Unique Equilibrium in Contests with Incomplete Information*

Christian Ewerhart† Federico Quartieri‡

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Abstract. For a large class of contests with incomplete information, it is shown that there exists at most one pure-strategy Nash equilibrium provided that contest success functions are "strictly concave" and cost functions are convex. In the considered class of contests, players may receive multidimensional private signals about strategically relevant aspects of the game, such as the number of contestants, the shape of the contest success function, valuations of the contest prize, cost functions, and financial constraints. Moreover, the state-dependent contest success function may be either continuous or discontinuous at the origin. Our results apply, in particular, to the rent-seeking game.

Keywords Contests · Equilibrium uniqueness · Private information

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†(Corresponding) University of Zurich, Winterthurerstrasse 30, CH-8006 Zurich, Switzerland; e-mail: christian.ewerhart@econ.uzh.ch

‡IULM University, Via Carlo Bo 1, 20143 Milan, Italy; e-mail: Federico.Quartieri@iulm.it

1 Introduction

In an interesting paper, Fey (2008) studies the problem of the existence of a pure-strategy Nash equilibrium in the symmetric two-player lottery contest with uniformly distributed, privately known marginal costs.[1] Fey (2008) conjectures that there is precisely one pure-strategy Nash equilibrium in this Bayesian game. In a subsequent article, Ryvkin (2010) examines a more general class of symmetric contests with independently distributed private costs, allowing for a wider class of contest success functions, for more general probability densities functions, and for more than two players. However, as Ryvkin (2010) notes, the fixed-point techniques used by Fey (2008) and by himself do not allow one to address the issue of equilibrium uniqueness.

In response to this research question, the present paper develops an approach to equilibrium uniqueness in contests that is both simple and general.[2] In fact, our arguments apply to many of the imperfectly discriminating contests of incomplete information that have been studied in the literature.[3] In particular, it is shown that the equilibria considered in Fey (2008) and Ryvkin (2010) are unique.

Our approach rests upon Rosen's (1965) uniqueness argument for concave N-person games with strategy spaces that are convex subsets of some Euclidean space. Rosen (1965) considers the Jacobian matrix J associated

[1] For a formal description of the lottery contest, see Section 4. For an introduction to the theory of contests, see Corchon (2007).

[2] By equilibrium uniqueness, we mean here the existence of at most one pure-strategy Nash equilibrium. The issue of the existence of at least one pure-strategy Nash equilibrium is not examined in the present paper.

[3] An overview of the literature on contests with incomplete information will be given at the end of this section.

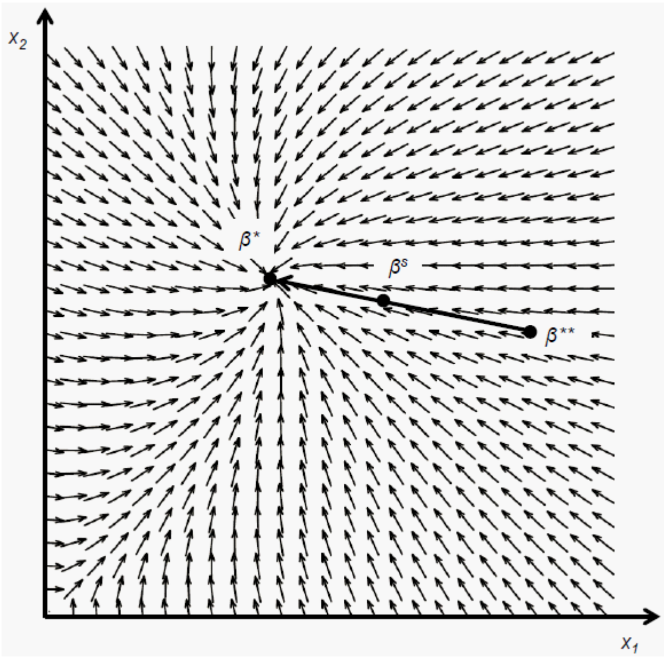

Figure 1: Illustration of Rosen's (1965) argument.

with players' marginal payoff functions, and requires $J + J^T$, i.e., the sum of J and its transpose, to be negative definite at all strategy profiles. To obtain some intuition, consider the pseudogradient associated with the payoff functions in an asymmetric two-player lottery contest. I.e., to each pair of bids $(x_1, x_2) \in \mathbb{R}^2_+ \backslash \{(0,0)\}$, one attaches a vector whose i's component corresponds to player i's marginal payoff, for $i = 1, 2$. Figure 1 shows the corresponding directional field, in which the length of the pseudogradient at each point is normalized to one.[4] At the unique interior equilibrium β^*, the pseudogradient vanishes. Suppose there was another interior equilibrium β^{**} that differs from β^*. Then the scalar product between the pseudogradient and the vector pointing from β^{**} to β^* would have to vanish at both β^* and

[4]In the example drawn, the value of the prize is $v = 1$, and marginal costs are $c_1 = 0.6$ for player 1, and $c_2 = 0.4$ for player 2.

β^{**}. But under Rosen's (1965) condition on the Jacobian, this scalar product turns out to be strictly declining as one moves along the straight line from β^{**} to β^*, which is impossible. The argument works, in fact, equally well for boundary equilibria. Hence, there is at most one equilibrium.

An extension of Rosen's theorem to Bayesian games is obtained by Ui (2004). Imposing Rosen's condition on the Jacobian in each state of the world, Ui (2004) shows that the Bayesian Nash equilibrium is essentially unique, in the sense that any two pure-strategy equilibria in which players maximize ex-ante expected payoffs must induce identical bid profiles in almost all states of the world. Ui (2004) applies his result to Bayesian potential games and team decision problems. However, as will be shown below, Ui's (2004) methods can be extended also to the case of contests.

Our analysis makes progress in five main dimensions. Firstly, it is noted that the condition on the Jacobian need not be imposed on the entire space of strategy profiles, but only on a strict subset thereof. This observation is important because, even with complete information, contests may not satisfy Rosen's condition at all strategy profiles.[5] Secondly, we identify a condition on how valuations may depend on the state of the world and on the players' private information without invalidating the general approach. Thirdly,

[5]For illustration, consider a lottery contest between three players with common valuation $v = 1$, and constant marginal costs. In this example,

$$J + J^T = \begin{pmatrix} -\frac{4(x_2+x_3)}{(x_1+x_2+x_3)^3} & -\frac{2x_3}{(x_1+x_2+x_3)^3} & -\frac{2x_2}{(x_1+x_2+x_3)^3} \\ -\frac{2x_3}{(x_1+x_2+x_3)^3} & -\frac{4(x_1+x_3)}{(x_1+x_2+x_3)^3} & -\frac{2x_1}{(x_1+x_2+x_3)^3} \\ -\frac{2x_2}{(x_1+x_2+x_3)^3} & -\frac{2x_1}{(x_1+x_2+x_3)^3} & -\frac{4(x_1+x_2)}{(x_1+x_2+x_3)^3} \end{pmatrix} \tag{1}$$

for a bid vector $x = (x_1, x_2, x_3)^T \in \mathbb{R}^3_+ \backslash \{(0,0,0)\}$. One notes that the matrix on the right-hand side of (1) is not negative definite for all x. For example, $z^T(J + J^T)z = 0$ for $z = x = (0,0,1)^T$. Hence, Rosen's condition does not hold in this example.

it is noted that the condition on the Jacobian may be replaced, using an argument due to Goodman (1980), by a set of more convenient conditions on the contest success function and the cost functions. Fourthly, we show that a discontinuity of the contest success function at the origin need not interfere with the uniqueness argument. This observation is particularly useful because some of the most popular contests, including the lottery contest, are discontinuous at the origin. Finally, we find a simple condition on the information structure under which a given pure-strategy Nash equilibrium is indeed unique (rather than essentially unique). In fact, that condition even seems to be crucial for uniqueness in the case of discontinuous contests.

Literature on contests with incomplete information. While the problem of equilibrium uniqueness in contests is well-understood in the case of complete information,[6] the existing literature offers only partial results for the case of incomplete information. Hurley and Shogren (1998a) consider a model with one-sided asymmetric information and private valuations. Assuming that the informed player is never discouraged from competing in the contest, they find a unique equilibrium. More generally, Hurley and Shogren (1998b) show that there is at most one interior equilibrium in any two-player lottery contest with private valuations and with two types for one player and three for the other, where types may be correlated. However, the index approach employed in that paper does not provide information about the possibility of boundary equilibria, in which some types would remain inactive (i.e., bid zero). Malueg and Yates (2004), Münster (2009), and Sui

[6] See, in particular, Pérez-Castrillo and Verdier (1992), Szidarovszky and Okuguchi (1997), Nti (1999), Cornes and Hartley (2005), and Yamazaki (2008, 2009).

(2009) study the unique equilibrium in a symmetric two-player lottery contest in which each player may have one of two valuations, and types may be correlated. Schoonbeek and Winkel (2006) characterize the unique equilibrium in an N-player contest with potential inactivity, where one player has private information about her valuation and all other players are identical. Wärneryd (2003, 2010) and Rentschler (2009) find a unique equilibrium in common-value contests between players each of which is either privately informed or completely uninformed. As mentioned above, the papers by Fey (2008) and Ryvkin (2010) allow for continuous and independent distributions of marginal costs, yet do not establish uniqueness. Based on a contraction argument, Wasser (2013a) finds a sufficient condition for uniqueness for the modified lottery contest with heterogeneous continuous distributions of marginal costs. Wasser (2013b) even allows for interdependent valuations and general continuous contest success functions, yet does not discuss uniqueness. Overall, however, as this overview shows, there is a lack of general results on equilibrium uniqueness.[7]

The rest of the paper is structured as follows. Section 2 contains preliminaries. Contests with continuous payoff functions are considered in Section 3. Section 4 deals with contests whose payoff functions are discontinuous at the origin. Section 5 concludes. An Appendix contains technical proofs and lemmas.

[7] Asymmetric information and uncertainty may take many forms in contests. For example, Lagerlöf (2007), Lim and Matros (2009), Münster (2006), and Myerson and Wärneryd (2006) examine the implications of introducing uncertainty about the number of players, whereas Baik and Shogren (1995), Bolle (1996), and Clark (1997) allow for incomplete information about a bias in the contest success function.

2 Preliminaries

This section introduces the basic set-up and our assumption on the information structure.

2.1 Set-up

We consider an N-player contest with incomplete information, where $N \geq 2$. All uncertainty is summarized in a state of the world ω, which is drawn ex ante from a compact Polish state space Ω according to some probability distribution μ on the Borel sets of Ω. Each player $i = 1, ..., N$ observes the realization of a signal or type $\theta_i = y_i(\omega)$, where y_i is a continuous mapping from Ω to some compact Polish space Θ_i. Signals are private information to the respective contestant, i.e., player $i = 1, ..., N$ does not observe the signal θ_j of any other player $j \neq i$. We write ν_i for the probability distribution on Θ_i induced by μ via y_i, for $i = 1, ..., N$.

Based on the private signal θ_i received, each player $i = 1, ..., N$ forms a posterior belief or conditional distribution μ_{i,θ_i} on the Borel sets of Ω,[8] and subsequently submits a bid $x_i \geq 0$, which may of course depend on the signal. For any profile of bids, $x_{-i} = (x_1, ..., x_{i-1}, x_{i+1}, ..., x_N) \in \mathbb{R}_+^{N-1}$, player i's payoff in state $\omega \in \Omega$ is given by $\Pi_i(x_i, x_{-i}, \omega) \equiv p_i(x_i, x_{-i}, \omega) v_i(\omega) - c_i(x_i, \omega)$, where $p_i : \mathbb{R}_+ \times \mathbb{R}_+^{N-1} \times \Omega \to [0, 1]$ is player i's state-dependent contest success function, $v_i : \Omega \to \mathbb{R}_+$ is player i's valuation function, and $c_i : \mathbb{R}_+ \times \Omega \to \mathbb{R}$ is player i's cost function.

We require that $p_0(x, \omega) \equiv 1 - \sum_{i=1}^{N} p_i(x_i, x_{-i}, \omega) \geq 0$, for any $x \in \mathbb{R}_+^N$ and

[8] Since Ω is Polish, posteriors exist. For details, see Kallenberg (1997, Ch. 5).

any $\omega \in \Omega$. Further assumptions on the contest technology will be imposed in Sections 3 and 4.

Our assumptions on the cost functions are as follows.

Convex costs (CC). *For any $i = 1, ..., N$ and any $\omega \in \Omega$, the function $c_i(\cdot, \omega)$ is twice differentiable with $\partial c_i/\partial x_i > 0$ and $\partial^2 c_i/\partial x_i^2 \geq 0$. Moreover, $\partial c_i/\partial x_i$ and $\partial^2 c_i/\partial x_i^2$ are continuous over $\mathbb{R}_+ \times \Omega$, for any $i = 1, ..., N$.*

Note that a player's cost function may depend on the state of the world, rather than only on the player's signal. Thus, costs expected at the time of bidding need not coincide with ex-post cost realizations.

The following assumption will be imposed on players' valuation functions.

Multiplicatively separable valuations (MS). *There is a continuous function $v : \Omega \to \mathbb{R}_{++}$ and, for each player $i = 1, ..., N$, a continuous function $\kappa_i : \Theta_i \to \mathbb{R}_{++}$ such that $v_i(\omega) = v(\omega) \cdot \kappa_i(y_i(\omega))$ for any $\omega \in \Omega$.*

This assumption is flexible enough to encompass the possibility of standard settings with private or common valuations of the contest prize. More specifically, in a private-value setting, $v \equiv 1$, while in a pure common-value setting, $\kappa_i \equiv 1$ for $i = 1, ..., N$. Additional settings are possible. For example, when the model captures an international conflict about the exclusive access to an oil field located under the Northern polar cap, then $v(\omega)$ might correspond to the size of that oil field, and $\kappa_i(y_i(\omega))$ to a country-specific valuation parameter.

Let $x_i^{\max} : \Theta_i \to \mathbb{R}_+$ be a measurable mapping that assigns a maximum bid to each type $\theta_i \in \Theta_i$, for each $i = 1, ..., N$. Note that $x_i^{\max}(\theta_i)$ may be zero, in which case type θ_i is forced to remain inactive. We will assume throughout that the function $x_i^{\max}$ is bounded, i.e., that there is a finite $\overline{x} > 0$ such that $x_i^{\max}(\theta_i) \leq \overline{x}$ for any $i = 1, ..., N$ and any $\theta_i \in \Theta_i$. By a bid function for player i, we mean a measurable mapping $\beta_i : \Theta_i \to \mathbb{R}_+$ such that $\beta_i(\theta_i) \in [0, x_i^{\max}(\theta_i)]$. Denote by B_i the set of all bid functions for player i. For a profile of bid functions $\beta_{-i} = \{\beta_j\}_{j \neq i} \in B_{-i} \equiv \prod_{j \neq i} B_j$, denote by $\beta_{-i}(y_{-i}(\omega)) = \{\beta_j(y_j(\omega))\}_{j \neq i} \in \mathbb{R}_+^{N-1}$ the corresponding profile of bids resulting in state $\omega \in \Omega$. Using this notation, expected payoffs for type $\theta_i \in \Theta_i$ are given by $\overline{\Pi}_i(x_i, \beta_{-i}, \theta_i) \equiv E[\Pi_i(x_i, \beta_{-i}(y_{-i}(\omega)), \omega)|y_i(\omega) = \theta_i]$, where $E[\cdot|y_i(\omega) = \theta_i]$ is the conditional expectation. A pure-strategy Nash equilibrium is then a profile of bid functions $\beta^* = \{\beta_i^*\}_{i=1}^N \in B \equiv \prod_{i=1}^N B_i$, such that $\overline{\Pi}_i(\beta_i^*(\theta_i), \beta_{-i}^*, \theta_i) \geq \overline{\Pi}_i(x_i, \beta_{-i}^*, \theta_i)$ for any $i = 1, ..., N$, any $\theta_i \in \Theta_i$, and any $x_i \in [0, x_i^{\max}(\theta_i)]$.

2.2 Information structure

The following assumption will be imposed on the information structure of the contest.

Absolute continuity (AC). *For any two players $i \neq j$, any $\theta_j \in \Theta_j$, and any ν_i-null set $\mathcal{N}_i \subset \Theta_i$, the set $y_i^{-1}(\mathcal{N}_i)$ is μ_{j,θ_j}-null.*

Intuitively, this assumption says that any set of signal realizations for some player i with prior probability zero has also a zero posterior probability for any player $j \neq i$ conditional on player j having observed any signal $\theta_j \in \Theta_j$. The following lemma validates condition (AC) for a number of informational settings that have been used in the literature.

Lemma 2.1 *Assumption (AC) holds in any of the following informational settings:*

(i) *For any $i = 1, ..., N$, the signal space Θ_i is finite and any signal realization $\theta_i \in \Theta_i$ has a positive probability.*

(ii) *There is a player $i_0 \in \{1, ..., N\}$ such that Θ_j is a singleton for any $j \neq i_0$.*

(iii) *There is a compact non-degenerate interval Ω_0 in some Euclidean space*[9] *such that $\Omega = \Omega_0 \times \Theta_1 \times ... \times \Theta_N$; for any $i = 1, ..., N$, the signal space Θ_i is a compact non-degenerate interval in some Euclidean space; for any $i = 1, ..., N$, the mapping y_i is the canonical projection from Ω to Θ_i; the probability distribution μ allows a positive density f with respect to the Lebesgue measure on Ω.*

Proof. See the Appendix. □

Lemma 2.1 covers, in particular, the cases of finite type distributions with or without correlation (Hurley and Shogren (1998a, 1998b), Malueg and Yates (2004), Schoonbeek and Winkel (2006)), continuous type distributions in which one player is informed about a common value and all others are completely uninformed (Wärneryd (2003), Rentschler (2009)), continuous type distributions with independence (Fey (2008), Ryvkin (2010), Wasser (2013a)), and continuous type distributions with interdependent valuations (Wasser (2013b)). The lemma also covers information structures such as the

[9] I.e., $\Omega_0 = [a_1, b_1] \times ... \times [a_m, b_m]$ for reals $a_1 < b_1, ..., a_m < b_m$, where $m \geq 1$ is the dimension of the Euclidean space.

mineral rights model that have been used in the literature on auctions, but less so in the literature on contests.

3 The uniqueness theorem

Our assumption of "strict concavity" on the contest technology will depend, to some extent, on the domain $S \subseteq \mathbb{R}_+^N$ of bid profiles over which the contest success function is continuous, and also on the domain $S_{-i} \subseteq \mathbb{R}_+^{N-1}$ of bid profiles for the opponents of every player i over which the contest success function is both strictly increasing and strictly concave in the own bid. Initially, we consider contest success functions that are continuous everywhere and both strictly increasing and strictly concave in the own bid regardless of the opponents' bid profile. Therefore, in this section, $S \equiv \mathbb{R}_+^N$ and $S_{-i} \equiv \mathbb{R}_+^{N-1}$ for all $i = 1, ..., N$.

Strictly concave technology (SC). *(i) For any* $i = 1, ..., N$ *and any* $\omega \in \Omega$*, the function* $p_i(\cdot, \cdot, \omega)$ *is twice differentiable on* $\mathbb{R}_+ \times S_{-i}$ *with* $\partial p_i / \partial x_i > 0$ *and* $\partial^2 p_i / \partial x_i^2 < 0$*. Moreover,* $\partial p_i / \partial x_i$ *and* $\partial p_i^2 / \partial x_i \partial x_j$ *are continuous on* $\mathbb{R}_+ \times S_{-i} \times \Omega$*, for any* $i, j = 1, ..., N$*. (ii) For any* $i = 1, ..., N$*, any* $x_i \geq 0$*, and any* $\omega \in \Omega$*, the function* $p_i(x_i, \cdot, \omega)$ *is convex over* S_{-i}*. (iii) The mapping* $p_0(\cdot, \omega)$ *is convex over* S*, for any* $\omega \in \Omega$*.*

In the continuous case, our uniqueness argument is summarized in the following result.

Theorem 3.1 *Impose (CC), (MS), (AC), and (SC). Then the* N*-player contest with incomplete information allows at most one pure-strategy Nash* *equilibrium.*

Proof. By (MS), one may divide each player i's payoff function by $\kappa_i(y_i(\omega)) > 0$ without changing the optimal bid of any $\theta_i \in \Theta_i$, and without affecting the validity of (CC). Hence, w.l.o.g., $v_i \equiv v$ for all $i = 1, ..., N$. Suppose there are two equilibria $\beta^* = (\beta_1^*, ..., \beta_N^*)$ and $\beta^{**} = (\beta_1^{**}, ..., \beta_N^{**})$ with $\beta^* \neq \beta^{**}$. Write $\beta^s = s\beta^* + (1-s)\beta^{**}$ for $s \in [0,1]$. Note that $\beta^1 = \beta^*$ and $\beta^0 = \beta^{**}$. By part (i) of Lemma A.1 in the Appendix, one may define, for any $s \in [0,1]$, the "scalar product"

$$\gamma_s \equiv \sum_{i=1}^{N} E_{\theta_i}\left[\overline{\pi}_i(s,\theta_i)(\beta_i^*(\theta_i) - \beta_i^{**}(\theta_i))\right], \tag{2}$$

where $E_{\theta_i}[\cdot]$ denotes the expectation with respect to ν_i, and $\overline{\pi}_i(s,\theta_i) \equiv \partial\overline{\Pi}_i(\beta_i^s(\theta_i), \beta_{-i}^s, \theta_i)/\partial x_i$. For $s = 0$ and $s = 1$, the necessary Kuhn-Tucker conditions at the equilibrium β^s imply $\beta_i^s(\theta_i) = 0$ if $\overline{\pi}_i(s,\theta_i) < 0$ and $\beta_i^s(\theta_i) = x_i^{\max}(\theta_i)$ if $\overline{\pi}_i(s,\theta_i) > 0$, for any $i = 1, ..., N$ and any $\theta_i \in \Theta_i$. It follows that $\gamma_0 \leq 0$ and $\gamma_1 \geq 0$. Plugging (14) into (2), the law of total expectation yields

$$\gamma_s = E\left[\sum_{i=1}^{N} \pi_i(s,\omega) z_i(\omega)\right] \tag{3}$$

for any $s \in [0,1]$, where $\pi_i(s,\omega) \equiv \partial\Pi_i(\beta_i^s(y_i(\omega)), \beta_{-i}^s(y_{-i}(\omega)), \omega)/\partial x_i$ and $z_i(\omega) \equiv \beta_i^*(y_i(\omega)) - \beta_i^{**}(y_i(\omega))$. We wish to show that $\gamma_1 - \gamma_0 < 0$. Consider some player $i \in \{1, ..., N\}$ and some state $\omega \in \Omega$. Since $\pi_i(\cdot,\omega)$ is continuously differentiable over the unit interval, the fundamental theorem of calculus implies

$$\pi_i(1,\omega) - \pi_i(0,\omega) = \int_0^1 \frac{\partial \pi_i(s,\omega)}{\partial s} ds. \tag{4}$$

Moreover,

$$\frac{\partial \pi_i(s,\omega)}{\partial s} z_i = v \sum_{j=1}^{N} \frac{\partial^2 p_i}{\partial x_j \partial x_i} z_i z_j - \underbrace{\frac{\partial^2 c_i}{\partial x_i^2}}_{\geq 0 \text{ by (CC)}} z_i^2, \tag{5}$$

where the arguments have been dropped on the right-hand side. Combining (3), (4) and (5), one arrives at

$$\gamma_1 - \gamma_0 \leq E\left[\int_0^1 v(\omega) z(\omega)^T J_p(\beta^s(y(\omega)),\omega) z(\omega) ds\right], \tag{6}$$

where $z(\omega) = (z_1(\omega), ..., z_N(\omega))^T$, and $J_p(x,\omega)$ is the $N \times N$ matrix whose elements are $\partial^2 p_i(x_i, x_{-i}, \omega)/\partial x_i \partial x_j$. By part (i) of Lemma A.2, $J_p(x,\omega) + J_p(x,\omega)^T$ is negative definite for any $x \in \mathbb{R}_+^N$ and any $\omega \in \Omega$. Therefore, $z(\omega)^T J_p(x,\omega) z(\omega) = \frac{1}{2} z(\omega)^T (J_p(x,\omega) + J_p(x,\omega)^T) z(\omega) < 0$ for any $x \in \mathbb{R}_+^N$ and for any $\omega \in \Omega$ with $z(\omega) \neq 0$. However, by part (i) of Lemma A.3, there is a player $i \in \{1, ..., N\}$ and a set $\mathcal{P} \subseteq \Omega$ of positive μ-measure such that $z_i(\omega) = \beta_i^*(y_i(\omega)) - \beta_i^{**}(y_i(\omega)) \neq 0$ for any $\omega \in \mathcal{P}$. Hence, inequality (6) implies $\gamma_1 - \gamma_0 < 0$, which is inconsistent with $\gamma_0 \leq 0$ and $\gamma_1 \geq 0$. The contradiction shows that there cannot be two distinct equilibria. □

In particular, Theorem 3.1 offers conditions for uniqueness in a setting with interdependent valuations, as considered by Wasser (2013b).

4 Discontinuous contests

In the popular rent-seeking game of Tullock (1980), the contest success function for player $i = 1, ..., N$ is state-independent, and given by

$$p_i(x_i, x_{-i}, \omega) = \begin{cases} \frac{x_i^R}{x_i^R + \sum_{j \neq i} x_j^R} & \text{if } (x_i, x_{-i}) \neq 0,^{10} \\ \frac{1}{N} & \text{if } (x_i, x_{-i}) = 0, \end{cases} \tag{7}$$

for some $R > 0$. A special case that has found particular attention in the literature is the lottery contest, where $R = 1$. Note that, as the contest success function (7) is discontinuous at the origin, Theorem 3.1 applies neither to the rent-seeking game in general nor to the lottery contest in particular. To nevertheless cover such cases, (SC) will be replaced in this section by a somewhat weaker condition:

Strictly concave technology ($\widetilde{\mathbf{SC}}$). *Properties (i) through (iii) of condition (SC) hold with* $S \equiv \mathbb{R}_+^N \backslash \{0\}$ *and* $S_{-i} \equiv \mathbb{R}_+^{N-1} \backslash \{0\}$ *for all* $i = 1, ..., N$.

Moreover, two new conditions will be added:

Well-behaved singularity (WS). *There is an* $\varepsilon > 0$ *such that* $p_i(x_i, 0, \omega) > p_i(0, 0, \omega) + \varepsilon$ *for any* $i = 1, ..., N$, *any* $x_i > 0$, *and any* $\omega \in \Omega$. *Moreover,* $p_i(\cdot, 0, \omega)$ *is constant over* $\mathbb{R}_{++}$, *for any* $i = 1, ..., N$ *and any* $\omega \in \Omega$.

No minuscule budgets (NM). *There is a* $\delta > 0$ *such that, for any* $i = 1, ..., N$ *and any* $\theta_i \in \Theta_i$, *either* $x_i^{\max}(\theta_i) = 0$ *or* $x_i^{\max}(\theta_i) \geq \delta$.

[10] For convenience, we will henceforth use 0 to denote the origin in Euclidean space, regardless of the dimension.

Condition (WS) concerns a player's probability of winning against a profile consisting exclusively of zero bids. The condition says that, in this case, marginally raising a zero bid enhances the chances of winning in a discontinuous way, and that raising a positive bid does not increase the probability of winning any further. Assumption (NM) says that financial constraints should either exclude a type from the contest altogether or allow a minimum flexibility in bidding.

Clearly, the properties collected in conditions ($\widetilde{\text{SC}}$) and (WS) are motivated by the example of the lottery contest. Indeed, it is straightforward to verify the following result.

Lemma 4.1 *Conditions ($\widetilde{SC}$) and (WS) hold for the lottery contest.*

Proof. See the Appendix. □

This lemma is more useful than it might appear at first glance. For example, in the rent-seeking game with $R < 1$, one may apply Lemma 4.1 to a modified contest in which each bidder i submits a transformed bid $\xi_i = x_i^R$. Similar arguments can be made in the more general case of logit contests (see, e.g., Ryvkin, 2010).

We arrive at the main uniqueness result for contests with payoff functions that are discontinuous at the origin.

Theorem 4.2 *The conclusion of Theorem 3.1 continues to hold when assumption (SC) is replaced by ($\widetilde{SC}$), (WS), and (NM).*

The proof is similar to that of Theorem 3.1, yet taking account of the two complications that, firstly, expected marginal profits for an inactive type may be unbounded off the equilibrium and, secondly, Rosen's condition on the Jacobian need not hold globally.

Proof. Suppose that there are two equilibria β^* and β^{**} with $\beta^* \neq \beta^{**}$, and define β^s as before. Then, by part (ii) of Lemma A.1, for $s = 0$ and $s = 1$, the mapping

$$\widetilde{\varphi}_i(s, \cdot) : \theta_i \mapsto \begin{cases} \overline{\pi}_i(s, \theta_i)(\beta_i^*(\theta_i) - \beta_i^{**}(\theta_i)) & \text{if } x_i^{\max}(\theta_i) > 0 \\ 0 & \text{if } x_i^{\max}(\theta_i) = 0 \end{cases} \tag{8}$$

is integrable over Θ_i. Hence, one may define the modified "scalar product"

$$\widetilde{\gamma}_s \equiv \sum_{i=1}^{N} E_{\theta_i} \left[\widetilde{\varphi}_i(s, \theta_i) \right], \tag{9}$$

where $\widetilde{\gamma}_0 \leq 0$ and $\widetilde{\gamma}_1 \geq 0$, as in the proof of Theorem 3.1. Combining (8), (9), and (14) leads to

$$\widetilde{\gamma}_s = E\left[\sum_{i=1}^{N} \widetilde{\psi}_i(s, \omega) \right] \tag{10}$$

for $s = 0$ and $s = 1$, where

$$\widetilde{\psi}_i(s, \omega) \equiv \begin{cases} \pi_i(s, \omega) z_i(\omega) & \text{if } x_i^{\max}(y_i(\omega)) > 0 \\ 0 & \text{if } x_i^{\max}(y_i(\omega)) = 0. \end{cases} \tag{11}$$

By part (iii) of Lemma A.4, it holds for μ-a.e. $\omega \in \Omega$ that, if $x_i^{\max}(y_i(\omega)) > 0$, then the function $\pi_i(\cdot, \omega)$ is continuously differentiable over the unit interval. If, however, $x_i^{\max}(y_i(\omega)) = 0$, then $z_i(\omega) = 0$. Therefore, as in the proof of

Theorem 3.1,

$$\widetilde{\gamma}_1 - \widetilde{\gamma}_0 \leq E\left[\int_0^1 v(\omega)z(\omega)^T J_p(\beta^s(y(\omega)),\omega)z(\omega)ds\right]. \tag{12}$$

It suffices to show that the right-hand side of (12) is negative. But by part (ii) of Lemma A.3, there are players $i \neq j$ and a set $\mathcal{P} \subseteq \Omega$ of positive μ-measure such that $z_i(\omega) \neq 0$ and $z_j(\omega) \neq 0$ for all $\omega \in \mathcal{P}$. Let $s \in (0,1)$. Since $z_i(\omega) \neq 0$ implies $\beta_i^s(y_i(\omega)) > 0$, and analogously, $z_j(\omega) \neq 0$ implies $\beta_j^s(y_j(\omega)) > 0$, the vector $x = \beta^s(y(\omega)) \equiv (\beta_1^s(y_1(\omega)), ..., \beta_N^s(y_N(\omega)))$ has two or more nonzero entries. Hence, by part (ii) of Lemma A.2, $J_p(x,\omega) + J_p(x,\omega)^T$ is negative definite. Therefore, $z(\omega)^T J_p(\beta^s(y(\omega)),\omega)z(\omega) < 0$ for any $\omega \in \mathcal{P}$. Since $s \in (0,1)$ was arbitrary, the right-hand side of (12) is indeed negative. □

It will be noted that Theorem 4.2 implies, in particular, that the equilibria studied by Fey (2008) and Ryvkin (2010) are unique. It also follows that there is at most one equilibrium in Hurley and Shogren's (1998b) setting with finitely many types for each player, even when allowing for equilibria with inactive types.

5 Concluding remarks

This paper has derived simple conditions for the existence of at most one pure-strategy Nash equilibrium in contests with incomplete information. While, in our view, it makes much sense to conjecture that one-shot contests with strictly concave technologies and convex costs should not cause coordination problems even under asymmetric information, the ultimate generality of the uniqueness result was still somewhat unexpected to us.

The findings of this paper should be desirable for several reasons. For example, in symmetric Bayesian contests, there is often a focus on symmetric equilibria (e.g., Myerson and Wärneryd, 2006). Given that equilibrium uniqueness in a symmetric game trivially implies the symmetry of the unique equilibrium, the present analysis offers a rationale for this approach. Further, uniqueness is a prerequisite for global stability (with respect to any dynamics for which Nash equilibria are stationary points). Finally, uniqueness may simplify comparative statics, revenue comparisons, and numerical analyses. E.g., Brookins and Ryvkin (2013) compare data obtained through laboratory experiments with numerical predictions that are derived under the hypothesis of uniqueness.

However, open questions remain. To start with, the approach developed in the present paper might extend to contests with multi-dimensional efforts and multiple prizes. We have not explored this possibility. Secondly, our results clearly do not apply when the contest technology is not strictly concave. Thirdly, there are settings in which condition (AC) is not satisfied, but the equilibrium is still unique. For example, this is the case for the common-value set-up in Wärneryd (2012), where two or more players are perfectly informed, while all others are completely uninformed. Finally, in Rosen's (1965) original framework, the unique equilibrium is globally stable and effectively computable. Exploring this final point might be particularly interesting.

Appendix

This appendix contains the proofs of Lemmas 2.1 and 4.1, as well as some technical lemmas.

Proof of Lemma 2.1. Fix $i \neq j$, $\theta_j \in \Theta_j$, and let $\mathcal{N}_i \subset \Theta_i$ be ν_i-null, i.e., $\nu_i(\mathcal{N}_i) \equiv \mu(y_i^{-1}(\mathcal{N}_i)) = 0$.

(i) Since any $\theta_i \in \Theta_i$ has a positive ν_i-probability, necessarily $\mathcal{N}_i = \varnothing$. Hence, $y_i^{-1}(\mathcal{N}_i) = \varnothing$ is μ_{j,θ_j}-null.

(ii) Assume first that Θ_i is a singleton. Then, either $y_i^{-1}(\mathcal{N}_i) = \varnothing$ or $y_i^{-1}(\mathcal{N}_i) = \Omega$. But the latter case is impossible because $\mu(y_i^{-1}(\mathcal{N}_i)) = 0$. Hence, $y_i^{-1}(\mathcal{N}_i) = \varnothing$ is μ_{j,θ_j}-null. Assume next that Θ_i is not a singleton. Then, $i = i_0$, and Θ_j is a singleton. Hence, player j's posterior equals the common prior, i.e., $\mu_{j,\theta_j} = \mu$ for the sole signal realization θ_j in Θ_j. Thus, $\mu_{j,\theta_j}(y_i^{-1}(\mathcal{N}_i)) = \mu(y_i^{-1}(\mathcal{N}_i)) = 0$.

(iii) By assumption, μ is equivalent to the Lebesgue measure λ on Ω, hence $y_i^{-1}(\mathcal{N}_i)$ is λ-null. Since y_i is the canonical projection, $y_i^{-1}(\mathcal{N}_i) = \Omega_0 \times \Theta_1 \times ... \times \Theta_{i-1} \times \mathcal{N}_i \times \Theta_{i+1} \times ... \times \Theta_N$ is a cylinder set, hence $\lambda_i(\mathcal{N}_i) = \lambda(y_i^{-1}(\mathcal{N}_i)) = 0$, where λ_i the Lebesgue measure on Θ_i. Moreover, since f is positive, Bayes' rule yields

$$\mu_{j,\theta_j}(y_i^{-1}(\mathcal{N}_i)) = \frac{\int I_{\mathcal{N}_i}(\theta_i) f(\omega_0, \theta_j, \theta_{-j}) d\lambda_{-j}}{\int f(\omega_0, \theta_j, \theta_{-j}) d\lambda_{-j}}, \tag{13}$$

where λ_{-j} denotes the Lebesgue measure on $\Omega_0 \times \Theta_{-j}$, and $I_{\mathcal{N}_i} : \Theta_i \to \{0, 1\}$ is the indicator function associated with the set $\mathcal{N}_i$. But the numerator in (13) vanishes. Hence, $y_i^{-1}(\mathcal{N}_i)$ is μ_{j,θ_j}-null. $\square$

Proof of Lemma 4.1. We check the properties of $(\widetilde{\text{SC}})$ first. Let $x_{-i} \in S_{-i}$. Then, $X_{-i} \equiv \sum_{j \neq i} x_j \neq 0$. Hence, $\frac{\partial}{\partial x_i} \frac{x_i}{x_i + X_{-i}} > 0$ and $\frac{\partial^2}{\partial x_i^2} \frac{x_i}{x_i + X_{-i}} = -\frac{2X_{-i}}{(x_i + X_{-i})^3} < 0$. Moreover, the first and second partial derivatives of p_i are obviously continuous over $\mathbb{R}_+ \times S_{-i} \times \Omega$. This proves property (i). As for property (ii), one notes that for any fixed $x_i \geq 0$, the mapping $X_{-i} \mapsto \frac{x_i}{x_i + X_{-i}}$ is convex over $\mathbb{R}_{++}$, and that the mapping $x_{-i} \mapsto X_{-i}$ is linear. Property (iii) is immediate because $p_0 \equiv 0$ in the lottery contest. As for (WS), it suffices to note that $p_i(0, 0, \omega) = \frac{1}{N} < 1$, while $p_i(x_i, 0, \omega) = 1$ for any $x_i > 0$. $\square$

The technical lemmas below are employed in the proofs of the uniqueness results, Theorems 3.1 and 4.2. The first lemma deals with the differentiability of expected payoffs and with integrability properties of the derivative.

Lemma A.1 **(i)** *Impose (CC) and (SC). Then, for any $s \in [0, 1]$ and any $\theta_i \in \Theta_i$, the derivative $\overline{\pi}_i(s, \theta_i) \equiv \partial \overline{\Pi}_i(\beta_i^s(\theta_i), \beta_{-i}^s, \theta_i)/\partial x_i$ is well-defined and finite, with*

$$\overline{\pi}_i(s, \theta_i) = E\left[\pi_i(s, \omega) | \, y_i(\omega) = \theta_i\right], \tag{14}$$

where $\pi_i(s, \omega) \equiv \partial \Pi_i(\beta_i^s(y_i(\omega)), \beta_{-i}^s(y_{-i}(\omega)), \omega)/\partial x_i$. Moreover, the mapping $\varphi_i(s, \cdot) : \theta_i \mapsto \overline{\pi}_i(s, \theta_i)(\beta_i^*(\theta_i) - \beta_i^{**}(\theta_i))$ is integrable over Θ_i. **(ii)** *Impose (CC), $(\widetilde{SC})$, (WS), and (NM). Then, for $s = 0$ and $s = 1$, and for any $\theta_i \in \Theta_i$ with $x_i^{\max}(\theta_i) > 0$, the derivative $\overline{\pi}_i(s, \theta_i)$ is well-defined and finite, with (14) holding true, where $\pi_i(s, \omega)$ is well-defined and finite for μ_{i,θ_i}-a.e. $\omega \in \Omega$. Moreover, the mapping $\widetilde{\varphi}_i(s, \cdot)$ defined in the proof of Theorem 4.2 is integrable over Θ_i.*

Proof. **(i)** From (CC) and (SC), $\partial \Pi_i / \partial x_i$ is continuous, hence bounded on the compact set $[0, \overline{x}] \times [0, \overline{x}]^{N-1} \times \Omega$. Therefore, by Billingsley (1995, Th. 16.8), $\overline{\pi}_i(s, \theta_i)$ is well-defined, and equation (14) holds. Clearly, $\overline{\pi}_i(s, \theta_i)$ is bounded over Θ_i. Since $\beta_i^*(\theta_i) - \beta_i^{**}(\theta_i)$ is likewise bounded, $\varphi_i(s, \cdot)$ is integrable over Θ_i.

(ii) Assume first that $\beta_i^s(\theta_i) > 0$. Then, for some compact neighborhood $K \subset \mathbb{R}_{++}$ of $\beta_i^s(\theta_i)$, the derivative $\partial \Pi_i / \partial x_i$ is continuous on $K \times [0, \overline{x}]^{N-1} \times \Omega$. Hence, $\overline{\pi}_i(s, \theta_i)$ is well-defined and finite, with (14) holding true, as in part (i) of this lemma. Assume next that $\beta_i^s(\theta_i) = 0$. Then, by part (ii) of Lemma A.4, the event $\beta_{-i}^s(y_{-i}(\omega)) = 0$ is μ_{i,θ_i}-null. Let $\omega \in \Omega$ with $\beta_{-i}^s(y_{-i}(\omega)) \neq 0$. Then, by ($\widetilde{\text{SC}}$), $\Pi_i(\cdot, \beta_{-i}^s(y_{-i}(\omega)), \omega)$ is concave, and differentiable at $\beta_i^s(\theta_i) = 0$. Hence, the difference quotient

$$\Delta^s(x_i, \omega) \equiv \frac{\Pi_i(x_i, \beta_{-i}^s(y_{-i}(\omega)), \omega) - \Pi_i(0, \beta_{-i}^s(y_{-i}(\omega)), \omega)}{x_i} \tag{15}$$

is monotone increasing as $x_i \downarrow 0$, with $\lim_{x_i \downarrow 0} \Delta^s(x_i, \omega) = \pi_i(s, \omega)$. Moreover, since marginal costs are bounded, there is a constant $\overline{c} > 0$ such that $\Delta^s(\overline{x}, \omega) \geq -\overline{c}$ for all $\omega \in \Omega$. By Beppo Levi's theorem, (14) holds. Moreover, from $x_i^{\max}(\theta_i) > 0$ and the equilibrium condition for $s = 0$ and $s = 1$, necessarily $\overline{\pi}_i(s, \theta_i) \leq 0$, so that $\overline{\pi}_i(s, \theta_i)$ is also finite. To prove that $\widetilde{\varphi}_i(s, \cdot)$ is integrable over Θ_i, one notes that $-\overline{c} \leq \overline{\pi}_i(s, \theta_i) \leq 0$ when $\beta_i^s(\theta_i) = 0$. Similarly, by (NM), there is a constant $\overline{p} > 0$ such that $0 \leq \overline{\pi}_i(s, \theta_i) \leq \overline{p}$ whenever $\beta_i^s(\theta_i) = x_i^{\max}(\theta_i) \geq \delta$. Finally, by the first-order condition, $\overline{\pi}_i(s, \theta_i) = 0$ when $\beta_i^s(\theta_i) \in (0, x_i^{\max}(\theta_i))$. Thus, $\overline{\pi}_i(s, \cdot)$ is indeed bounded over the domain where $x_i^{\max}(\theta_i) > 0$. $\square$

The next lemma is a straightforward variant of Goodman's (1980) Lemma.

Lemma A.2 (Goodman, 1980) (i) *Under (SC),* $J_p(x,\omega) + J_p(x,\omega)^T$ *is negative definite for any* $x \in \mathbb{R}^N_+$ *and any* $\omega \in \Omega$. **(ii)** *The conclusion of part (i) remains true if (SC) is replaced by* $(\widetilde{SC})$ *and* x *is required to possess two or more nonzero entries.*

Proof. **(i)** Let H^*, H^{**}, and M^k, with $k = 1, ..., N$, be the $N \times N$ matrices whose respective elements are $h^*_{ij} = \sum_{k=1}^N \frac{\partial^2 p_k(x_k, x_{-k}, \omega)}{\partial x_i \partial x_j} = -\frac{\partial^2 p_0(x,\omega)}{\partial x_i \partial x_j}$, $h^{**}_{ij} = \frac{\partial^2 p_i(x_i, x_{-i}, \omega)}{\partial x_i^2}$ for $i = j$ and $h^{**}_{ij} = 0$ otherwise, and $m^k_{ij} = \frac{\partial^2 p_k(x_k, x_{-k}, \omega)}{\partial x_i \partial x_j}$ for $k \neq i, j$ and $m^k_{ij} = 0$ otherwise. Then H^* is negative semidefinite, H^{**} is negative definite, and each M^k is positive semidefinite. Hence, $J_p(x,\omega) + J_p(x,\omega)^T = H^* + H^{**} - \sum_{k=1}^N M^k$ is negative definite. **(ii)** If $x \in \mathbb{R}^N_+$ has two or more nonzero entries, then $x_{-i} \neq 0$ for all $i = 1, ..., N$, so that the proof proceeds as before. □

The following lemma says that when the assumption of absolute continuity holds and expected payoffs are strictly concave w.r.t. the own bid, then any two distinct equilibria must differ in a "substantial" way.

Lemma A.3 (i) *Impose (CC), (AC), and (SC). Suppose there are two equilibria* β^* *and* β^{**} *with* $\beta^* \neq \beta^{**}$. *Then there exist two players* $i \neq j$ *and a set* $\mathcal{P} \subseteq \Omega$ *of positive* μ*-measure such that* $\beta^*_i(y_i(\omega)) \neq \beta^{**}_i(y_i(\omega))$ *and* $\beta^*_j(y_j(\omega)) \neq \beta^{**}_j(y_j(\omega))$ *for all* $\omega \in \mathcal{P}$. **(ii)** *The conclusion of part (i) remains true if (SC) is replaced by* $(\widetilde{SC})$ *and (WS).*

Proof. **(i)** By contradiction. Write $\mathcal{N}_j = \{\theta_j \in \Theta_j | \beta^*_j(\theta_j) \neq \beta^{**}_j(\theta_j)\}$, and suppose that there exists some player $i \in \{1, ..., N\}$ such that $y_j^{-1}(\mathcal{N}_j)$ is μ-null for any $j \neq i$. Fix some $\theta_i \in \Theta_i$ for the moment. Then, by

(AC), $y_j^{-1}(\mathcal{N}_j)$ is μ_{i,θ_i}-null for any $j \neq i$. Hence, also $\bigcup_{j\neq i} y_j^{-1}(\mathcal{N}_j) = \{\omega \in \Omega | \beta^*_{-i}(y_j(\omega)) \neq \beta^{**}_{-i}(y_j(\omega))\}$ is μ_{i,θ_i}-null. Thus, $\overline{\Pi}_i(\cdot, \beta^*_{-i}, \theta_i) = \overline{\Pi}_i(\cdot, \beta^{**}_{-i}, \theta_i)$. By (CC) and (SC), $\overline{\Pi}_i(\cdot, \beta^*_{-i}, \theta_i)$ is an integral over strictly concave functions, hence strictly concave. Hence, from the equilibrium condition, $\beta^*_i(\theta_i) = \beta^{**}_i(\theta_i)$. Since $\theta_i \in \Theta_i$ was arbitrary, $\beta^*_i = \beta^{**}_i$. Repeating the argument with i replaced by any $j \neq i$ shows that, in fact, $\beta^* = \beta^{**}$.

(ii) By part (i) of Lemma A.4, $\beta^*_{-i}(y_{-i}(\omega)) \neq 0$ is never a μ_{i,θ_i}-null event. Hence, $\overline{\Pi}_i(\cdot, \beta^*_{-i}, \theta_i)$ is strictly concave, and the proof proceeds as before. □

The final lemma is used in the proofs of Lemmas A.1 and A.3, and also in the proof of Theorem 4.2.

Lemma A.4 *Impose (CC), ($\widetilde{SC}$), and (WS), and let $i \in \{1, ..., N\}$.* **(i)** *For any $\theta_i \in \Theta_i$ with $x_i^{\max}(\theta_i) > 0$, the event $\beta^*_{-i}(y_{-i}(\omega)) \neq 0$ is not μ_{i,θ_i}-null.* **(ii)** *For any $\theta_i \in \Theta_i$ with $x_i^{\max}(\theta_i) > 0$ and $\beta^*_i(\theta_i) = 0$, the event $\beta^*_{-i}(y_{-i}(\omega)) = 0$ is μ_{i,θ_i}-null.* **(iii)** *There is a μ-null set $\mathcal{N}_i \subset \Omega$ such that for any $\omega \in \Omega \backslash \mathcal{N}_i$ with $z_i(\omega) \neq 0$, we have $(\beta^s_i(y_i(\omega)), \beta^s_{-i}(y_{-i}(\omega))) \neq 0$ for any $s \in [0, 1]$.*

Proof. **(i)** By contradiction. Suppose that the event $\beta^*_{-i}(y_{-i}(\omega)) \neq 0$ is μ_{i,θ_i}-null. Then, $\overline{\Pi}_i(\cdot, \beta^*_{-i}, \theta_i)$ is strictly decreasing over $\mathbb{R}_{++}$ by (WS) and (CC). However, from $x_i^{\max}(\theta_i) > 0$ and (WS), $x_i = 0$ cannot be a maximizer of $\overline{\Pi}_i(\cdot, \beta^*_{-i}, \theta_i)$. Therefore, β^* cannot be an equilibrium.

(ii) Suppose that the event $\beta^*_{-i}(y_{-i}(\omega)) = 0$ is not μ_{i,θ_i}-null. Then $\overline{\Pi}_i(\cdot, \beta^*_{-i}, \theta_i)$ jumps up at $x_i = 0$ by (CC), ($\widetilde{\text{SC}}$) and (WS). Moreover, $x_i^{\max}(\theta_i) > 0$. Hence, $\beta^*_i(\theta_i) > 0$.

(iii) For $s \in [0,1]$, write

$$\mathcal{N}_i^s = \{\omega \in \Omega | x_i^{\max}(y_i(\omega)) > 0 \text{ and } (\beta_i^s(y_i(\omega)), \beta_{-i}^s(y_{-i}(\omega))) = 0\}. \quad (16)$$

By the law of total probability, $\mu(\mathcal{N}_i^0) = E_{\theta_i}[\mu_{i,\theta_i}(\mathcal{N}_i^0)]$. But by part (ii) of this lemma, $\mathcal{N}_i^0$ is μ_{i,θ_i}-null for any $\theta_i \in \Theta_i$ with $x_i^{\max}(\theta_i) > 0$. Hence, $\mathcal{N}_i^0$ is μ-null. By analogy, $\mu(\mathcal{N}_i^1) = 0$. But $\mathcal{N}_i^s = \mathcal{N}_i \equiv \mathcal{N}_i^0 \cap \mathcal{N}_i^1$ for any $s \in (0,1)$, which proves the assertion. □

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