Perfect Competition in Differential Information Economies: Consistency of Incentive Compatibility and Efficiency*

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Abstract

The idea of perfect competition for an economy with differential information is formalized via an idiosyncratic signal process in which the private signals of almost every individual agent can influence only a negligible group of agents, and the individual agents' relevant signals are essentially pairwise independent conditioned on the true states of nature. The existence of incentive compatible, individually rational and Pareto efficient allocations is shown for such a perfectly competitive differential information economy with or without "common values" via simple measure-theoretic methods. Thus, the conflict between incentive compatibility and Pareto efficiency is resolved exactly, and its asymptotic version derived for a sequence of large, but finite private information economies.

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*This work was initiated in October 2003 while Yeneng Sun was visiting the University of Illinois at Urbana-Champaign. This version was finished in May 2004.
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1 Introduction

The classical Arrow-Debreu-McKenzie model of perfect competition is obviously at odds with itself as the finitude of economy size implies that individuals can exercise some influence on the prices at which goods are either sold or bought in the economy. Aumann [3] resolves this issue by introducing an economy with an atomless measure space of agents. In such an economy, each individual agent has non-negligible consumption in general, but with negligible impact on the aggregate demand, and therefore takes prices as given. Thus, the formulation of an atomless measure space of agents captures precisely the meaning of perfect competition.\(^1\)

The Aumann model is deterministic as each agent’s characteristics are non-random. Thus, in this model, contracts (trades) are made under complete information. It is not an exaggeration to say that all economic activities or all contacts among individuals in an economy are made under conditions of uncertainty or incomplete information. To this end, it is of interest to know whether or not one can introduce asymmetric or private information\(^2\) on the Aumann economy, and still be able to capture the meaning of perfect competition. Notice that once private information is introduced in the Aumann model, an agent may have monopoly power on her information, and thus may have an incentive to manipulate her information to become better off. This poses the following question: can one model the idea of perfect competition in an economy with asymmetric information? To put differently, can one model the concept of negligible private information?

It is well-known that there is a conflict between incentive compatibility and Pareto efficiency in a finite-agent differential information economy. However, intuition suggests that a perfectly competitive market should still perform efficiently since no single agent has monopoly power on information. In seminar papers, Gul-Postlewaite [6] and McLean-Postlewaite [12] formalized this intuitive idea as independent replicas of a fixed differential information economy with finitely many agents and showed the consistency of incentive compatibility and efficiency in an approximate sense. The key point in this approach is that though an individual agent’s information is not becoming more accurate, its overall influence is diminishing when the number of agents goes to infinity.\(^3\) In a way, the models considered in [6] and [12] can be viewed as

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\(^1\)See [7] for a systematic development of large economies and extensive references.

\(^2\)When it is appropriate, we shall use the terminologies of private information, differential information, incomplete information and asymmetric information interchangeably.

\(^3\)For an economy with a fixed finite number of agents, McLean and Postlewaite also showed in [12] that the conflict between incentive compatibility and efficiency can be made arbitrarily small when the agents are able to predict the true states of nature with sufficient accuracy in terms of small information size. Krasa and Shafer [9] considered similar questions in terms of convergence of equilibria in a sequence of incomplete information economies of fixed size to equilibria of a complete information economy when the noise in the signals converges to zero.
capturing the idea of approximate perfect competition in a differential information economy.

The main purpose of this paper is to formulate precisely the idea of perfect competition for a differential information economy so that the conflict between incentive compatibility and Pareto efficiency can be resolved exactly. A heuristic way to capture the idea of perfect competition for such an economy is that the private signal of an individual agent can only influence a negligible corner of the market, and the signals associated with the individual agents (for example, used in their utility functions) are essentially independent of each other conditioned on the true states of nature. An immediate technical difficulty with a formal formulation of this idea in an atomless economy is the so-called measurability problem of independent processes, first noted by Doob in [5]. In our context, a signal process that is essentially independent, conditioned on the true states of nature may not be measurable at all; in fact, it follows from Proposition 1 of [14] that it is never jointly measurable in the usual sense except for trivial cases.

In [13] and [16], an extension of the usual measure-theoretic product with the Fubini property is used to resolve the measurability problem and to prove the exact law of large numbers that removes individual-level uncertainty through aggregation. We shall use this measure-theoretic tool to model the above heuristic idea of perfect competition in a differential information economy and to resolve exactly the incompatibility of incentive compatibility and Pareto efficiency. In addition, our new exact results can be translated to obtain asymptotic results on the consistency of incentive compatibility and efficiency for a general sequence of large, but finite private information economies.

The rest of the paper is organized as follows. After presenting the basic measure-theoretic framework in Section 2, a perfectly competitive differential information economy is considered in Section 3 in the setting of a “common value” model, as in [12], where agents’ types are purely informational in the sense that they do not enter the utility functions. It is proved in Theorem 1 that any allocation in the corresponding complete information economy can be implemented as an incentive compatible allocation in the private information economy that transforms exactly the usual Pareto efficiency and Walrasian equilibrium to their ex post versions via very simple measure-theoretic methods. The existence of incentive compatible and ex post Walrasian (and hence ex post individually rational and ex post efficient) allocations follows easily (part (3) of Theorem 1), and thus the conflict between incentive compatibility and Pareto efficiency is resolved exactly in this setting.

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4 This is the case, when uncertainty only stems from characteristics of the objects being traded. As noted in [12] and [6], typical examples include the utility from a used car, and finance problems (such as the utility of a share of an oil field or a company engaging in a R&D project).
The same type of existence result as in part (3) of Theorem 1 is shown in Theorem 2 of Section 4 for the more general case that agents’ types are allowed to enter the utility functions. Theorem 3 of Section 5 presents the asymptotic version of Theorem 2 for a general sequence of large, but finite private information economies.

In Section 6, we compare our results with those of Gul-Postlewaite [6] and McLean-Postlewaite [12]. In comparison with the approximate results in [6] and [12] for independent replicas, we obtain exact results with a transparent proof for perfectly competitive private information economies and approximate results with a proof that follows standard procedures for a very general sequence of large, but finite private information economies. We also dispense with the strict concavity and concavity assumptions on the utility functions respectively in [6] and [12]. In addition, the type of regularity condition\(^5\) imposed in [6] for a private information economy with type dependent utility functions is not needed in our limit and asymptotic models.

Section 7 contains some concluding remarks. Section 8 is an appendix that includes the proofs of all the results plus a statement of the exact law of large numbers and associated definitions, and a brief description on a construction of the information structure that satisfies the required conditions. All our definitions and results in the main text as well as the proofs in Sections 8.2 - 8.4 of the appendix are stated in common measure-theoretic or asymptotic terms, which can be read by a reader with some background in probability theory. Nonstandard analysis is only used in two sections of the appendix: the proof of Theorem 3 in Section 8.5 uses the standard procedures of lifting, pushing-down and transfer in nonstandard analysis.\(^6\) Finally, a construction of the information structure using Loeb probability spaces is given in Section 8.6.\(^7\)

2 Some basic definitions

We fix an atomless probability space\(^8\) \((I, \mathcal{I}, \lambda)\) representing the space of economic agents, and \(S = \{s_1, s_2, \ldots, s_K\}\) the space of true states of nature (its power set denoted by \(S\)), which are not known to the agents. One can take \(I\) to be the unit interval if one prefers. Let \(T^0 = \{q_1, q_2, \ldots, q_L\}\) be the space of all the possible signals (types) for individual agents, \((T, T)\) a measurable space that model the private signal profiles for all the agents, and thus \(T\) is a

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\(^5\)It requires the demands of two different types for an agent to be never identical for all prices in some open ball for every realization of the relevant uncertainty; see page 1277 of [6].

\(^6\)See [2] and [8] for additional references and for some detailed discussion on recent applications of these standard procedures in economics.

\(^7\)The reader can pick up some basic knowledge of nonstandard analysis by reading Chapters 1, 2, 3 and 5 in the book [11].

\(^8\)We use the convention that all probability spaces are countably additive.
space of functions from $I$ to $T^0$. Thus, $t \in T$, as a function from $I$ to $T^0$, represents a private signal profile for all agents in $I$. For agent $i \in I$, $t(i)$ (also denoted by $t_i$) is the **private signal** of agent $i$ while $t_{-i}$ the restriction of the signal profile $t$ to the set $I \setminus \{i\}$ of agents different from $i$; let $T_{-i}$ be the set of all such $t_{-i}$. For simplicity, we shall assume that $(T, \mathcal{T})$ has a product structure so that $T$ is a product of $T_{-i}$ and $T^0$, while $\mathcal{T}$ is the product algebra of the power set $\mathcal{T}^0$ on $T^0$ with a $\sigma$-algebra $\mathcal{T}_{-i}$ on $T_{-i}$. We shall adopt the usual notation $(t_{-i}, t'_i)$ to denote the signal profile whose value is $t'_i$ for agent $i$ ($t'_i \in T^0$), and the same as $t$ for other agents.

Let $(\Omega, \mathcal{F}, P)$ be a probability space representing all the uncertainty on the true states as well as on the signals for all the agents, where $(\Omega, \mathcal{F})$ is the product measurable space $(S \times T, \mathcal{S} \otimes \mathcal{T})$. Let $P^S$ and $P^T$ be the marginal probability measures of $P$ respectively on $(S, \mathcal{S})$ and on $(T, \mathcal{T})$. Let $\tilde{s}$ and $\tilde{t}_i$, $i \in I$ be the respective projection mappings from $\Omega$ to $S$ and from $\Omega$ to $T^0$ with $\tilde{t}_i(s, t) = t_i$. For each true state $s \in S$, we assume without loss of generality that the state is essential in the sense that $\pi_s = P^S(\{s\}) > 0$; let $P^T_s$ be the conditional probability measure on $(T, \mathcal{T})$ when the random variable $\tilde{s}$ takes value $s$. Thus, for each $B \in \mathcal{T}$, $P^T_s(B) = P(\{s\} \times B)/\pi_s$. It is obvious that $P^T = \sum_{s \in S} \pi_s P^T_s$. Note that the conditional probability measure $P^T_s$ is often denoted as $P(\cdot|s)$ in the literature.

One can also introduce the conditional probability measure$^{11}$ $P^S(\cdot|t)$ on $S$ such that $P^S(\{s\}|t)$ forms a probability weights in $s \in S$ for a fixed $t \in T$, is $\mathcal{T}$-measurable in $t \in T$ for a fixed $s \in S$, and for each $B \in \mathcal{T}$, $P(\{s\} \times B) = \int_B P^S(\{s\}|t)dP^T(t)$. Let $p_s(\cdot)$ be the density function of $P^T_s$ with respect to $P^T$; it is easy to see that $P^S(\{s\}|t) = \pi_s p_s(t)$ for $P^T$-almost all $t \in T$. For $i \in I$, let $\tau_i$ be the signal distribution of agent $i$ on the space $T^0$, and $P^{S \times T_{-i}}(\cdot|t_i)$ the conditional probability measure on the product measurable space $(S \times T_{-i}, \mathcal{S} \otimes \mathcal{T}_{-i})$ when the signal of agent $i$ is $t_i \in T^0$. If $\tau_i(\{t_i\}) > 0$, then it is clear that for $D \in \mathcal{S} \otimes \mathcal{T}_{-i}$, $P^{S \times T_{-i}}(D|t_i) = P(D \times \{t_i\})/\tau_i(\{t_i\})$.

As noted in the introduction, in order to work with a signal process that is independent conditioned on the true states $s \in S$, we need to work with a joint agent-probability space $(I \times T, \mathcal{I} \otimes \mathcal{T}, \lambda \otimes P^T_s)$ that extends the usual measure-theoretic product $(I \times T, \mathcal{I} \otimes \mathcal{T}, \lambda \otimes P^T_s)$ of the agent space $(I, \mathcal{I}, \lambda)$ and the probability space $(T, \mathcal{T}, P^T_s)$, and retains the Fubini property.$^{13}$

$^{9}$In the literature, one usually assumes that different agents have possibly different sets of signals and require that the agents take all their own signals with positive probability. For notational simplicity, we choose to work with a common set $T^0$ of signals, but allow zero probability for some of the signals. There is no loss of generality in this latter approach.

$^{10}$$\tilde{t}_i$ can also be viewed as a projection from $T$ to $T^0$.

$^{11}$Note that a conditional probability measure is uniquely defined up to a null set.

$^{12}$For $q \in T^0$, $\tau_i(\{q\})$ is the probability $P(\tilde{t}_i = q)$.

$^{13}$\(\mathcal{I} \otimes \mathcal{T}\) is a $\sigma$-algebra that contains the usual product $\sigma$-algebra $\mathcal{I} \otimes \mathcal{T}$, and the restriction of the countably additive probability measure $\lambda \otimes P^T_s$ to $\mathcal{I} \otimes \mathcal{T}$ is $\lambda \otimes P^T_s$. 

Its formal definition is given in Definition 7 of the Appendix.

Let $\mathcal{I} \otimes \mathcal{F}$ be the collection of all subsets $E$ of $I \times \Omega$ such that there are sets $A \in \mathcal{I} \otimes \mathcal{T}$, $C \in \mathcal{S}$ such that $E = \{(i, s, t) \in I \times \Omega : (i, t) \in A, s \in C \}$. By abusing the notation, we can denote $E$ by $A \times C$ and $\mathcal{I} \otimes \mathcal{F}$ by $(\mathcal{I} \otimes \mathcal{T}) \otimes \mathcal{S}$. Define $\lambda \otimes P$ on $\mathcal{I} \otimes \mathcal{F}$ by letting $\lambda \otimes P(A \times C) = \sum_{s \in C} \pi_s \lambda \otimes P_s^T(A)$. Thus, one can view $\lambda \otimes P_s^T$ as the conditional probability measure on $I \times T$, given $s = s$.

3 Economies with common values

3.1 A large deterministic economy

Let $\mathcal{E}_0$ be a large (Aumann) deterministic economy with the atomless probability space $(I, \mathcal{I}, \lambda)$ as the space of agents and $\mathbb{R}^m_+$ as the common consumption set, $u^0$ a function from $I \times \mathbb{R}^m_+$ to $\mathbb{R}$ such that for any given $i \in I$, $u^0(i, x)$ is the utility of agent $i$ at consumption bundle $x \in \mathbb{R}^m_+$. Assume that $u^0(i, x)$ is $\mathcal{I}$-measurable in $i \in I$, continuous and monotonic in $x \in \mathbb{R}^m_+$. Let $e$ be a $\lambda$-integrable function from $I$ to $\mathbb{R}^m_+$, where $e(i)$, (also denoted by $e_i$) is the initial endowment of agent $i$. We can represent $\mathcal{E}_0$ by $\{(I, \mathcal{I}, \lambda), u^0, e\}$. Let $\Delta_m$ be the unit simplex in $\mathbb{R}^m_+$.

For the purpose of readability, we present in the following definition several standard concepts.

Definition 1 1. An allocation for the economy $\mathcal{E}_0$ is simply an integrable function $x$ from $(I, \mathcal{I}, \lambda)$ to $\mathbb{R}^m_+$.

2. An allocation $x$ is said to be individually rational if for $\lambda$-almost $i \in I$, $u^0(i, x(i)) \geq u^0(i, e(i))$.

3. An allocation $x$ is feasible in $\mathcal{E}_0$ if $\int_I x(i) d\lambda(i) = \int_I e(i) d\lambda(i)$.

4. A feasible allocation $x$ is efficient in $\mathcal{E}_0$ if there does not exist any other feasible allocation $y$ such that $u^0(i, y(i)) > u^0(i, x(i))$ for $\lambda$-almost all $i \in I$.\textsuperscript{15}

5. A feasible allocation $x$ is a Walrasian allocation (competitive equilibrium allocation) in $\mathcal{E}_0$ if there is a price system $p \in \Delta_m$ such that for $\lambda$-almost all $i \in I$, $x(i)$ is a maximal element in the budget set $\{z \in \mathbb{R}^m_+ : p \cdot z \leq p \cdot e(i)\}$ under the utility function $u^0(i, \cdot)$.

6. A coalition $A$ (i.e., a set in $\mathcal{I}$ with $\lambda(A) > 0$) is said to block an allocation $x$ if there exists an allocation $y$ such that $\int_A y(i) d\lambda(i) = \int_A e(i) d\lambda(i)$, and for $\lambda$-almost all $i \in A$,

\textsuperscript{14}This means that if $x, y \in \mathbb{R}^m_+$, $x \geq y$ with $x \neq y$, then $u^0(i, x) > u^0(i, y)$.

\textsuperscript{15}The monotonicity assumption implies that the efficiency of $x$ is equivalent to the nonexistence of a feasible allocation $y$ such that $u^0(i, y(i)) \geq u^0(i, x(i))$ for $\lambda$-almost all $i \in I$ with a strict inequality for a set of agents $i$ with $\lambda$-positive measure.
\[ u^0(i, y(i)) > u^0(i, x(i)). \] A feasible allocation \( x \) is said to be in the core of \( \mathcal{E}_0 \) if there is no coalition that blocks \( x \).

### 3.2 The economic model

We shall now follow the definition and notation in Section 2. We consider an atomless economy with asymmetric information, which corresponds to the asymptotic replica economies considered in [12]. The common consumption set is the positive orthant \( \mathbb{R}^m_+ \). Let \( u \) be a function from \( I \times \mathbb{R}^m_+ \times S \) to \( \mathbb{R}_+ \) such that for any given \( i \in I \), \( u(i, x, s) \) is the utility of agent \( i \) at consumption bundle \( x \in \mathbb{R}^m_+ \) and true state \( s \in S \).\(^{16}\) For any given \( s \in S \), assume that \( u(i, x, s) \), (also denoted by \( u_s(i, x) \)),\(^{17}\) is \( \mathcal{I} \)-measurable in \( i \in I \), continuous and monotonic in \( x \in \mathbb{R}^m_+ \). As in [12], the utility of agent \( i \) does not depend on her or any other agents’ signals. Let \( e \) be a \( \lambda \)-integrable function from \( I \) to \( \mathbb{R}^m_+ \) such that \( \int_I e(i) d\lambda \) is in the strictly positive cone\(^{18}\) \( \mathbb{R}^m_+ \), where \( e(i) \) is the initial endowment of agent \( i \).

For each \( s \in S \), \( \mathcal{E}_s^c = \{(I, \mathcal{I}, \lambda), u_s, e\} \) is a large deterministic economy. The collection \( \mathcal{E}^c = \{\mathcal{E}_s^c : s \in S\} \) is called a Complete Information Economy (CIE). The following definition adapts Definition 1 to the CIE setting.

**Definition 2**  
1. An allocation for the CIE is a function \( x^c \) from \( I \times S \) to \( \mathbb{R}^m_+ \) such that for each \( s \in S \), \( x^c_s \) is \( \lambda \)-integrable. Let \( \mathcal{A}^c \) be the collection of all the allocations for the CIE.

2. A CIE allocation \( x^c \) is said to be individually rational if for each \( s \in S \), \( x^c_s \) is individually rational in \( \mathcal{E}_s^c \).

3. A CIE allocation \( x^c \) is feasible if for each \( s \in S \), \( x^c_s \) is feasible in \( \mathcal{E}_s^c \).

4. A feasible CIE allocation \( x^c \) is said to be efficient if for each \( s \in S \), \( x^c_s \) is efficient in \( \mathcal{E}_s^c \).

5. A feasible CIE allocation \( x^c \) is said to be a Walrasian allocation (competitive equilibrium allocation) if for each \( s \in S \), there is a price system \( p_s \in \Delta_m \) such that \( (x^c_s, p_s) \) is a competitive equilibrium in \( \mathcal{E}_s^c \).

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\(^{16}\) We assume that the utility functions take non-negative values to avoid stating various integrability conditions explicitly. In fact, one can impose the condition of linear growth on the utilities to guarantee that the relevant expected utilities as used in this paper are finite. A real-valued function \( v \) on \( \mathbb{R}^m_+ \) is said to satisfy the condition of linear growth if there exist positive numbers \( \alpha \) and \( \beta \) such that \( v(x) \leq \alpha \|x\| + \beta \) for all \( x \in \mathbb{R}^m_+ \). When a continuous function \( v \) satisfies that condition, \( v(y(\cdot)) \) is integrable on \( (T, \mathcal{T}, P_T) \) whenever \( y(\cdot) \) is so. It is obvious that any concave function on \( \mathbb{R}^m_+ \) always satisfies the condition of linear growth.

\(^{17}\) In the sequel, we shall often use subscripts to denote some variable of a function that is viewed as a parameter in a particular context.

\(^{18}\) A vector \( x \) is in \( \mathbb{R}^m_+ \) if and only if all its components are positive.
6. A feasible CIE allocation \( x^c \) is said to be in the core of the CIE if for each \( s \in S \), \( x^c_s \) is in the core of \( \mathcal{E}_s^c \).

In the complete information economy, the agents are informed with the true state. We shall now consider a corresponding Private Information Economy, where the agents are informed with their signals but not the true state. In this case, the agents will use the conditional probability measure \( P^S(\cdot | t) \) on \( S \) to compute their expected utilities. For \( t \in T \), the ex post utility \( U_i(x | t) \) of agent \( i \) (also denoted by \( U(i, x, t) \)) for her consumption bundle \( x \in \mathbb{R}^m_+ \) with the given signal profile \( t \) is \( \sum_{s \in S} u_i(x, s) P^S(\{s\} | t) \). It is obvious that for any fixed \( x \in \mathbb{R}^m_+ \), \( U(i, x, t) \) is \( \mathcal{I} \otimes \mathcal{T} \)-measurable in \( (i, t) \in I \times T \) and continuous in \( x \in \mathbb{R}^m_+ \). The collection \( \mathcal{E}^P = \{(I \times \Omega, \mathcal{I} \otimes \mathcal{F}, \lambda \otimes P), u, e, (\tilde{t}_i, i \in I), \tilde{s}\} \) is called a Private Information Economy (PIE). For each fixed \( t \in T \), \( \mathcal{E}^P_t = \{(I, \mathcal{I}, \lambda), U(\cdot, \cdot, t), e\} \) is a large deterministic economy. The following is an analog of Definition 2 in the setting of a PIE.

**Definition 3**

1. An allocation for the PIE is an integrable function \( x^p \) from \((I \times T, \mathcal{I} \otimes \mathcal{T}, \lambda \otimes P^T)\) to \( \mathbb{R}^m_+ \). Let \( \mathcal{A}^p \) be the collection of all the allocations for the PIE.

2. A PIE allocation \( x^p \) is said to be ex post individually rational if for \( P^T \)-almost all \( t \in T \), \( x^p_t \) is individually rational in \( \mathcal{E}^p_t \).

3. A PIE allocation \( x^p \) is ex post feasible if for \( P^T \)-almost all \( t \in T \), \( x^p_t \) is feasible in \( \mathcal{E}^p_t \).

4. A feasible PIE allocation \( x^p \) is said to be ex post efficient if for \( P^T \)-almost all \( t \in T \), \( x^p_t \) is efficient in \( \mathcal{E}^p_t \).

5. A feasible PIE allocation \( x^p \) is said to be an ex post Walrasian allocation (ex post competitive equilibrium allocation) if there is a measurable price function \( p \) from \((T, \mathcal{T})\) to \( \Delta_m \) such that for \( P^T \)-almost all \( t \in T \), \((x^p_t, p_t)\) is a competitive equilibrium in \( \mathcal{E}^p_t \).

6. A feasible PIE allocation \( x^p \) is said to be in the ex post core of the PIE if for \( P^T \)-almost all \( t \in T \), \( x^p_t \) is in the core of \( \mathcal{E}^p_t \).

In the Private Information Economy, each agent \( i \) is privately informed with her signal \( t_i \). A major issue is whether the agent will have any incentive to mis-report that signal. The following definition of incentive compatibility is standard.

**Definition 4** For a PIE allocation \( x^p \), an agent \( i \in I \), a signal profile \( t \in T \), and a signal \( t'_i \in T^0 \), let

\[
U_i(x^p, t'_i | t_i) = \int_{S \times T^{-i}} u_i(x^p(t_{-i}, t'_i), s)dP^{S \times T^{-i}(\cdot | t_i)},
\]
which is the expected utility of agent $i$ when she receives private signal $t_i$ but mis-reports as $t'_i$. The PIE allocation $x^p$ is said to be incentive compatible if $\lambda$-almost all $i \in I$,

$$U_i(x^p_i, t_i | t_i) \geq U_i(x^p_i, t'_i | t_i)$$

holds for $\tau_i$-almost all $t_i, t'_i \in T^0$.

### 3.3 Perfect competition in a large economy with asymmetric information

The fundamental idea of perfect competition is that there are many economic agents, and that each individual agent has negligible influence in the market. Though each individual agent has non-negligible consumption in general, her share of consumption in the aggregate in terms of per capita consumption is negligible; that property can be guaranteed by using an atomless measure space as the space of agents.

When individuals have asymmetric information, a heuristic way to capture the idea of perfect competition is that the private signal of an individual agent can only influence a negligible set of agents, and moreover those signals associated with the individual agents that play a particular role in the model (for example, used in the utility functions or in calculating the aggregate signal distribution in some sense) are essentially independent of each other. The following definition formalizes this intuitive idea.

**Definition 5** Let $G^0$ be a finite set $\{g_1, g_2, \ldots, g_M\}$, (with power set $G^0$), and $F$ be a measurable process from $(I \times T, I \otimes T)$ to $G^0$. For agent $i \in I$, $F(i, t)$ is the derived signal of agent $i$ from the signal profile $t$. The process $F$ is called an idiosyncratic signal process if it has the following two properties.

1. The process $F$ is a signal process with negligible influence from private signals. That is, for $\lambda$-almost all $i \in I$, there is a set $A_i \in \mathcal{I}$ with $\lambda(A_i) = 1$ such that for any $t \in T$ and $t'_i \in T^0$, $F(j, (t_{-i}, t_i)) = F(j, (t_{-i}, t'_i))$ holds for each $j \in A_i$.

2. The process $F$ is essentially pairwise independent conditioned on $\tilde{s}$.

Condition (1) means that agent $i$’s private signal $t_i$ can only possibly influence the value of $F(j, t)$ for a null set of agents $j \in I - A_i$. Thus, whenever agent $i$ mis-reports her private signal $t_i$ has no effect on $F(j, t)$ for almost all agents $j \in I$. Condition (2) says that when a true state $s$ is realized, agent $i$’s derived signal $F(i, \cdot)$ is independent of agent $j$’s derived signal $F(j, \cdot)$ for almost all agents $i, j \in I$. A formal definition of essential pairwise independence is given in Definition 8 of the appendix.

Notice that $t_i$ is the private signal of agent $i$, and we simply call $F(i, t)$ her signal.
Note that the property in Definition 5 (1) can also be defined for the case when $I$ has finitely many agents and $\lambda$ is the counting probability measure. Since any single agent is not negligible, the validity of (1) implies that for any $i, j \in I$, $t \in T$ and $t'_i \in T^0$, $F(j,(t_{-i},t_i)) = F(j,(t_{-i},t'_i))$, which implies that for any $i \in I$, $F(i,\cdot)$ is constant. Thus, in order for the property in (1) to be meaningful, one has to work in a model with an atomless measure space of agents.

Our idiosyncratic signal process is a general function of the agents’ announcements satisfying the above two conditions.\(^\text{19}\) We shall consider two special cases in the following two remarks; one involves only agents’ private signals and the other replication of signals.

**Remark 1** When $F(i,t) = t_i$ for all $(i,t) \in I \times T$, i.e. $F(i,\cdot)$ only takes agent $i$’s private signal as its value, one can simply take $A_i = I \setminus \{i\}$ for any $i \in I$. Since $\lambda$ is assumed to be atomless, any single agent is negligible, and hence $\lambda(A_i) = 1$. It is obvious that $F(j,(t_{-i},t_i)) = t_j = F(j,(t_{-i},t'_i))$ for $j \in A_i$. Therefore, $F$ is a signal process with negligible influence from private signals. If, in addition, $F$ is essentially pairwise independent conditioned on $s$, then $F$ is an idiosyncratic signal process. Note that the property in Definition 5 (1) can only guarantee that the private signal of an agent has negligible influence in the functional form. Some underlying correlations conditioned on $s$ may still exist in a non-trivial way. For example, one can construct $P$ so that for a non-negligible set $A$ of agents $i \in I$, $t_i$, $i \in A$ are correlated conditioned on $s$; then an individual agent may still have non-negligible influence. Thus, condition (2) is needed.

**Remark 2** (a) When a differential information economy with $n$ agents is replicated as in [6], [12], the agents are divided into many cohorts of $n$ agents and the signals within each cohort may be used in the utility functions or used for calculating the joint distributions within the cohort. As an analog in the continuum setting, we assume that the space of agents is in the form $(I,I,\lambda) = (I^n \times I',I^n \otimes I',\lambda_n \otimes \lambda')$, where $I^n = \{1,2,\ldots,n\}$, $I^n$ the power set on $I^n$, $\lambda_n$ the counting probability measure, and $(I',I',\lambda')$ is an atomless probability space. For $i' \in I'$, the agents $(1,i'),(2,i'),\ldots,(n,i')$ are said to be in the same cohort. For an agent $i = (k,i') \in I^n \times I'$, $t \in T$, let $F((k,i'),t) = (t_{(1,i')},t_{(2,i')},\ldots,t_{(n,i')})$. Then, $F$ is a process from $(I \times T,I \otimes T)$ to $C^0 = (T^0)^I$ that takes the signals of the agents in the same cohort. For $i = (k,i') \in I^n \times I'$, let $A_i = I \setminus \{(1,i'),(2,i'),\ldots,(n,i')\}$; then, $j = (l,j') \in A_i$ implies that $j' \neq i'$, $F(j,(t_{-i},t_i)) = (t_{(1,j')},\ldots,t_{(n,j')}) = F(j,(t_{-i},t'_i))$ for any $t \in T$ and $t'_i \in T^0$. Since finitely many agents are still negligible, $\lambda(A_i) = 1$, and $F$ is a signal process with negligible influence from private signals.

\(^{19}\)This kind of general function is considered to be desirable in the second paragraph of page 2440 in [12]; see also footnote 32 below.
(b) We can define another process $F'$ from $(I' \times T, \mathcal{F} \otimes T)$ to $G^0$ by letting $F'(i', t) = (t_{(1,i')}, t_{(2,i')}, \cdots, t_{(n,i')})$ for $(i', t) \in I' \times T$. An analogous property to that of independent replicas is that for all $i', j' \in I'$ with $i' \neq j'$, $F'(i', \cdot)$ and $F'(j', \cdot)$ are independent with identical distributions conditioned on $\tilde{s}$. We do not need this strong condition in order for $F$ to be an idiosyncratic signal process. It is easy to see that $F$ is essentially pairwise independent conditioned on $\tilde{s}$ if and only if so is $F'$; thus, if $F$ or $F'$ has this property, then $F$ is an idiosyncratic signal process.

When the true state is $s$, the signal distribution of agent $i$ conditioned on the true state is $P^T_s F^{-1}_i$, i.e., the probability for agent $i$ to have $g_t$ as her signal is $P^T_s (F^{-1}_i \{g_t\})$ for each $1 \leq l \leq M$, where $F_i = F(i, \cdot)$. Let $\mu_s$ be the agents’ average signal distribution conditioned on the true state $s$, i.e.,

$$\mu_s(\{g_t\}) = \int_I P^T_s (F^{-1}_i \{g_t\}) d\lambda = \int_I \int_T 1_{\{g_t\}}(F(i, t)) dP^T_s d\lambda,$$

where $1_{\{g_t\}}$ is the indicator function of the singleton set $\{g_t\}$. By the Fubini property for $(I \times T, \mathcal{F} \otimes T, \lambda \otimes P^T_s)$, $\mu_s$ is actually the distribution $(\lambda \otimes P^T_s) F^{-1}_i$ of $F$, viewed as a random variable on the product space $I \times T$.

From now on, we shall impose the following non-triviality assumption on the process $F$:

$$\forall s, s' \in S, s \neq s' \Rightarrow \mu_s \neq \mu_{s'}.$$  \hspace{1cm} (1)

This says that different true states of nature correspond to different average conditional distributions of agents’ signals. It corresponds to the non-triviality condition in [6] and [12] for the independent replica models.

Next, we define the following sets

$$\forall s \in S, L_s = \{t \in T : \lambda F^{-1}_t = \mu_s\}; \quad L_0 = T - \bigcup_{s \in S} L_s.$$  \hspace{1cm} (2)

The non-triviality assumption implies that for any $s, s' \in S$ with $s \neq s'$, $L_s \cap L_{s'} = \emptyset$. The measurability of the sets $L_s, s \in S$ and $L_0$ follows from the measurability of $F$. Thus, the collection $\{L_0\} \cup \{L_s, s \in S\}$ forms a measurable partition of $T$. That partition will play a central role in later sections.

---

20. For a formal definition of the Fubini property, see Definition 7.

21. As we will see in the appendix, under the condition of essential pairwise independence, the exact law of large numbers in [13] and [16] (see Lemma 1 in the appendix) implies that $(\lambda \otimes P^T_s) F^{-1}_i = \lambda F^{-1}_i$ for $P^T_s$-almost all $t \in T$.

22. See condition (iii) on page 1277 in [6], condition (c) on page 2434 in [12].

23. As noted in footnote 21, under the condition of essential pairwise independence, the exact law of large numbers in [13] and [16] implies that $P^T_s(L_s) = 1$ for each $s \in S$.
3.4 Incentive compatibility and ex post efficient, Walrasian and core allocations

Define a mapping $\Phi$ from the set $A^c$ of CIE allocations to the set $A^p$ of PIE allocations as follows. For any CIE allocation $x^c \in A^c$, define

$$
\Phi(x^c)(i, t) = \begin{cases} 
  e(i) & \text{if } t \in L_0, \\
  x^c(i, s) & \text{if } t \in L_s, s \in S 
\end{cases}
$$

for $(i, t) \in I \times T$. It is obvious that $\Phi(x^c)$ is integrable on $(I \times T, \mathcal{I} \otimes \mathcal{T}, \lambda \otimes P^T)$, (and thus integrable on the extension $(I \times T, \mathcal{I} \boxtimes \mathcal{T}, \lambda \boxtimes P^T)$), and consequently is a PIE allocation. This means that $\Phi$ is indeed a mapping from $A^c$ to $A^p$.

Theorem 1 below shows that $\Phi$ plays a central role between the two economies $E^c$ and $E^p$. In particular, under assumption that $F$ is an idiosyncratic signal process, any CIE allocation $x^c$ can be transformed to an incentive compatible PIE allocation $\Phi(x^c)$, and the ex post efficiency of $\Phi(x^c)$ is equivalent to the efficiency of $x^c$. The same type of equivalence also holds for core and Walrasian allocations. Thus, the existence of incentive compatible, ex post Walrasian, (and thus ex post individually rational and ex post efficient) allocations, follows from the usual existence result on Walrasian allocations, as in [4] and [7]. The theorem below is proved in Section 8.3 of the Appendix.

**Theorem 1**

(1) If $F$ is a signal process with negligible influence from private signals, then the PIE allocation $\Phi(x^c)$ is always incentive compatible for any CIE allocation $x^c$.

(2) Assume that the process $F$ is essentially pairwise independent conditioned on $\bar{s}$. Let $x^c$ be any CIE allocation. Then $x^c$ is individually rational, or feasible, or efficient, or a Walrasian allocation, or a core allocation in the CIE if and only if $\Phi(x^c)$ has the corresponding ex post version of the property in the PIE.\(^{24}\) In addition, we have

$$
\int_T u_i(\Phi(x^c)(i, t), s)dP^T_s(t) = u_i(x^c(i, s), s),
$$

which means that the expected utility of $\Phi(x^c)(i, \cdot)$ conditioned on the true state $s$ is always the utility of $x^c(i, s)$.

(3) If $F$ is an idiosyncratic signal process, then there exists an incentive compatible PIE allocation $x^p$ that is an ex post Walrasian allocation (and thus ex post individually rational and ex post efficient).

\(^{24}\)By the usual core equivalence theorem in [3] and [7], a PIE allocation is in the ex post core if and only if it is an ex post Walrasian allocation. Thus, core equivalence is still valid in this perfectly competitive framework.
4 Economies with type dependent utility functions

4.1 The economic model

Section 3 focuses on an atomless economy with asymmetric information, where no agents’ types enter utility functions. In this section, we shall consider the more general case that allows agents’ types to appear in the utility functions.

We shall follow the definition and notation in Sections 2 and 3.3. The common consumption set is $\mathbb{R}^n_+$. Let $v$ be a function from $I \times \mathbb{R}^n_+ \times S \times G^0$ to $\mathbb{R}_+$ such that for any given $i \in I$, $v(i, x, s, g)$ is the utility of agent $i$ at consumption bundle $x$, true state $s$, and the agent’s signal $g$. For any given $s \in S$, and $g \in G^0$, assume that $v(i, x, s, g)$ is $\mathcal{I}$-measurable in $i \in I$, continuous and monotonic in $x \in \mathbb{R}^m_+$. For given $(s, t)$, let $u(i, x, s, t) = v(i, x, s, F(i, t))$. It can be easily checked that for any fixed $x \in \mathbb{R}^m_+$, $s \in S$, $u(i, x, s, t)$ is $\mathcal{I} \otimes \mathcal{T}$-measurable.\footnote{For any fixed $x \in \mathbb{R}^m_+$, $s \in S$, and $r \in \mathbb{R}$, one can simply observe that
$$\{ (i, t) \in I \times T : u(i, x, s, t) < r \} = \bigcup_{g \in G^0} \left[ F^{-1}(\{g\}) \cap \{ (i, t) \in I \times T : v(i, x, s, g) < r \} \right].$$} Let $e$ be an integrable function from $I$ to $\mathbb{R}^m_+$ with $\int_I e(i) d\lambda(i) \in \mathbb{R}^{m+}$, where $e(i)$ is the initial endowment of agent $i$.

As in [6] and Section 3 here, we can define a Private Information Economy, where the agents are informed with their signals but not the true state. The ex post utility $U_i(x|t)$ of agent $i$, (also denoted by $U(i, t, x)$), for the agent’s consumption $x \in \mathbb{R}^m_+$ with the given signal profile $t$ is $\sum_{s \in S} u(i, x, s, t) dP^S(\{s\}|t)$. It is obvious that for any fixed $x \in \mathbb{R}^m_+$, $U(i, t, x)$ is $\mathcal{I} \otimes \mathcal{T}$-measurable. For each fixed $t \in T$, $\mathcal{E}^p = \{(I, \mathcal{I}, \lambda), U(\cdot, \cdot, t), e\}$ is a large deterministic economy. The collection $\mathcal{E}^p = \{(I \times \Omega, \mathcal{I} \otimes \mathcal{F}, \lambda \otimes P), u, e, F, (i, i \in I), s\}$ is called a Private Information Economy (PIE). Definition 3 is still applicable for the PIE in this section.

The following definition of incentive compatibility is the same as Definition 4 except that the utility functions $u_i$ are now signal dependent.

**Definition 6** For a PIE allocation $x^p$, an agent $i \in I$, a signal profile $t \in T$, and a signal $t'_i \in T^0$, let
$$U_i(x^p_i, t'_i|t_i) = \int_{S \times T_{-i}} u_i(x^p_i(t_{-i}, t'_i), s_{-i}(t_{-i}, t_i)) dP^{S \times T_{-i}}(\cdot|t_i),$$
which is the expected utility of agent $i$ when she receives private signal $t_i$ but mis-reports as $t'_i$. The PIE allocation $x^p$ is said to be incentive compatible if $\lambda$-almost $i \in I$,
$$U_i(x^p_i, t'_i|t_i) \geq U_i(x^p_i, t'_i|t_i)$$
holds for $\tau_i$-almost all $t_i, t'_i \in T^0$.\footnote{For any fixed $x \in \mathbb{R}^m_+$, $s \in S$, and $r \in \mathbb{R}$, one can simply observe that
$$\{ (i, t) \in I \times T : u(i, x, s, t) < r \} = \bigcup_{g \in G^0} \left[ F^{-1}(\{g\}) \cap \{ (i, t) \in I \times T : v(i, x, s, g) < r \} \right].$$}
### 4.2 Consistency of incentive compatibility and efficiency

For each $s \in S$, consider the large deterministic economy $E_s = \{(I \times T, \mathcal{I} \otimes \mathcal{T}, \lambda \otimes P_T), u_s, e\}$, where the utility function for agent $(i, t) \in I \times T$ is $u_s(i, t, \cdot) = u(i, \cdot, s, t)$ and the initial endowment for agent $(i, t)$ is $e(i)$. When the signal process $F$ is essentially pairwise independent conditioned on $\tilde{s}$, we can restate part of Theorem 3 in [15] as Lemma 3 in the appendix for the convenience of the reader. For this purpose, we assume that the underlying measure space $(I \times T, \mathcal{I} \otimes \mathcal{T}, \lambda \otimes P_T)$ is a Loeb product space for each $s \in S$.\footnote{Note that Loeb product spaces provide a big class of measure spaces that extend the usual measure-theoretic products and retain the Fubini property. For the purposes of modeling limits of large, but finite phenomena, there is no loss of generality to work with Loeb product spaces since they themselves can be viewed as the limiting objects for sequences of product measure spaces (see Section 8.5 on how this is the case for the context of this paper). See Section 8.6 for a brief description of Loeb product spaces, and [11] and [13] for more details.}

Based on Lemma 3, we can prove the following theorem that corresponds to the result in Part (3) of Theorem 1 (the proof is given in Section 8.4 of the Appendix).

**Theorem 2** If $F$ is an idiosyncratic signal process, then there exists an incentive compatible allocation $x^p$ in the Private Information Economy such that $x^p$ is an ex post Walrasian allocation (and thus ex post individually rational, and ex post efficient).

### 5 Asymptotic interpretation

In this section, we translate Theorem 2 to an asymptotic setting. Fix $n \geq 1$. We shall first define the $n$-th Private Information Economy $E^n_0$. Let $I^n$ be $\{1, 2, \ldots, n\}$ with the counting probability measure $\lambda_n$ on its power set $\mathcal{T}^n$; $(I^n, \mathcal{T}^n, \lambda_n)$ represents the space of agents for the $n$-th economy.\footnote{We shall use both superscript and subscript $n$ to index objects in the $n$-th economy.} The sets $S$ and $T_0$ have the same meanings as in Section 2, i.e., $S = \{s_1, s_2, \ldots, s_K\}$, (with power set $\mathcal{S}$), is the space of true states that are not known to the agents; and $T^0 = \{q_1, q_2, \ldots, q_L\}$ is the space of all the possible private signals for individual agents.

Let $T^n = (T^0)^n$ be the space of all the functions from $I^n$ to $T^0$ with its power set $\mathcal{T}^n$, and $(\Omega^n, \mathcal{F}^n)$ the product of $(S, \mathcal{S})$ and $(T^n, \mathcal{T}^n)$. Let $(\Omega^n, \mathcal{F}^n, P_n)$ be a probability space representing all the uncertainty (on the true states as well as on the private signals for all the agents) in the $n$-th economy, $P^n_S$ and $P^{T^n}_n$ the marginal probability measures of $P_n$ respectively on $(S, \mathcal{S})$ and on $(T^n, \mathcal{T}^n)$. Let $\tilde{s}^n$ be the projection mapping from $\Omega^n$ to $S$.

For $t^n \in T^n$, $t^n$ is a function from $I^n$ to $T_0$ representing a signal profile for all the agents in $I^n$, and $t^n(i^n)$, also denoted by $t^n_i$, the private signal received by agent $i^n \in I^n$. Let $\tilde{t}^n_i$ be the projection mapping from $\Omega^n$ to $T^0$ with $\tilde{t}^n_i(s, t^n) = t^n_i$. For $t^n \in T^n$ and $i^n \in I^n$, let $t^n_{i^n}$.
be the restriction of the signal profile \( t^n \) to the set \( I^n \setminus \{i^n\} \); \( T^n_{-i^n} \) denotes the set of all such \( t^n_{-i^n} \).

Let \( P^n_{T^n_s} \) denote the conditional probability measure \( P^n_{T^n_s}(\cdot|s) \) on \((T^n, T^n)\) when the random variable \( \bar{s}^n \) takes value \( s \). Let \( P^n_s(\cdot|\bar{t}^n) \) be the conditional probability measure on \( S \) given the signal profile \( \bar{t}^n \in T^n \). For \( i^n \in I^n \), let \( \tau^n_{i^n} \) be the signal distribution \( P^n_{i^n}(\bar{t}^n_{i^n})^{-1} \) of agent \( i^n \) on the space \( T^n \), and \( P^n_{S \times T^n_{-i^n}}(\cdot|\bar{t}^n_{i^n}) \) the conditional probability measure on \( S \times T^n_{-i^n} \) when the signal for agent \( i^n \) is \( \bar{t}^n_{i^n} \in T^n \).

Let \( F^n \) be a signal process from \((I^n \times T^n, T^n \otimes T^n)\) to a finite space \( G^0 = \{g_1, g_2, \ldots, g_M\} \) of derived signals for the agents; \( F^n(i^n, t^n) \) will enter the utility function of agent \( i^n \).

For \( s \in S \), let \( \mu^n_s = (\lambda_n \otimes P^n_{T^n_s})(F^n)^{-1} \). Assume that there exists a positive number \( \delta_0 \) such that for each \( n \geq 1 \),
\[
\forall s \in S, \pi^n_s = P^n_s(\{ s \}) \geq \delta_0; \forall s, s' \in S, s \neq s' \Rightarrow \| \mu^n_s - \mu^n_{s'} \| \geq \delta_0, \tag{5}
\]
where \( \mu^n_s \) and \( \mu^n_{s'} \) are viewed as points in the unit simplex \( \Delta_K \) in \( \mathbb{R}^K \), and \( \| \mu^n_s - \mu^n_{s'} \| \) is their Euclidean distance. This condition corresponds to the non-triviality assumption on the measures \( P^n_s \) and \( \mu_s, s \in S \) in the limit model in Sections 2 and 3.

We shall now define the utilities and endowments for the \( n \)-th Private Information Economy. For simplicity, we take a compact subset \( U_0 \) of the space \( U(\mathbb{R}^m_+) \) of non-negative continuous and monotonic functions on \( \mathbb{R}^m_+ \) satisfying the condition of linear growth\(^{28}\) that is endowed with the supnorm topology, and a compact subset \( E_0 \) of \( \mathbb{R}^m_+ \). Let \( v^n \) and \( e^n \) be mappings\(^{29}\) respectively from \( I^n \times S \times G^0 \) to \( U_0 \) and from \( I^n \) to \( E_0 \), where \( v^n(i^n, s, g) \) is the utility function of agent \( i^n \) at state \( s \) in \( S \) and her signal \( g \), and \( e^n(i^n) \) the initial endowment of agent \( i^n \). For \( t^n \in T^n \), the ex post utility \( U^n_{-i^n}(x|t^n) \) of agent \( i^n \), (also denoted by \( U^n(i^n, x, t^n) \)), for her consumption bundle \( x \in \mathbb{R}^m \) with the given signal profile \( t^n \) is \( \sum_{s \in S} u^n_{i^n}(x, s, t^n)P^n_s(\{s\}|t^n) \), where \( u^n_{i^n}(x, s, t^n) = v^n(i^n, s, F^n(i^n, t^n))(x) \).

The \( n \)-th Private Information Economy is simply the collection \( \mathcal{E}^p_n = \{(I^n \times \Omega^n, T^n \otimes T^n, \lambda_n \otimes P_n), u^n, e^n, (i^n, i^n \in I^n), \bar{s}^n \} \). An allocation for \( \mathcal{E}^p_n \) is a function from \((I^n \times T^n, T^n \otimes T^n, \lambda_n \otimes P^n_{T^n_s})\) to \( \mathbb{R}^m_+ \).

For an allocation \( x^n_i \) of the PIE \( \mathcal{E}^p_n \), an agent \( i^n \in I^n \), a signal profile \( t^n \in T^n \), and a signal \( (\bar{t}^n_{i^n})' \in T^n \), let
\[
U^n_{i^n}(x^n_i(i^n), (\bar{t}^n_{i^n})'|t^n) = \int_{S \times T^n_{-i^n}} u^n_{i^n}(x^n_i(i^n, (\bar{t}^n_{i^n})', (\bar{t}^n_{i^n})'), s, t^n) dP^n_{S \times T^n_{-i^n}}(\cdot|\bar{t}^n_{i^n}),
\]
\(^{28}\)The purpose of including the condition of linear growth as defined in Footnote 16 is to guarantee the relevant expected utilities in the limiting case to have finite values. Note that the utility functions in [6] are assumed to be bounded (p. 1277).

\(^{29}\)The compactness assumption on both \( U_0 \) and \( E_0 \) can be relaxed respectively to a tightness condition on the induced distribution of \( u^n \) on \( U(\mathbb{R}^m_+) \) and to a uniform integrability condition on \( e^n \).
which is the expected utility of agent $i^n$ when she receives private signal $t^n_{in}$ but mis-reports as $(t^n_{in}')$.

The following theorem is an asymptotic analog of Theorem 2. The proof can be found in Section 8.5 of the Appendix.

**Theorem 3** For the sequence $\mathcal{E}_n^n, n \geq 1$ of PIEs, assume that the signal processes $F^n, n \geq 1$ are asymptotically idiosyncratic in the sense that for any $\delta > 0$, and $s \in S$, both of the following sequences converge to one as $n$ goes to infinity:

\[
\lambda_n \land \lambda_n \left( \{(i^n, j^n) \in I^n \times I^n : \| P_{ns}^{T_n} (F^n, F^n)^{-1} - P_{ns}^{T_n} (F^n) - P_{ns}^{T_n} (F^n)^{-1} \| \leq \delta \} \right),
\]

\[
\lambda_n \land \lambda_n \left( \{(i^n, j^n) \in I^n \times I^n : \forall t^n \in T^n, (t^n_{in})' \in T^n, F^n(j^n, t^n) = F^n(j^n, (t^n_{in}', (t^n_{in}'))) \} \right).
\]

Then for any given $\varepsilon > 0$, there is a positive integer $N$ such that for any $n > N$, there exists an allocation $x^n_0$ for the PIE $\mathcal{E}_n^n$, a price function $p^n$ from $T^n$ to the price simplex $\Delta_m$, and sets $B_n \subseteq I^n$ and $C^n \subseteq T^n$ with $\lambda_n (B^n) > 1 - \varepsilon$ and $P_{ns}^{T_n} (C^n) > 1 - \varepsilon$ satisfying the following properties.

(a) For each $i^n \in B_n$,

\[
U^n_{i^n} ((x^n_0)_{i^n}, t^n_{i^n'}, t^n_{i^n}) + \varepsilon \geq U^n_{i^n} ((x^n_0)_{i^n}, (t^n_{i^n'})')
\]

holds for all $t^n_{i^n}, (t^n_{i^n'})' \in T^n$ with $\tau^n_0 (\{t^n_{i^n}\}) \geq \varepsilon$ and $\tau^n_0 (\{t^n_{i^n} \}) \geq \varepsilon$.

(b) For any $(i^n, t^n) \in I^n \times T^n$, $p^n(t^n)x^n_0(i^n, t^n) = p^n(t^n)e^n(i^n)$.

(c) For all $t^n \in C^n$,

(i) $\| \int I^n x^n_0(i^n, t^n) d\lambda_n - \int I^n e^n(i^n) d\lambda_n (i^n) \| \leq \varepsilon$;

(ii) $\lambda_n (\{i^n \in I^n : \forall y \in \mathbb{R}^+_n, \lambda p^n(t^n) y \leq \lambda p^n(t^n) e^n(i^n) \Rightarrow U^n_{i^n} (x^n_0(i^n, t^n)|t^n) + \varepsilon \geq U^n_{i^n} (y|t^n) \}) \geq 1 - \varepsilon$;

(iii) $\lambda_n (\{i^n \in I^n : U^n_{i^n} (x^n_0(i^n, t^n)|t^n) + \varepsilon \geq U^n_{i^n} (e^n(i^n)|t^n) \}) \geq 1 - \varepsilon$;

(iv) There does not exist an allocation $y_n$ from $I^n$ to $\mathbb{R}^+_n$ such that $\int I^n y_n(i^n) d\lambda_n (i^n) = \int I^n e^n(i^n) d\lambda_n (i^n)$, and for all $i^n \in I^n$, $U^n_{i^n} (y_n(i^n)|t^n) > U^n_{i^n} (x^n_0(i^n, t^n)|t^n) + \varepsilon$.

Theorem 3 (a) says that the PIE allocation $x^n_0$ for $\mathcal{E}_n^n$ is approximately incentive compatible; (b), (i) and (ii) of (c) mean that $x^n_0$ is an approximate ex post Walrasian allocation; (iii) and (iv) of (c) show that $x^n_0$ is both ex post individually rational and ex post efficient in an approximate sense. Note that the approximate ex post individual rationality in (iii) of (c) clearly follows from the approximate optimality with budget constraints in (ii) of (c). However, the statement in (iv) of (c) on approximate ex post efficiency may not follow from the approximate

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30In this paper, $\| \cdot \|$ denotes the Euclidean norm.
ex post Walrasian property. Since the common value model can be regarded as a special case of the model with type dependent utilities, exactly the same result in Theorem 3 also provides an asymptotic version of Part (3) of Theorem 1.

6 Relationship to Gul-Postlewaite [6] and McLean-Postlewaite [12]

This paper is mainly motivated by the independent replica models studied in Gul-Postlewaite [6] and McLean-Postlewaite [12]. We shall discuss below the relationship of this paper with [6] and [12] in detail.

The conclusion of the main theorem in [6] is the consistency of incentive compatibility, ex post individual rationality and ex post efficiency (ex post Walrasian equilibrium) in an approximate sense for the independent replica model of a fixed private information economy with finitely many agents, where the agent types enter the utility functions. Our Theorem 2 shows that incentive compatibility, ex post individual rationality and ex post efficiency (ex post Walrasian equilibrium) can be achieved exactly in a perfectly competitive private information economy with an atomless measure space of agents and type dependent utility functions. We also note that a viable model for a limit economy that corresponds to the type of asymptotic models with an independence assumption as in [6] is made possible in the measure-theoretic framework adopted here. As noted in the introduction, our framework is free of the measurability problem, and thus the limit results obtained are mathematically meaningful, and cannot be obtained in a framework based on the usual product measure spaces.

One advantage of our approach is that our results on limiting economies in Theorem 2 can be re-interpreted for the asymptotic large, but finite economies via common procedures in nonstandard analysis. Thus, we obtain in Theorem 3 the consistency of incentive compatibility, ex post individual rationality and ex post efficiency (ex post Walrasian equilibrium) all in an approximate sense for a general sequence of large, but finite private information economies. While the conclusion in Theorem 3 is weaker than that of the main theorem in [6], it is noted below that the assumptions used to derive the conclusion in Theorem 3 are also much less stringent.

The model in [6] relies on a regularity condition that requires the demands of two different types for an agent to be never identical for all prices in some open ball for every realization of the relevant uncertainty. It is not clear what type of utility functions will produce this regularity condition. This condition is not needed in our exact model in Section 4 and in the asymptotic

31 Suppose that there exists an allocation \( y_n \) satisfying the required properties in (iv) of (c). Then, (ii) of (c) implies that for any \( t^n \in C^n \), \( \lambda_n \{ i^n \in I^n : p_n(t^n)y_n(i^n) > p_n(t^n)e^n(i^n) \} \geq 1 - \varepsilon \); one may not be able to derive a contradiction from that condition with \( \int_{I^n} p_n(t^n)y_n(i^n) d\lambda_n = \int_{I^n} p_n(t^n)e^n(i^n) d\lambda_n \).
model in Section 5. The utility functions in this paper are not assumed to be concave while strict concavity of utility functions is assumed in [6].

Next, we compare the restrictions on the information structures. In the independent replica model in [6], the private signal of an individual agent has influence over $n$ agents of the same cohort for a fixed positive integer $n$, and the discrete parameter process that takes the signals of all the $n$ agents in the relevant cohort as its values are mutually independent and identically distributed (iid) conditioned on the true states. As noted in Remark 2 in Section 3.3, one can also work with a similar version for atomless limit economy with an iid assumption on a signal process $F'$ to give a particular idiosyncratic signal process $F$.

On the other hand, our general idiosyncratic signal process is a general function of the agents’ announcements, which is considered to be desirable in [12]. It is more general than the replica case in two aspects. First, it allows the private signal of an individual agent to influence a negligible corner of the market (in particular, any finitely many agents in an atomless market). Second, the signal process is not assumed to be iid but essentially pairwise independent (which is much weaker than iid) conditioned on the true states. As for the asymptotic case, our Theorem 3 only imposes a version of asymptotic approximate independence on a general sequence of large, finite private information economies while [6] works with the independent replica model that is a very special sequence of large, but finite private information economies.

While the regularity condition in [6] requires that an individual agent’s utility cannot be independent of her type and the common value model in [12] is thus ruled out in the model of [6], we do not need the type of regularity condition in this paper. This means that the common value model as in [12] can indeed be regarded as a special case for the more general models with type dependent utilities. Thus, part (3) of our Theorem 1 is covered by our Theorem 2 and most of the above comparisons between the main theorem in [6] and our Theorems 2 and 3 here are still valid for the comparisons between Theorem 2 in [12] and our Theorems 1 and 3.

One can also say something beyond the consistency of incentive compatibility and efficiency about the more special common value model in Section 3. Part (1) of our Theorem 1 says that any allocation in the complete information economy can be transformed to an in-

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32 It is suggested on page 2440 of [12] that one could estimate the true state of nature using a general function of the agents’ announcements. Our idiosyncratic signal process is indeed a general function satisfying the two conditions in Definition 5. In particular, it can use just the announcements from a coalition $A$ of agents by letting $F_i(t) = t_i$ for $i \in A$ and $F_i(t) = g_0$ for $i \notin A$, where $g_0$ is a point in $G^0$ (for this case, we take $G^0 = \{g_0\} \cup T^0$); $F$ is an idiosyncratic signal process when the private signals for agents in coalition $A$ are essentially independent conditioned on the true states (see the suggestion on page 2440 of [12]).

33 See the discussion on page 2434 of [12].

34 Instead of using the strict concavity assumption on utility functions in [6], the concavity assumption is used for the common value model in [12], which is not needed in the common value model in Section 3 of this paper.
centive compatible allocation with the same conditional expected utility as in equation (4) in our Theorem 1 (2), which, corresponds to the approximate results in Corollary 2 in [12] for an economy with a fixed finite number of agents. Our Theorem 1 together with the definition of $\Phi$ in equation (3) and Lemma 2 in the appendix corresponds to the approximate results in Theorem 1 in [12].

Part (2) of our Theorem 1 shows that many properties are preserved under the transformation $\Phi$, including individual rationality and efficiency (which corresponds to the approximate result in Corollary 1 in [12]). Here we note that approximate results for an economy with a fixed finite number of agents as in [9] and [12] may not say much about a sequence of large, but finite economies since the accuracy of approximation typically depends on the number of agents in the economy. In comparison, our exact results for an atomless economy can be re-interpreted as some approximate results for an asymptotic large, but finite economies.

Finally, we compare our proofs with those in [6] and [12]. The proofs for the approximate results in [6] and [12] require intricate and ingenious computations. In comparison, the proofs of our exact results are simple and transparent in measure-theoretic terms while the proof of our approximate results follows standard procedures in nonstandard analysis.

7 Concluding remarks

This paper shows that the introduction of a suitable mathematical model to capture the meaning of perfect competition in a differential information economy has a high reward. We are able to go beyond the results in Gul-Postlewaite [6] and McLean-Postlewaite [12] in several aspects. In particular, not only for the first time we model the idea of perfect competition in a differential information economy, and therefore generalize the Aumann model, but also we resolve exactly the incompatibility of incentive compatibility and Pareto efficiency. Furthermore, our results for the limit economies guarantee the corresponding results for large but finite economies, and also a number of assumptions needed in [6] and [12] can be dispensed with in our general setting.

8 Appendix

8.1 The exact law of large numbers

In order to work with independent processes constructed from signal profiles, we need to work with an extension of the usual measure-theoretic product having the Fubini property. Here is

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35Corollary 1 in [12] requires the existence of a strictly individually rational efficient allocation for the CIE, which precludes the case that the allocation of initial endowments is already efficient.

36As noted in the last paragraph on page 2434 of [12], Theorem 2 of [12] does not follow from Theorem 1 of [12].
a formal definition.

**Definition 7** Let \((I, \mathcal{I}, \lambda)\) and \((\Omega, \mathcal{F}, P)\) be probability spaces. A probability space \((I \times \Omega, \mathcal{W}, Q)\) is said to be a Fubini extension of the usual product space \((I \times \Omega, \mathcal{I} \otimes \mathcal{F}, \lambda \otimes P)\) if it is an extension of \((I \times \Omega, \mathcal{I} \otimes \mathcal{F}, \lambda \otimes P)\), and for any real-valued \(Q\)-integrable function \(g\) on \((I \times \Omega, \mathcal{W})\), the two functions \(g_i = g(i, \cdot)\) and \(g_\omega = f(\cdot, \omega)\) are integrable respectively on \((\Omega, \mathcal{F}, P)\) for \(\lambda\)-almost all \(i \in I\) and on \((I, \mathcal{I}, \lambda)\) for \(P\)-almost all \(\omega \in \Omega\); moreover, \(\int_\Omega g dP\) and \(\int_I g_\omega d\lambda\) are integrable respectively on \((I, \mathcal{I}, \lambda)\) and on \((\Omega, \mathcal{F}, P)\), with \(\int_{I \times \Omega} g dQ = \int_I (\int_\Omega g_i dP) d\lambda = \int_\Omega (\int_I g_\omega d\lambda) dP\). The space \((I \times \Omega, \mathcal{W}, Q)\) is denoted by \((I \times \Omega, \mathcal{I} \otimes \mathcal{F}, \lambda \otimes P)\).

We shall now follow the notation of Section 2. When the probability space \((I \times T, \mathcal{I} \otimes \mathcal{T}, \lambda \otimes P^T_s)\) is a Fubini extension of the usual product space \((I \times T, \mathcal{I} \otimes \mathcal{T}, \lambda \otimes P^T_s)\), for each \(s \in S\), it can be checked that \((I \times \Omega, \mathcal{I} \otimes \mathcal{F}, \lambda \otimes P)\), defined in the last paragraph of Section 2, is a Fubini extension of the usual product space \((I \times \Omega, \mathcal{I} \otimes \mathcal{F}, \lambda \otimes P)\).

We shall now introduce the following crucial independence condition. We state the definition using a complete separable metric space \(X\) for the sake of generality; in particular, a finite space or an Euclidean space is a complete separable metric space.

**Definition 8** A process \(G\) from \(I \times T\) to a complete separable metric space \(X\) is said to be essentially pairwise independent conditioned on the true state random variable \(\tilde{s}\) if

1. \(G\) is \(\mathcal{I} \otimes \mathcal{T}\)-measurable;

2. for each \(s \in S\), the random variables \(G_i\) from \((T, \mathcal{T}, P^T_s)\) to \(X\) are essentially pairwise independent in the sense that for \(\lambda\)-almost all \(i \in I\), \(G_i\) and \(G_j\) are independent for \(\lambda\)-almost all \(j \in I\).

The following is an exact law of large numbers for a continuum of independent random variables shown in [13] and [16], which is stated as a lemma here using our notation for the convenience of the reader.\(^{37}\)

**Lemma 1** If a process \(G\) from \(I \times T\) to a complete separable metric space \(X\) is essentially pairwise independent conditioned on \(\tilde{s}\), then for each \(s \in S\), the cross-sectional distribution \(\lambda G^{-1}_s\) of the sample function \(G_t(\cdot) = G(t, \cdot)\) is the same as the distribution \((\lambda \otimes P^T_s)G^{-1}\) of the process \(G\) viewed as a random variable on \((I \times T, \mathcal{I} \otimes \mathcal{T}, \lambda \otimes P^T)\) for \(P^T\)-almost all \(t \in T\).

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\(^{37}\) This result was originally stated on the Loeb measure spaces in [13] (Theorem 5.2). However, it is noted in [16] that the result can be proved for an extension of the usual product with the Fubini property (Theorem 3.5). See also Chapter 7 in [11] (and in particular, Section 7.5), written by Y. N. Sun. We state this version for the convenience of the reader.
8.2 The conditional probability $P^S(\cdot|t), t \in T$

Let $\delta_s$ be the Dirac measure on $S$ that gives probability one to the point $s$ and zero to other points. Define a function $H$ from $T$ to the space of probability measures on $S$ by letting

$$H(t) = \begin{cases} \delta_s & \text{for } t \in L_s, s \in S, \\ \delta_{s_0} & \text{for } t \in L_0. \end{cases}$$

Lemma 2 below shows that the conditional independence of the signal process $F$ together with the associated exact law of large numbers as stated in Lemma 1 says that $P^T(\cup_{s' \in S} L_{s'}) = 1,$ and $H(t), t \in T$ is a version of the conditional probability $P^S(\cdot|t), t \in T$. Note that using $H$ as the conditional probability $P^S(\cdot|t), t \in T$ means the following. When all the signals are reported by the agents to form a signal profile $t$, the agents will be able to determine the true state to be $s$ if the cross-sectional signal distribution $\lambda F_t^{-1}$ is observed to be $\mu_s$.

**Lemma 2** If $F$ is essentially pairwise independent conditioned on $\tilde{s}$, then $P^T(\cup_{s' \in S} L_{s'}) = 1,$ and $H(t), t \in T$ is a version of $P^S(\cdot|t), t \in T$.

**Proof:** The exact law of large numbers as stated in Lemma 1 says that the set $L_s = \{t \in T : \lambda F_t^{-1} = \mu_s\}$ has $P^T_s$-probability one. Thus, $P^T_s(L_{s'}) = 0$ for $s \neq s' \in S$, $P^T_s(\cup_{s' \in S} L_{s'}) = 1$, and $P^T(\cup_{s' \in S} L_{s'}) = \sum_{s \in S} \pi_s P^T_s(\cup_{s' \in S} L_{s'}) = 1$. Hence, $P^T$ is a convex combination of mutually singular probability measures $P^T_s, s \in S$.

Fix any $s' \in S, B \in T$. Since $B \setminus L_{s'}$ is a subset of $T \setminus L_{s'}$, which is a $P^T_{s'}$-null set, we have $P^T_{s'}(B \setminus L_{s'}) = 0$. Thus, $P^T_{s'}(B \cap L_{s'}) = P^T_{s'}(B) - P^T_{s'}(B \setminus L_{s'}) = P^T_{s'}(B)$. Hence, for any $s \in S$, we have the following identities

$$\int_B H(t)(\{s\})dP^T(t) = \sum_{s' \in S} \pi_{s'} \int_B H(t)(\{s\})dP^T_{s'}(t) = \sum_{s' \in S} \pi_{s'} \int_{B \cap L_{s'}} H(t)(\{s\})dP^T_{s'}(t)$$

$$= \sum_{s' \in S} \pi_{s'} \int_{B \cap L_{s'}} \delta_{s'}(\{s\})dP^T_{s'}(t) = \pi_s P^T_{s'}(B \cap L_s) = \pi_s P^T_{s'}(B)$$

$$= P(\{s\} \times B) = \int_B P^S(\{s\}|t)dP^T_{s}(t)$$

which implies that $H(t), t \in T$ is indeed a version of $P^S(\cdot|t), t \in T,$ by the arbitrary choices of $s \in S$ and $B \in T$.

8.3 Proof of Theorem 1

(1): By Definition 5 (1), we know that there is a set $A^* \in \mathcal{I}$ with $\lambda(A^*) = 1$ such that for any $i \in A^*$, there is a set $A_i \in \mathcal{I}$ with $\lambda(A_i) = 1$ such that for any $t \in T$ and $t'_i \in T^0$, the sample functions $F_{(t_{-i}, t_i)}(\cdot)$ and $F_{(t_{-i}, t'_i)}(\cdot)$ agree on $A_i$. Since the society’s signal distribution cannot
be influenced by a negligible set of agents outside the set $A_t$, we have $\lambda F_{(t-i,t_i)}^{-1} = \lambda F_{(t-i,t'_i)}^{-1}$. This means that for any $i \in A^t$, $t \in T$, $t'_i \in T^0$, and $s \in S$,

$$t \in L_s \Leftrightarrow \lambda F_{(t-i,t_i)}^{-1} = \mu_s \Leftrightarrow \lambda F_{(t-i,t'_i)}^{-1} = \mu_s \Leftrightarrow (t-i,t'_i) \in L_s. \quad (7)$$

Since $L_0$ is $T \setminus \bigcup_{s \in S} L_s$, we also know that $t \in L_0 \Leftrightarrow (t-i,t'_i) \in L_0$.

Denote $\Phi(x^c)$ by $x^p$. We have for any $i \in A^t$, $x^p(i,t) = x^p(i,(t_i,t'_i))$ for any $t \in T$ and $t'_i \in T^0$. Therefore, the condition of incentive compatibility in Definition 4 is satisfied by $x^p$.

Thus, part (1) is shown.

(2): Assume that $F$ is essentially pairwise independent conditioned on $\tilde{s}$. The exact law of large numbers as stated in Lemma 1 says that the set $L_s = \{t \in T : \lambda F_{(t-i,t_i)}^{-1} = \mu_s\}$ has $P_s^T$-probability one. Since, for any $s \in S$, $i \in I$, one always has $x^p(i,t) = \Phi(x^c)(i,t) = x^c(i,s)$ for $t \in L_s$, and then by integration, we obtain that

$$\int_T u_i(\Phi(x^c)(i,t),s)dP_s^T(t) = \int_{L_s} u_i(\Phi(x^c)(i,t),s)dP_s^T(t) = u_i(x^c(i,s),s),$$

which is equation (4).

Lemma 2 says that $H$ is a version of the conditional probability $P^S(\cdot \mid t)$. Thus, for any version of the conditional probability $P^S(\cdot \mid t)$, we always have $P^S(\cdot \mid t) = \delta_s$ for $P^T$-almost all $t \in L_s$. Hence, for $P^T$-almost all $t \in L_s$,

$$U(i,\cdot,t) = \sum_{s' \in S} u(i,\cdot,s')P^S(\{s'\} \mid t) = \sum_{s' \in S} u(i,\cdot,s')\delta_s(\{s'\}) = u(i,\cdot,s)$$

for all $i \in I$, and $\mathcal{E}_t^p = \mathcal{E}_s^c$. Let $B_s = \{t \in L_s : \mathcal{E}_t^p = \mathcal{E}_s^c\}$; then $P^T(L_s \setminus B_s) = 0$. Hence, for any $t \in B_s$, the ex post economy-allocation pair $(\mathcal{E}_t^p, x^p(\cdot,t))$ is exactly the same as the complete information economy-allocation pair $(\mathcal{E}_s^c, x^c(\cdot,s))$, which means that they must have the same properties.

Thus, if $x^c$ is efficient, then for each fixed $s \in S$, $x^c(\cdot,s)$ is efficient for $\mathcal{E}_s^c$. This means that $x^p(\cdot,t)$ is efficient for $\mathcal{E}_t^p$ for any $t \in B_s$. Since $P^T(L_s \setminus B_s) = 0$ for each $s \in S$, $P^T(\cup_{s \in S} B_s) = P^T(\cup_{s \in S} L_s) = 1$. Hence $x^p(\cdot,t)$ is efficient for $\mathcal{E}_t^p$ for $P^T$-almost all $t \in T$. Hence $x^p$ is ex post efficient.

For the other direction, fix any $s \in S$. If $x^p$ is ex post efficient, then $x^p(\cdot,t)$ is efficient for $\mathcal{E}_t^p$ for $P^T$-almost all $t \in T$, and in particular for $P^T$-almost all $t \in L_s$. Since $P^T(B_s) = P^T(L_s) = \sum_{s' \in S} \pi_s P^T(L_s) = \pi_s P^T(L_s) = \pi_s > 0$, we can certainly find a $t \in B_s$ such that $x^p(\cdot,t)$ is efficient for $\mathcal{E}_t^p$. For such a $t \in B_s$, since the economy-allocation pairs $(\mathcal{E}_t^p, x^p(\cdot,t))$ and $\mathcal{E}_t^c = \mathcal{E}_s^c$ have the same properties.

\footnote{Here we note that the Dirac measure $\delta_s$ has probability one at the point $s$ and zero at those points $s' \in S$ with $s' \neq s$.}
are the same, we obtain that \( x^c(\cdot, s) \) is efficient for \( E^c_s \). Since \( s \) is arbitrarily chosen in \( S \), we know that \( x^c \) is efficient.

The rest of the proof for part (2) follows clearly from the definition of each of the properties in Definition 2 and their ex post versions in Definition 3 by using the argument adopted for the proof of efficiency.

(3): By the usual existence result on Walrasian allocations in [4] or [7], there exists an allocation \( x^c \) that is a Walrasian allocation for the CIE. Since \( s \) is arbitrarily chosen in \( S \), we know that \( x^c \) is efficient.

The rest of the proof for part (2) follows clearly from the definition of each of the properties in Definition 2 and their ex post versions in Definition 3 by using the argument adopted for the proof of efficiency.

8.4 Proof of Theorem 2

Assume that the underlying measure space \((I \times T, I \otimes T, \lambda \otimes P^T_s)\) is a Loeb product space for each \( s \in S \). To make those variables that are given parameters in the particular context clear, we use subscripts extensively in this section. The following lemma is part of Theorem 3 in [15].

**Lemma 3** Assume that the signal process \( F \) is essentially pairwise independent conditioned on \( \tilde{s} \). Then, for each fixed \( s \in S \), there is a Walrasian allocation \( x_s \) with a strictly positive price system \( p_s \) for the economy \( E_s \) such that for \( P^T_T \)-almost all \( t \in T \), \( x_s(s,t) \) is a Walrasian allocation with a price system \( p_s \) for the large deterministic economy \( E_{(s,t)} = \{(I, I, \lambda), u(s,t), e\} \).

**Proof of Theorem 2:**

By Lemma 3, there is a Walrasian allocation \( x_s \) with a strictly positive price system \( p_s \) for the economy \( E_s \) such that for \( P^T_s \)-almost all \( t \in T \), \((p_s, x_s(s,t))\) is a Walrasian equilibrium for the large deterministic economy \( E_{(s,t)} = \{(I, I, \lambda), u(s,t), e\} \). Without loss of generality, assume that for every agent \((i, t) \in I \times T \), \( x_s(i, t) \) is the maximal element within her budget set. Thus, for each \( i \in I \), \( t, t' \in T \), agents \((i, t)\) and \((i, t')\) have the same endowment, and consequently the same budget set. Hence the utility of agent \((i, t)\) at \( x_s(i, t) \) is greater than or equal to her utility at \( x_s(i, t') \) since \( x_s(i, t') \) belongs to the budget set of agent \((i, t)\). This means that

\[
\forall i \in I, s \in S, t, t' \in T, u(i, x_s(i, t), s, t) \geq u(i, x_s(i, t'), s, t). \tag{8}
\]

\[39\]Note that preferences are used in [15] while we use utility functions here. It is easy to show that when we work with the preferences induced by the utility functions \( u(i, \cdot, s, t) \), we can still obtain measurability and essential independence for the relevant mappings into the space of preferences so that Theorem 3 in [15] is applicable in our context.

\[40\]When \( S \) is the trivial singleton set, Lemma 3 is actually a statement on the existence of an ex post Walrasian equilibrium with a common price system.
As in equation (3), define a mapping $x^p$ from $I \times T$ to $\mathbb{R}_+^m$ by letting

$$
x^p(i, t) = \begin{cases} 
e(i) & \text{if } t \in L_0, \\
x_s(i, t) & \text{if } t \in L_s, s \in S 
\end{cases}
$$

for $(i, t) \in I \times T$. It is obvious that $x^p$ is integrable on $(I \times T, \mathcal{I} \otimes \mathcal{T}, \lambda \otimes P^T)$, and hence a PIE allocation.

By the same argument in the first paragraph of the proof of Theorem 1 in Section 8.3, we obtain that for $\lambda$-almost all $i \in I$, for any $t \in T$, $t'_i \in T^0$ and $s \in S$, $t \in L_s$ if and only if $(t_i - t'_i) \in L_s$. Thus, for $\lambda$-almost all $i \in I$, for any $t \in L_s$ and $t'_i \in T^0$, we have

$$u_i(x_i^p(t_i - t'_i), s, (t_i - t'_i)) = u_i(x_i^p(i, t), s, t) \geq u_i(x_i^p(i, (t_i - t'_i)), s, (t_i - t'_i)) = u_i(x_i^p(t_i, t'_i), s, (t_i - t'_i));$$

when $t \in L_0$, we also have $(t_i, t'_i) \in L_0$; hence

$$u_i(x_i^p(t_i - t'_i), s, (t_i - t'_i)) = u_i(e(i), s, t) = u_i(x_i^p(t_i - t'_i), s, (t_i - t'_i)).$$

Thus, for $\lambda$-almost all $i \in I$, we have $U_i(x_i^p, t_i | t_i) \geq U_i(x_i^p, t'_i | t_i)$ for $\tau_i$-almost all $t_i, t'_i \in T^0$. Therefore, $x^p$ is an incentive compatible PIE allocation.

Fix any $s \in S$. Lemma 2 says that $P^S(\cdot | t) = \delta_s$ for $P^T$-almost all $t \in L_s$. Since $P^T = \sum_{s' \in S} \pi_{s'} P^T_{s'}$ with $\pi_{s'} > 0$, a $P^T$-null set is a $P^T_{s'}$-null set; hence $P^S(\cdot | t) = \delta_s$ for $P^T_{s'}$-almost all $t \in L_s$. Let $E_s$ be the set of all $t \in L_s$ such that $P^S(\cdot | t) = \delta_s$, and $(p_s, x(s, t))$ is a Walrasian equilibrium for $\mathcal{E}(s, t)$. Then, the above argument and Lemma 3 imply that $P^T_s(E_s) = 1$, and for any $t \in E_s$, we have $\int_T x(s, t)(i) d\lambda(i) = \int_T e(i) d\lambda(i)$.

Let $E = \cup_{s \in S} E_s$. Then

$$P^T(E) = \sum_{s' \in S} \pi_{s'} P^T_{s'}(\cup_{s \in S} E_s) = \sum_{s' \in S} \pi_{s'} P^T_{s'}(E_s') = \sum_{s' \in S} \pi_{s'} = 1.$$

Since the sets $L_s, s \in S$ are disjoint, so are $E_s, s \in S$. For any $t \in E$, there is a unique $s \in S$ such that $t \in E_s$ and

$$\int_T x^p(i, t) d\lambda(i) = \int_T x(s, t)(i) d\lambda(i) = \int_T e(i) d\lambda(i),$$

which implies that $x^p$ is feasible for the PIE.

Define a measurable function $p^*$ from $(T, \mathcal{T})$ to $\mathbb{R}_+^m$ by letting

$$p^*(t) = \begin{cases} e^* & \text{if } t \in L_0, \\
p_s & \text{if } t \in L_s, s \in S 
\end{cases}
$$

where $e^*$ is the vector whose components are $1/m$.

Fix any $s \in S$ and $t \in E_s$. Then, $U_t = u(s, t)$, $p^*(t) = p_s$, $\mathcal{E}_t^p = \mathcal{E}(s, t)$ and $x^p_t = x(s, t)$. Since $(p_s, x(s, t))$ is a Walrasian equilibrium for $\mathcal{E}(s, t)$, $(p^*(t), x^p_t)$ is a Walrasian equilibrium for $\mathcal{E}_t^p$.

Hence, for any $t \in E$, $(p^*(t), x^p_t)$ is a Walrasian equilibrium for $\mathcal{E}_t^p$. Since $P^T(E) = 1$, $x^p$ is thus an incentive compatible and ex post Walrasian allocation in the PIE, and therefore ex post individually rational, and ex post efficient.
8.5 Proof of Theorem 3

We transfer the PIE sequence $\mathcal{E}_n^p, n \in \mathbb{N}$ to the nonstandard universe to obtain an internal sequence $\mathcal{E}_n^p, n \in \mathbb{N}^*$ of PIEs.\(^{41}\)

Fix $n \in \mathbb{N}^*$. We shall round off the infinitesimals in the internal PIE $\mathcal{E}_n^p$ to obtain a standard PIE $\mathcal{E}_n^p$ using Loeb measures. Let $I = I^n$, $T = T^n$ and $\Omega = \Omega^n$. Let $(I, \mathcal{I}, \lambda)$, $(\Omega, \mathcal{F}, P)$ and $(I \times \Omega, \mathcal{I} \boxtimes \mathcal{F}, \lambda \boxtimes P)$ be the Loeb spaces\(^{42}\) of the respective internal probability spaces $(I^n, \mathcal{I}^n, \lambda_n)$, $(\Omega^n, \mathcal{F}^n, P_n)$ and $(I^n \times \Omega^n, \mathcal{I}^n \boxtimes \mathcal{F}^n, \lambda_n \otimes P_n)$. It is clear that $F_n$ (to be denoted $F$) is still a mapping from $(I^n \times T^n, \mathcal{I}^n \boxtimes \mathcal{F}^n)$ to $G^0 = \{g_1, g_2, \ldots, g_M\}$. Denote $s^n$ by $\tilde{s}$. The rest of the definition related to the information structure $(\Omega, \mathcal{F}, P)$ can be found in Section 2. By equation (5), we have $\mu_s \neq \mu_{s'}$ for all $s \neq s' \in S$, and $\pi_s = P^S \{ \{s\} \} > 0$ for all $s \in S$, where $\mu_s = (\lambda \boxtimes P_s^T) F^{-1}$. Hence, for any $s \in S$ and $s' \in S$, $\mu_s \neq \mu_{s'}$.

Since $v^n$ and $e^n$ take values respectively in $*\mathcal{U}_0$ and $*\mathcal{E}_0$, the compactness of $\mathcal{U}_0$ and $\mathcal{E}_0$ implies that the mappings $v^n$ and $e^n$ have standard parts $v = \circ (v^n)$ and $e = \circ (e^n)$ respectively,\(^{43}\) where $v$ is a measurable mapping from $(I \times G^0, \mathcal{I} \boxtimes S \otimes \mathcal{G}^0)$ to $\mathcal{U}_0$, and $e$ a measurable mapping from $(I, \mathcal{I})$ to $\mathcal{E}_0$. It is obvious that for any given $s \in S$, $g \in G^0$, $v(i, x, s, g)$ is $\mathcal{I}$-measurable in $i \in I$, continuous and monotonic in $x \in \mathbb{R}^m$. Let $u(i, x, s) = v(i, x, s, F_i(t))$. Thus, we obtain the PIE $\mathcal{E}_n^p = \{(I \times \Omega, \mathcal{I} \boxtimes \mathcal{F}, \lambda \boxtimes P), u, e, (\tilde{t}, i \in I), \tilde{s}\}$.

For any standard positive integer $r$, take $\delta = 1/r$. Both of the following measures

$$\lambda_n \otimes \lambda_n \left( \left\{ (i^n, j^n) \in I^n \times I^n : \| P_{ns}^{T^n} (F_{i^n}^n, F_{j^n}^n) - 1 \| \leq \frac{\delta}{r} \right\} \right),$$

$$\lambda_n \otimes \lambda_n \left( \left\{ (i^n, j^n) \in I^n \times I^n : \forall t^n \in T^n, (t^n_i, t^n_j) \in T^0, F^n(j^n, t^n) = F^n(j^n, (t^n_i, t^n_j)) \right\} \right)$$

are greater than or equal to $1 - \delta$ for all $s \in S$; and hence the same inequalities hold for some positive infinitesimal $\delta = 1/r$ with $r \in \mathbb{N}^*$ by the spillover principle.\(^{44}\) Thus, for $\lambda \otimes \lambda$-almost all $(i, j) \in I \times I$, $P_{s}^T(F_i, F_j) - 1 = P_{s}^T F_i^{-1} \otimes P_{s}^T F_j^{-1}$ holds for all $s \in S$, and $F(j, (t_{-i}, t_i)) = F(j, (t_{-i}, t'_i))$ for any $t \in T$ and $t'_i \in T^0$. The Fubini property implies that for $\lambda$-almost all $i \in I$ and for $\lambda$-almost all $j \in I$, $F(j, (t_{-i}, t_i)) = F(j, (t_{-i}, t'_i))$ for any $t \in T$ and $t'_i \in T^0$, and $F_i$ and $F_j$ are independent conditioned on $\tilde{s}$. Thus, $F$ is an idiosyncratic signal process.

By Theorem 2 and its proof, there exists an incentive compatible PIE allocation $x^p$ that is an ex post Walrasian allocation with strictly positive price system $p(t) = p_s \in \Delta_m$ for $t \in L_s$, where $p_s$ is an equilibrium price of a corresponding large deterministic economy with true state

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\(^{41}\)As in [11], the nonstandard extension of an object $O$ is denoted by $*O$.

\(^{42}\)To avoid the possibility of generating different classes of null sets on an underlying space, we only work with the $\sigma$-algebras generated by the relevant internal algebras.

\(^{43}\)If $v^n(\cdot)$ is clearly a bounded function.

\(^{44}\)See part (i) of Theorem 2.8.11 in [11].
$s \in S$. Note that the monotonicity assumption on the utilities implies that for $P^T$-almost all $t \in T$, $p(t)x^p(i, t) = p(t)e(i)$ for $\lambda$-almost all $i \in I$, which also implies the essential boundedness of $x^p$. Let $p_n$ be an internal lifting of $p$ that only takes values in $\{p_s : s \in S\}$, and $x^p_n$ be an $S_{\lambda_n \otimes P^T_n}$-integrable lifting of $x^p$. Fix any $\varepsilon \in \mathbb{R}_+$. 

It is clear that for $\lambda \otimes P^T$-almost all $(i^n, t^n) \in I^n \times T^n$, $p_n(t^n)x^p_n(i^n, t^n) \simeq p(t^n)x^p(i^n, t^n)$ and $p_n(t^n)e^n(i^n) \simeq p(t^n)e(i^n)$. One can modify the values of $x^p_n$ to retain its property of being an $S_{\lambda_n \otimes P^T_n}$-integrable lifting of $x^p$ such that for all $(i^n, t^n) \in I^n \times T^n$,

$$
p_n(t^n)x^p_n(i^n, t^n) = p_n(t^n)e^n(i^n).$$  \hspace{1cm} (9)

Without loss of generality, we can assume that both $x^p$ and $x^p_n$ are bounded, and $x^p(i^n, t^n)$ is the standard part $x^p_n(i^n, t^n)$ for all $(i^n, t^n) \in I^n \times T^n$.

An internal version of Keisler’s Fubini theorem\(^{46}\) says that for $P^T$-almost all $t^n \in T^n$, $x^p_n(\cdot, t^n)$ is an $S_{\lambda_n}$-integrable lifting of $x^p(\cdot, t^n)$, which implies that $\int_{I^n} x^p_n(i^n, t^n)d\lambda_n \simeq \int_{I^n} x^p(i^n, t^n)d\lambda$. The compactness of $E_0$ certainly implies that $\int_{I^n} e^n(i^n)d\lambda_n \simeq \int_{I^n} e(i^n)d\lambda$. By the feasibility of $x^p$, we obtain that for $P^T$-almost all $t^n \in T^n$,

$$
\left\| \int_{I^n} x^p_n(i^n, t^n)d\lambda_n(i^n) - \int_{I^n} e^n(i^n)d\lambda_n(i^n) \right\| \leq \varepsilon.
$$  \hspace{1cm} (10)

For $P^T$-almost all $t^n \in T^n$, the standard part $p(t^n)$ of $p_n(t^n)$ is strictly positive and $(p(t^n), x^p(\cdot, t^n))$ is an equilibrium for the economy $E^p_n$; take any such $t^n \in T^n$. For $\lambda$-almost all $i \in I$, $x^p(i, t^n)$ is the maximal element in the budget set of agent $i$; take any such $i \in I$. Then, for any $y \in \ast \mathbb{R}_+^n$ with $p_n(t^n)y \leq p_n(t^n)e^n(i)$ or $p_n(t^n)y \simeq p_n(t^n)e^n(i)$, $y$ has a standard part $\,^0y$ with $p(t^n)^0y \leq p(t^n)e(i)$. By the continuity of the payoffs and compactness of $U_0$, it is easy to see that both $U^n_i(x^p_n(i^n, t^n)|t^n) \simeq U_i(x^p(i, t^n)|t^n)$ and $U^n_i(y|t^n) \simeq U_i(\,^0y|t^n)$ hold. Hence, we can obtain that for $P^T$-almost all $t^n \in T^n$,

$$
\lambda_n(\{t^n \in I^n : \forall y \in \ast \mathbb{R}_+^n, p_n(t^n)y \leq p_n(t^n)e^n(i^n) \}) \Rightarrow U^n_i_{\lambda_n}(x^p_n(i^n, t^n)|t^n) + \varepsilon \geq U^n_{\lambda_n}(y|t^n) \geq 1 - \varepsilon,
$$  \hspace{1cm} (11)

which also implies that

$$
\lambda_n(\{i^n \in I^n : U^n_{\lambda_n}(x^p_n(i^n, t^n)|t^n) + \varepsilon \geq U^n_{\lambda_n}(e^n(i^n)|t^n) \}) \geq 1 - \varepsilon.
$$  \hspace{1cm} (12)

Let $D_n$ be the set of $t^n \in T^n$ such that there exists an internal allocation $y_n$ from $I^n$ to $\ast \mathbb{R}_+^n$ with the properties (i) $\int_{I^n} y_n(t^n)d\lambda_n(i^n) = \int_{I^n} e^n(i^n)d\lambda_n(i^n)$, and (ii) $U^n_{\lambda_n}(y_n(i^n)|t^n) > \hspace{1cm} \text{See [11] for the definition and properties of $S$-integrability.} \hspace{1cm} \text{See Proposition 5.3.12 in [11].} \hspace{1cm} 26$
Let $U^n_i(x^n_i(t^n), t^n_i)|t^n_i) + \varepsilon$ holds for all $i^n \in I^n$. Suppose that $P^{T^n}(D_n)$ is a positive real number. By the argument in the above paragraph, there exists $t^n \in D_n$ such that for $\lambda$-almost all $i^n \in I^n$, if $U^n_i(y^n_i(i^n)|t^n) > U^n_i(x^n_i(i^n), t^n)|t^n) + \varepsilon$, then $p_n(t^n_i)y_n(i^n) > p_n(t^n_i)e^n(i^n)$ and $p_n(t^n_i)y_n(i^n) \neq p_n(t^n_i)e^n(i^n)$; fix such a $t^n \in D_n$. We thus know that for $\lambda$-almost all $i^n \in I^n$, $p_n(t^n_i)y_n(i^n) - p_n(t^n_i)e^n(i^n)$ is a non-infinitesimal number in $\ast \mathbb{R}_+$; let $E$ be the internal set of all $i^n \in I^n$ such that $p_n(t^n_i)y_n(i^n) \leq p_n(t^n_i)e^n(i^n)$. Then $\lambda_n(E) \simeq 0$. It is clear that $\alpha = \int_{I^n \setminus E} (p_n(t^n_i)y_n(i^n) - p_n(t^n_i)e^n(i^n))d\lambda_n(i^n)$ has a standard part $\circ \alpha \in \mathbb{R}_+$. On the other hand, the feasibility condition (i) implies that $\int_{I^n} (p_n(t^n_i)y_n(i^n) - p_n(t^n_i)e^n(i^n))d\lambda_n(i^n) = 0$. Hence, we have

$$\alpha = \int_{E} (p_n(t^n_i)e^n(i^n) - p_n(t^n_i)y_n(i^n))d\lambda_n(i^n) \leq \int_{E} p_n(t^n_i)e^n(i^n)d\lambda_n(i^n) \simeq 0. \quad (13)$$

Thus, $\circ \alpha \leq 0$, which contradicts the fact that $\circ \alpha \in \mathbb{R}_+$. Therefore, $P^{T^n}(D_n) = 0$, and consequently $P^{T^n}_n(D_n) \simeq 0$.

By the continuity of the payoffs again, the incentive compatibility of $x^p$ implies that for $\lambda$-almost all $i^n \in I^n$,

$$U^n_{i^n}((x^n_{i^n})_{i^n}, t^n_{i^n}|t^n_{i^n}) + \varepsilon \geq U^n_{i^n}((x^n_{i^n})|i^n), (t^n_{i^n})|t^n_{i^n}) \quad (14)$$

holds for all $t^n_{i^n}, (t^n_{i^n})' \in T^n$ with $\tau^n_{i^n}(\{t^n_{i^n}\}) \geq \varepsilon$ and $\tau^n_{i^n}(\{(t^n_{i^n})\}) \geq \varepsilon$.

Let $B_n$ be the set of all $i^n \in I^n$ such that equation (14) holds for $i^n$, and $C_n$ the set of all $t^n \in T^n - D^n$ such that equations (10) and (11) (and thus equation (12) also) hold for $t^n$. Then, $\lambda_n(B^n) > 1 - \varepsilon$ and $P^{T^n}_n(C^n) > 1 - \varepsilon$ for any $n \in \ast \mathbb{N}_\infty$. The rest follows from the spillover principle.47

### 8.6 A construction of the information structure

In this section, we shall show that the information structure used in Sections 2 and 3.3 does exist via the Loeb measure construction in nonstandard analysis (see [10]). The reader is referred to [11] for basic nonstandard analysis.

Let $I$ be a hyperfinite set with its internal power set $\mathcal{I}_0$, $\lambda_0$ the internal counting probability measure on $(I, \mathcal{I}_0)$, $\mathcal{I}$ the $\sigma$-algebra $\sigma(\mathcal{I}_0)$ generated by $\mathcal{I}_0$, and $\lambda$ the corresponding Loeb measure on $(I, \mathcal{I})$.48 The atomless probability space $(I, \mathcal{I}, \lambda)$ is used as the space of agents. Since $I$ is simply an equivalence class of a sequence of finite sets in an ultrapower construction,49 the external cardinality of $I$ is the cardinality of the continuum. Thus, one can indeed take $I$ as the unit interval $[0, 1]$ with a suitable measure structure.

47 See part (ii) of Theorem 2.8.11 in [11].
48 The basic intuition for a hyperfinite set is that it is defined by a sequence of finite sets.
Let $T^0 = \{q_1, q_2, \ldots, q_L\}$ be the space of all (at least two) possible signals for the individual agents with its power set $\mathcal{T}^0$. Let $T = (T^0)^I$ be the space of all internal functions from $I$ to $T^0$ with its internal power set $\mathcal{T}_0$. For $i \in I$, let $T_{-i} = (T^0)^{I \setminus \{i\}}$ be the set of all internal functions from $I \setminus \{i\}$ to $T^0$ with its internal power set $(\mathcal{T}_{-i})_0$. Let $\mathcal{T} = \sigma(\mathcal{T}_0)$ and $\mathcal{T}_{-i} = \sigma((\mathcal{T}_{-i})_0)$. Then $(T, \mathcal{T})$ is indeed the product of $(\mathcal{T}_{-i}, \mathcal{T}_{-i})$ with $(T^0, \mathcal{T}^0)$, and hence it does have the desired product structure as stated in Section 2.

Let $S = \{s_1, s_2, \ldots, s_K\}$ be the space of true states with its power set $\mathcal{S}$, and $\Omega = S \times T$ with its internal power set $\mathcal{F}_0$ (and thus $\mathcal{F}_0 = \mathcal{S} \otimes \mathcal{T}_0$). Let $\mathcal{F} = \sigma(\mathcal{F}_0)$. It is obvious that $\mathcal{F} = \mathcal{S} \otimes \mathcal{T}$. Thus, $(\Omega, \mathcal{F})$ is the product of $(S, \mathcal{S})$, $(T^0, \mathcal{T}^0)$ and $(T_{-i}, \mathcal{T}_{-i})$.

Let $(I \times \Omega, \mathcal{I}_0 \otimes \mathcal{F}_0)$ be the internal product of $(I, \mathcal{I}_0)$ and $(\Omega, \mathcal{F}_0)$ and $\mathcal{I} \otimes \mathcal{F} = \sigma(\mathcal{I}_0 \otimes \mathcal{F}_0)$. For the purpose of illustration, we only consider the simple case that the signal process $F$ takes the private signals as its values, i.e., for $(i, t) \in I \times T$, $F(i, t) = t(i)$, where $t(i)$ is the private signal received by agent $i$ for a signal profile $t$. Then $F$ is $\mathcal{I}_0 \otimes \mathcal{T}_0$-measurable (and hence $\mathcal{I} \otimes \mathcal{F}$-measurable).

For each $s \in S$, let $\rho^s_0$ be an internal measure on $(T, \mathcal{T}_0)$ such that

$$
\lambda_0 \otimes \lambda_0 \left( \{(i, j) \in I \times I : \forall q, q' \in T^0, \rho^s_0(\{t \in T : F_i(t) = q & F_j(t) = q'\}) \right) \\
\simeq \rho^s_0(\{t \in T : F_i(t) = q\}) \rho^s_0(\{t \in T : F_j(t) = q'\}) \simeq 1.
$$

(15)

The purpose of equation (15) is to guarantee independence for $F$ conditioned on the true states.\(^{50}\) Let $\pi_s, s \in S$ be a positive probability weight function on $S$. Define an internal probability measure $P_0$ on $(\Omega, \mathcal{F}_0)$ by letting $P_0(\{s\} \times B) = \pi_s \rho^s_0(B)$ for any $s \in S$ and $B \in \mathcal{T}_0$. Let $P$ be the corresponding Loeb measure on $(\Omega, \mathcal{F})$. It is clear that the marginal probability measure $P^S$ of $P$ on $(S, \mathcal{S})$ has the property that $P^S(\{s\}) = \pi_s$ for each $s \in S$. Let $\tilde{s}$ be the projection mapping from $\Omega$ to $S$. It is easy to see that for each $s \in S$, the conditional probability measure $P^T_s$ on $(T, \mathcal{T})$, given $\tilde{s} = s$ is the Loeb measure $\rho^s$ of $\rho^s_0$.

Fix $s \in S$. Let $\lambda \otimes P$ and $\lambda \otimes P^T_s$ be the corresponding Loeb measures of $\lambda_0 \otimes P_0$ and $\lambda_0 \otimes \rho^s_0$ respectively on $\mathcal{I} \otimes \mathcal{F}$ and on $\mathcal{I} \otimes \mathcal{T}$. As first noted by Anderson in [1], $(I \times \Omega, \mathcal{I} \otimes \mathcal{F}, \lambda \otimes P)$ and $(I \times T, \mathcal{I} \otimes \mathcal{T}, \lambda \otimes P^T_s)$ are respective extensions of the usual products $(I \times \Omega, \mathcal{I} \otimes \mathcal{F}, \lambda \otimes P)$ and $(I \times T, \mathcal{I} \otimes \mathcal{T}, \lambda \otimes P^T_s)$. These extensions are actually Fubini extensions by Keisler’s Fubini theorem (see Corollary 5.3.14 in [11]). Equation (15) implies that the random variables $F_i, i \in I$ on $(T, \mathcal{T}, P^T_s)$ are essentially pairwise independent conditioned on $\tilde{s}$. Thus, $F$ is an idiosyncratic signal process. It is also easy to make sure that the non-triviality assumption in equation (1) is satisfied.\(^{51}\)

\(^{50}\)There are many ways to construct such $\rho^s_0$; an easy way is to take the internal product measure on $(T, \mathcal{T}_0)$ of a hyperfinite sequence $(\tau^i_t)_{t \in I}$ of internal probability measures on $(T^0, \mathcal{T}^0)$.

\(^{51}\)Let $\mu_s$ be a standard probability distribution on $(T^0, \mathcal{T}^0)$ such that $\mu_s \neq \mu_{s'}$ if $s \neq s' \in S$. One simple way
Finally, we note that the purpose to take only the \( \sigma \)-algebras generated by the relevant internal algebras is to make sure that we work on the same \( \sigma \)-algebra \( \mathcal{T} \). Notice that the Loeb measures \( P^s_\mathcal{T} \) on \( \mathcal{T} = \sigma(\mathcal{T}_0) \) may have different completions.

**References**


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to guarantee the non-triviality assumption is to take \((T; \mathcal{T}_0, \rho^0_\mathcal{T})\) to be the internal product measure space of \(|I|\) copies of \((T^0, \mathcal{T}^0, \mu^0)\). In this case, \(F\) is an iid process conditioned on \(\tilde{s}\). However, there are also many other ways to construct \(\rho^0_\mathcal{T}\) so that \(F\) is non-iid conditioned on \(\tilde{s}\).