Subgame Perfect Cooperation in an Extensive Game

by

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Subgame Perfect Cooperation in an Extensive Game

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Abstract

This paper brings together two of the most important solution concepts of game theory – subgame-perfect Nash equilibrium of a non-cooperative game and the core of a cooperative game. Our approach rests on two fundamental ideas: (1) Given an extensive game, the formation of a coalition leads to a new game where all the members of the coalition become one player. (2) At the origin of any subgame, the only possible coalitions consist of players who have decision nodes in the subgame. We introduce a concept of subgame perfect cooperative equilibrium, which we label the γ-core of an extensive game. We provide a necessary and sufficient condition for the existence of the γ-core of an extensive game of perfect information. As a motivating example, we formulate the problem of global warming as a dynamic game with simultaneous moves and show that if the payoff functions are quadratic, then the γ-core of the game is nonempty.

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1 Introduction

This paper brings together two of the most important solution concepts of game theory – subgame-perfect Nash equilibrium of a non-cooperative game and the core of a cooperative game. Nash equilibrium and Selten's subgame perfection are at the foundation of non-cooperative game theory and their analysis in specific environments has attracted research across the social and behavioral sciences. Since von-Neumann and Morgenstern's seminal volume, cooperative game theory has also been of great interest; particularly important concepts are the core (the set of outcomes that are stable against cooperation within groups), introduced by Gillies (1953), and the Shapley value (1953), which assigns to each player his expected marginal contribution. Nash's classic 1953 paper on bargaining suggests the unification of non-cooperative and cooperative game theory; this suggestion has come to be known as the Nash Program. Numerous papers have contributed to this program including Rubinstein (1982), Perry and Reny (1994), Pérez-Castrillo (1994), Compte and Jehiel (2009), and Serrano (1995). See Serrano (2008) for a brief survey. Our work also contributes to this program by integrating the two solution concepts.

Another related line of literature has introduced concepts of cooperation within the framework of a non-cooperative game. In particular, strong Nash equilibrium (Aumann (1959)) requires that the strategy choices of players are stable against cooperative deviations by coalitions of players who can write binding agreements. A coalition-proof Nash equilibrium (Bernheim, Peleg, and Whinston (1987)) requires that no coalition of players can improve upon the equilibrium with some joint deviation by the coalition members that is itself immune to deviations by subcoalitions. With the exception of Bernheim, Peleg, and Whinston (1987), which we discuss further in our concluding section, we are not aware of any other papers introducing concepts of cooperation in general extensive games.

In this paper, we propose embedding coalition formation into the extensive form of a game and a notion of subgame perfect cooperation. Underlying our notion is the idea that a coalition becomes a single player; given a game in extensive form with player set N, when a coalition S forms, a new game is created in which the players in S become one single player. Another fundamental idea of our approach is that, at any point in the extensive game, only those players who still have decisions to make can form coalitions and they can only coordinate their decisions from that point onwards. These two ideas are applied to define and study cooperation in extensive games. They can be applied to games of both perfect and imperfect information; in this paper we treat only games of perfect information.

An issue that arises in the treatment of cooperation within coalitions in a non-cooperative game is what is the response of the players in the complementary coalition? In our approach, since a coalition is simply a player in a game derived from the original game, it is appealing to take the remaining set of players as singletons. In this paper, we define a concept of subgame perfect cooperation, that we label the γ-core of an extensive game. Recall that a feasible payoff vector is in the γ-core (Chander (2010)) of a *strategic* game if no coalition can improve upon its payoff by deviating from any strategy profile that generates that payoff vector.[1] The subgame-perfect γ-core of an *extensive* game differs in that, it is the set of feasible payoff vectors such that no coalition can improve upon its payoff by deviating not only at the origin of the game but also at *any decision node* as the game unfolds along the history generated by any strategy profile that leads to that set of feasible payoff vectors. As will be made clear below, the γ-core of an extensive game takes into account interactions of coalitions through the solution concept, analogously to how Nash and subgame-perfect Nash equilibrium take account of interactions of players.

For the game-theoretic results of this paper, introducing subgame-perfect cooperation, we assume transferable utility so that the utility of a coalition becomes the sum of the utilities of the coalition members. As a motivating example, we treat a dynamic game of global warming. We conclude this introduction with some further discussion of the γ-core. Discussion of the related literature is included in the concluding section of the paper.

Observe that the γ-core takes into account opportunities for higher coalitional payoffs that may arise at any point in the game, i.e., a strategy profile that induces a γ-core payoff of an extensive game is immune to deviations by coalitions as the game unfolds along the history generated by the strategy profile. In order to define the γ-core, we must first specify the payoffs that a coalition can credibly obtain at each decision node of the game. Such payoffs are

[1] See also Chander (2007) and, for an earlier form of the γ-core in application, Chander and Tulkens (1997).

determined as follows. A player is *active* in a subgame if some decision node of the subgame is a decision node of that player. Similarly, a coalition is *active* in a subgame if all its members are active in the subgame. Suppose some players who are active in a subgame with origin at x have formed a coalition to coordinate their decisions in all their decision nodes in the subgame.[2] For such a coalition, consider the *induced* subgame which has origin at x and in which the player set consists of the coalition and the remaining individual active players. Since a subgame-perfect Nash equilibrium (SPNE) of an extensive game is a Nash equilibrium of each subgame of the extensive game, a subgame-perfect Nash equilibrium strategy of the coalition in the induced subgame prescribes a play that is optimal for the coalition, given the optimal strategies of the remaining players (i.e. the individual active players). Thus, a subgame-perfect Nash equilibrium payoff of the coalition in the induced subgame is a payoff that the coalition can credibly obtain if the game reaches node x. In fact, this is exactly how the payoff of an active player is determined in a subgame of an extensive game. The only difference here is that one of the players in the induced subgame is a coalition, and therefore the induced subgame is not identical to the original subgame unless the coalition consists of a single player.

Given a subgame, it is straightforward to define the payoff that an active coalition can obtain by deviating in the subgame. Let S be an active coalition in the subgame with origin at x. If S deviates, then its resulting payoff is a SPNE payoff that it obtains in the induced subgame with origin at x and in which the player set consists of S and remaining individual active players. We refer to this payoff simply as a payoff that coalition S can credibly obtain by deviating at decision node x of the game.[3] Note that the payoff to a coalition from a deviation may be different in different subgames in which the coalition is active. Example 1 in the next section illustrates this important fact.

Given the payoffs that coalitions can credibly obtain by deviating at each decision node, the γ-core of an extensive game is defined as the set of feasible payoff vectors with the property that no coalition (including the grand coalition) can credibly obtain a higher payoff by deviating as the game unfolds along the history generated by any strategy profile that leads to that set of feasible payoff vectors. As will become clear below, the γ-core of an extensive game is a

[2] Note that x need not be a decision node of the coalition.

[3] We consider deviations *only* by coalitions that are active at a decision node. An alternative core concept that takes into account deviations by *all* coalitions at each decision node is in our view too strong.

refinement of the γ-core of the strategic form representation of the extensive game in the same sense as the set of subgame-perfect Nash equilibria is a refinement of the set of Nash equilibria of the extensive game.

Proposing a solution concept without assurance about its existence for some general class of games may not be of much interest. Thus, we provide a necessary and sufficient condition for the existence of the γ-core of an extensive game with perfect information. Bondareva (1963) and Shapley (1967) show that the core of a characteristic function game is nonempty if and only if the game is balanced. An obvious extension of the Bondareva–Shapley condition to extensive games may seem to be that the characteristic function game representation of each subgame is balanced. However, we construct below examples to show that this condition is neither sufficient nor necessary. Thus, we introduce what we call the *characteristic function representation* of an extensive game of perfect information. In the characteristic function representation of the extensive game, the payoff of a coalition is equal to the highest subgame perfect payoff that the coalition can credibly obtain at any decision node along any history that leads to a terminal node at which the payoff of the grand coalition is maximized. We show that the extensive game has a nonempty γ-core if and only if the characteristic function representation of the game is balanced.

The paper is organized as follows. Section 2 presents the definition of the γ-core of a *strategic* game. Section 3 introduces the concept of the γ-core of an *extensive* game of perfect information. It also provides a necessary and sufficient condition for the existence of the γ-core of an extensive game. Section 4 introduces the dynamic game formulation of the problem of global warming and shows that if the payoff functions are quadratic, then the γ-core is nonempty. Section 5 further discusses the γ-core and related literature.

2 The γ-core of a strategic game

It is convenient to first take note of the concept of γ-core of a *strategic* game (Chander (2010)).[4] We denote a strategic game with transferable utility by (N, T, u) where $N = (1, \dots, n)$ is the

[4] In contrast, the core concepts introduced by Maskin (2003) and Huang and Sjöström (2006) are based on a partition function. These core concepts, therefore, ignore the strategic interactions that are behind the payoffs of the coalitions. Similarly, the conventional α- and β- cores, by definition, rule out interesting strategic interactions

player set, $T = T_1 \times \cdots \times T_n$ is the set of strategy profiles, T_i is the strategy set of player i, $u = (u_1, \dots, u_n)$ is the vector of payoff functions, and u_i is the payoff function of player i. A strategy profile is denoted by $t = (t_1, \dots, t_n) \in T$. We denote a coalition by S and its complement by $N \backslash S$. Given $t = (t_1, \dots, t_n) \in T$, let $t_S \equiv (t_i)_{i \in S}, t_{-S} \equiv (t_j)_{j \in N \backslash S}$, and $(t_S, t_{-S}) \equiv t = (t_1, \dots, t_n)$.

Given a coalition $S \subset N$, the *induced* strategic game (N^S, T^S, u^S) is defined as follows:

- The player set is $N^S = \{S, (j)_{j \in N \backslash S}\}$, i.e., coalition S and all $j \in N \backslash S$ are the players (thus the game has $n - s + 1$ players[5]);
- The set of strategy profiles is $T^S = T_S \times_{j \in N \backslash S} T_j$ where $T_S = \times_{i \in S} T_i$ is the strategy set of player S and T_j is the strategy set of player $j \in N - S$;
- The vector of payoff functions is $u^S = (u_S^S, (u_j^S)_{j \in N \backslash S})$ where $u_S^S(t_S, t_{-S}) = \sum_{i \in S} u_i\,(t_S, t_{-S})$ is the payoff function of player S and $u_j^S(t_S, t_{-S}) = u_j(t_S, t_{-S})$ is the payoff function of player $j \in N \backslash S$, for all $t_S \in T_S$ and $t_{-S} \in \times_{j \in N \backslash S} T_j$.

Observe that if $(\tilde{t}_S, \tilde{t}_{-S}) = \tilde{t}$ is a Nash equilibrium of the induced game (N^S, T^S, u^S), then $u_S^S(\tilde{t}_S, \tilde{t}_{-S}) = \sum_{i \in S} u_i(\tilde{t}_S, \tilde{t}_{-S}) \geq \sum_{i \in S} u_i\,(t_S, \tilde{t}_{-S})$ for all $t_S \in T_S$.[6] Thus, for each $S \subset N$, a Nash equilibrium of the induced game (N^S, T^S, u^S) assigns a payoff to S that it can obtain without cooperation from the remaining players. If the induced game (N^S, T^S, u^S) has multiple Nash equilibria, then a Nash equilibrium with the highest payoff for S can be selected.[7] In this way, a unique payoff can be assigned to the coalition. (Other selections in the case of multiple equilibria are possible. Our selection of the highest payoff makes the conditions for non-emptiness of the γ-core more stringent.)

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between the players. See Chander (2010) for a comparison of the γ-core of a *strategic* game with the α- and β-cores.

[5] The small letters n and s denote the cardinality of sets N and S, respectively.

[6] Notice that if S is a singleton coalition, a Nash equilibrium of the induced game (N^S, T^S, u^S) is a Nash equilibrium of the game (N, T, u), a strong Nash equilibrium of (N, T, u) is a Nash equilibrium of every induced game $(N^S, T^S, u^S), S \subset N$. Conversely, if a strategy $\bar{t} \in T$ is a Nash equilibrium of every induced game $(N^S, T^S, u^S), S \subset N$, then $\bar{t}$ is a strong Nash equilibrium of the game (N, T, u). The set of strong Nash equilibrium payoffs is not *necessarily* equal to the set of γ-core payoffs.

[7] Such a payoff will surely exist if the strategy sets are compact (or finite) and the payoff functions are continuous.

The γ-characteristic function of a strategic game (N, T, u) is the function $w^{\gamma}(S) = \sum_{i \in S} u_i(\tilde{t}_S, \tilde{t}_{-S})$, $S \subset N$, where $(\tilde{t}_S, \tilde{t}_{-S}) \in T$ is a Nash equilibrium of the induced game (N^S, T^S, u^S) with the highest payoff for coalition S.

The pair (N, w^{γ}) is a *characteristic function* game representation of the strategic game (N, T, u). The γ-core of the strategic game (N, T, u) or, equivalently, the core of the characteristic function game representation (N, w^{γ}) of (N, T, u) is the set of payoff vectors p such that (i) $\sum_{i \in N} p_i = w^{\gamma}(N)$ and (ii) for each $S \subset N, \sum_{i \in S} p_i \geq w^{\gamma}(S)$.[8]

3 The γ-core of an extensive game

We denote an extensive game of perfect information by $\Gamma = (N, K, P, u)$ where $N = \{1, \dots, n\}$ is the player set and K is the game tree with origin denoted by 0. Let Z denote the set of terminal nodes of game tree K and let X denote the set of non-terminal nodes, i.e., the set of decision nodes. The player partition of X is given by $P = \{X_1, \dots, X_n\}$ where X_i is the set of all decision nodes of player $i \in N$. The payoff function is $u: Z \to R^n$ where $u_i(z)$ denotes the payoff of player i at terminal node z. Since there is a one-one correspondence between the game tree and the strategy sets of the players, we do not explicitly state the strategy sets.

3.1 *The induced extensive games*

Given $\Gamma = (N, K, P, u)$ and a coalition $S \subset N$, the *induced* extensive game $\Gamma^S = (N^S, K^S, P^S, u^S)$ is defined as follows:

- The player set is $N^S = \{S, (i)_{i \in N \setminus S}\}$, i.e., coalition S and all $j \in N \setminus S$ are the players (thus the game has $n - s + 1$ players);
- The game tree is $K^S = K$ (thus the set of decision nodes is X);
- The player partition of X is $P^S = \{X_S, (X_i)_{i \in N \setminus S}\}$ where $X_S = \cup_{j \in S} X_j$;
- The profile of payoff functions is $u^S = (u_S^S, (u_i^S)_{i \in N \setminus S})$ where $u_S^S(z) = \sum_{j \in S} u_j(z)$ is the payoff function of S and $u_i^S(z) = u_i(z)$ is the payoff function of $i \in N \setminus S$, for all $z \in Z$.

[8] Note that it is efficient for the grand coalition to form, since the grand coalition can choose at least the same strategies as the players in any coalition structure (i.e. any partition of the total player set into coalitions).

Note that if S is a singleton coalition, then $\Gamma^S = \Gamma$. For each $S \subset N$, the induced game $\Gamma^S = (N^S, K^S, P^S, u^S)$ represents the situation in which the players in S have formed a coalition to coordinate their decisions in all their decision nodes. Example 1 below illustrates the definitions so far.

Example 1 Let Γ be the extensive game depicted in Fig.1. Then, x_1 is the origin of the game tree $K, N = \{1,2\}, Z = \{z_1, z_2, z_3\}, X = \{x_1, x_2\}, P = \{\{x_1\}, \{x_2\}\}$ and $u: Z \to R^2$ is given by $u(z_1) = (2, 1), u(z_2) = (4, 2)$, and $u(z_3) = (1, 3)$.

The induced extensive game Γ^N when players 1 and 2 form a coalition to coordinate their decisions in all their decision nodes is depicted in Fig. 2. The game tree is the same, but now we have a one-player game with player set $\{N\}$. So $P^N = \{\{x_1, x_2\}\}$, $u_N^N(z_1) = 3, u_N^N(z_2) = 6$, and $u_N^N(z_3) = 4$. Notice that each strategy of player N in game Γ^N generates a history of game Γ.

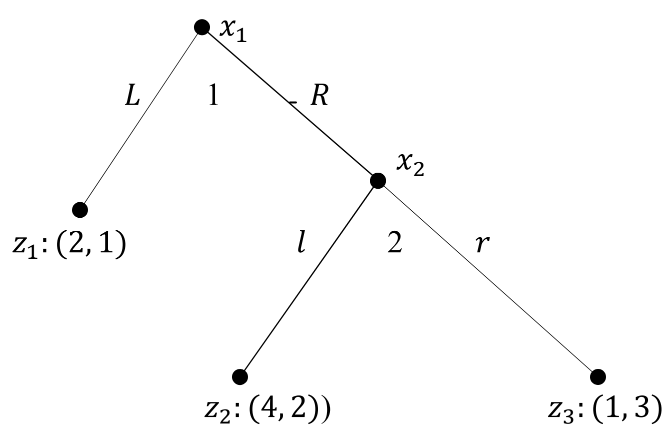

Figure 1

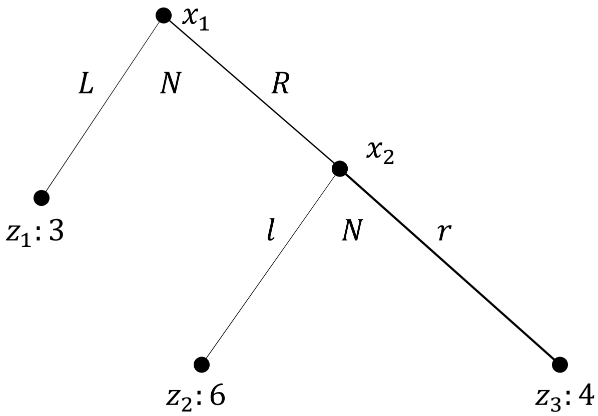

Figure 2

In the following, we will often not distinguish between player i and coalition $\{i\}$. Given $x \in X$, let Γ_x denote the subgame with origin at x. Since the origin of Γ is denoted by 0, $\Gamma_0 = \Gamma$ and if $x \neq 0$, Γ_x is a proper subgame of Γ. Note that the player set of a proper subgame Γ_x may be smaller than the set N (but is not necessarily so). A player is *active* in subgame Γ_x if some decision node in Γ_x is a decision node of the player. Similarly, a coalition is *active* in subgame Γ_x if all its members are active in the subgame Γ_x. Let S be an active coalition in subgame Γ_x. Then, the induced game Γ_x^S is defined from Γ_x in exactly the same way as the induced game Γ^S is defined from Γ. Clearly, $\Gamma_0^S = \Gamma^S$. Since Γ is a game of perfect information, so is each game Γ_x^S, $x \in X$ and S an active coalition in Γ_x. In what follows, it will be often convenient to refer to "a coalition that is active in the subgame with origin at x" simply as "an active coalition at x".

A subgame-perfect Nash equilibrium of an extensive game is a Nash equilibrium of each subgame of the extensive game. Therefore, for each coalition S which is active at x, a SPNE strategy of S in the game Γ_x^S prescribes a play that is optimal for S from point x onwards, given the optimal strategies of the remaining active players. Thus, a SPNE payoff of a coalition S in the induced game Γ_x^S is a payoff that it can credibly obtain if the game reaches node x.

The subgame-perfect Nash equilibria of the family of extensive games Γ_x^S, $x \in X$ and S an active coalition in Γ_x, determine the payoffs that coalitions can credibly obtain at each decision node of the game Γ. If the induced game Γ_x^S has more than one SPNE, then a SPNE with the highest payoff for the coalition can be selected.[9]

Let us use Example 1 again to illustrate the additional definitions introduced. Since the game Γ in Example 1 has only two players, $\Gamma_{x_1}^{\{1\}} = \Gamma_{x_1}^{\{2\}} = \Gamma$. The SPNE payoff of coalition $\{1\}$ in the game $\Gamma_{x_1}^{\{1\}}$ is 2 and its SPNE strategy is L. Similarly, the SPNE payoff of $\{2\}$ in game $\Gamma_{x_1}^{\{2\}}$ is 1 and its SPNE strategy is rR ($\equiv r$ if player 1 plays R). The SPNE payoff of player N in the game $\Gamma^N (= \Gamma_{x_1}^N)$ in Fig. 2 is 6 and its SPNE strategy is (R, lR)($\equiv R$; l if played R). Notice that the SPNE strategy (R, lR) of coalition N is not compatible with the SPNE strategies L and rR of coalitions $\{1\}$ and $\{2\}$, respectively. This plays a crucial role in what follows.

If players 1 and 2 decide to form a coalition, the payoff of the coalition is 6, as implied by the SPNE of $\Gamma_{x_1}^N$. If coalition $\{1\}$ decides to deviate in the beginning of the game, its resulting payoff is 2, as implied by the SPNE of $\Gamma_{x_1}^{\{1\}}$. Similarly, if $\{2\}$ decides to deviate in the beginning of the game, its resulting payoff is 1, as implied by the SPNE of $\Gamma_{x_1}^{\{2\}}$. In sum, the coalitions $\{1\}, \{2\}$ and N which are active at x_1 can obtain payoffs of 2, 1 and 6, respectively. Thus, none of them can improve upon a payoff vector (p_1, p_2) such that $p_1 \geq 2, p_2 \geq 1, p_1 + p_2 = 6$. E. g., given the feasible payoff vector (4, 2), no coalition can obtain a higher payoff by deviating from the grand coalition's strategy (R, lR) in the *beginning* of the game. Yet, we claim that the strategy profile (R, lR) and the feasible payoff vector (4, 2) are not a sensible prediction for the game. That is because if the strategy profile (R, lR) and the feasible payoff vector (4, 2) are implemented, the game would reach node x_2 and therefore the strategy profile (R, lR) and the payoff vector (4, 2) should also be immune to deviations by all active coalitions at x_2. The only active coalition at x_2 is $\{2\}$ and it can obviously obtain a higher payoff of 3 by taking action R once the game reaches

[9] This is obviously not the only way to handle multiplicity of equilibria.

x_2.[10] Thus, the strategy profile (R, lR) is not immune to deviations by all active coalitions along the history generated by the strategy profile unless the payoff vector (p_1, p_2) is such that $p_1 + p_2 = 6, p_1 \geq 2$, and $p_2 \geq 3$.

Some important points emerge from the above discussion of Example 1. First, it demonstrates that the relative bargaining power of coalitions may change as the game unfolds along the history generated by a strategy profile. For instance, coalition $\{2\}$ can obtain a payoff of only 1 by deviating at x_1, but a payoff of 3 by deviating at x_2. That could happen, despite the fact that coalition $\{2\}$ continues with its SPNE strategy, because x_2 is not reached in the history generated by the SPNE of the induced game $\Gamma_{x_1}^{\{2\}}$. In more general terms, this could happen because a SPNE strategy of a coalition is not a SPNE strategy of a proper subcoalition. To conclude, only the feasible payoff vectors which are immune to deviations by all active coalitions not only at the origin of the game but also at all decision nodes along the history leading to the payoff vectors are a sensible prediction of the game.

3.2 *The γ-core of an extensive game*

Given a game $\Gamma = (N, K, P, u)$, let z denote the terminal node of the history generated by a strategy profile. A payoff vector $(p_1, \dots, p_n)$ is *feasible* for the strategy profile if $\sum_{i \in N} p_i = u_N^N(z)$. A payoff vector is *feasible* if it is feasible for some strategy profile. Note that a payoff vector may be feasible for more than one strategy profile and the histories generated by the strategy profiles may be different. By a history leading to the payoff vector $(p_1, \dots, p_n)$ we mean a history with terminal node z such that $u_N^N(z) = \sum_{i \in N} p_i$.

For each $x \in X$, let $w^\gamma(S; x)$ denote the highest subgame-perfect Nash equilibrium payoff of a coalition S which is active at x.[11] For each $x \in X$, we shall refer to the function $w^\gamma(S; x)$, S an active coalition at x, as the γ-characteristic function of the subgame with origin at x. Notice that

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[10] Notice that action R is consistent with the SPNE strategy of $\{2\}$ in the game $\Gamma_{x_1}^{\{1\}}$ and is a SPNE strategy of $\{2\}$ in the game $\Gamma_{x_1}^{\{2\}}$. Yet, the payoff that $\{2\}$ can obtain at x_2 is higher. This could happen because the node x_2 is not reached in the history generated by the SPNE of game $\Gamma_{x_1}^{\{2\}}$.

[11] Such a payoff can obviously be found by backward induction in the induced game Γ_x^S, even with infinite strategy sets if the strategy sets are compact.

the history generated by any strategy profile begins at the origin of game Γ and all coalitions including coalition N are active at least at the origin . Given the payoffs $w^{\gamma}(S;x)$, $x \in X$ and S an active coalition at x, the γ-core consists of feasible payoff vectors with the property that no coalition can improve upon its payoff by deviating not only at the origin of the game but also at any decision node along the histories leading to the feasible payoff vectors.[12]

Definition 1 The γ-core of an extensive game $\Gamma = (N, K, P, u)$ is the set of all feasible payoff vectors $(p_1, \dots, p_n)$ such that for each coalition $S \subset N,\ w^{\gamma}(S;x) \leq \sum_{i \in S} p_i$ for all decision nodes x along the histories generated by the strategy profiles for which the payoff vector $(p_1, \dots, p_n)$ is feasible.[13]

Let $z^* \in Z$ be a terminal node such that $u_N^N(z^*) \geq u_N^N(z)$ for all $z \in Z$. Such a terminal node exists if the extensive game Γ is finite or if the strategy sets are compact and the payoff functions are continuous. Definition 1 implies that the γ-core of the extensive game Γ must be a subset of the set of feasible payoff vectors $(p_1, \dots, p_n)$ such that $\sum_{i \in N} p_i = u_N^N(z^*)$. That is because the origin of the extensive game Γ is a decision node along every history of the game and there are no other feasible payoff vectors that are immune to deviations by coalition N, which is active at least at the origin. Note that Definition 1 takes into account the possibility that the terminal node at which the total payoff $u_N^N(z)$ is highest may not be unique and that the payoffs that coalitions can obtain along the nodes of different histories leading to the highest total payoff may be different.

3.3 *A necessary and sufficient condition for the existence of the γ-core*

Bondareva (1963) and Shapley (1967) show that the core of a characteristic function game is nonempty if the game is balanced. An obvious extension of this sufficient condition to extensive games may seem to be that every characteristic function game $w^{\gamma}(S;x), x \in X,$ is balanced.

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[12] As Example 1 illustrates, the subgame perfect payoffs that a coalition can obtain as the game unfolds along the history generated by a strategy profile may be higher.

[13] Remember that for each coalition S, the function $w^{\gamma}(S,x)$ is defined for only those decision nodes x at which coalition S is active.

However, as examples 2 and 3 below show, this condition is neither sufficient nor necessary for the existence of the γ-core of an extensive game.

Example 2 Let Γ be the game depicted in Fig. 3. Then,

$$w^{\gamma}(\{1,2\};x_1) = 19, \qquad w^{\gamma}(\{1\};x_1) = 8, \qquad w^{\gamma}(\{2\};x_1) = 7.$$

Thus, a γ-core payoff (p_1, p_2) must satisfy the inequalities $p_1 + p_2 = 19, p_1 \geq 8, p_2 \geq 7$. Furthermore, in the subgame with origin at x_2,

$$w^{\gamma}(\{2\};x_2) = 12.$$

Thus, a γ-core payoff vector (p_1, p_2) must also satisfy $p_2 \geq 12$. Since there is no (p_1, p_2) such that $p_1 + p_2 = 19, p_1 \geq 8$, and $p_2 \geq 12$, the γ-core of the game Γ is empty. However, the characteristic function games $w^{\gamma}(S;x_1)$ and $w^{\gamma}(S;x_2)$ have nonempty cores and are therefore balanced. This shows that the γ-core of a game may be empty even if the characteristic function game representations of all its subgames are balanced.

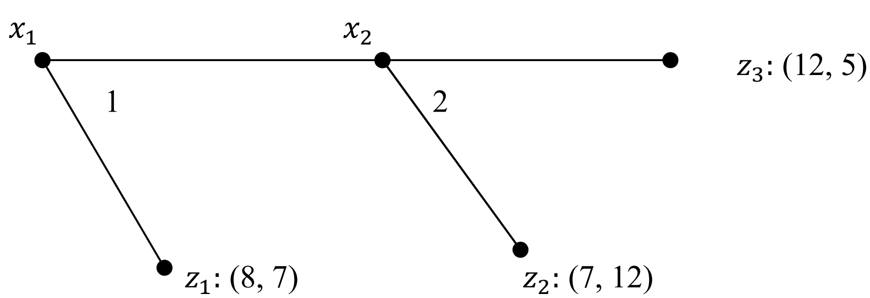

Figure 3

The condition discussed above fails to ensure the existence of the γ-core because it does not fully take in to account the dynamic structure of the game. A sufficient condition should make connections between the characteristic functions of the various subgames. A closer examination of Example 2 suggests the following sufficient condition.

Let $Z^* \subset Z$ be such that if $z^* \in Z^*$, then $u_N^N(z^*) \geq u_N^N(z)$ for all $z \in Z$. Let $X(z^*)$ denote the set of decision nodes along the history that leads to the terminal node z^*. Let $X^* = \cup X(z^*)$ where the union is taken over all $z^* \in Z^*$. For each $S \subset N$, let $w^\gamma(S) = \max_x w^\gamma(S, x)$ where the maximum is taken over all nodes $x \in X^*$at which S is active. Note that the origin $0 \in X^*$ and therefore each coalition S is active at least at one $x \in X^*$. Clearly, $w^\gamma(N) = u_N^N(z^*)$, for all $z^* \in Z^*$. We shall refer to the function $w^\gamma(S)$ as the *characteristic function* of the extensive game Γ. Definition 1 implies that the γ-core of game Γ consists of payoff vectors from the set $\{ (p_1, \dots, p_n) \colon \sum_{i \in N} p_i = u_N^N(z^*)\}$ that are immune to deviations by all active coalitions in the subgames with origin at the nodes in the set X^*. Since the origin $0 \in X^*$ and all coalitions are active at the origin, the subgame perfect payoffs obtainable by all coalitions in the game Γ play a role in determining the initial set of potential γ-core payoff vectors. The subgame perfect payoffs that the coalitions can obtain in the proper subgames with origins at nodes in X^* lead to refinements (as in Example 1) of the initial set of potential γ-core payoff vectors.

Proposition 1 The γ-core of an extensive game Γ is nonempty if and only if the characteristic function game representation (N, w^γ) of Γ is balanced.

<u>Proof</u>: We first prove sufficiency of the condition. Since (N, w^γ) is balanced, the Bondareva-Shapley theorem implies that there exists a payoff vector $(p_1, \dots, p_n)$ such that $\sum_{i \in N} p_i = w^\gamma(N)$ and $\sum_{i \in S} p_i \geq w^\gamma(S)$, $S \subset N$. By definition of game (N, w^γ), for each $S \subset N, w^\gamma(S) \geq w^\gamma(S, x)$ at each decision node $x \in X^*$ at which S is active. Furthermore, $w^\gamma(N) = u_N^N(z^*)$, for all $z^* \in Z^*$. The above inequalities imply that for each $S \subset N, \sum_{i \in S} p_i \geq w^\gamma(S) \geq w^\gamma(S, x)$ at each $x \in X^*$ such that S is active at x, and $\sum_{i \in N} p_i = u_N^N(z^*)$, that is, $(p_1, \dots, p_n)$ is a feasible payoff vector. Hence, $(p_1, \dots, p_n)$ meets all conditions of a payoff vector in the γ-core of Γ. This proves the sufficiency of the condition. .

Next, if the extensive game Γ has a nonempty γ-core, then there exists a feasible payoff vector $(p_1, \dots, p_n)$ such that for each $S \subset N$, $w^\gamma(S; x) \le \sum_{i \in S} p_i$ at each decision node x along the history generated by any strategy profile for which the payoff vector $(p_1, \dots, p_n)$ is feasible. Since the origin 0 is a decision node of the history generated by any strategy profile and coalition N is active at the origin, $\sum_{i \in N} p_i \ge w^\gamma(N, 0)$. Furthermore, since $(p_1, \dots, p_n)$ is a feasible payoff vector, $\sum_{i \in N} p_i \le w^\gamma(N, 0) = w^\gamma(N)$. Thus, $\sum_{i \in N} p_i = w^\gamma(N)$ and $(p_1, \dots, p_n)$ is a feasible payoff vector for any history of the game leading to a $z^* \in Z^*$. Therefore, for each $S \subset N$, $w^\gamma(S; x) \le \sum_{i \in S} p_i$ at each $x \in X^*$. Thus, by definition, $\sum_{i \in S} p_i \ge w^\gamma(S)$ for each $S \subset N$, and the payoff vector $(p_1, \dots, p_n)$ is in the core of the characteristic function game (N, w^γ). Hence, the core of the characteristic function game(N, w^γ) is nonempty, and therefore the game (N, w^γ) is balanced. This proves the necessity of the condition. ■

Note that the necessary and sufficient condition does not imply that the γ-core of each subgame of the extensive game is nonempty. This can be seen from Example 3, which is an extension of Example 2.

Example 3 Let Γ be the game depicted in Fig. 4. Then, in the subgame with origin at x_1,

$$w^\gamma(N; x_1) = 30;$$

$$w^\gamma(\{1,2\}; x_1) = 16, \qquad w^\gamma(\{1,3\}; x_1) = 18, \qquad w^\gamma(\{2,3\}; x_1) = 10;$$

$$w^\gamma(\{1\}; x_1) = 9, \qquad w^\gamma(\{2\}; x_1) = 5, \qquad w^\gamma(\{3\}; x_1) = 5.$$

Thus, a γ-core payoff vector (p_1, p_2, p_3) must satisfy at least the inequalities $p_1 + p_2 + p_3 = 30$, $p_1 + p_2 \ge 16, p_1 + p_3 \ge 18, p_2 + p_3 \ge 10, p_1 \ge 9, p_2 \ge 5, p_3 \ge 5$. Similarly, in the subgame with origin at x_2,

$$w^\gamma(\{2,3\}; x_2) = 19;$$

$$w^\gamma(\{2\}; x_2) = 8, \qquad w^\gamma(\{3\}; x_2) = 7.$$

Finally, in the subgame with origin at x_3, $w^\gamma(\{3\}; x_3) = 12$. It is easily verified that the payoff vector (10, 8, 12) belongs to the γ-core of game Γ. Hence the γ-core of game Γ is nonempty and a characteristic function game representation of Γ is balanced. However, as in Example 2, the γ-core of the subgame with origin at x_2 is empty, as there is no payoff vector (p_2, p_3) such that $p_2 + p_3 = 19, p_2 \geq 8$ and $p_3 \geq 12$.

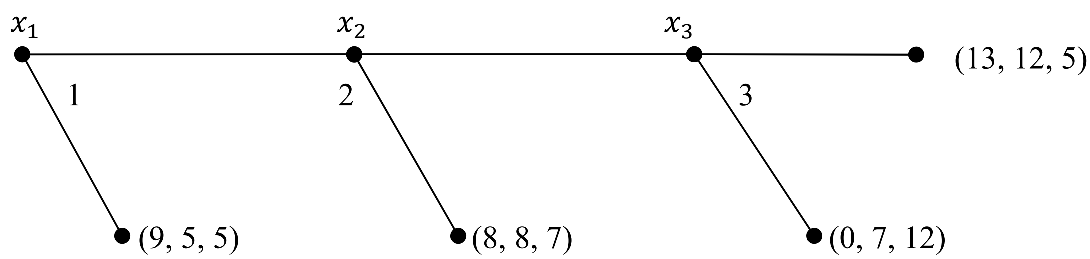

Figure 4

Core concepts in dynamic games have attracted the interest of economists for many years. Douglas Gale (1978) explores the issue of time consistency in the Arrow-Debreu model with dated commodities when agents distrust the forward contracts signed at the first date. He introduces the sequential core which consists of allocations that cannot be improved upon by anyone at any date. Similarly, Forges, Mertens, and Vohra (2002) propose the ex ante incentive compatible core. Becker and Chakrabarti (1995) propose the recursive core as an allocation such that no coalition can improve upon its consumption stream at any time, given its accumulation of assets up to that time.

As an illustration of our core concept, we formulate the problem of global warming as a dynamic game and show that if the payoff functions are quadratic, then the γ-core of the dynamic game is nonempty.[14] We shall make use of the fact that subgame perfect payoffs of coalitions can be found by backward induction. Since the dynamic game involves simultaneous moves and the strategy sets are not finite, the non-emptiness of the γ-core demonstrates that our core concept can be applied to a wider class of games than considered so far.

4 A dynamic game of global warming

There are n countries, indexed by $i = 1, \dots, n$. Time is treated as discrete and indexed $t = 1, \dots, T$, where T is finite. The variables $x_{it} \geq 0$ and $y_{it} \geq 0$ denote the consumption and production, respectively, of a composite private good in country i at time t. Similarly $e_{it} \geq 0$ and $z_t \geq 0$ denote, respectively, the level of emissions and the amount of ambient pollutant at time t. While x_{it}, y_{it} and e_{it} are flow variables, z_t is an accumulating stock as formally defined in equation (3) below.

Production and utility at time t are specified as $y_{it} = g_i(e_{it})$ and $u_i(x_{it}, z_t) = x_{it} - v_i(z_t)$, respectively. The function $g_i(e_{it})$ is the production function and $v_i(z_t)$ is the damage function. A discrete time path $(x_{1t}, \dots, x_{nt}; y_{1t}, \dots, y_{nt}; e_{1t,} \dots, e_{nt}; z_t)_{t=1}^{T}$ is *feasible* if for every $t = 1, \dots, T$,

$$y_{it} = g_i(e_{it}) \quad \text{for all} \quad i \in N, \tag{1}$$

$$\textstyle\sum_{i \in N} x_{it} = \sum_{i \in N} y_{it} \tag{2}$$

$$z_t = (1 - \delta) z_{t-1} + \textstyle\sum_{i \in N} e_{it}, \; z_0 > 0 \text{ given.} \tag{3}$$

Here $0 \leq \delta < 1$ is the natural rate of decay of the stock z_t. Note that transfers of the composite private good are allowed across the countries in each period t, but not across the periods. Given the quasi-linearity of the utility functions $u_i(x_{it}, z_t)$, this is not really an assumption as there is no gain from postponing consumption and there is no possibility of borrowing against the future consumption. Given a feasible path $(x_{1t}, \dots, x_{nt}; y_{1t}, \dots, y_{nt}; e_{1t,} \dots, e_{nt}; z_t)_{t=1}^{T}$, the aggregate utility

[14]Dynamic game formulations of the problem of global warming have been studied previously by Dutta and Radner (2005), Germain, Toint, Tulkens and de Zeeuw (2003), and Dockner and Long (1993) among others. Reinganum and Stokey (1985) study a dynamic game of resource extraction with a similar structure.

of country i is $V_i = \sum_{t=1}^{T} \beta^{t-1} u_i(x_{it}, z_t) = \sum_{t=1}^{T} \beta^{t-1} [x_{it} - v_i(z_t)]$, where $0 < \beta \leq 1$ is the discount factor. In the optimal control literature, the emissions $e_i = (e_{it})_{t=1}^{T}$ are called *control variables* and the resulting stocks $z_t, t = 1, \dots, T$, are the *state variables*. While the latter are not strategies in the dynamic game, they are induced by the former and appear in the payoff functions. In fact, z_t represents a decision node of the dynamic game at time t.

In what follows we assume that the production functions, $g_i(e_{it})$, are strictly increasing and strictly concave, and the damage functions, $v_i(z_t)$, are strictly increasing and convex. In addition we assume that there exists an $e^0 > 0$ such that $g_i'(e^0) < v_i'(e^0)$ for each $i \in N$. The assumption means that beyond a certain level of emissions, e^0, the marginal product of emissions for each country i, $g_i'(e^0)$, is less than the marginal damage, $v_i'(z)$, since the ambient pollution z, according to (3), is at least e^0 if the emissions of country i alone are e^0 and the marginal damage function is non-decreasing. That is, $v_i'(e^0) \leq v_i'(z)$, since $z \geq e^0$. The assumption thus ensures that no country will ever emit more than e^0, i.e., the emissions of the countries are such that $0 \leq e_i \leq e^0$ for each $i \in N$.

We do not formally define the dynamic game associated with the above dynamic model of global warming. Instead, it is sufficient for our purpose and analytically convenient to use a reduced form of each of its subgames starting from the last but one period $T - 1$. The reduced form of the subgame in period t $(= 1, \dots, T - 1)$ with origin at $z_{t-1} > 0$ is the strategic game (N, E, u_t) such that

- $N = \{1, \dots, n\}$ is the set of players,
- $E = E_1 \times \cdots \times E_n$ is the set of joint strategies and $E_i = \{e_{it}: 0 \leq e_{it} \leq e^0\}$ is the set of strategies of player i,
- $u_t = (u_{1t}, \dots, u_{nt})$ is the vector of payoff functions such that for each $e_t = (e_{1t}, \dots, e_{nt}) \in E$, $u_{it}(e_t) = g_i(e_{it}) - v_i(z_t) + p_i(z_t)$, $z_t = (1 - \delta) z_{t-1} + \sum_{j \in N} e_{jt}$.

Here $(p_1(z_t), \dots, p_n(z_t))$ is a γ-core allocation of the reduced form of the subgame in the next period, i.e., the game with origin at $z_t (= (1 - \delta) z_{t-1} + \sum_{j \in N} e_{jt})$. We aim to show that each reduced game has a nonempty γ-core if each function $p_i(z_t)$, like the negative of each damage function, $-v_i(z_t)$, is strictly decreasing and concave.

In order to derive solutions of these subgames analytically, we assume specific functional forms of production and damage functions.[15] In particular, we assume that the production functions $g_i(e_{it})$ are quadratic,

$$g_i(e_{it}) = c_i e_{it} - \frac{1}{2} e_{it}^2, \tag{4}$$

where $c_i > 0$ is sufficiently large, and the damage functions

$$v_i(z_t) = \frac{1}{2} z_t^2. \tag{5}$$

Note that this allows some degree of asymmetry as we do not require $c_i = c_j$, for $i \neq j$.

As noted above, it is possible to find the SPNE payoffs of each coalition and a γ-core allocation of the dynamic game by backward induction. Thus, we begin with the subgame in the last period T with origin at $z_{T-1} > 0$. That is, with the strategic game (N, E, u_T) where

- $N = \{i = 1, 2, \ldots, n\}$ is the set of players,
- $E = E_1 \times \cdots \times E_n$ is the set of joint strategies and $E_i = \{e_{iT}: 0 \leq e_{iT} \leq e^0\}$ is the set of strategies of player i.
- $u_T = (u_{1T}, \ldots, u_{nT})$ is the vector of payoff functions such that $u_{iT}(e_T) = c_i e_{iT} - \frac{1}{2} e_{iT}^2 - \frac{1}{2}[(1-\delta) z_{T-1} + \sum_{j \in N} e_{jT}]^2$, z_{T-1} given.

The last term of the payoff function is decreasing and quadratic in $\sum_{j \in N} e_{jT}$. Let

$$p_i(z_{T-1}) = \frac{1}{2} c_i^2 - \frac{1}{1+n^2}\left(1 + \frac{1}{2}\frac{n^2}{1+n^2}\right)[(1-\delta) z_{T-1} + \sum_{j \in N} c_j]^2, i \in N. \tag{6}$$

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[15] The restriction to specific functional forms is dictated by the fact that only for specific classes of such dynamic games the subgame-perfect Nash equilibria can be derived analytically. Dockner and van Long (1995) do likewise and furthermore restrict to only two identical players.

Note that each $p_i(z_{T-1})$ is quadratic in z_{T-1}. As will be seen below this plays a crucial role in finding a solution of the game in the previous period, i.e., the game (N, E, u_{T-1}).

Proposition 2 The γ-core of the strategic game (N, E, u_T) is nonempty. In particular, the payoff allocation $(p_1(z_{T-1}), \ldots, p_n(z_{T-1}))$ as defined in (6) is feasible and in the γ-core of the strategic game (N, E, u_T) representing the subgame of the dynamic game with origin at z_{T-1}.

Given a γ-core allocation of the subgame in period T, we find by backward induction a γ-core allocation of the subgame in period $T-1$ with origin at, say, z_{T-2}. Substituting $z_{T-1} = (1-\delta)z_{T-2} + \sum_{j\in N} e_{jT-1}$ in equation (6), $p_i(z_{T-1}) = p_i((1-\delta)z_{T-2} + \sum_{j\in N} e_{jT-1})$ is decreasing and quadratic in $\sum_{j\in N} e_{jT-1}$. Then, in the the reduced game (N, E, u_{T-1}) with origin at z_{T-2},

$$u_{iT-1}(e_{T-1}) = c_i e_{iT-1} - \frac{1}{2} e_{iT-1}^2 - \frac{1}{2}[(1-\delta)z_{T-2} + \textstyle\sum_{j\in N} e_{jT-1}]^2 + \beta p_i((1-\delta)z_{T-2} + \textstyle\sum_{j\in N} e_{jT-1}), \qquad (7)$$

for each $i \in N$. Thus, the payoff functions $u_{iT-1}, i \in N$, in game (N, E, u_{T-1}) have exactly the same functional form in terms of the strategies of the players and the stock as the payoff functions $u_{iT}, i \in N$, in game(N, E, u_T). Therefore, as in Proposition 2, the γ-core of the reduced game (N, E, u_{T-1}) is nonempty. In particular, a payoff allocation $\left(p_1(z_{T-2}), \ldots, p_n(z_{T-2})\right)$ belongs to the γ-core of the reduced game (N, E, u_{T-1}) and has exactly the same functional form as the functions $\left(p_1(z_{T-1}), \ldots, p_n(z_{T-1})\right)$ in the game in the next period. By induction, therefore, the γ-core of each of the reduced games $(N, E, u_t), t = 1, \ldots, T$, is nonempty. To prove the non-emptiness of the γ-core of the dynamic game with origin at z_0, it is sufficient to show that the allocation $\left(p_1(z_{T-2}), \ldots, p_n(z_{T-2})\right)$ is in the γ-core of the subgame with origin at z_{T-2}. The rest follows from induction.

Proposition 3 Given $z_{T-2} > 0$, let $(p_1(z_{T-2}), \ldots, p_n(z_{T-2}))$ be a core allocation of the reduced game (N, E, u_{T-1}). Then, $(p_1(z_{T-2}), \ldots, p_n(z_{T-2}))$ is a γ-core allocation of the subgame of the dynamic game with origin at z_{T-2}.

The proofs of both propositions 2 and 3 are relegated to the Appendix at the end of the paper. Both proofs use the fact that the reduced form of each subgame of the dynamic game is a strategic game with a non-empty γ-core.

5. Concluding remarks

The γ-core concept introduced in this paper allows each coalition complete freedom in choosing its strategy. An alternative concept would be to require that a coalition can choose only those strategies that are immune to deviations by subcoalitions of the coalition in the same self-enforcing manner as in a coalition-proof Nash equilibrium (Bernheim, Peleg, and Whinston (1987)) vis-à-vis a strong Nash equilibrium. In the strategic game framework, as in Section 2, that amounts to requiring that for each S, the Nash equilibrium $(\tilde{t}_S, \tilde{t}_{-S})$ of the induced game (N^S, T^S, u^S) is such that $\tilde{t}_S$ is self-enforcing in the component game with the strategies of the complement fixed at $\tilde{t}_{-S}$ and $\sum_{i \in S} u_i(\tilde{t}_S, \tilde{t}_{-S}) \geq \sum_{i \in S} u_i(t_S, \tilde{t}_{-S})$ for all $t_S \in T_S$ that are *self-enforcing* in the component game.[16] The so-defined alternative γ-core is a fully non-cooperative concept, but it is a weaker concept in the sense that the core is larger. That is because under the alternative concept, the payoff assigned to each coalition is lower, for coalition S can always choose $\tilde{t}_S$ even if its strategy is not required to be self-enforcing. The assumption that the players in deviating coalitions can write binding agreements makes the condition for non-emptiness of the γ-core more stringent.

Another alternative concept of core to consider is due to Maskin (2003). Maskin assumes that if a coalition deviates, then the remaining players also form a coalition. Maskin uses a partition function form game to introduce his core concept and thereby ignores the strategic interactions behind the payoffs of the coalitions. However, our approach can be used to extend the basic idea behind Maskin's core concept to an extensive game. More specifically, the induced subgames now have only two players: if S is the set of all active players at a decision node x, then for each $S' \subset S$, the player set of the induced subgame with origin at x consists of $\{S', S - S'\}$. As in the case of γ-core, the highest SPNE payoffs of the induced subgame can be used to assign payoffs to S and S'. However, this raises a new conceptual difficulty. If the induced game at decision

[16] Note that such an equilibrium is not necessarily a coalition-proof Nash equilibrium of the game (N^S, T^S, u^S) .

node x has multiple subgame-perfect Nash equilibria, then which one is to be selected for assigning payoffs to coalitions S' and $S - S'$? If the payoffs are highest for both coalitions at the same SPNE of the subgame, then the choice is clear. But if not, then there does not seem to be any accepted method for choosing between the alternative equilibria.[17] This conceptual difficulty does not arise in the case of γ-core because the subgame-perfect Nash equilibria of the induced game are used to assign payoff to only one coalition S', and therefore one that gives the highest payoff to S' is selected.

Nevertheless, the assumption underlying the γ-characteristic function that the players outside a coalition all remain separate, and do not form a coalition of their own, may seem arbitrary. Such criticism changes implicitly the strategic choices available to players from those of taking different actions to those of forming different coalitions. Such issues have been treated within the theory of endogenous coalition formation (see e.g. Bloch (1996) and Ray and Vohra (1999)). With such considerations in mind, Chander (2007, 2010) formulates the problem of coalition formation as an infinitely repeated game in which the players choose whether to cooperate. He shows that if a coalition forms, then breaking apart into singletons is an equilibrium strategy of the remaining players. Moreover, the γ-core payoff vectors are an outcome of the repeated game. In other words, the γ-core can be interpreted as a solution of a non-cooperative game of coalition formation. We follow Chander in taking the complementary subset of players as individuals, not as coalitions.

Our approach differs from that of Chander (2007, 2010) and other literature in that we consider extensive games and subgame perfection. Our approach rests on the two fundamental ideas discussed in the introduction: coalitions become players and, at the origin of any subgame, only those players who still have decisions to make can become part of a coalition. Possibilities for coalition actions are taken into account through the equilibrium notion – in this, paper, the γ-core.

Appendix

[17] A similar conceptual difficulty arises in Biran and Forges (2010) who study cooperation in Bayesian games.

Proof of Proposition 2 The Nash equilibrium of the game (N, E, u_T) can be found from the first order conditions of maximization. Since the payoff functions u_{iT} are strictly increasing and strictly concave, the first order conditions lead to a unique solution $\bar{e}_T = (\bar{e}_{1T}, \dots, \bar{e}_{nT})$ such that

$$c_i - \bar{e}_{iT} = (1-\delta)z_{T-1} + \sum_{j\in N} \bar{e}_{jT}, i \in N. \quad (8)$$

Similarly, the payoff maximizing cooperative strategy of all players is $e_T^* = (e_{1T}^*, \dots, e_{nT}^*)$ such that

$$c_i - e_{iT}^* = n\left[(1-\delta)z_{T-1} + \sum_{j\in N} e_{jT}^*\right], i \in N. \quad (9)$$

Let $z_T^* = (1-\delta)z_{T-1} + \sum_{j\in N} e_{jT}^*$. Then, $v_i'(z_T^*)/\sum_{j\in N} v_j'(z_T^*) = 1/n$. From the Corollary to Proposition 5 in Chander and Tulkens (1997), it follows that the allocation given by

$$p_i(z_{T-1}) = g_i(\bar{e}_{iT}) - \tfrac{1}{n}\left(\sum_{j\in N}[g_j(\bar{e}_{jT}) - g_j(e_{jT}^*)]\right) - v_i[(1-\delta)z_{T-1} + \sum_{j\in N} e_{jT}^*] \quad (10)$$

is in the γ-core of the game (N, E, u_T). Then, the proof follows by substituting from (4), (5), (8), and (9) and then comparing (10) with (6). ■

Proof of Proposition 3 Given a coalition S and the subgame (of the dynamic game) with origin at z_{T-2}, let $(\tilde{e}_{T-1}, \tilde{e}_T)$ = denote the SPNE between coalition S and the other individual players. Then, the payoff that coalition S can credibly achieve is

$$w^\gamma(S; z_{T-2}) \equiv c_i\tilde{e}_{iT-1} - \tfrac{1}{2}\tilde{e}_{iT-1}^2 - \tfrac{1}{2}[(1-\delta)z_{T-2} + \sum_{j\in N} \tilde{e}_{jT-1}]^2 + \beta\,[c_i\tilde{e}_{iT} - \tfrac{1}{2}\tilde{e}_{iT}^2 - \tfrac{1}{2}[(1-\delta)\left[(1-\delta)z_{T-2} + \sum_{j\in N} \tilde{e}_{jT-1}\right] + \sum_{j\in N} \tilde{e}_{jT}]^2].$$

Let, $\tilde{z}_{T-1} = (1-\delta)z_{T-2} + \sum_{j\in N} \tilde{e}_{jT-1}$. Then, by definition, $\tilde{e}_T$ is a Nash equilibrium between coalition S and the other individual players in the strategic game (N, E, u_T) with origin at $\tilde{z}_{T-1}$. Thus, by definition of the γ-core of the game (N, E, u_T) with origin at $\tilde{z}_{T-1}$,

$$w^\gamma(S; z_{T-2}) \le c_i\tilde{e}_{iT-1} - \tfrac{1}{2}\tilde{e}_{iT-1}^2 - \tfrac{1}{2}[(1-\delta)z_{T-2} + \sum_{j\in N} \tilde{e}_{jT-1}]^2 + \beta \sum_{i\in S} p_i((1-\delta)z_{T-2} + \sum_{j\in N} \tilde{e}_{jT-1}),$$

where $(p_1((1-\delta)z_{T-2} + \sum_{j\in N} \tilde{e}_{jT-1}), \dots, p_n((1-\delta)z_{T-2} + \sum_{j\in N} \tilde{e}_{jT-1}))$ is a γ-core allocation of the subgame in period T. However, since $(p_1(z_{T-2}), \dots, p_n(z_{T-2}))$ is a γ-core allocation of the reduced game (N, E, u_{T-1}) with the payoff function $(u_{1T-1}, \dots, u_{nT-1})$ as defined in (7), the right hand side expression, by definition, cannot be higher than $\sum_{i \in S} p_i(z_{T-2})$. Hence, $w^{\gamma}(S; z_{T-2}) \leq \sum_{i \in S} p_i(z_{T-2})$. ■

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