\alpha x_t+x_t'\Psi x_t\right)+\frac{\beta}{1-\gamma}\log E_t\left[e^{(1-\gamma)(v_{t+1}+\mu^c+\phi^c x_t+\sigma_0^c\nu_{t+1})}\right]\\
=&(1-\beta)\frac{\varepsilon}{\varepsilon-1}\left(\mu^\alpha+\phi^\alpha x_t+x_t'\Psi x_t\right)\\
&+E_t\left[e^{(1-\gamma)(D+F\delta x_t+F\sigma_0^x\nu_{t+1}+(\delta x_t+\sigma_0^x\nu_{t+1})'H(\delta x_t+\sigma_0^x\nu_{t+1})+\mu^c+\phi^c x_t+\sigma_0^c\nu_{t+1})}\right]\\
=&(1-\beta)\frac{\varepsilon}{\varepsilon-1}\mu^\alpha+\beta(D+\mu^c)\\
&+\left((1-\beta)\frac{\varepsilon}{\varepsilon-1}\phi^\alpha+\beta F\delta+\beta\phi^c\right)x_t\\
&+x_t'\left((1-\beta)\frac{\varepsilon}{\varepsilon-1}\Psi+\beta\delta' H\delta\right)x_t\\
&+\frac{\beta}{1-\gamma}\log E_t\left[e^{\underbrace{(1-\gamma)(F\sigma_0^x+\sigma_0^c+2x_t'\delta' H\sigma_0^x)}_{a'}\nu_{t+1}+\nu_{t+1}'\underbrace{(1-\gamma)\sigma_0^{x\prime}H\sigma_0^x}_{\Lambda}\nu_{t+1}}\right]
\end{aligned}
$$

To solve the expectation I use the fact that ν_{t+1} is normally distributed and solve the expected value:

$$
\begin{aligned}
E_t\left[e^{a'\nu_{t+1}+\nu_{t+1}'\Lambda\nu_{t+1}}\right]=&\int e^{a'\nu_{t+1}+\nu_{t+1}'\Lambda\nu_{t+1}}\frac{1}{(2\pi)^{n/2}}e^{-\frac{1}{2}\nu_{t+1}\nu_{t+1}'}d\nu_{t+1}\\
=&\frac{1}{(2\pi)^{n/2}}\int e^{a'\nu_{t+1}+\nu_{t+1}'(\Lambda-\frac{1}{2}I)\nu_{t+1}}d\nu_{t+1}
\end{aligned}
$$

By completing squares, I have to obtain inside the exponential $-\frac{1}{2}(\nu_{t+1}'+m')S^{-1}(\nu_{t+1}+m)$ to have a normal distribution with mean $-m$ and variance S. For

$$
\begin{aligned}
S=&-(2\Lambda-I)^{-1}\\
m'=&a'(2\Lambda-I)^{-1}
\end{aligned}
$$

the only part to complete the square is $-\frac{1}{2}m'S^{-1}m$, which I add and subtract, and remains outside of the integral since it does not depend on the shock ν_{t+1}:

$$
\begin{aligned}
E_t\left[e^{a'\nu_{t+1}+\nu'_{t+1}\Lambda\nu_{t+1}}\right] =& \frac{1}{(2\pi)^{n/2}}\int e^{a'\nu_{t+1}+\nu'_{t+1}(\Lambda-1/2I)\nu_{t+1}}d\nu_{t+1} \\
&\times\int e^{\frac{1}{2}a'(2\Lambda-I)^{-1}a-\frac{1}{2}a'(2\Lambda-I)^{-1}a}d\nu_{t+1} \\
=&\frac{1}{(2\pi)^{n/2}}\int e^{-\frac{1}{2}(\nu'_{t+1}-\underbrace{a'(2\Lambda-I)^{-1}}_{\mu})'S^{-1}(\nu'_{t+1}-a'(2\Lambda-I)^{-1})}d\nu_{t+1} \\
&\times\int e^{-\frac{1}{2}a'(2\Lambda-I)^{-1}a}d\nu_{t+1} \\
=&\left|S\right|^{1/2}e^{-\frac{1}{2}a'(2\Lambda-I)^{-1}a}\frac{1}{(2\pi)^{n/2}\left|S\right|^{1/2}}\int e^{-\frac{1}{2}(\nu_{t+1}-\mu)'S^{-1}(\nu_{t+1}-\mu)}d\nu_{t+1} \\
=&\left|S\right|^{1/2}e^{-\frac{1}{2}a'(2\Lambda-I)^{-1}a}
\end{aligned}
$$

Using this result in the value function, I obtain

$$
\begin{aligned}
v_t =&(1-\beta)\frac{\varepsilon}{\varepsilon-1}\mu^{\alpha}+\beta(D+\mu^{c}) \\
&+\left((1-\beta)\frac{\varepsilon}{\varepsilon-1}\phi^{\alpha}+\beta F\delta+\beta\phi^{c}\right)x_t \\
&+x'_t\left((1-\beta)\frac{\varepsilon}{\varepsilon-1}\Psi+\beta\delta' H\delta\right)x_t \\
&+\frac{\beta}{1-\gamma}\log\left(\left|S\right|^{1/2}e^{-\frac{1}{2}a'(2\Lambda-I)^{-1}a}\right)
\end{aligned}
\tag{A-1}
$$

Finally, substituting in (a', S, Λ), I obtain the implicit expression for the value function, linear-quadratic in the state, as the guess was implying:

$$
\begin{aligned}
v_t =&(1-\beta)\frac{\varepsilon}{\varepsilon-1}\mu^{\alpha}+\beta(D+\mu^{c})+\beta\frac{1-\gamma}{2}\ln\left|(I-2(1-\gamma)\sigma_0^{x\prime}H\sigma_0^{x})^{-1}\right| \\
&-\frac{\beta(1-\gamma)}{2}(F\sigma_0^{x}+\sigma_0^{c})(2(1-\gamma)\sigma_0^{x\prime}H\sigma_0^{x}-I)^{-1}(F\sigma_0^{x}+\sigma_0^{c})' \\
&+\left(\frac{(1-\beta)\varepsilon}{\varepsilon-1}\phi^{\alpha}+\beta(F\delta+\phi^{c}-2(1-\gamma)(F\sigma_0^{x}+\sigma_0^{c})(2(1-\gamma)\sigma_0^{x\prime}H\sigma_0^{x}-I)^{-1}\sigma_0^{x\prime}H'\delta)\right)x_t \\
&+x'_t\left(\frac{(1-\beta)\varepsilon}{\varepsilon-1}\Psi+\beta\delta' H\delta-2\beta(1-\gamma)\delta' H\sigma_0^{x}(2(1-\gamma)\sigma_0^{x\prime}H\sigma_0^{x}-I)^{-1}\sigma_0^{x\prime}H'\delta\right)x_t
\end{aligned}
\tag{A-2}
$$

The parameters of the value function guess result as follow (implicitly defined):

$$
\begin{aligned}
D =& \frac{\varepsilon}{\varepsilon-1}\mu^{\alpha} + \frac{\mu^{c}}{1-\beta} + \beta\frac{1-\gamma}{2}\ln|(I-2(1-\gamma)\sigma_0^{x\prime}H\sigma_0^{x})^{-1}| \\
&+ \frac{\beta}{1-\beta}\frac{(1-\gamma)}{2}(F\sigma_0^{x}+\sigma_0^{c})(I-2(1-\gamma)\sigma_0^{x\prime}H\sigma_0^{x})^{-1}(F\sigma_0^{x}+\sigma_0^{c})' && \text{(A-3)} \\
F =& \left((1-\beta)\frac{\varepsilon}{\varepsilon-1}\phi^{\alpha} + \beta\phi^{c} + 2\beta(1-\gamma)\sigma_0^{c}(I-2(1-\gamma)\sigma_0^{x\prime}H\sigma_0^{x})^{-1}\sigma_0^{x\prime}H'\delta\right) \\
&\times \left(I-\beta\delta-2\beta(1-\gamma)\sigma_0^{x}(I-2(1-\gamma)\sigma_0^{x\prime}H'\sigma_0^{x})^{-1}\sigma_0^{x\prime}H'\delta\right)^{-1} && \text{(A-4)} \\
H =& (1-\beta)\frac{\varepsilon}{\varepsilon-1}\Psi + \beta\delta'H\delta \\
&+ 2\beta(1-\gamma)\delta'H\sigma_0^{x}(I-2(1-\gamma)\sigma_0^{x\prime}H'\sigma_0^{x})^{-1}\sigma_0^{x\prime}H'\delta && \text{(A-5)}
\end{aligned}
$$

Working out a bit more the expression for F, we can obtain a simpler expression. The term

$$(I-2(1-\gamma)\sigma_0^{x\prime}H\sigma_0^{x})^{-1}$$

will appear repeated times, so it is convenient to define

$$\zeta \equiv (I-2(1-\gamma)\sigma_0^{x\prime}H\sigma_0^{x})^{-1},$$

and D, F, and H become:

$$
\begin{aligned}
D =& \frac{\varepsilon}{\varepsilon-1}\mu^{\alpha} + \frac{\mu^{c}}{1-\beta} + \beta\frac{1-\gamma}{2}\ln|\zeta| + \frac{\beta}{1-\beta}\frac{(1-\gamma)}{2}(F\sigma_0^{x}+\sigma_0^{c})\zeta(F\sigma_0^{x}+\sigma_0^{c})' && \text{(A-6)} \\
F =& \left((1-\beta)\frac{\varepsilon}{\varepsilon-1}\phi^{\alpha} + \beta\phi^{c} + 2\beta(1-\gamma)\sigma_0^{c}\zeta\sigma_0^{x\prime}H\delta\right)(I-\beta\zeta\delta)^{-1} && \text{(A-7)} \\
H =& (1-\beta)\frac{\varepsilon}{\varepsilon-1}\Psi + \beta\delta'H\delta + 2\beta(1-\gamma)\delta'H\sigma_0^{x}\zeta\sigma_0^{x\prime}H'\delta && \text{(A-8)}
\end{aligned}
$$

A.3 Stochastic Discount Factor

By homogeneity of degree 1, I can express the value function as

$$V_t = \frac{\partial V_t}{\partial \mathcal{C}_t}\mathcal{C}_t + E_t\left[\frac{\partial V_t}{\partial V_{t+1}}V_{t+1}\right] \tag{A-9}$$

I derive the stochastic discount factor from the shadow valuation of an stream of future value expressed in terms of marginal value of non-housing consumption. I scale equation (A-9) by marginal value of non-housing consumption, to obtain

$$\frac{V_t}{\frac{\partial V_t}{\partial C_t}} = \frac{\frac{\partial V_t}{\partial \mathcal{C}_t}\mathcal{C}_t}{\frac{\partial V_t}{\partial \mathcal{C}_t}\frac{\partial \mathcal{C}_t}{\partial C_t}} + E_t\left[\frac{\frac{\partial V_t}{\partial V_{t+1}}\frac{\partial V_{t+1}}{\partial \mathcal{C}_{t+1}}\frac{\partial \mathcal{C}_{t+1}}{\partial C_{t+1}}}{\frac{\partial V_t}{\partial \mathcal{C}_t}\frac{\partial \mathcal{C}_t}{\partial C_t}}\frac{V_{t+1}}{\frac{\partial V_{t+1}}{\partial C_{t+1}}}\right]. \tag{A-10}$$

The first term inside the expected value is the shadow valuation of an stream of future value expressed in terms of marginal value of non-housing consumption. Therefore it is a valid stochastic discount factor, and can be expressed as (11).

To collect the different derivatives,

$$\frac{\partial V_t}{\partial \mathcal{C}_t} = (1-\beta)V_t^{\rho}\mathcal{C}_t^{-\rho} \tag{A-11}$$

$$\frac{\partial V_t}{\partial V_{t+1}} = \beta V_t^{\rho} E_t\left[V_{t+1}^{1-\gamma}\right]^{\frac{\gamma-\rho}{1-\gamma}} V_{t+1}^{-\gamma} \tag{A-12}$$

$$\frac{\partial \mathcal{C}_t}{\partial C_t} = \left(\frac{\mathcal{C}_t}{C_t}\right)^{\frac{1}{\varepsilon}} \tag{A-13}$$

Now I can express (A-9) as

$$\begin{aligned}
\frac{V_t}{\frac{\partial V_t}{\partial C_t}} &= \frac{\frac{\partial V_t}{\partial \mathcal{C}_t}C_t}{\frac{\partial V_t}{\partial C_t}} + E_t\left[\frac{\frac{\partial V_t}{\partial V_{t+1}}V_{t+1}}{\frac{\partial V_t}{\partial C_t}}\right] \\
&= \frac{\frac{\partial V_t}{\partial \mathcal{C}_t}C_t}{\frac{\partial V_t}{\partial \mathcal{C}_t}\frac{\partial \mathcal{C}_t}{\partial C_t}} + E_t\left[\frac{\frac{\partial V_t}{\partial V_{t+1}}\frac{\partial V_{t+1}}{\partial C_{t+1}}}{\frac{\partial V_t}{\partial C_t}}\frac{V_{t+1}}{\frac{\partial V_{t+1}}{\partial C_{t+1}}}\right] \\
&= \frac{\frac{\partial V_t}{\partial \mathcal{C}_t}C_t}{\frac{\partial V_t}{\partial \mathcal{C}_t}\frac{\partial \mathcal{C}_t}{\partial C_t}} + E_t\left[\frac{\frac{\partial V_t}{\partial V_{t+1}}\frac{\partial V_{t+1}}{\partial \mathcal{C}_{t+1}}\frac{\partial \mathcal{C}_{t+1}}{\partial C_{t+1}}}{\frac{\partial V_t}{\partial \mathcal{C}_t}\frac{\partial \mathcal{C}_t}{\partial C_t}}\frac{V_{t+1}}{\frac{\partial V_{t+1}}{\partial C_{t+1}}}\right]
\end{aligned} \tag{A-14}$$

The first term inside the expected value is the shadow valuation of an stream of future value expressed in terms of marginal value of non-housing consumption, therefore it is a valid one-period stochastic discount factor (SDF_{t+1}):

$$SDF_{t+1} = \frac{\frac{\partial V_t}{\partial V_{t+1}} \frac{\partial V_{t+1}}{\partial \mathcal{C}_{t+1}} \frac{\partial \mathcal{C}_{t+1}}{\partial C_{t+1}}}{\frac{\partial V_t}{\partial \mathcal{C}_t} \frac{\partial \mathcal{C}_t}{\partial C_t}} = \frac{\beta V_t^{\rho} E_t \left[V_{t+1}^{1-\gamma} \right]^{\frac{\gamma-\rho}{1-\gamma}} V_{t+1}^{-\gamma} (1-\beta) V_{t+1}^{\rho} \mathcal{C}_{t+1}^{-\rho} \left(\frac{\mathcal{C}_{t+1}}{C_{t+1}} \right)^{\frac{1}{\varepsilon}}}{(1-\beta) V_t^{\rho} \mathcal{C}_t^{-\rho} \left(\frac{\mathcal{C}_t}{C_t} \right)^{\frac{1}{\varepsilon}}}$$

$$= \beta \left(\frac{V_{t+1}}{E_t \left[V_{t+1}^{1-\gamma} \right]^{\frac{1}{1-\gamma}}} \right)^{\rho-\gamma} \left(\frac{\mathcal{C}_{t+1}}{\mathcal{C}_t} \right)^{\frac{1}{\varepsilon}-\rho} \left(\frac{C_{t+1}}{C_t} \right)^{-\frac{1}{\varepsilon}}$$

$$= \beta \left(\frac{V_{t+1}}{E_t \left[V_{t+1}^{1-\gamma} \right]^{\frac{1}{1-\gamma}}} \right)^{\rho-\gamma} \left(\frac{\mathcal{C}_{t+1}}{\mathcal{C}_t} \right)^{-\rho} \left(\frac{1 + \left(w_{t+1} \frac{S_{t+1}}{C_{t+1}} \right)^{\frac{\varepsilon-1}{\varepsilon}}}{1 + \left(w_t \frac{S_t}{C_t} \right)^{\frac{\varepsilon-1}{\varepsilon}}} \right)^{\frac{1}{\varepsilon-1}}$$

$$= \beta \left(\frac{V_{t+1}}{E_t \left[V_{t+1}^{1-\gamma} \right]^{\frac{1}{1-\gamma}}} \right)^{\rho-\gamma} \left(\frac{C_{t+1}}{C_t} \right)^{-\rho} \left(\frac{1 + \left(w_{t+1} \frac{S_{t+1}}{C_{t+1}} \right)^{\frac{\varepsilon-1}{\varepsilon}}}{1 + \left(w_t \frac{S_t}{C_t} \right)^{\frac{\varepsilon-1}{\varepsilon}}} \right)^{\frac{1-\varepsilon\rho}{\varepsilon-1}}$$

$$= \beta \left(\frac{V_{t+1}}{E_t \left[V_{t+1}^{1-\gamma} \right]^{\frac{1}{1-\gamma}}} \right)^{\rho-\gamma} \left(\frac{C_{t+1}}{C_t} \right)^{-\rho} \left(\frac{\alpha_{t+1}}{\alpha_t} \right)^{\frac{1-\varepsilon\rho}{1-\varepsilon}} \tag{A-15}$$

Multiplying and dividing by C_{t+1} and C_t, we re-scale V_{t+1} and V_t, so that we can use the solution we obtained in (A-5).

$$SDF_{t+1} = \beta \left(\frac{\frac{V_{t+1}}{C_{t+1}} \frac{C_{t+1}}{C_t}}{E_t \left[\left(\frac{V_{t+1}}{C_{t+1}} \frac{C_{t+1}}{C_t} \right)^{1-\gamma} \right]^{\frac{1}{1-\gamma}}} \right)^{\rho-\gamma} \left(\frac{C_{t+1}}{C_t} \right)^{-\rho} \left(\frac{\alpha_{t+1}}{\alpha_t} \right)^{\frac{1-\varepsilon\rho}{1-\varepsilon}} \tag{A-16}$$

and the consumption growth factor remains to the power of $-\gamma$ after rearranging and canceling

$$SDF_{t+1} = \beta \left(\frac{\frac{V_{t+1}}{C_{t+1}}}{E_t \left[\left(\frac{V_{t+1}}{C_{t+1}} \frac{C_{t+1}}{C_t} \right)^{1-\gamma} \right]^{\frac{1}{1-\gamma}}} \right)^{\rho-\gamma} \left(\frac{C_{t+1}}{C_t} \right)^{-\gamma} \left(\frac{\alpha_{t+1}}{\alpha_t} \right)^{\frac{1-\varepsilon\rho}{1-\varepsilon}} \tag{A-17}$$

Taking logarithms of (A-17), we obtain the following log stochastic discount factor:

$$
\begin{aligned}
sdf_{t+1} = &\log \beta - \gamma(c_{t+1} - c_t) + (\rho - \gamma) v_{t+1} - \\
&- \frac{\rho - \gamma}{1 - \gamma} \log E_t \left[e^{(1-\gamma)(v_{t+1} + c_{t+1} - c_t)} \right] + \frac{1 - \varepsilon\rho}{1 - \varepsilon} (\log \alpha_{t+1} - \log \alpha_t)
\end{aligned} \tag{A-18}
$$

Now, taking the approximation $\rho \to 1$, (A-18) becomes

$$
\begin{aligned}
sdf_{t+1} = &\log \beta - \gamma(c_{t+1} - c_t) + (1 - \gamma) v_{t+1} - \\
&- E_t \left[e^{(1-\gamma)(v_{t+1} + c_{t+1} - c_t)} \right] + \log \alpha_{t+1} - \log \alpha_t
\end{aligned} \tag{A-19}
$$

And following the above steps to solve for the expectation, we finally obtain

$$
\begin{aligned}
sdf_{t+1} =& \log\beta - \gamma(c_{t+1} - c_t) + \log\alpha_{t+1} - \log\alpha_t - \frac{1}{2}\log|\zeta| \\
& - \frac{(1-\gamma)^2}{2}(F\sigma_0^x + \sigma_0^c)\zeta(F\sigma_0^x + \sigma_0^c)' - (1-\gamma)\mu^c - (1-\gamma)\phi^c x_t \\
& - 2(1-\gamma)^2(F\sigma_0^x + \sigma_0^c)\zeta\sigma_0^{x\prime}H'\delta x_t - x_t'(2(1-\gamma)^2)\delta' H\sigma_0^x\zeta\sigma_0^{x\prime}H'\delta x_t \\
& + (1-\gamma)F\sigma_0^x\nu_{t+1} + 2(1-\gamma)x_t'\delta' H\delta\nu_{t+1} + \nu_{t+1}'(1-\gamma)\sigma_0^{x\prime}H\sigma_0^x\nu_{t+1} = \\
=& \log\beta - \frac{(1-\gamma)^2}{2}(F\sigma_0^x + \sigma_0^c)\zeta(F\sigma_0^x + \sigma_0^c)' - \frac{1}{2}\log|\zeta| \\
& - 2(1-\gamma)^2(F\sigma_0^x + \sigma_0^c)\zeta\sigma_0^{x\prime}H'\delta x_t - x_t' 2(1-\gamma)^2\delta' H\sigma_0^x\zeta\sigma_0^{x\prime}H'\delta x_t \\
& - (c_{t+1} - c_t) + \log\alpha_{t+1} - \log\alpha_t \\
& + \left((1-\gamma)(F\sigma_0^x + \sigma_0^c) + 2(1-\gamma)x_t'\delta' H\delta\right)\nu_{t+1} + \nu_{t+1}'(1-\gamma)\sigma_0^{x\prime}H\sigma_0^x\nu_{t+1}
\end{aligned}
\tag{A-20}
$$

This is the form where the factors can be identified. If we substitute in the processes to leave the

sdf_{t+1} as a function of the state x_t and ν_{t+1}, to evaluate the price of risk, this is what is obtained:

$$
\begin{aligned}
&= \log\beta - \mu^c - \frac{(1-\gamma)^2}{2}(F\sigma_0^x + \sigma_0^c)(I - 2(1-\gamma)\sigma_0^{x\prime} H\sigma_0^x)^{-1}(F\sigma_0^x + \sigma_0^c)' - \frac{1}{2}\log|\zeta| + \\
&+ \left(-\phi^c - (1-\gamma)^2(F\sigma_0^x + \sigma_0^c)\zeta\sigma_0^{x\prime} H'\delta + \phi^\alpha(I-\delta)\right) x_t + \\
&+ x_t'\left(-2(1-\gamma)^2\delta' H\sigma_0^x\zeta\sigma_0^{x\prime} H'\delta + \Psi - \delta'\Psi\delta\right) x_t + \\
&+ \left((1-\gamma)(F\sigma_0^x + \sigma_0^c) - \sigma_0^c + 2x_t'\delta((1-\gamma)H - \Psi)\sigma_0^x - \phi^\alpha\sigma_0^x\right)\nu_{t+1} + \\
&+ \nu_{t+1}'\sigma_0^{x\prime}\left((1-\gamma)H - \Psi\right)\sigma_0^x\nu_{t+1} \qquad \text{(A-21)}
\end{aligned}
$$

This is a summary of the variables' impulse responses. $\sigma(1)$ means the long-run impulse response, i.e. the sum of all the impulse responses in the infinite horizon, whilst $\sigma(\beta)$ is the discounted infinite sum of all the responses.

$$
\begin{aligned}
\sigma^x(1) &= (I-\delta)^{-1}\sigma_0^x && \text{(A-22)} \\
\sigma^x(\beta) &= (I-\beta\delta)^{-1}\sigma_0^x && \text{(A-23)} \\
\sigma^c(1) &= \sigma_0^c + \phi^c(I-\delta)^{-1}\sigma_0^x && \text{(A-24)} \\
\sigma^c(\beta) &= \sigma_0^c + \beta\phi^c(I-\beta\delta)^{-1}\sigma_0^x && \text{(A-25)} \\
\sigma^\alpha(1) &= \phi^\alpha\sigma^x(1) + \sigma_0^{x\prime} S\sigma_0^x, \quad \text{where } S - \delta' S\delta = \Psi && \text{(A-26)} \\
\sigma^\alpha(\beta) &= \phi^\alpha\sigma^x(\beta) + \sigma_0^{x\prime} S_\beta\sigma_0^x, \quad \text{where } S_\beta - \beta^2\delta' S_\beta\delta = \Psi && \text{(A-27)}
\end{aligned}
$$

The term $F\sigma_0^x + \sigma_0^c$ can be expressed as

$$
\begin{aligned}
F\sigma_0^x + \sigma_0^c =& \sigma^c(\beta\zeta) + (1-\beta)\frac{\varepsilon}{\varepsilon-1}\sigma^\alpha(\beta\zeta) \\
&- (1-\beta)\frac{\varepsilon}{\varepsilon-1}\sigma_0^{x\prime} S_{\beta\zeta}\sigma_0^x + 2\beta(1-\gamma)\sigma_0^c\zeta\sigma_0^{x\prime} H\delta(I-\beta\zeta\delta)^{-1}\sigma_0^x \qquad \text{(A-28)}
\end{aligned}
$$

A.4 Eigenfunction

The solution to the eigenfunction problem

$$\varphi(x_t) = e^{-ax_t - \frac{1}{2}x_t' b x_t} \tag{A-29}$$

is given by

$$\begin{aligned} a = & \big(K + 2(-\sigma_0^c - \phi^\alpha \sigma_0^x + (F\sigma_0^x + \sigma_0^c) + \pi)(\Sigma^{-1} - 2(N + \sigma_0^x{}' b \sigma_0^x))^{-1} \sigma_0^x{}'((1-\gamma)H - \Psi + b)'\delta\big) \\ & \big(I - \delta - 2\sigma_0^x(\Sigma^{-1} - 2(N + \sigma_0^x{}' b \sigma_0^x))^{-1} \sigma_0^x{}'((1-\gamma)H - \Psi + b)\delta\big)^{-1} \end{aligned} \tag{A-30}$$

$$b = L + \delta' b \delta + 2\delta'((1-\gamma)H - \Psi + b)\sigma_0^x(\Sigma^{-1} - 2(N + \sigma_0^x{}' b \sigma_0^x))^{-1} \sigma_0^x{}'((1-\gamma)H - \Psi + b)'\delta \tag{A-31}$$

which is solved numerically.

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