THE CENTRALISED SOLUTION OF THE UZAWA-LUCAS MODEL WITH EXTERNALITIES

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The paper solves the centralised version of the Uzawa-Lucas model when an externality is present. By means of a transformation that uses the average output per unit of physical capital and the consumption to physical capital ratio, the paper gets information on the characteristics of the transitional dynamics of the model, both locally and (exceptionally for this kind of very complex models) globally. It is shown that the equation of motion of the average output per unit of physical capital can be easily solved in closed form to generate a logistic function converging to the steady state. Furthermore, also the equation of motion for the consumption over physical capital ratio can be explicitly solved. Then, all the equations of motion of the original variables in the model can be recursively solved.

1. Introduction

After Romer [10],[1] Lucas [7] has greatly contributed to the debate on the ultimate determinants of the growth process. This author, following the literature on the role of human capital, particularly Uzawa [15], has developed a two sector model of economic growth with externalities that explains the evolution of economic systems without any intervention of exogenous factors.[2]

In fact, the Uzawa-Lucas model has become very popular in the literature. One of its characteristics is that it is a two sector model, with two production functions devoted respectively to produce physical and human capital. Agents are homogeneous and have the same level of work qualification and expertise (human capital). Moreover, they devolve a fraction of non leisure time to produce the final good, and dedicate the remaining to training and studying. The first sector produces a consumption good by mean of a constant returns to scale Cobb-Douglas technology, whereas the second one produces human capital. The main peculiarity of the model stays in the functional form of the technology used in the second sector. In fact, in this sector, a linear technology is specified in which physical capital does not appear.

Another important aspect worth stressing concerns the presence of externalities that introduces a wedge between the return on capital as it is perceived by the representative agent, and the return on capital from a social point of view. The relevance of this difference is twofold. First of all, one of the usual results of a two sectors growth model is no more obvious, as long as it is no more guaranteed that the optimal trajectories monotonically converge towards the steady state equilibrium along the saddle path in the phase diagram. Indeed, as shown in the recent literature (Benhabib-Perli [2]), when some form of market imperfection is introduced in this kind of models, a *continuum* of different equilibrium trajectories, all of them attracted by the same steady state, may easily emerge. This fact implies that two economies, with the same initial conditions in terms of state variables, will asymptotically have the same growth rate; however, the transitional dynamics, and hence the steady state level of the variables, will depend upon the initial values selected for the control variables.

Secondly, the presence of an externality lead us to distinguish between "decentralised" or market solution, and "centralised" or social planner solution. As a matter of fact, when a distortion

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[1] See also Romer D. [11], p. 116 and f., and Barro R. J.-Sala i Martin X. [1], p. 146 and f.

[2] Besides the pioneering contribution of Solow R. [12], the most known models of exogenous growth are Cass D. [3] and Koopmans T. C. [6].

due to such an externality occurs, the market equilibrium solution differs from the optimal solution that would be chosen by an hypothetical social planner. According with the most recent results on endogenous growth theory, this work acknowledges the crucial importance, usually neglected in the literature, of studying the centralised solution of the model when the externality is not zero.

The paper is organised as follows. In section 2, the standard Uzawa-Lucas model and its decentralised solution is summarised. In section 3, the original system of four differential equations is reduced into a three equations system, by means of an algebraic transformation of the variables involved in the model. The new variables we introduce are two ratios: the physical to human capital ratio and the consumption to physical capital ratio. It can be shown, in fact, that these ratios stay constant along the balanced growth path (BGP henceforth). The BGP properties can be studied by using the Jacobian matrix of the model. Indeed, the eigenvalues signs of this matrix reveal that a stable branch exists, that brings the system to the long-run equilibrium.

In section 4, the system is further reduced into two differential equations in two unknowns, the first one being the average output per unit of physical capital, while the second is, once again, the consumption to physical capital ratio. It will be demonstrated that this system has the same properties as the previous one in R^3. Anyway, while the system in R^2 is surely less informative than those in R^3 and R^4 (for instance, in R^2 the difference between state and control variables is no more distinguishable), at the same time its dynamics can be represented in a phase diagram, which shows the trajectories of the involved variables.

In section 5, it is suggested how the explicit solution of the reduced two-dimensional model with externalities can be obtained. This solution, never presented until now in the literature on the Uzawa-Lucas model, represents the most original and innovative contribution of this work. Some policy implications of this solution are also illustrated at the end of the section. Finally, section 6 contains some brief conclusions.

2. The centralised solution of the model

2.1. The standard Uzawa-Lucas model[3]

In the Uzawa-Lucas model, two economic sectors are considered: in the first one, output is produced by means of physical capital and human capital, whereas in the second sector human capital is produced by means of human capital. This means that the production of human capital involves no physical capital. The model maintains the assumption that education is relatively intensive in human capital.

In this model, to get endogenous growth it is not necessary that any externality be present. As a matter of fact, endogenous growth is guaranteed by the fact that human capital has a constant marginal productivity.

To each sector applies a distinct production function, which both exhibit constant returns to scale. Thus, in steady state, the rates of return remain constant, and the economy can grow at a constant rate. We define the production function in the physical sector as follows:

$$Y = AK^{a} (uhL)^{1-a} \tag{1}$$

where Y is the level of output, A is the constant technological level of the economy, K is physical capital, L is labour, which is kept constant, u (with $0<u<1$) is the fraction of labour time devolved to

[3] A complete version of the Uzawa-Lucas model is presented in Barro R. J.-Sala i Martin X. [1]. Useful comments are also found in Solow R. [13, 14].

produce output, so (1-u) is the fraction of labour time devolved to produce human capital, h is an index of the average quality of labour, that is the schooling level of workers, therefore hL is the quantity of labour measured in efficiency units, and finally uhL is the quantity of labour expressed in efficiency units employed to produce output.

Even if the model could exhibit endogenous growth without the help of any externality, in fact Lucas introduces this in the form of the average human capital of the working force. In other words, such an externality can be defined as the average schooling level of workers. Therefore, the production function in the first sector is completed as follows:

$$Y = AK^{a} (uhL)^{1-a} h_a^{b} \tag{2}$$

where $h_a{}^{b}$ is a measure of the externality.

Dividing by L, equation (2) takes the following form:

$$y = Ak^{a} (uh)^{1-a} h_a^{b} \tag{3}$$

The physical capital accumulation function defines net investment, which is given, ignoring depreciation, by the difference between production and consumption, that is:

$$\dot{K} = AK^{a} (uhL)^{1-a} h_a^{b} - C \tag{4}$$

or, considering that L is constant, we can write (4) in per capita terms:

$$\dot{k} = Ak^{a} (uh)^{1-a} h_a^{b} - c \tag{5}$$

where the coherence of the model requires that $h_a = h$.

In the second sector, the human capital accumulation function is:

$$\dot{h} = fh(1-u) \tag{6}$$

where f is a constant that indicates schooling productivity. So, marginal productivity of human capital is constant and it does not depend on the level of human capital already accumulated. In fact:

$$\frac{\dot{h}}{h} = f(1-u) \tag{7}$$

that is to say, human capital growth rate is given by its own productivity, which is constant in steady state. As a consequence, also human capital growth rate is constant.

2.2. The maximisation problem

We can, therefore, define the constrained maximisation problem to be solved. The consumer maximises an isoelastic utility function (CIES) taking into account the two constraints given by $\dot{k}$ and $\dot{h}$, with given initial level for k and h. We should recall that, with an isoelastic utility function, the elasticity of marginal utility with respect to consumption (call it s) is constant, therefore it does not depend on the consumption level. Such a constant is given by the inverse of the instantaneous elasticity of substitution, which in turn is given by $E = -\frac{U^{'}(c)}{cU^{''}(c)} = \frac{1}{s}$.

The maximisation problem is then specified as follows:

$$\text{Max } U_0 = \int_0^{\infty} \frac{c_t^{1-s} - 1}{1-s} e^{-rt} \quad (8)$$

subject to:

$$\dot{k} = Ak^a u^{1-a} h^{1-a+b} - c$$

$$\dot{h} = fh(1-u)$$

$k(0) = k_0$ given, $h(0) = h_0$ given, k, h, c>0 $\forall\, t$

where r is the instantaneous discount rate.

In this subsection, we derive the centralised solution of the model, when the central planner takes into account the externality. As for such an externality to be coherent with respect to *h*, the production function must be written as follows:

$$y = Ak^a u^{1-a} h^{1-a+b} \quad (9)$$

which is the form used in the physical capital accumulation function above.

To solve this problem, we need to define the Hamiltonian function:

$$H = \left[\frac{c^{1-s} - 1}{1-s}\right] + l_1[Ak^a u^{1-a} h^{1-a+b} - c] + l_2[hf(1-u)] \quad (10)$$

where l_1 and l_2 are the two costate variables or auxiliary variables. From (10) it is possible to get the following first order conditions for a maximum:

$$H_c = 0 \quad \Rightarrow \quad c^{-s} = l_1 \quad (11)$$

$$H_u = 0 \quad \Rightarrow \quad l_1[Ak^a (1-a) u^{-a} h^{1+b-a}] - l_2 hf = 0 \quad (12)$$

$$H_k = -\dot{l}_1 + r\,l_1 \quad \Rightarrow \quad l_1\left[aAk^{a-1}u^{1-a}h^{1-a+b}\right] = -\dot{l}_1 + r\,l_1 \quad (13)$$

$$H_h = -\dot{l}_2 + r\,l_2 \quad \Rightarrow \quad l_1\left[(1-a+b)Ak^a u^{1-a} h^{b-a}\right] + l_2\left[f(1-u)\right] = -\dot{l}_2 + r\,l_2 \quad (14)$$

$$H_{l_1} = \dot{k} \quad \Rightarrow \quad \dot{k} = Ak^a u^{1-a} h^{1-a+b} - c \quad (15)$$

$$H_{l_2} = \dot{h} \quad \Rightarrow \quad \dot{h} = fh(1-u) \quad (16)$$

with the transversality condition:

$$\lim_{t\to\infty} e^{-rt}\left[l_1 k + l_2 h\right]=0 \qquad (17)$$

2.3. The Balanced Growth Path (BGP) analysis

From the system given by equations (11)-(17), it is possible to obtain the growth rates of the variables along the balanced growth path, that is the growth rates for *c*, *k*, *h* and *u*:

$$\frac{\dot{c}}{c}=\frac{1}{s}\left[aAk^{1-a}u^{1-a}h^{1-a+b}-r\right] \qquad (18)$$

$$\frac{\dot{k}}{k}=Ak^{a-1}u^{1-a}h^{1-a+b}-\frac{c}{k} \qquad (19)$$

$$\frac{\dot{h}}{h}=f(1-u) \qquad (20)$$

$$\frac{\dot{u}}{u}=\frac{f(1-a+b)}{a}-\frac{c}{k}+\frac{f(1-a+b)}{1-a}u \qquad (21)$$

where the growth rate of *u* is obtained applying logs and deriving with respect to time equation (12).

Now, we look for the existence of optimal growth paths for the variables involved in the model. To do this, first of all we need to fix the limits of the parametric space such that $u^*\in(0,1)$. The value of u^* along its balanced growth path is given by:

$$u^*=1-\frac{(1-a)(f-r)+fb}{fs(1-a+b)} \qquad (22)$$

In order to guarantee that this expression for u^* be bounded between zero and one, we need to study the following two inequalities jointly:

$$\frac{(1-a)(f-r)+fb}{fs(1-a+b)}<1 \qquad (23)$$

$$\frac{(1-a)(f-r)+fb}{fs(1-a+b)}>0 \qquad (24)$$

Thus, we obtain the parametric space r and s, given by:

$$r\in(0,fy) \qquad (25)$$

and

$$s>1-\frac{r}{fy} \qquad (26)$$

where $y = \frac{1-a+b}{1-a}$.

Once the parametric space has been defined, it is possible to find the steady state value of the common growth rate for consumption, output, and physical capital. Such a growth rate is given by:

$$g_y = \frac{f-r}{s} + \frac{f\,b}{(1-a)s} \qquad (27)$$

In addition, in steady state we have:

$$g_y = a g_k + (1-a+b) g_h = y g_h \qquad (28)$$

from which, since in steady state both consumption and physical capital grow at the same rate as output, it is also possible to obtain the following growth rate for human capital:

$$g_h = \frac{(1-a)(f-r)+b f}{s(1-a+b)} \qquad (29)$$

which is different from (27), that is from the common growth rate of consumption, output and physical capital.

The difference between the two growth rates is due to the externality. If this is zero (b=0), the growth rate of human capital is exactly the same as the common growth rate of the other variables, and this common value collapses to:

$$g = \frac{(f-r)}{s} \qquad (30)$$

3. The reduction of the dynamic system in three dimensions

3.1. The transitional dynamics in R^3

In order to reduce the original system of four differential equations in four unknowns into a three equations system in three unknowns, we need to define two new variables, obtained by the following ratios between the original variables:

$$p = kh^{\frac{(1-a+b)}{(a-1)}} \qquad (31)$$

$$q = \frac{c}{k} \qquad (32)$$

In steady state, c and k grow at the same rate, hence their ratio stays constant. As for h, on the contrary, we know that its growth rate, when the externality is taken into account, is lower than k's: in fact g_h is equal to g_k multiplied by $\frac{(1-a)}{(1-a+b)}$.

It follows that, in order to keep the ratio between these two variables constant, h must be raised to the power $\frac{(1-a+b)}{(a-1)}$. It is obvious that, if $b=0$, the last expression reduces to –1, therefore the ratio between the two variables simply becomes $\frac{k}{h}$. This ratio stays constant because, in this case, both variables grow at the same common rate.

Using these two new variables, we can reduce by one the dimension of the original system, which was formed by four differential equations. The new system can be specified as follows:

$$g_p = Ap^{a-1}u^{1-a} - q - f\frac{1-a+b}{1-a}(1-u) \tag{33}$$

$$g_u = \frac{f(1-a+b)}{a} - q + \frac{f(1-a+b)}{1-a}u \tag{34}$$

$$g_q = q + A\frac{a-s}{s}p^{a-1}u^{1-a} - \frac{r}{s} \tag{35}$$

Solving this system, we obtain the steady state values of p and q, plus the value of u^* just determined in section 2.3. by equation (22). The growth rates of p and q must be zero in steady state, being them defined as ratios between variables that are growing at the same rate.

Putting the three equations equal to zero, we then obtain the following steady state values for p and q:

$$p^* = \left[\frac{fy}{aA}\right]^{\frac{1}{a-1}} u^* \tag{36}$$

$$q^* = \frac{f(1-a+b)}{a} + fy\, u^* \tag{37}$$

Substituting for u^* its steady state value given by (22), we find the following value for q^*:

$$q^* = \frac{r}{s} - \frac{fyj}{a} \tag{38}$$

where $j = \frac{a-s}{s}$.

3.2. Local stability analysis of the steady state

In order to analyse the local stability properties of the fixed point generated by the system (33)–(35),[4] it is necessary to evaluate the Jacobian matrix J at the stationary point. The computation

[4] It is worth noticing that in R^3 the BGP becomes a point in the phase space.

of the eigenvalues of this matrix, obtained applying standard mathematical theorems, yields very interesting information on the movement of the variables in a neighbourhood of the steady state.

In the dynamical system generated by the centralised solution of the Uzawa-Lucas model, we find that:

$$J\left(p^*,u^*,q^*\right)=\begin{bmatrix} \frac{\partial \dot{p}_t}{\partial p_t} & \frac{\partial \dot{p}_t}{\partial u_t} & \frac{\partial \dot{p}_t}{\partial q_t} \\ \frac{\partial \dot{u}_t}{\partial p_t} & \frac{\partial \dot{u}_t}{\partial u_t} & \frac{\partial \dot{u}_t}{\partial q_t} \\ \frac{\partial \dot{q}_t}{\partial p_t} & \frac{\partial \dot{q}_t}{\partial u_t} & \frac{\partial \dot{q}_t}{\partial q_t} \end{bmatrix}_{BGP} \quad (39)$$

The elements of the matrix are given as follows:

$$J_{11}=\frac{\partial \dot{p}_t}{\partial p_t}=Aa\, p_t^{a-1}u_t^{1-a}-q_t-f\,\frac{1-a+?}{1-a}\left(1-u_t\right)=\frac{\dot{p}_t}{p_t}+\left(a-1\right)Ap_t^{a-1}u_t^{1-a}$$

$$J_{12}=\frac{\partial \dot{p}_t}{\partial u_t}=A\left(1-a\right)p_t^{a}u_t^{-a}+f\,\frac{1-a+?}{1-\mathrm{a}}\,p_t$$

$$J_{13}=\frac{\partial \dot{p}_t}{\partial q_t}=-p_t$$

$$J_{21}=\frac{\partial \dot{u}_t}{\partial p_t}=0$$

$$J_{22}=\frac{\partial \dot{u}_t}{\partial u_t}=\frac{f\left(1-a+?\right)}{a}-q_t+2f\,\frac{\left(1-a+?\right)}{1-a}u_t=\frac{\dot{u}_t}{u_t}+f\,\frac{\left(1-a+?\right)}{1-a}u_t$$

$$J_{23}=\frac{\partial \dot{u}_t}{\partial q_t}=-u_t$$

$$J_{31}=\frac{\partial \dot{q}_t}{\partial p_t}=\frac{a-s}{s}\left(a-1\right)Ap_t^{a-1}u_t^{1-a}\,q_t$$

$$J_{32}=\frac{\partial \dot{q}_t}{\partial u_t}=\frac{a-s}{s}\left(1\text{-}a\right)Ap_t^{a}u_t^{-a}\,q_t$$

$$J_{33}=\frac{\partial \dot{q}_t}{\partial q_t}=2q_t+\frac{a-s}{s}\,Ap_t^{a-1}u_t^{a-1}-\frac{?}{s}=\frac{\dot{q}_t}{q_t}+q_t$$

It follows that the Jacobian, evaluated along the BGP, can be written as:

$$J^*=\begin{bmatrix} J_{11}^* & \frac{p^*}{u^*}\left(-J_{11}^*+f?u^*\right) & -p^* \\ 0 & f?u^* & -u^* \\ \frac{f J_{11}^* q^*}{p^*} & -\frac{f J_{11}^* q^*}{u^*} & q^* \end{bmatrix} \quad (40)$$

where $J_{11}^*=-\left(1-a\right)A\left(p^*\right)^{a-1}\left(u^*\right)^{1-a}$.

In order to study the stability properties of the steady state, we need to calculate the eigenvalues of this matrix. The computation is simplified because it is possible to factorise the characteristic equation of the matrix. The resulting equation is, in fact, the following:

$$\left[J_{11}^{*}-?_{1}\right]\left[f\,y\,u^{*}-?_{2}\right]\left[q^{*}-?_{3}\right]=0 \qquad (41)$$

To get the eigenvalues we only need to equalise to zero the three factors of the equation. Hence, we obtain:

$$?_{1}=J_{11}^{*} \qquad (42)$$

$$k_{2}=f\,y\,u^{*} \qquad (43)$$

$$?_{3}=q^{*} \qquad (44)$$

Now, substituting the expressions for p^{*} and u^{*} into the first eingevalue, we obtain its value as a function of the following parameters:

$$k_{1}=\frac{-f\left(1-a+b\right)}{a} \qquad (45)$$

We can therefore prove the following proposition:

Proposition 3.1. *The characteristic equation associated with the Jacobian matrix J, evaluated at the steady state, has a negative real eingevalue and two positive real eigenvalues: this implies that the steady state is locally unique and shows the saddle path stabilities properties.*

Proof. The analysis of the signs of (42), (43) and (44) allows us to mark an unambiguous sign to the eigenvalues. Standard mathematical reasoning leads to the uniqueness result and the stability properties of the steady state.

Proposition 3.1 implies that, for given initial values of physical and human capital, there exists only one value for c and u that allows the system to settle on the stable branch. To this regards, given the explicit form for the eigenvalues, we have the following

Corollary 3.1. *The speed of convergence of the variables near the steady state is given by* $k_{1}=\left[-f(1-a+b)/a\right]$*. It increases (in absolute value) as* f *increases and decreases as* a *increases.*

Proof. Standard mathematical reasoning proves that it is the negative eingevalue that regulates the speed of convergence near the steady state. The sign of partial derivatives with respect to parameters proves the second part of the corollary.

Moreover, it can be shown that the speed of convergence is a technical relation which is not influenced by preference parameters.

4. The further reduction of the dynamic system into two dimensions

4.1. The transitional dynamics in R^2

In the centralised solution, the differential equations system of the Uzawa-Lucas model can be further reduced to only two differential equations.[5] For this purpose the following proposition holds:

Proposition 4.1. *The transitional dynamics of Uzawa-Lucas model with externalities is fully described by the following differential equation system:*

$$\frac{\dot{z}_1}{z_1} = (1-a)\left(\frac{f y}{a} - z_1\right) \tag{46}$$

$$\frac{\dot{z}_2}{z_2} = z_2 + j\ z_1 - \frac{r}{s} \tag{47}$$

Proof. Let us consider the two new variables $z_1 = \frac{y}{k}$ and $z_2 = q = \frac{c}{k}$. Applying logs and time derivatives we obtain the two differential equation given in the proposition.

The first variable z_1, normalising the technological level A to one, is the average output per unit of physical capital.[6] The second variable z_2, which has already been defined in section 3.1., is again the ratio between consumption and physical capital.

This new portrayal of the system does not allow to read directly the original values of the variables. Also the distinction between state and control variables disappears. However, the reduction of the model to only two equations permits to study the phase diagram of the system. Moreover, this system in R^2 is saddle path stable.

Corollary 4.1. *Given the initial conditions, the growth rate of* z_1 *is independent from* z_2.

Proof. Simply reading the equation of motion of z_1 leads to the conclusion in the corollary.

We can find the steady state values of z_1 and z_2 equalising to zero their growth rates. We therefore obtain:

$$z_1^* = \frac{f y}{a} \tag{48}$$

$$z_2^* = \frac{r}{s} - \frac{f y j}{a} \tag{49}$$

Obviously, z_2^* has the same value we already found in equation (38) for q^*.

4.2. Local stability analysis of the BGP

At this point, we can analyse the stability of the BGP generated by the centralised solution of the model in the (z_1, z_2) space. To do this, we can follow the same procedure as in section 3.2. Therefore, after determining the Jacobian matrix (evaluated at the BGP), we can determine the signs of the eigenvalues of the system.

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[5] The techniques used here are taken from Mattana P. [8, 9].

[6] Substitute for y the value given by the production function and put A=1, we obtain $z_1 = \left(\frac{u h^{\frac{(1-a+b)}{(1-a)}}}{k}\right)^{1-a}$.

To write the Jacobian, we need to evaluate the partial derivatives of the two differential equations with respect to each of the independent variables. We therefore have:

$$\frac{\partial \dot{z}_1}{\partial z_1} = \frac{\dot{z}_1}{z_1} - (1-a)z_1 \tag{50}$$

$$\frac{\partial \dot{z}_1}{\partial z_2} = 0 \tag{51}$$

$$\frac{\partial \dot{z}_2}{\partial z_1} = j\ z_2 \tag{52}$$

$$\frac{\partial \dot{z}_2}{\partial z_2} = \frac{\dot{z}_2}{z_2} + z_2 \tag{53}$$

from which, recalling that in the steady state the growth rates of z_1 and z_2 are zero, we can write the Jacobian evaluated at the steady state as follows:

$$J^* = \begin{bmatrix} -(1-a)z_1^* & 0 \\ j\ z_2^* & z_2^* \end{bmatrix} \tag{54}$$

Now, we need the eigenvalues of the characteristic equation associated with this Jacobian matrix, which takes the form:

$$\left[-(1-a)z_1^* - k_1\right]\left[z_2^* - k_2\right] = 0 \tag{55}$$

Setting both factors of this equation equal to zero, we get the following eigenvalues:

$$k_1 = -(1-a)z_1^* \tag{56}$$

$$k_2 = z_2^* \tag{57}$$

The first eingevalue is negative, being the two terms $(1-a)$ and z_1^* both positive, while the second eingevalue is given by z_2^*. The previously found value for z_2^* does not allow us to know its sign. Nevertheless, we can trace it back to the value we found for q^* in section 3.1., given that z_2 and q depict the same variable. From (37) we find that the value of q^* is undoubtedly positive, hence the second eingevalue is positive as well.

Proposition 4.2. *The characteristic equation associated with the matrix J, evaluated at the steady state, has a negative real eigenvalue and a positive real eigenvalue, given respectively by* $k_1 = -(1-a)z_1^* < 0$ *and* $k_2 = z_2^* > 0$. *Therefore, the steady state is saddle path stable and locally unique.*

Proof. Standard mathematical reasoning leads to the results given in the proposition.

We have therefore proved that reducing the dynamic system to only two equations yields the same results already proved for the three equations system presented in section 3.

4.3. The phase diagram

As we know, a system in R^2 can be portrayed and studied in the corresponding phase diagram. This particular kind of representation shows the dynamic movements of the variables of the model (in this case z_1 and z_2).

In order to draw the phase diagram, it is necessary to set $\dot{z}_1 = 0$ and $\dot{z}_2 = 0$. Putting the growth rate of z_1 equal to zero, after simple algebra, we obtain that $\dot{z}_1 = 0$ when:

$$z_1 = \frac{fy}{a} \tag{58}$$

Following the same procedure for the growth rate of z_2, we find that $\dot{z}_2 = 0$ whenever:

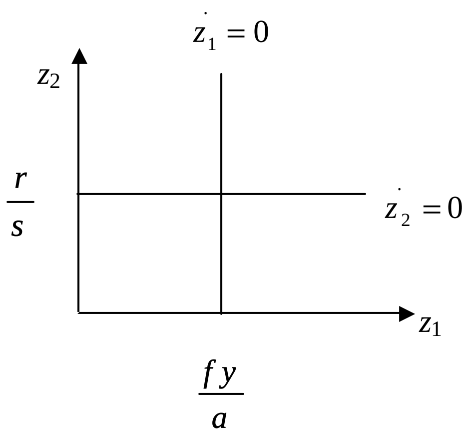
$$z_2 = \frac{r}{s} - \frac{a-s}{s} z_1 \tag{59}$$

We can therefore conclude that the locus of stationary points of z_1, i.e. the locus of the points where $\dot{z}_1 = 0$, is a perpendicular straight line to the z_1 axis at the $\frac{fy}{a}$ point. Such an expression is nothing else than the stationary value of z_1.

As regards to the stationary points of z_2, we need to distinguish three cases. In fact, the function (59) is a straight line, with the slope depending on the values of a and s. Therefore, the three cases are:

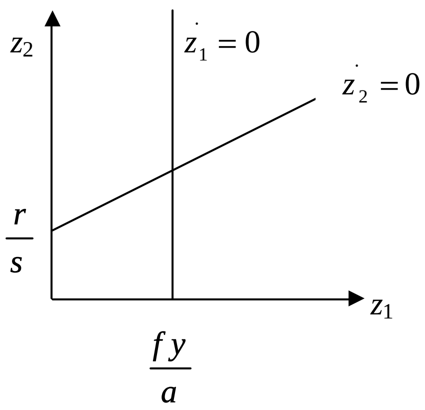
<u>*case* 1</u>: $a = s \Rightarrow z_2 = \frac{r}{s}$. In this case, as shown in Figure 1, the function is perpendicular to the z_2 axis at the point $\frac{r}{s}$.

Figure 1. *Stationary points for* $s = a$



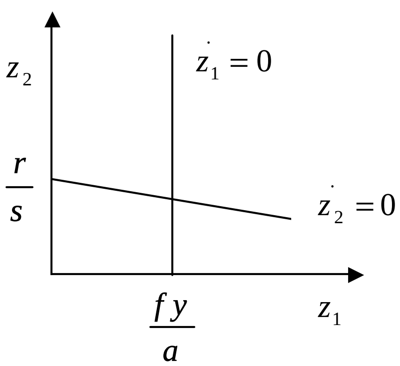
<u>*case* 2</u>: $a < s \Rightarrow$ the angular coefficient of the straight line $z_2 = \frac{r}{s} - \frac{a-s}{s} z_1$ is positive. This case is depicted in Figure 2.

Figure 2. *Stationary points for* $s > a$



<u>*case* 3</u>: $a > s \;\; \Rightarrow$ the angular coefficient of the straight line $z_2 = \frac{r}{s} - \frac{a-s}{s} z_1$ is negative. In such a case, we have the situation depicted in Figure 3.

Figure 3. *Stationary points for* $s < a$



Hence, the slope of the straight line $\dot{z}_2 = 0$ depends on the ratio between the physical capital share on output and the elasticity of marginal utility with respect to consumption.

Once we know the stationary points of z_1 and z_2, we can study the motion of the two variables in the other points of the diagram. To do that, we need to know the signs of the two equations $\dot{z}_1$ and $\dot{z}_2$. We know that:

$$\dot{z}_1 = (1-a)\left(\frac{fy}{a} - z_1\right)z_1 \tag{60}$$

and

$$\dot{z}_2 = \left(z_2 + j\, z_1 - \frac{r}{s}\right)z_2 \tag{61}$$

The first equation is positive until z_1 reaches its steady state value, and then it becomes negative. The opposite is true for the second equation, which is negative until z_2 reaches its steady state value.

Figure 4. *The phase diagram for* $s > a$

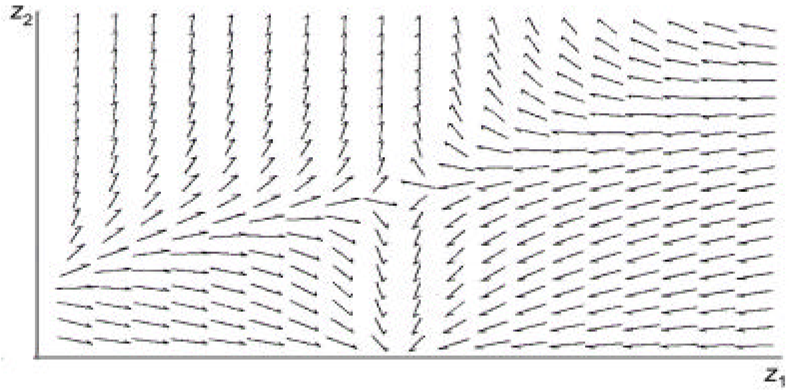


Figure 5. *The phase diagram for* $s = a$

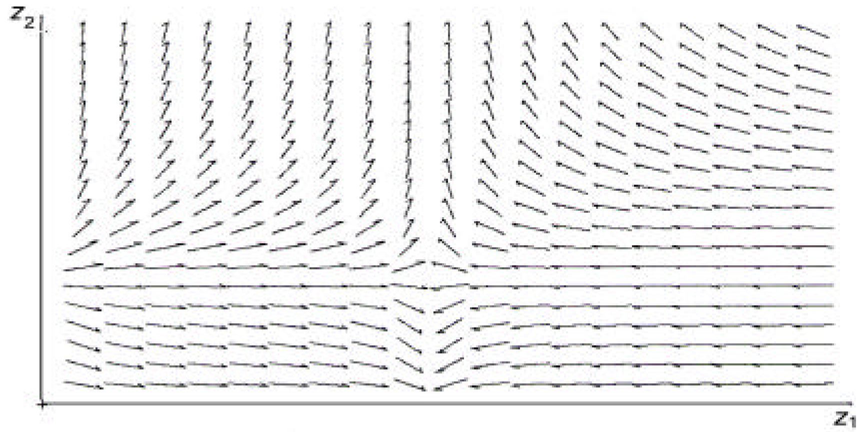
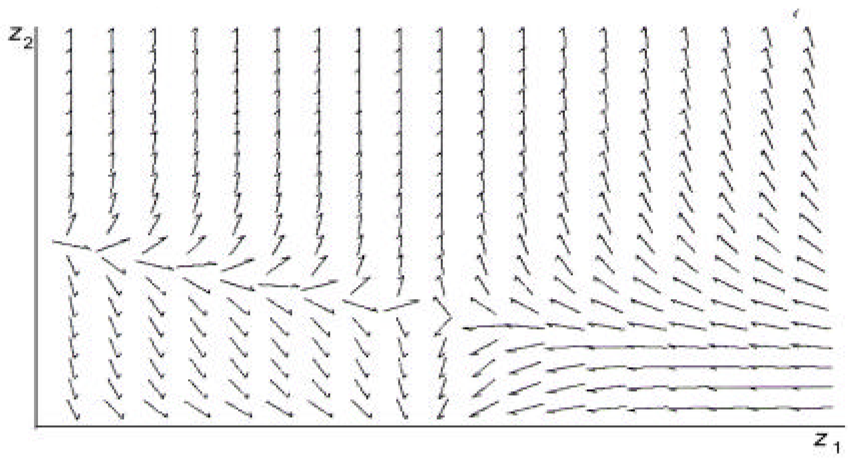


Figure 6. *The phase diagram for* $s < a$



Therefore, we can formulate the following

Proposition 4.3. *As z_1 increases, it ranges from a region in which $\dot{z}_1>0$ to a region in which $\dot{z}_1<0$. The boundary between these two regions is the straight line $z_1=\frac{fy}{a}$, which also is the steady state value for z_1. As z_2 increases, it passes from a region in which $\dot{z}_2<0$ to a region in which $\dot{z}_2>0$. The boundary between these two regions is the straight line $z_2=\frac{r}{s}-\frac{a-s}{s}z_1$.*

Proof. The study of the signs of the functions $\dot{z}_1$ and $\dot{z}_2$ yields the results in proposition.

The three cases depicted in Figures 4, 5 and 6, which are computer generated for numerical values of the parameters, confirm proposition 4.3.

5. The explicit solution of the model

5.1. The optimal path for the average output per unit of physical capital

The solution of the following reduced system of two differential equations in the two variables z_1 and z_2:

$$\frac{\dot{z}_1}{z_1}=(1-a)\left(\frac{fy}{a}-z_1\right) \tag{62}$$

$$\frac{\dot{z}_2}{z_2}=z_2+jz_1-\frac{r}{s} \tag{63}$$

gives the optimal paths that these variables follow along the transition towards the steady state.

The *general solution* of the model will contain two arbitrary constants. These constants will depend on the initial conditions, i.e. on the values given to the variables at time zero. In fact, on the basis of these initial values, the two variables will evolve differently in time. This is the reason why it is necessary that the initial values for z_1 and z_2 must be chosen in such a way that the steady state can be reached. This is guaranteed if the initial values for z_1 and z_2 are both chosen on the stable branch of the phase diagram. This choice gives the *definite solution* of the model. If, on the contrary, the initial values are not chosen on the stable branch, then the time evolution of the two variables leads to points far away from the steady state.

Then, for the first order differential equations system (62)-(63), the following proposition holds:

Proposition 5.1. *The trajectory of z_1 is given by*

$$z_1(t)=\frac{z_1^*}{1+b\exp(-qz_1^*t)}. \tag{64}$$

Proof. The equation of motion for z_1 can be rewritten as $\dot{z}_1=(z_1^*-z_1)z_1q$, where $q=(1-a)$ and $b=\frac{z_1^*-z_1(0)}{z_1(0)}$ are control parameters and $z_1^*=\frac{fy}{a}$ is the steady state value for z_1.

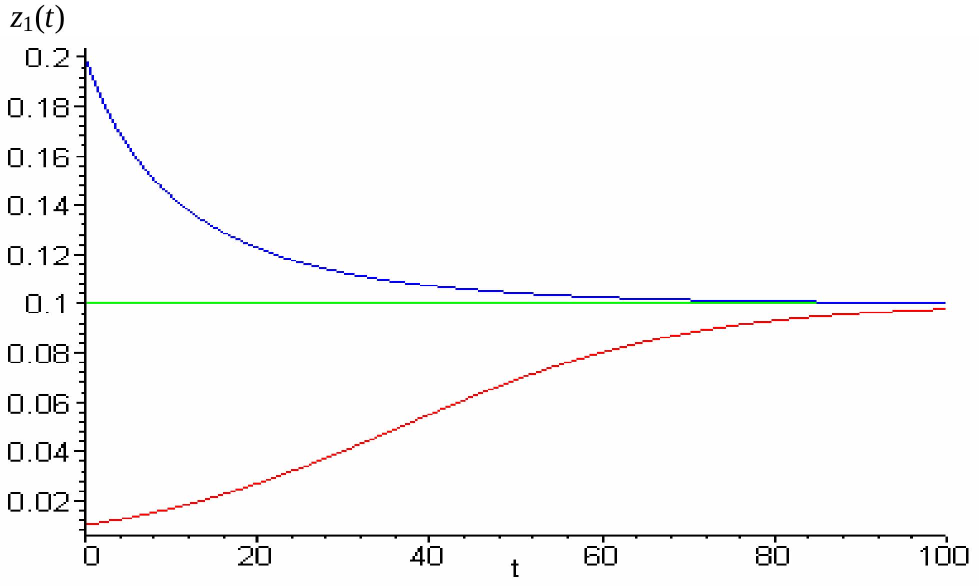
It is worth noticing that the time path of z_1 depends on its steady state value, the initial condition $z_1(0)$, the physical capital share on output a and, finally, the externality b, but it does not depend on time preference parameters. Moreover, it should be noticed that the time trajectory of z_1, as well as the corresponding differential equation, does not depend on the position of z_2.

Other characteristics of the time path of z_1 can be described as follows. When the value of z_1 is close to zero, its growth rate, in absolute value, will be close to $\frac{f(1-a+b)}{a}$. On the contrary, when the value of z_1 grows, the growth rate asymptotically tends to zero and reaches this value in the steady state.

It is possible to draw Figure 7 that shows the time path of z_1 in three different cases, depending on whether the initial value of z_1 be, respectively, greater than, equal to or less than $z_1^* = \frac{f y}{a}$.

It is very interesting to note that the time path of z_1 follows a logistic law in the interesting case when the level of human capital is lower than the optimal one (lower curve in Figure 7). Anyway, the standard case applies whenever it is the physical capital that is lower than the optimal level, which makes z_1 decrease along with decreasing growth rates (upper curve in Figure 7).

Figure 7. *Different cases of convergence of $z_1(t)$ towards the steady state*



5.2. The optimal path of the consumption/physical capital ratio

After the solution for z_1 is found, we can also find the solution for z_2. This depends on the initial choice of the value of z_1.

Proposition 5.2. *The trajectory of* $z_2 = q = \frac{c}{k}$ *is given by*

$$z_2(t) = \frac{\exp\left(\frac{-r\,t}{s}\right)\left[b + \exp\left(q\, z_1^* t\right)\right]^z}{R - \int \exp\left(\frac{-r\,t}{s}\right)\left[b + \exp\left(q\, z_1^* t\right)\right]^z dt} \tag{65}$$

where R is an arbitrary constant and $z = \frac{a - s}{s(1-a)}$.

Proof. The time path of z_2, after substituting for the value of z_1, is ruled by a Bernoulli equation that can be solved.

Note that the complexity of the differential equation does not allow to obtain an explicit solution for z_2. This implies that, depending on the sign and the value of z, the integral at the denominator of (65) takes a different form. In order to find the solution of the integral in the denominator of equation (65), we need therefore to clearly separate two cases: the first when $s < a$, hence z is positive; the second when $s > a$, hence z is negative.

Depending on the relation between these two parameters, the time path of z_2 will be different. This also depends on b, i.e. the ratio of the difference between the steady state value of z_1 and its initial value over the same initial value.

In the next subsection, therefore, we consider two numerical cases, respectively with $z > 0$ and $z < 0$, and solve the integral in $z_2(t)$ to obtain the form of the time path of this variable in the two cases. It should also be noticed that when $s = a$, then $z_2 = z_2(0) = \frac{?}{s}$. In this case, there is no transitional dynamics of z_2.

5.3. Some numerical simulations of explicit trajectories for z_2

In order to investigate the shapes of alternative trajectories for z_2, we assign explicit values to the parameter z. In particular, we evaluate the symmetric cases of $z = \pm 2$. In fact, the numerical analysis shows that when $z=2$ the solution is purely algebraic, whereas the case in which z is negative implies a form of the solution that is partly algebraic and partly logarithmic.

In addition, as it comes out from (65), the time path of $z_2(t)$ also depends on b, that is on the position of $z_1(0)$ with respect to its steady state level, and hence on the sign of the same b. As a consequence, in what follows, the values of $z = \pm 2$ will be associated to two opposite values for b, the first one associated to an excess of physical capital [$z_1(0) < z_1^* = \frac{fy}{a} \Rightarrow b<0$], and the second one associated to an excess of human capital [$z_1(0) > z_1^* = \frac{fy}{a} \Rightarrow b>0$].

a) <u>First case: $z=2$, for $s < a$</u>

First of all, let us consider the case when $z=2$. In this case,[7] we have:

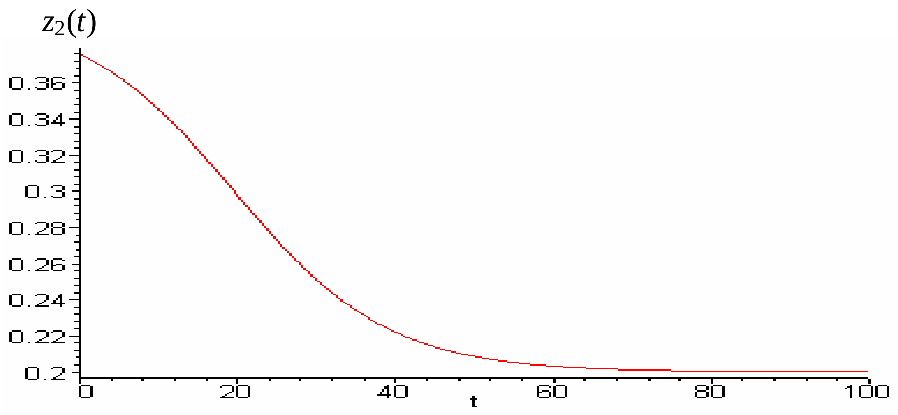
Proposition 5.3. *The transitional dynamics of z_2 is decreasing if we start with an excess of physical capital* ($b<0$), *and increasing if we start with an excess of human capital* ($b>0$).

Proof. The numerical simulations invariably yield the results in the proposition.

Two particular cases are depicted in Figures 8 and 9. Once more, it should be noticed the logistic form of time trajectories of z_2, in the case in which the transition is due to an economic shock that determines an excessive physical capital. On this regard, the numerical simulations show that the logistic form of the trajectory increases with the value of b.

[7] The results are confirmed also if we choose positive values for p different from 2.

Figure 8. *The evolution of $z_2(t)$ associated to an excess of physical capital when* $s < a$



The computer simulations show more complex dynamics (hardly compatible with the real evolution of an economy) for values of *b* smaller than –1. In this case, in fact, it happens that z_2 first increases, then decreases and finally tends to a monotonic convergence towards the stationary state. Figure 8, therefore, is obtained for *b* greater than –1 and smaller than zero.

b) <u>Second case: z = -2, for $s > a$</u>

The case when $s > a$, associated with a value of z = -2, is more interesting from the empirical point of view. The numerical simulations performed allow to assert the following

Proposition 5.4. *The transitional dynamics of z_2 is increasing if we start with an excess of physical capital* (*b*<0), *and decreasing if we start with an excess of human capital* (*b*>0).

Proof. The numerical simulations invariably yield the result in the proposition.

These cases are depicted in Figures 10 and 11. It should be stressed that, in the case when we start with an excess of physical capital with respect to human capital, the time path of z_2 still follows a logistic law: in a first period z_2 grows more than proportionally, then in a second period it grows less than proportionally, getting asymptotically close to its steady state value.

Differently from the previous case, the arbitrary constant associated to the integration process is not zero, but has to be chosen exactly in such a away to assure the convergence towards the stationary state.

Finally, the case in which *b*<0 is, once more, more complicated to deal with. In fact, when z= -2, the presence of an asymptote restricts the choice of *b*.

Figure 9. *The evolution of $z_2(t)$ associated to an excess of human capital when* $s < a$

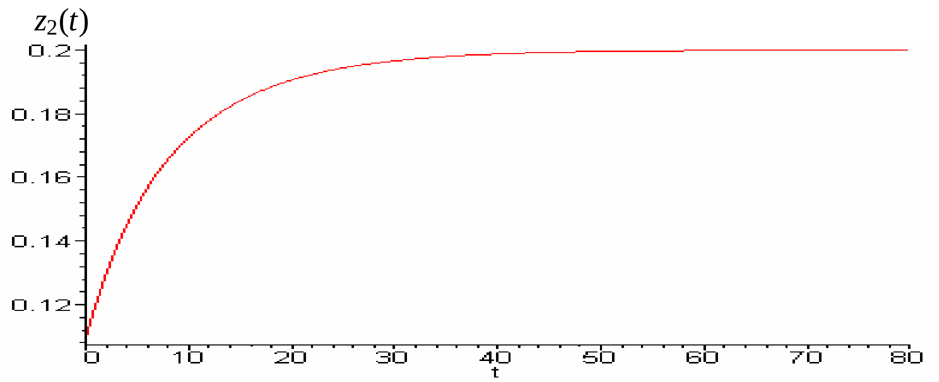


Figure 10. *The evolution of $z_2(t)$ associated to an excess of physical capital when $s > a$*

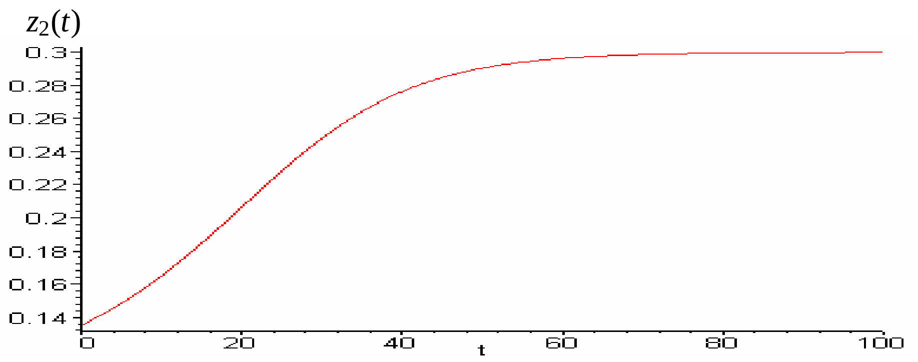
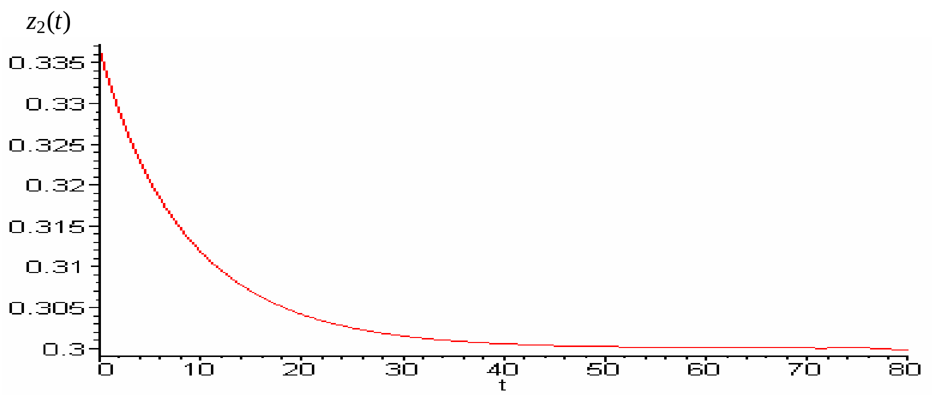


Figure 11. *The evolution of $z_2(t)$ associated to an excess of human capital when $s > a$*



5.3. Some further implications for the optimal time paths of the original variables of the model

The depicted evolution of the two variables z_1 and z_2 are rich of implications in terms of the solution of the model in the original variables. To see that, let us recall the system of differential equations obtained by the maximisation problem performed by the social planner:

$$\frac{\dot{c}}{c} = \frac{1}{s}\left[aAk^{1-a}u^{1-a}h^{1-a+b} - r\right] \tag{66}$$

$$\frac{\dot{k}}{k} = Ak^{a-1}u^{1-a}h^{1-a+b} - \frac{c}{k} \tag{67}$$

$$\frac{\dot{h}}{h} = f(1-u) \tag{68}$$

$$\frac{\dot{u}}{u} = \frac{f(1-a+b)}{a} - \frac{c}{k} + \frac{f(1-a+b)}{1-a}u \tag{69}$$

As should be noticed, these dynamic equations are all expressed in terms of z_1 and z_2. Although a deeper analysis of this point is beyond the scope of this paper, some numerical simulations allow us to prove the following interesting:

Proposition 5.5. *Given the explicit trajectories for z_1 and z_2, all the dynamic laws originated by the centralised solution of the Uzawa-Lucas model can be recursively solved.*

Proof. The proposition derives from the numerical simulations depicted above.

Proposition 5.5. has many important effects from the normative economics point of view. If, as proved by numerical simulations, it is possible to solve the model, then the social planner has important instruments to manage the economy in order to maximise social welfare.

5.4. Interpretation of a and s and policy implications

As we know, a is the share of physical capital and s is the inverse of the elasticity of substitution, which is also interpreted as a coefficient of risk aversion. If the agent is risk averse (high value of s), he prefers to consume more income today instead of moving consumption (by savings) to tomorrow. Thus, the dynamics in the Uzawa-Lucas model depends on the fact that the share of physical capital is larger or smaller than the coefficient of risk aversion.

Proposition 5.5 implies that just one couple of optimal control variables, given by an initial optimal consumption $c^*(0)$ and an initial optimal time devoted to work $u^*(0)$, corresponds to each initial couple of physical capital $k(0)$ and human capital $h(0)$. Furthermore, this choice depends on the value of the externality b .

Thus, given the externality b , as the ratio $k(0)/h(0)$ varies, c and u take their optimal initial values according to whether a is bigger or smaller than s . It is easy to show that, given b and $h(0)$, when $k(0)$ grows, $c^*(0)$ and $u^*(0)$ must increase if $a < s$ and decrease if $a > s$. Then, the policy maker can make initial c and u assume their optimal values. A way to do this is to adopt fiscal policies that reduce or raise both initial consumption (by a consumption tax) and/or the time devoted to work (by a labour tax). The measure of these policies depends on the effect of the externality b on the human capital accumulation[8].

6. Conclusions

The recent literature has stressed the importance of endogenous growth models, which explain the evolution of economic systems without turning to the role of exogenous factors. Starting from Lucas [7], who resumes some of the initial intuitions of Uzawa [15], a large body of this literature has built on models with two capital goods (or two sectors), with or without the presence of distorting factors such as externalities.

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[8] Studies on this kind of policies can be found in Judd K. [5] and Garcia-Castillo P., Sanso M. [4].

The peculiarities, in this literature, of the Uzawa-Lucas model, are many. The first one is the explicit consideration of human capital, in the form of learning. Secondly, the Uzawa-Lucas model takes into account an externality springing from human capital. This has remarkable effects with regard both to the evolution of the economy (private or social/centralised) and, more generally, to the properties of the transitional dynamics and the steady state. A great importance, in the light of the most recent results on endogenous growth literature, is here attributed to the difference between centralised and decentralised (or competitive) solution of the maximization problem. In this framework, the main aim of this work was to obtain and study the centralised solution of the Uzawa-Lucas model with externalities. As far as we know, the results here derived have not yet been fully analysed in the literature.

These results are very interesting. Starting from the description of the characteristics of the original model, we defined the balanced growth paths for the four variables that enter into the model, i.e. consumption, physical capital, human capital and the fraction of labour time devolved to produce output. The first result, fairly standardised in the literature of two sector models, is that in the steady state the growth rates of physical capital, consumption and output are the same, while human capital, if an externality is present, grows at a lower rate. Only if there is no externality does the human capital grow at the same rate as the other variables of the model.

A further step exploits the possibility to reduce the dimension of the model from 4 to 3 equations, by mean of a transformation of variables. In this way, it becomes possible to study the stability properties of the balanced growth path (BGP). However, the most interesting results are obtained when the model is further reduced to only two equations. By mean of another transformation that uses the average output per unit of physical capital and the consumption to physical capital ratio, we get more information on the characteristics of the transitional dynamics, both locally (through the phase diagram) and (exceptionally for this kind of very complex models) globally.

As a matter of fact, it is this last point that represents the innovative conclusion of this work. It is shown that the equation of motion of the average output per unit of physical capital can be easily solved in closed form (to generate a logistic function converging to the steady state); once this result has been obtained, also the equation of motion for the consumption over physical capital ratio can be explicitly solved. Then, all the equations of motion of the original variables in the model (consumption, output, physical capital, and human capital) can be recursively solved.

The paper ends up with some policy hints about the role of fiscal policy in implementing the optimal trajectories of the involved variables. Once again, the measure of these policies depends on the effect of the externality on the human capital accumulation.

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