Scalable games

Peter Eccles* Nora Wegner†

November 3, 2014

Abstract

We propose conditions on games of incomplete information, under which solving the game reduces to solving a corresponding game of complete information. These games are referred to as scalable games. After establishing a link between (i) scalable games, (ii) games of complete information and (iii) games with strategy restrictions, we present two distinct applications. The first application demonstrates how scalable games can be used to model complex situations in a tractable manner. For instance we provide a tractable model of an asymmetric private value first price auction with a reserve price and risk averse bidders. The second application shows how certain deterministic all pay auctions with incomplete information are strategically equivalent to stochastic contests with complete information. In particular for the two player case we show that a contest is strategically equivalent to an all-pay auction whenever the relevant contest success function is homogenous of degree 0 and other mild conditions are satisfied.

*Universidad Carlos III de Madrid, Department of Economics, C/Madrid 126, 28903 Getafe (Madrid), Spain. Email addresses: `eccles.peter@gmail.com`, `norawegner@gmail.com`. We would like to thank our supervisors Natalia Fabra and Ángel Hernando-Veciana for their support, encouragement and many discussions. We also thank Marco Celentani, Laura Doval, Mikhail Drugov, Alessandro Pavan, Pablo Schenone, Ron Siegel, Asher Wolinsky as well as participants at CEMFI-UC3M Workshop 2013, RES Conference 2014, Stony Brook Festival of Game Theory 2014, EEA Congress 2014, JEI 2014 and seminar participants at Universidad Carlos III, Northwestern University and the University of Mannheim for useful comments and suggestions. .

1 Introduction

In this paper we study certain games of incomplete information - which we call *scalable games* - where players observe their type before choosing their action. Scalable games are of interest because they provide a link between three different ways of modeling the same phenomena. We show that scalable games are strategically equivalent to corresponding games of complete information where players do not observe any private information. Moreover scalable games are strategically equivalent to other games of incomplete information where players choose a strategy profile from a restricted set before their type is revealed. We now discuss each of these games in turn.

Many situations where players face incomplete information can be modeled directly using scalable games, including settings such as firms competing on price or on quantity, auction environments and beauty contests. We show that such models (i) can capture complex situations and (ii) can be solved by analysing a game of complete information. Therefore using the scalable game structure provides an economist with a tractable way of solving incomplete information games for a wide range of problems, which are difficult to solve in general. For instance consider the case of an asymmetric private value auction with a reserve price and risk averse bidders. This problem does not have a simple general solution. However if the auction considered has the scalable games structure, pure strategy equilibria can be found by solving a set of simple equations.

For every scalable game there exists a strategically equivalent game of complete information. In particular this is interesting when the resulting complete information game has been previously studied in the literature and hence is of interest in its own right. In order to demonstrate this, we show that there is a mapping between the equilibria of certain all-pay auctions and the equilibria of complete information contests with stochastic allocation rules introduced by Lazear & Rosen (1981) and Tullock (1980). We further illustrate this point by characterizing the two player contest success functions which are strategically equivalent to some all pay auction where players have private values. In the contest the prize is allocated according to a stochastic allocation rule and players know each others valuations of the prize;

meanwhile in the all-pay auction the prize is allocated according to a deterministic allocation rule and players do not know each others valuations of the prize. Hence the result shows that two seemingly distinct games are strategically equivalent. In addition to this general result covering two player contests, we give more specific results for n-player contests. First we show that any n-player Tullock contest is strategically equivalent to some all-pay auction where players have private values. A similar result links a common value all pay auction and the contest success function with draws considered in Yildizparlak (2013). This shows a surprising connection between contests with draws and common value auctions.

In a scalable game players observe their type $t_i \in T_i$ and then choose an action $a_i \in A_i$ as is the case in standard models. On the other hand in games of strategy restrictions as proposed by Compte & Postlewaite (2013) players choose a strategy profile $\sigma_i : T_i \mapsto A_i$ from a restricted set Σ before observing their type. For instance Compte & Postlewaite (2013) study the case of auctions where $\sigma_i \in \Sigma$ whenever $\sigma_i(t_i) = t_i - e_i$. We also study games of strategy restrictions and argue that - depending on the application - certain restricted sets Σ should be preferred to others. In the case of auctions where $\sigma_i \in \Sigma$ whenever $\sigma(t_i) = t_i \times e_i$ and show that such a game of strategy restrictions is equivalent to some scalable game without strategy restrictions. This gives justification for considering such a strategy restriction since it can be shown to be strategically equivalent to a game without strategy restrictions.[1]

Scalable games borrow the information structure used in global games introduced by Carlsson (1991). A state of the world θ is realized and players observe a noisy signal t_i of this state, which we refer to as the type of player i. All players then simultaneously choose an action determining the outcome of the game. More precisely scalable games are those games that satisfy two key assumptions namely that (i) the payoff function is scalable and (ii) the information structure satisfies an invariance property. The first assumption - scalability of the payoff function - captures a number of settings where the structure of the game remains unchanged if all variables are scaled by a constant. In particular this holds when utility functions are homogeneous of degree α, or are additively invariant. This captures a large

[1] We have not been able to find similar justification for the strategy restriction considered by Compte & Postlewaite (2013).

family of games studied in the literature including (i) models with quadratic utility, (ii) auctions and procurement contests, (iii) certain public good problems and many others. Many situations of incomplete information can be described using a scalable payoff function.

The second assumption is equivalent to requiring that the shape of the distribution of types is common knowledge among players, but - after observing his type - a player receives no information about what quantile of the distribution his type was drawn from. This ensures that a player's type does not provide the player with information about his rank. Such an assumption is approximately satisfied in the model considered by Abreu & Brunnermeier (2003) and in the model considered by Klemperer (1999). Although the assumption does not hold in models with a proper prior, it does hold in several models using a diffuse prior.

Together these two assumptions imply that the game - and hence the decision problem - solved by a player does not look fundamentally different depending on his type. In these games incomplete information arises from uncertainty over types and over the state of the world. The main contribution of this paper is to show that equilibria of a scalable game can be found by studying a related complete information game. First this provides a simple way to find the equilibrium of the original game of incomplete information. Secondly - in the case where the complete information game has independent interest - the original game of incomplete information gives an alternative description of the strategic situation studied in the game of complete information.

In this paper we primarily focus on the Nash equilibria of scalable games where players use certain scalable strategies. These strategies are scaled in the same way as the information structure and the payoff structure. Therefore in the equilibria considered, players not only consider a very similar decision problem, they also act in very similar ways whatever their type: for instance in an auction setting all players might bid a constant proportion of their valuation. In these equilibria both the decision problem and the action chosen are scaled by the player's type.

In section 7 we consider the assumption of a scalable information structure in combination with the improper prior in some more detail. We show that together they give rise to a key property that both provides intuition for the result and plays a central role in our analysis. This key property is referred to as *maximal rank uncertainty* and means that a player's type does not provide him with any information about his rank compared to the types of other players. The probability a player assigns to being in any rank is the same for all of his types. A related property is studied by Morris & Shin (2007) in the case of global games.

The remainder of the paper is structured as follows. We first discuss related literature and present an illustrating example. In section two we present the model and formally introduce the class of scalable games. Section three provides the analysis of scalable games and establishes the link between scalable games and the corresponding games of complete information. Section four establishes the link between scalable games and corresponding games with strategy restrictions. Section five and six cover cover the applications discussed above. In section seven we discuss the key property, while section eight concludes.

1.1 Related Literature

In the literature, models which either satisfy the definition of a scalable game or can easily be modified to become a scalable game, have been used to model specific situations of uncertainty. As mentioned above, the formation of asset price bubbles studied by Abreu & Brunnermeier (2003) is one such model. Other examples include the clock games considered by Brunnermeier & Morgan (2010), a recent paper on double auctions by Satterthwaite *et al.* (2014), as well as supply function competition studied by Vives (2011). While these papers provide models for specific situations, we aim at providing a general tool to model situations of incomplete information using a scalable information structure.

The information structure of the proposed class of games has close links with the literature on *global games* introduced by Carlsson & Van Damme (1993) and considered in Morris & Shin (2002) among others. As in global games, players face uncertainty about the state of the world θ which is drawn from a diffuse prior. Moreover each player does not observe θ but instead receives a partially informative signal t_i about the state of the world, where $t_i = \theta + z_i$

and z_i can be interpreted as a noise term. However, in global games the main objective is equilibrium selection which arises since coordination is more difficult when the state of the world is unknown. Moreover the games considered in this paper do not necessarily have dominance regions and a player's signal typically enters his payoff function directly. Above all the focus of this paper lies on the characterization of equilibria rather than equilibrium selection.

The framework proposed also has close ties with the literature on *quadratic utility models* a comprehensive treatment of which is provided in Angeletos & Pavan (2007) for a continuum of players, while Ui & Yoshizawa (2014) consider a discrete number of players. In these games there is also uncertainty about the state of the world and players receive a noisy signal of the state. Quadratic utility models typically focus on the social value of information and the role of information acquisition.[2] Applications to Cournot competition are provided by Vives (1988) and Myatt & Wallace (2013).

As in this paper, players receive a signal about the state of the world which can be interpreted to be a player's type and may enter a players payoff function directly. On the one hand, scalable games make stronger distributional assumptions on the state and the signals: the information structure in a quadratic utility model is affine, satisfying the assumption that $E[\theta|t_i] = \alpha t_i + \beta$; in the related scalable game in additive form we require the shape of the distribution to be the same for all types and hence $E[\theta|t_i] = t_i + \beta$. On the other hand, scalable games make weaker assumptions on the payoff function. While the payoff function in most quadratic utility models depend on the actions of others only through the aggregate, the payoff assumption in this paper is substantially weaker and allows for a much wider range of applications.

Although equilibria in scalable strategies arise in both scalable games and in models with quadratic utility, these equilibria in scalable strategies are driven by different factors. In a quadratic utility model a player observing a higher signal, knows that in expectation the

[2]For models with endogenous information structures see for example Colombo & Pavan (2014) Myatt & Wallace (2012) and Pavan (2014).

state has gone up and so has the aggregate action of other players. Since his utility function is quadratic he wants to raise his action in the same way. Due to the affine relationship between signals, the player uses a scalable strategy profile to increase his action. On the other hand, in the model outlined below, scalable strategies arise, because both the information structure and the payoff structure are scaled for any type. This different mechanism is suitable for different applications across a variety of type spaces.

Finally considering a translation from one game to a strategically equivalent game, which is easier to solve, has also been proposed by Baye & Hoppe (2003) in the case of rent seeking and patent races. However they consider relationships between games of complete information, while we consider translations from an incomplete information game to a corresponding complete information game. To fix ideas we now consider an example

1.2 Example

We now introduce a simple example to illustrate the strategic equivalence of certain games that are closely related, but have a different information structure. Three cases are distinguished (i) players face uncertainty about the types of other players, (ii) players face uncertainty because the translation from actions to outcomes is noisy and (iii) a complete information game.

Consider a world with two competing countries labeled $\{1, 2\}$ who actively exert their influence in a certain region. At time θ a new militant group emerges, which threatens the security of one country but furthers the interests of the other. It is assumed that countries have no prior information about when the new group will emerge, and this is modelled by θ being drawn from a diffuse prior with $g(\theta) = 1$ for all $\theta \in \mathbb{R}$.

Each country does not immediately learn of this new development, but rather learns at some time $t_i = \theta + z_i$, where each z_i is independent of θ and is distributed uniformly over the interval $[0, 1]$. After learning of the event each country must choose a time $a_i \geq t_i$ at which to decide upon a response. It is assumed that decisions are immediately put into action.

Since better intelligence will lead to more effective intervention, it is assumed that the payoff associated with executing an action at time a_i is $\overline{u}_i(a_i, t_i) = a_i - t_i$. However so that the two countries do not enter into direct conflict, only the first action chosen is executed, and the second mover receives a payoff of $\underline{u}_i(a_i, t_i) = 0$. As will be clear from the formal definition that follows this game is a scalable game.

In order to solve this scalable game we look for a symmetric equilibrium in scalable strategies of the form $\sigma_i(t_i) = t_i + e^*$. Since the strategy of country j is assumed to be monotonic, the maximization problem of country i can be written as follows:

$$\max_{a_i} \Big\{ V_i(a_i|t_i) \Big\} = \max_{a_i} \Big\{ F\Big(\sigma_j^{-1}(a_i)|t_i\Big)(a_i - t_i) \Big\}$$

Differentiating this equation leads to the following first order condition:

$$\frac{dV_i(a_i|t_i)}{da_i} = F\Big(\sigma_j^{-1}(a_i)|t_i\Big) + \frac{1}{\sigma'\Big(\sigma^{-1}(a_i)\Big)} f\Big(\sigma_j^{-1}(a_i)|t_i\Big)(a_i - t_i)$$

In equilibrium the following first order condition must be equal to 0 when $a_i = \sigma_i(t_i)$. Using the fact that for all t_i (i) $\sigma_i(t_i) = \sigma_j(t_i)$, (ii) $\sigma'(t_i) = 1$ and (iii) $\sigma_i(t_i) - t_i = e^*$ we reach:

$$F(t_i|t_i) + f(t_i|t_i)e^* = 0$$

It can be calculated that $F(t_i|t_i) = 0.5$ while $f(t_i|t_i) = 1$. Hence $e^* = 0.5$ and indeed $\sigma_i(t_i) = t_i + 0.5$ corresponds to an equilibrium in scalable strategies. This completes the analysis of the scalable game.

Consider now that instead of delay both countries learn of the emergence of the new militant group immediately and so $t_i = \theta$. Again each country chooses a time a_i at which to decide upon a response. However in this version of the game there is a delay between the decision to act and the implementation of the action itself. Indeed the action only comes into effect at a time $a_i + z_i$ where again z_i is drawn from a uniform distribution over the interval $[0, 1]$ for each $i \in \{1, 2\}$. Country i is the first mover only if $a_i + z_i < a_j + z_j$ and in this case country i receives a payoff of $\overline{u}_i(a_i, t_i) = a_i - t_i$. The second mover again receives a payoff of

$\underline{u}(a_i, t_i) = 0$. Although this second game is a game with uncertainty over the state of nature rather than over types, it bears a strong resemblance to the first game.

In order to solve this game we look again for an equilibrium in scalable strategies of the form $\sigma_i(t_i) = t_i + e^*$. The maximization problem of player i can be written as follows:

$$\max_{a_i} \Big\{ V_i(a_i|t_i) \Big\} = \max_{a_i} \Big\{ P\Big(a_i + z_i < a_j + z_j\Big)(a_i - t_i) \Big\}$$

Note that $a_i = e_i - t_i$ and $a_j = e_j - t_j$. Making e_i the choice variable and re-writing the maximization problem accordingly leads to:

$$\max_{e_i} \Big\{ V_i(e_i + t_i|t_i) \Big\} = \max_{e_i} \Big\{ P\Big(e_i + z_i < e_j + z_j\Big) e_i \Big\}$$

Writing the left-hand side in terms of an integral leads to:

$$\max_{e_i} \Big\{ V_i(e_i + t_i|t_i) \} = \max_{e_i} \Big\{ e_i \int_0^1 \int_0^1 \mathbf{1}_{\{} e_i + z_i < e_j + z_j \} dz_i dz_j \Big\}$$

Note that the right-hand side depends only on (e_i, e_j). This motivates us to consider the game of complete information without types.

Finally suppose that each country learns of the emergence of the new militant group at time 0 and so $t_i = \theta = 0$. Each country then chooses a time to wait $e_i = a_i - t_i$. After choosing a time to wait payoffs are realised and are given as follows:

$$\phi_i(e_i, e_j) = e_i \int_0^1 \int_0^1 \mathbf{1}_{\{} e_i + z_i < e_j + z_j \} dz_i dz_j$$

Here we have defined a game of complete information using the maximization problem of the previous game. After evaluating the integral, the payoff function can be written as follows:

$$\phi_i(e_i, e_j) = \begin{cases} 0 & \text{if} \quad e_i > e_j + 1 \\ \frac{(1-e_i+e_j)^2}{2} e_i & \text{if} \quad e_j < e_i < e_j + 1 \\ \left[1 - \frac{(1-e_j+e_i)^2}{2}\right] e_i & \text{if} \quad e_j - 1 < e_i < e_j \\ e_i & \text{if} \quad e_i < e_j - 1 \end{cases}$$

Again we look for a symmetric pure strategy equilibrium of the form (e^*, e^*). Differentiating with respect to e_i and using the fact that $e_i = e_j = e^*$ leads to:

$$\frac{\delta \phi(e_i, e_j)}{\delta e_i} = -e^* + 0.5$$

Since at equilibrium this first order condition must be equal to 0 it follows that $e^* = 0.5$ is the only candidate equilibrium. Indeed it can be shown that it is a symmetric pure strategy equilibrium. This proves a particular case of the theorem showing that $\sigma_i(t_i) = t_i + e^*$ is an equilibrium of the initial scalable game if and only if (e^*, e^*) is an equilibrium of the corresponding complete information game. The next section now formalizes this claim in a general setting.

2 Model

First consider a finite set of players $I = \{1, ..., n\}$, each of whom have a type $t_i \in (\underline{t}_i, \bar{t}_i) = T_i$ and choose actions $a_i \in (\underline{a}_i, \overline{a}_i) = A_i$. Note that it is assumed that A_i and T_i are open intervals for all $i \in I$. We use $\mathbf{t} = (t_1, \ldots, t_n)$ and $\mathbf{a} = (a_1, \ldots, a_n)$ to denote the vector of types and the vector of actions respectively. Moreover t_{-i} (or a_{-i}) is used to mean the vector $\mathbf{t}$ (or $\mathbf{a}$) excluding the i'th element. Secondly define a domain to be a set $(\underline{\theta}, \overline{\theta}) = \Theta \subseteq \mathbb{R}$, from which the state θ is drawn.[3] Without loss of generality we consider the case where $A_i = T_i = \Theta$ for all $i \in I$. Thirdly consider a strictly increasing differentiable function $G(\theta)$ - referred to as a generator - where G is a bijection from Θ to $\mathbb{R}$. An environment $\{I, \Theta, G\}$ is made up of a set of players I, a domain Θ and a generator G.

[3] We allow for the case where $\underline{\theta} = -\infty$ or $\overline{\theta} = \infty$

The generator G is associated with (i) an improper prior $g(\theta)$, (ii) binary operators $\oplus_G$ and $\ominus_G$ and (iii) a set of strategies Σ_G. We now introduce each of these components. The derivative of the generator $G(\theta)$ is denoted by $g(\theta)$, which corresponds to an improper prior over the state space Θ. Given a generator G, we define the operators $a \oplus_G b = G^{-1}\Big(G(a) + G(b)\Big)$ and $a \ominus_G b = G^{-1}\Big(G(a) - G(b)\Big)$. The two important examples are given in the table below:[4]

Θ	$G(\theta)$	$g(\theta)$	$a \oplus_G b$	$a \ominus_G b$
$\mathbb{R}$	θ	1	$a + b$	$a - b$
$\mathbb{R}_{++}$	$\ln(\theta)$	$\frac{1}{\theta}$	$a \times b$	$a \div b$

The first case of a uniform improper prior where $\Theta = \mathbb{R}$ and $g(\theta) = 1$, has been used in the global games and auction literature.[5] Meanwhile the second case where $\Theta = \mathbb{R}_{++}$ and $g(\theta) = \frac{1}{\theta}$ plays an important role in our applications. Other cases shed light on the extent to which strategy restrictions can be justified by appealing to games without strategy restrictions. Results obtained using improper priors closely correspond to results obtained using suitably chosen proper priors; a formal discussion of improper priors can be found in Hartigan (1983).[6]

Remembering that $T_i = A_i = \Theta$, a strategy for player i is a mapping $\sigma_i : \Theta \mapsto \Theta$. We define the set of strategies associated with a generator $G : \Theta \mapsto \mathbb{R}$ as follows:

$$\Sigma_G := \Big\{\sigma_i \Big| \text{ for some } \;\; e_i \in \Theta, \sigma_i(t_i) = t_i \oplus_G e_i \;\; \text{for all} \;\; t_i \in \Theta\Big\}$$

In the special case where $\Theta = \mathbb{R}$ and $G(\theta) = \theta$, the set Σ_G contains the additively scalable strategies of the form $\sigma_i(t_i) = t_i + e_i$. Similarly when $\Theta = \mathbb{R}_{++}$ and $G(\theta) = \ln(\theta)$, the set Σ_G contains the multiplicatively scalable strategies of the form $\sigma_i(t_i) = t_i \times e_i$. Throughout

[4] Note that $(a \oplus_G b) \oplus_G c = a \oplus_G (b \oplus_G c)$. Moreover it is easy to check that $(a \oplus_G b) \ominus_G c = a \oplus_G (b \ominus_G c)$. This ensures standard addition and subtraction can be used.

[5] See for example Klemperer (1999) and Morris & Shin (2002)

[6] In the appendix it is shown that when δ is small, results obtained under the uniform prior $g(\theta) = 1$ closely correspond to results obtained using the proper prior $g(\theta) = \frac{1}{2\delta} e^{-\delta|\theta|}$. This result is easily generalised to other improper priors.

the paper we will use $\sigma(\mathbf{t}) = (\sigma_1(t_1), ..., \sigma_n(t_n))$ to denote the vector of strategy profiles and $\mathbf{x} \oplus_G k$ to mean $(x_1 \oplus_G k, x_2 \oplus_G k, \ldots, x_n \oplus_G k)$ where k is a constant. Similarly, $\mathbf{x} \oplus_G \mathbf{y}$ means $(x_1 \oplus_G y_1, x_2 \oplus_G y_2, \ldots, x_n \oplus_G y_n)$. Finally we define $0_G = G^{-1}(0)$ and note that $0_G \in \Theta$.

2.1 The Game

In this paper we consider one shot simultaneous move games in incomplete information. The game is described by the environment $\{\Theta, I, G\}$ along with a utility function for each player given by $u_i(\mathbf{a}, \theta, t_i)$ and a conditional distribution function of a player's type given each state $\theta \in \Theta$, denoted by $F_i(t_i|\theta)$. These functions are assumed to be independent conditional on θ.

The timing can be thought of as follows: first the state θ is drawn from the domain Θ according to the improper prior $g(\theta)$, but is not observed by the players. For each player $i \in I$, a type t_i is drawn from the conditional distribution function $F_i(t_i|\theta)$. Each player privately observes his type but does not observe the state nor the types of his opponents. We assume that conditional on their type, players form beliefs about the state according to Bayesian updating on the uninformative prior. Hence the conditional probability that player i assigns the to the state being less than θ can be written as follows:

$$G_i(\theta|t_i) = \frac{\int_{-D}^{\theta} f_i(t_i|\tilde{\theta})g(\tilde{\theta})d\tilde{\theta}}{\int_{\Theta} f_i(t_i|\tilde{\theta})g(\tilde{\theta})d\tilde{\theta}} \tag{1}$$

These conditional probabilities about the state are relevant for the player when determining his expected payoff and hence when choosing an action.

Having observed his type each player chooses an action a_i and receives a payoff $u_i(\mathbf{a}, \theta, t_i)$. To summarise, a game with an improper prior is composed of the following elements:

$$\Gamma = \{\Theta, I, g, (F_i)_{i\in I}, (u_i)_{i\in I}\}$$

2.2 Scalable payoff structure

We now define a scalable payoff structure which is composed of an environment $\{I, \Theta, G\}$ and a set of payoff functions $(u_i)_{i \in I}$. The payoff function of player i is given by $u_i : (A_j)_{j \in I} \times \Theta \times T_i \mapsto \mathbb{R}$ and maps (i) the actions $(a_1, ..., a_n)$ of all players, (ii) the state θ and (iii) the type t_i of player i to a payoff. Informally a payoff function is scalable if when the inputs of u_i are scaled the corresponding payoff is scaled in a similar way. For instance an auction without an entry fee described by a scalable payoff function since scaling the valuation and the bids of all players leaves the payoffs agents receive unchanged except for a scaling factor. However an auction with entry costs is not described by a scalable payoff function, since scaling the valuation and bids of all players changes the burden of the entry cost relative to the potential reward. Formally we now introduce a scalabel payoff structure as follows:

Assumption 1 (Scalable payoff structure)**.** *There exist functions $C_i(t_i) : \Theta \mapsto \mathbb{R}_{++}$ and $D_i(t_i) : \Theta \mapsto \mathbb{R}$, such that for all $i \in I$, for all $k, t_i, \theta \in \Theta$ and for all $\mathbf{a} \in \mathbb{R}^n$:*

$$\frac{u_i(\mathbf{a}; \theta; t_i) - D_i(t_i)}{C_i(t_i)} = \frac{u_i(\mathbf{a} \oplus_G k; \theta \oplus_G k; t_i \oplus_G k) - D_i(t_i \oplus_G k)}{C_i(t_i \oplus_G k)}$$

Note that this assumption refers to the opeartor $\oplus_G$ and hence depends on the generator function G. For this reason we say that the payoff structure is scalable with respect to G if the utility functions satisfy assumption 1. To show that this payoff assumption can capture several economic environments, we now prove two lemmas and provide a number of examples:

Lemma 2.1. *Suppose $\Theta = \mathbb{R}$ and $G(\theta) = \theta$. Moreover suppose for all $i \in I$, for all $\theta, k, t_i \in \Theta$ and for all $\mathbf{a} \in \Theta^n$:*

$$u_i(\mathbf{a}; \theta; t_i) = u_i(\mathbf{a} + k; \theta + k; t_i + k)$$

Then the payoff structure is scalable.

The payoff structure in this case is homogeneous of degree zero in the log transform. It can easily be seen that Assumption 1 is satisfied by setting $C_i(t_i) = 1$ and $D_i(t_i) = 0$ for all

$t_i \in \Theta$, for all $i \in I$. One environment that satisfies this case is a beauty contest where agents want their move a_i both to be close to the true state θ and to be close to the average move. Such a contest can be summarised by the following payoff function, where $r \in [0, 1]$ captures the relative importance of being close to the true state and of being close to the average move:

$$u_i(\mathbf{a}; \theta; t_i) = -(1-r)\Big(a_i - \theta\Big)^2 - r\Big(a_i - \frac{1}{|I|}\sum_{j \in I} k_j\Big)^2$$

Note here that t_i does not directly enter the payoff function and is simply a signal player i uses to gain information about the value of θ and inform his decision. Beauty contests with similar payoff structures have been considered by Morris & Shin (2002) and Myatt & Wallace (2012).

Lemma 2.2. *Suppose $\Theta = \mathbb{R}_{++}$ and $G(\theta) = \ln\theta$. Moreover suppose that for some $\alpha \in \mathbb{R}_+$ for all $\theta, t_i, k \in \Theta$ and for all $i \in I$, for all $\mathbf{a} \in \Theta^n$:*

$$u_i(\mathbf{a}; \theta; t_i) = k^{\alpha} u_i(\mathbf{a}.\mathbf{k}; \theta k; t_i k)$$

Then the payoff structure is scalable.

This lemma shows that - by considering a suitable domain Θ and suitable distribution function G - any payoff function which is homogenous of degree α can be captured. One example that satisfies this structure is a first price auction with a combination of private values and common values. Let t_i^{β} capture the private value element of a player's valuation and $\theta^{1-\beta}$ capture the common value element of a player's valuation. Player i submits a bid a_i and if he submits the highest bid he wins the object and pays his bid. If he submits the lower bid he does not win the object and pays nothing. This is summarised by the function below, where $\beta \in [0, 1]$ captures the relative importance of private values and common values:

$$u_i(a_i, a_j; \theta; t_i) = \begin{cases} t_i^{\beta}\theta^{1-\beta} - a_i & \text{if} \quad a_i > a_j \\ 0 & \text{otherwise} \end{cases}$$

As well as being able to model several auction environments, quadratic utility models - which are homogenous of degree two - can be captured in this setting. One example of a quadratic utility model that can be captured is a model of Cournot competition with linear demand. Here a_i captures the quantity player i produces, and θ represents a demand shock about which agents are imperfectly informed. The price is given by $\Big(\theta - \sum_{j \in I} a_j\Big)$ and hence the payoff of agents becomes:

$$u_i(a_i; \theta; t_i) = a_i\Big(\theta - \sum_{j \in I} a_j\Big)$$

Further applications to auctions are studied in section 6. Having given examples of the environments that can be captured by scalable payoff functions, we now turn attention to the second assumption.

2.3 Scalable information structure

We now define a scalable information structure which is composed of an environment $\{\Theta, I, G\}$ and a set of conditional distributions $(F_i)_{i \in I}$. The conditional distribution associated with player i is given by $F_i : T_i \times \Theta \mapsto [0, 1]$, where $F_i(t_i|\theta)$ captures the probability that - given the state is θ - the type of player i is less than or equal to t_i. It is assumed throughout that $F_i(t_i|\theta)$ is differentiable with respect to t_i, with derivative $f_i(t_i|\theta)$. Moreover we assume that t_i and t_j are independent conditional on θ whenever $i \neq j$. With this in mind, we define a scalable information structure as follows:

Assumption 2 (Scalable information structure)**.** *For all $i \in I$ and for all $k, \theta, t_i, \in \Theta$:*

$$F_i(t_i|\theta) = F_i(t_i \oplus_G k|\theta \oplus_G k)$$

Similarly this assumption refers to the operator $\oplus_G$ and depends on the generator function G. Therefore we say that the information structure is scalable with respect to G if the conditional distribution functions satisfy assumption 2. This assumption captures the fact that the conditional distribution of types has a similar shape when θ is changed. When

$a \oplus_G b = a + b$ this implies that conditional beliefs are additively invariant: that is to say the shape of the distribution is common knowledge but players do not know their position in the distribution. For instance this holds when players know that the distribution is uniform over the interval $[\theta - 1, \theta + 1]$, but do not necessarily know the value of the state θ. This is illustrated in Figure 1.

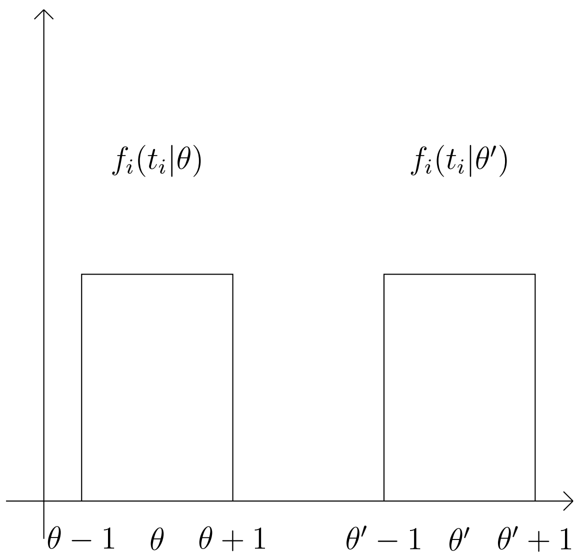

Figure 1: Uniform:Additive

Meanwhile when $a \oplus_G b = a \times b$ this assumption implies that conditional beliefs are homogenous of degree 0. For instance this holds when players know that the distribution is uniform over the interval $[0, 2\theta]$, but do not necessarily know the value of the median θ. This is illustrated in Figure 2.

2.4 Scalable games

Above we have introduced all components and assumptions necessary to define a scalable game $\Gamma = \{\Theta, I, G, (F_i)_{i \in I}, (u_i)_{i \in I}\}$. If players' utility functions and the information structure are scalable with respect to G, then we say that the game is scalable. Formally:

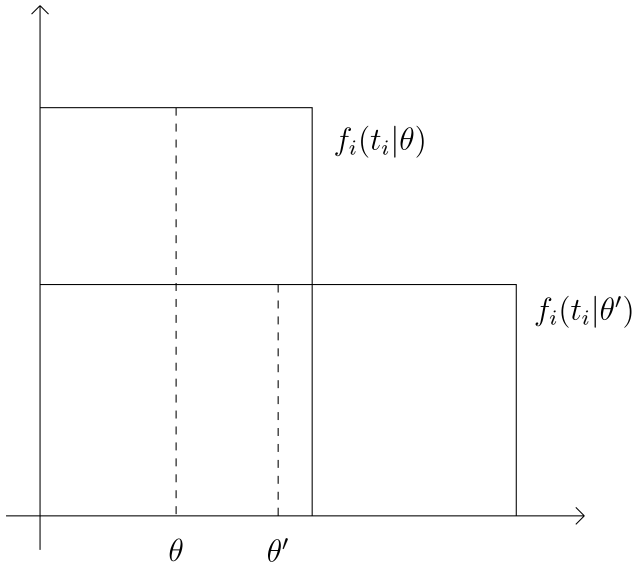

Figure 2: Uniform: Multiplicative

Definition 1. *Scalable Game The game* Γ *is a scalable game, if it is a one shot simultaneous move game and satisfies Assumptions 1 and 2, of the payoff structure and the information structure being scalable.*

2.5 Equilibrium

The equilibrium concept used when analysing scalable games is Bayesian Nash equilibrium. Using $\sigma(\mathbf{t}) = (\sigma_1(t_1), \ldots, \sigma_n(t_n))$ to denote a strategy profile, formally an equilibrium is defined as follows:

Definition 2 (Equilibrium)**.** *The strategy profile* $\sigma(\mathbf{t})$ *is an equilibrium of the scalable game* Γ*, if and only if for all players* $i \in I$ *and for all types* $t_i \in \Theta$ *and all deviations* $a_i \in \Theta$ *it holds that:*

$$\int_{\Theta^n} g_i(\theta|t_i) \prod_{j \neq i}^{n} f_j(t_j|\theta) u_i\Big(\sigma_i(t_i), \sigma_{-i}(t_{-i}); \theta; t_i\Big) dt_{-i} d\theta \geq \int_{\Theta^n} g_i(\theta|t_i) \prod_{j \neq i}^{n} f_j(t_j|\theta) u_i\Big(a_i, \sigma_{-i}(t_{-i}); \theta; t_i\Big) dt_{-i} d\theta$$

This definition says that in equilibrium player i with type t_i has no incentive to deviate from his prescribed strategy $\sigma_i(t_i)$ to another strategy a_i. To calculate his expected utility from playing a certain strategy, players consider the likelihood of the state being θ and the opponents' types being t_{-i}. This is captured in the expression $g_i(\theta|t_i)\prod_{j\neq i}^{n} f_j(t_j|\theta)$.

2.6 Expected payoff function against scalable strategies (EPFASS)

The focus of this paper lies on a special class of equilibria, where all players follow strategies of the form $\sigma_j(t_j) = t_j \oplus_G e_j$, which we call scalable strategies. If $G(\theta) = \theta$, then scalable strategies are additively linear of the form $\sigma_j(t_j) = t_j + e_j$. Meanwhile if $G(\theta) = \ln(\theta)$, then scalable strategies are multiplicatively linear of the form $\sigma_j(t_j) = t_j.e_j$. In order to simplify notation when considering these equilibria, we introduce the expected payoff function against scalable strategies (henceforward EPFASS) describing a player's expected payoff when he has a certain type t_i and all players j use scalable strategies of the form $\sigma_j(t_j) = t_j \oplus_G e_j$. The EPFASS is given as follows:

Definition 3 (EPFASS)**.**

$$V_i(e_i|e_{-i}, t_i) = \int_{\mathbf{D}^n} g_i(\theta|t_i) \prod_{j\neq i} f_j(t_j|\theta) u_i\Big(e_i \oplus t_i, e_{-i} \oplus t_{-i}; \theta; t_i\Big) d\theta \prod_{j\neq i} dt_j$$

This additional notation completes the description of a scalable game and we now turn to the analysis.

3 Analysis

First we introduce the complete information game Γ_N induced by a scalable game. This game is in normal form, and hence there is no uncertainty over the types of each player. The formal definition of the complete information game induced by a scalable game is given below:

Definition 4 (Complete information game)**.** *The complete information game* $\Gamma_N = \{I, (A_i)_{i\in I}, (\phi_i)_{i\in I}\}$ *induced by a scalable game*
$\Gamma = \{\Theta, I, G, (F_i)_{i\in I}, (u_i)_{i\in I}\}$ *has the following payoff function:*

$$\phi_i(e_i, e_{-i}) = \int_{\Theta^n} \prod_{j=1}^{n} f_j(z_j|0_G) u_i\Big(e_i, e_{-i} \oplus_G z_{-i} \ominus_G z_i; 0_G \ominus_G z_i; 0_G\Big) \prod_{j=1}^{n} dz_j$$

We refer to the variables of integration $(z_1, ..., z_n)$ as payoff shocks. Taking expectations over these payoff shocks leads to the formation of the new payoff functions $\phi_i(e_i, e_{-i})$. Note that these payoff functions induce a complete information game without uncertainty.

The following proposition formally links the payoff function of the complete information game to the EPFASS of the original scalable game. This result is central to the analysis and is formally stated as follows:

Proposition 3.1. *For all i there exists functions $C_i : \Theta \mapsto \mathbb{R}_+$ and $D_i : \Theta \mapsto \mathbb{R}$ such that for all $t_i \in \Theta$*

$$\phi_i(e_i, e_{-i}) = \frac{1}{C_i(t_i)}\Big[V_i(e_i|e_{-i}, t_i) - D_i(t_i)\Big]$$

This proposition shows that a scalable game with *ex-ante uncertainty* over types is closely related to the corresponding game of complete information with *interim uncertainty* over payoff shocks. In particular there is a close correspondence when players choose scalable strategies $\sigma = (\sigma_i)_{i \in I}$ where $\sigma_i(t_i) = t_i \oplus_G e_i$ in the scalable game and pure strategies $\mathbf{e} = (e_1, ..., e_n)$ in the complete information game.

This proposition links the EPFASS of the incomplete information game to the payoff function of the complete information game. Focusing on scalable strategies, the assumptions of the payoff structure and the information structure being scalable (assumptions 1 and 2), allow us to reduce the dimensionality of the decision problem. Up to a normalization, the maximization problem looks the same for all types and can hence be written as the game of complete information defined in equation 4. Although this proposition is the central part of the result, the proof is relegated to the appendix, as it is notationally cumbersome. Using Proposition 3.1 we can now state the main result of the paper:

Theorem 3.2. *Consider a scalable game Γ. The strategy profile $\sigma^* = (\sigma_1^*, ..., \sigma_n^*)$ where $\sigma_i(t_i) = t_i \oplus e_i^*$ is a Nash equilibrium of this game , if and only if the strategy profile $\mathbf{e}^* = (e_1^*, ..., e_n^*)$ is a Nash equilibrium of the corresponding complete information game Γ_N.*

The proof can be found in the appendix.

This theorem shows that there is a correspondence between the equilibrium in scalable strategies of a scalable game and the pure strategy Nash equilibrium of the corresponding complete information game. This is useful because it means that an economist can model uncertainty using a scalable game, but in order to find an equilibrium it is sufficient to analyse a corresponding complete information game. Using scalable games enables the difficult problem of finding equilibria of models of incomplete information to be reduced to the simpler problem of finding equilibria of models of complete information.

Using a scalable game to analyse a situation with uncertainty in a tractable manner is similar to the approach taken by Abreu & Brunnermeier (2003), who analyse asset-pricing bubbles. Indeed minor changes in modelling choices[7] ensure that the model considered by Abreu & Brunnermeier (2003) can be written as a scalable game. The equilibrium in scalable strategies found in this paper could be found by examining a game of complete information: the choice problem is essentially the same to that of a complete information game. This example shows that scalable games are a useful tool for modelling a wide range of uncertainties in a tractable manner.

4 Strategy restrictions

In this section we investigate a link between scalable games and games with strategy restrictions. In a game with strategy restrictions, players are required to choose their strategy from a limited set of strategies. Recently Compte & Postlewaite (2013) consider strategy restrictions in auctions. Their relationship with their approach is discussed in more detail below.

In this section we highlight a relationship between such games with strategy restrictions and scalable games. This provides a foundation for games with strategy restrictions as it demonstrates that under certain conditions a game in which strategy restrictions are imposed is equivalent to a scalable game without strategy restrictions. Moreover we show what type

[7]In particular using an improper prior rather than a specially designed proper prior

of strategy restrictions can be justified in this way, which depends on the game.

A game with strategy restrictions is defined by a set of players I, a an action space for each player $A_i = (\underline{a}, \overline{a})$, a set of types for each player $T_i = (\underline{t}_i, \overline{t}_i)$, a payoff function for each player $(\pi_i)_{i \in I}$ a probability distribution function h and a set of strategies Σ. Here each π_i is a mapping $\pi_i : (A_i)_{i \in I} \times T_i \mapsto \mathbb{R}$, while $h : (T_i)_{i \in I} \mapsto \mathbb{R}$. As before we apply normalisations if necessary, and without loss of generality assume $A_i = T_i = (\underline{\theta}, \overline{\theta}) = \Theta$ for all $i \in I$.

A strategy for player i is again a mapping $\sigma_i : \Theta \mapsto \Theta$ and Σ denotes a collection of such strategies. Therefore after appropriate normalizations a game with strategy restrictions can be captured by $\Gamma^R = \Big\{ I, \Theta, (\pi_i)_{i \in I}, h, \Sigma \Big\}$. It is assumed that types are distributed independently with $h(\mathbf{t}) = \prod_{i \in I} h_i(t_i)$.

In order to ensure that players indeed obey the strategy restriction, each player chooses a strategy $\sigma_i \in \Sigma$ *before* observing his type.[8] This means that the maximization problem that players face can be written as follows:

$$\max_{\sigma_i \in \Sigma} \Big\{ \int_{\Theta^n} h(\mathbf{t}) u_i \Big(\sigma_i(t_i), \sigma_{-i}(t_{-i}) \Big) d\mathbf{t} \Big\} \tag{2}$$

Equilibrium requires that no player can gain in expectation by deviating from one strategy σ_i^* to another strategy $\hat{\sigma}_i$. As well as taking expectations over the types of other players, each player also takes expectations over his own type. This captures the fact that a player does not know his type when choosing his strategy:

Definition 5. *The strategy profile $\sigma^* = (\sigma_1^*, ..., \sigma_n^*)$ is an equilibrium if for all $i \in I$ and $\hat{\sigma}_i$*

$$\int_{\Theta^n} h(\mathbf{t}) u_i \Big(\sigma_i^*(t_i), \sigma_{-i}^*(t_{-i}) \Big) d\mathbf{t} \geq \int_{\Theta^n} h(\mathbf{t}) u_i \Big(\hat{\sigma}_i(t_i), \sigma_{-i}^*(t_{-i}) \Big) d\mathbf{t}$$

Let S be the set of all strategies $\sigma_i : \Theta \mapsto \Theta$. In the special case where $\Sigma = S$, players are unrestricted. Therefore the interest in games of strategy restrictions lies in the case where $\Sigma \subset S$. The following result shows how certain games with strategy restrictions correspond to scalable games without strategy restrictions.

[8] Note that if players choose their strategy conditioning on their type, they may choose a different strategy profile for each realisation of the type, hence creating a new strateg profile not necessarily in Σ.

Theorem 4.1. *Suppose G is a bijection between Θ and $\mathbb{R}$ and consider a game with strategy restrictions $\Gamma^R = \left\{I, \Theta, (\pi_i)_{i\in I}, H, \Sigma_G\right\}$. Define the corresponding scalable game $\Gamma = \left\{I, \Theta, (u_i)_{i\in I}, F, g\right\}$ where*

1. $g(t) = G'(t)$
2. $F(\mathbf{t}|\theta) = H(\mathbf{t} \ominus_G \theta)$
3. $u_i(\mathbf{a}; t_i; \theta) = \pi_i(\mathbf{a} \ominus_G \theta; t_i \ominus_G \theta)$

Then σ^* is an equilibrium of the game with strategy restrictions Γ^R if and only if σ^* is an equilibrium of the scalable game Γ.

Therefore when $A_i = T_i = \Theta$ for all $i \in I$ and $\Sigma = \Sigma_G$, then a game with strategy restrictions has the same equilibria as some corresponding scalable game. This means that strategy restrictions can be justified whenever Theorem 4.1 can be applied. Theorem 4.1 shows that any equilibrium of Γ^R reached by applying the strategy restriction Σ_G corresponds to an equilibrium of a scalable game Γ with (i) no strategy restrictions, (ii) a similar payoff function and (iii) a different information structure. In many cases the payoff functions directly correspond:[9] for instance the case of private value auctions and multiplicative strategy restrictions corresponds to the case of private value auctions with a scalable information structure.

To summarize this analysis provides foundations for certain games of strategy restrictions and gives a modeler guidance on which family of strategy restrictions is most appropriate for which problem. In particular additive strategy restrictions are likely to be more appropriate when $A_i = T_i = \Theta = \mathbb{R}$, while multiplicative strategy restrictions are likely to be more appropriate when $A_i = T_i = \Theta = \mathbb{R}_{++}$. For this reason we believe that in the context of auctions when each valuation $t_i \in \mathbb{R}_{++}$, the multiplicative strategy restrictions may be have stronger foundations than the additive strategy restrictions considered by Compte & Postlewaite (2013).

[9] Suppose for some function $\psi_i : \Theta^n \mapsto \mathbb{R}$ that $\pi_i(\mathbf{a}; t_i) = \psi_i(\mathbf{a} \ominus_G t_i)$ then $u_i(\mathbf{a}; t_i; \theta) = \pi_i(\mathbf{a}; t_i)$. This makes the games directly comparable: for instance a private value auction with strategy restrictions will be associated with a private value auction under a scalable information structure.

5 Application - modeling complex situations

The aim of this section is to show how scalable games can be used to build tractable models of complex situations. Examples of incomplete information games that can sensibly be stated in a way to fit the suggested framework include various auction formats including both private and value auctions in either a first price, all pay or fractional all pay setting. Indeed the double auction framework considered by Satterthwaite *et al.* (2014) is an example of an existing model in the literature, where the scalable games structure ensures tractability of an otherwise complex situation. Another situation that can be modelled using a scalable games structure is the formation of asset price bubbles. The setting described by Abreu & Brunnermeier (2003) can be easily adapted to fit assumptions 1 and 2 and to illustrate the same effects. The advantage of using scalable games is that the procedure for finding an equilibrium is particularly simple.

Both the applications considered by Abreu & Brunnermeier (2003) and Satterthwaite *et al.* (2014) require the domain to be $\Theta = \mathbb{R}$ and the function $G(\theta) = \theta$. Our applications consider a different domain namely $\Theta = \mathbb{R}_{++}$ and the function $G(\theta) = \ln(\theta)$. This is particularly appropriate when it is natural to restrict the type space to the positive numbers, such as when a player's type represents his valuation for an object. When $\Theta = \mathbb{R}_{++}$ and $G(\theta) = ln(\theta)$ it follows that $a \oplus_G b = ab$. It follows that the complete information game corresponding to a scalable game of this form is given as follows:

$$\phi_i(\mathbf{e}) := \int_{\mathbf{z}\in\mathbb{R}^n} \left(\prod_{j=1}^{n} f_j(z_j|1)\right) u_i\Big(\frac{1}{z_i}\mathbf{e}.\mathbf{z}; \frac{1}{z_i}, 1\Big) \prod_{j=1}^{n} dz_i \tag{3}$$

This form will be used repeatedly in the applications that follow.

5.1 Auctions with risk aversion and a reserve price

Here we consider risk averse players competing in a first price auction with a reserve price. The variable θ denotes the reserve price, while the parameter $r_i \in [1, \infty)$ captures the level of risk aversion of player i. In the auction examples considered here no symmetry assumptions are made: players may be drawn from different distributions with $F_i \neq F_j$ for $i \neq j$; moreover players may have different levels of risk aversion with $r_i \neq r_j$ when $i \neq j$. The utility function of the scalable game is given as follows:

$$u_i(\mathbf{a}; \theta; t_i) = \begin{cases} (t_i - a_i)^{r_i} & \text{if} \quad a_i > a_j \text{ whenever } i \neq j \text{ and } a_i > \theta \\ 0 & \text{otherwise} \end{cases} \tag{4}$$

In this case the complete information game is $\{I, (\phi_i)_{i \in I}\}$ where for each $i \in I$:

$$\phi_i(e_i, e_{-i}) = (1 - e_i)^{r_i} \int_{\frac{1}{e_i}}^{\infty} \int_{\mathbb{R}^{n-1}_{++}} \prod_{j \in I} f_j(z_j|1) \mathbf{1}_{\{e_i z_i > e_j z_j\}} \prod_{j \neq i} dz_j dz_i \tag{5}$$

Here the complete information game can be interpreted as follows. The term $(1 - e_i)^{r_i}$ corresponds to the utility that player i will receive in the case that he wins. Meanwhile the term inside the integral represents the probability with which player i will win the auction given that he bids according to a multiplicative strategy $\sigma_i(t_i) = t_i e_i$. If player i is to win a number of inequalities must hold: first player i must submit the highest bid with $a_i > a_j$ for all j which corresponds to the inequality $e_i z_i > e_j z_j$; secondly player i must bid above the reserve price with $a_i > \theta$ which corresponds to the inequality $\frac{1}{z_i} > e_i$. This shows that the link between the scalable game and corresponding complete information game.

5.2 Fractional all-pay auctions with private and common values

We now turn attention to all-pay auctions, where the parameter γ denotes the extent to which costs are incurred by a bidder regardless of the outcome of the auction. At the extremes, $\gamma = 1$ captures the case of a first price auction and $\gamma = 0$ captures the case of an all-pay auction. Intermediate values of $\gamma \in (0, 1)$ capture intermediate cases where a player pays some fraction of his bid in the event of losing the auction. In this case θ is used to model a common value component of the good. In such a model players' valuations t_i could be considered as noisy estimates of the common value θ. In this model we allow for the utility of a player to depend on some combination of the common value θ and their noisy estimate t_i.

The parameter $\beta \in [0, 1]$ captures the importance of the common value component compared to the private value. The utility function of the scalable game is given as follows:

$$u_i(\mathbf{a}; \theta; t_i) = \begin{cases} t_i^{\beta}\theta^{1-\beta} - a_i & \text{if} \quad a_i > a_j \text{ whenever } i \neq j \\ (1-\gamma)a_i & \text{otherwise} \end{cases} \tag{6}$$

This leads to the following game of complete information

$$\phi_i(e_i, e_{-i}) = \int_0^{\infty} f_i(z_i|1) \prod_{j \notin I} F_j\Big(\frac{e_i z_i}{e_j}\Big)(z_i^{\beta-1} - \gamma e_i)dz_i - (1-\gamma)e_i \tag{7}$$

The complete information game in this case is somewhat harder to interpret, because the probability of winning and the valuation player i places on the good are related: indeed if the idiosyncratic shock z_i is higher, then player i is on the one hand more likely to win the auction but on the other hand will value the object less. This relationship is reflected in the fact that the term representing the valuation $(z_i^{\beta-1} - 1)$ now enters the integral and is decreasing in z_i.

5.3 Second price auctions with common values

Second price auctions with a common value component are an important class of auctions and have been studied by Milgrom (1981), Milgrom & Weber (1982) and Liu (2014)amongst others. This class of auctions can also be modelled in this framework. In this case we consider a pure common value auction with two players where each player has a common value θ for the good and each receives a noisy signal t_i which is correlated with the common value θ.[10] The utility function of the scalable game is given as follows:

$$u_i(\mathbf{a}; \theta; t_i) = \begin{cases} \theta - a_j & \text{if} \quad a_i > a_j \text{ whenever } i \neq j \\ 0 & \text{otherwise} \end{cases} \tag{8}$$

In this case the corresponding game of complete information is given as follows:

[10]The n-player case and case with private and common values are similar.

$$\phi_i(e_i, e_j) = \int_0^\infty \int_0^\infty f_i(z_i|1) f_j(z_j|1) \mathbf{1}_{\{e_i z_i \geq e_j z_j\}} \Big(\frac{1}{z_i} - \frac{e_j z_j}{z_i} \Big) dz_j dz_i \tag{9}$$

$$= \int_0^\infty \int_0^{\frac{e_i z_i}{e_j}} f_i(z_i|1) f_j(z_j|1) \Big(\frac{1}{z_i} - \frac{e_j z_j}{z_i} \Big) dz_j dz_i \tag{10}$$

We now give some guidance on how to find equilibria to these games of complete information, and argue that such a task would be much easier than trying to solve the more general problem without scalability assumptions.

5.4 Finding pure strategy equilibria

When finding pure strategy equilibria of the complete information games given above, the first order conditions $\frac{\delta \phi_i(e_i, e_{-i})}{\delta e_i}\Big|_{\mathbf{e}=\mathbf{e}^*} = 0$ is a necessary condition for $\mathbf{e}^*$ to be a pure strategy equilibrium: since the action space is an open set there can be no boundary solutions and the first order conditions must equal 0. Therefore the pure strategy equilibria of the games above can be found by setting the first order conditions to 0 and considering the resulting system of equations (one for each player). We call a solution to these equations a candidate pure strategy equilibrium, and each candidate must be checked against all possible deviations. This either could be done mechanically or assumptions could be made ensuring candidate pure strategy equilibria are indeed equilibria.

While solving the relevant system of equations and checking the resulting solutions is not always straightforward, it is normally far easier than tackling the more general problem without scalability assumptions directly. For instance the analysis in Maskin & Riley (2000) demonstrates the complexity of solving a simple two-bidder first price auction where bidders have private values: in general such a problem requires solving a system of differential equations (one for each player) and then checking additional conditions hold.[11] On the other hand finding solutions to the complete information games outlined here involves solving a

[11] Moreover Maskin & Riley (2000) assume valuations are drawn independently which is arguably as restrictive as the scalability assumption made here.

system of simple equations (one for each player) and then checking additional conditions hold. Using scalability assumptions drastically reduces the potential set of strategies that must be considered - it effectively reduces the set of strategies to Σ_G - and enables modeling of complex situations in a tractable way.

6 Application - Contest success functions

In this section we explore the relationship between all pay auctions and stochastic contest success functions. We assume a contest success function to be a mapping $\Psi : \mathbb{R}^n_{++} \mapsto [0,1]^n$ which maps a vector of positive efforts denoted $\mathbf{E} = (E_1, ..., E_n)$ to a vector of probabilities. The term $\Psi_i(E_i, E_{-i})$ is used to denote the $i'th$ argument of Ψ and captures the probability with which player i wins the prize. Contest success functions were first modelled in this way by Tullock (1980) and since then they have been extensively investigated.

Our contribution is to provide robust foundations for a number of complete information contests, by showing that they are the corresponding complete information game of some all-pay auction. This provides a bridge between the literature on contests and the literature on all-pay auctions, allowing results proved in one framework to be translated to the other. We assume that V_i captures player i's valuation of the prize and $\mathbf{V} = (V_1, ..., V_n)$ denotes the vector of valuations. This payoff is denoted by $\Phi_i(E_i, E_{-i}, \mathbf{V})$ and given as follows:

$$\Phi_i(E_i, E_{-i}, \mathbf{V}) = \Psi_i(E_i, E_{-i})V_i - E_i \tag{11}$$

This represents that player i wins the prize valued V_i with probability $\Psi_i(E_i, E_{-i})$ and must pay a cost of E_i. It is convenient to consider strategically equivalent contests written in terms of relative efforts - that is measuring a player's effort relative to his valuation for the prize. Let $e_i = \frac{E_i}{V_i}$ denote a player's relative effort and define:

$$\begin{aligned} \psi_i(e_i, e_{-i}) &= \Psi_i(e_i V_i, e_{-i}.V_{-i}) = \Psi_i(E_i, E_{-i}) && (12) \\ \phi_i(e_i, e_{-i}) &= \frac{1}{V_i}\Phi_i(e_i V_i, e_{-i} V_i, \mathbf{V}) = \frac{1}{V_i}\Phi_i(E_i, E_{-i}, \mathbf{V}) && (13) \\ & && (14) \end{aligned}$$

Therefore $\phi_i(e_i, e_{-i}) = \psi_i(e_i, e_{-i}) - e_i$ defines a strategically equivalent contest, where the contest success function ψ maps a vector of (relative) efforts to a vector of probabilities.[12] Having written the contest in terms of relative efforts, we now make the following definition:

Definition 6. *An all pay auction Γ is associated with a contest $\{I, \Phi\}$ if*

1. *The contest $\{I, \phi\}$ is the corresponding contest in terms of relative effort*

2. *The contest $\{I, \phi\}$ is the corresponding complete information game induced by Γ*

In this section we show that a number of contests are representations of certain scalable all-pay auctions. Throughout we assume that $\Theta = \mathbb{R}_{++}$ and $G(\theta) = \ln(\theta)$.

6.1 Tullock contest success function

Consider the case where n players $I = \{1, ..., n\}$ participate in a sealed bid private value all-pay auction. The players' utility functions are thus given by equation 6 with $\gamma = 1$ and $\beta = 1$. Associated with each player is a constant V_i and uncertainty is distributed on the interval $[0, \infty)$ according to the following distribution:

$$F_i(t_i|\theta) = exp\Big[- \Big(\frac{V_i\theta}{t_i}\Big)^{\alpha}\Big] \quad \text{on} \quad [0, \infty) \tag{15}$$

The corresponding complete information game is given as follows:

$$\phi_i(e_i, e_{-i}) = \frac{(V_i e_i)^{\alpha}}{\sum_{j\in I}(V_j e_j)^{\alpha}} - e_i \tag{16}$$

[12] The advantage of this formulation is it avoids repeated writing of $\mathbf{V}$.

An axiomatization of this contest success function is given by Clark & Riis (1998), who interpret E_i as a contestant's absolute level of effort and the parameters V_i^{α} as a measure of how far the contest is skewed towards player i. Considering the corresponding contest with absolute efforts leads to the following:

$$\Phi_i(E_i, E_{-i}) = \Big[\frac{(E_i^{\alpha})}{\sum_{j\in I}(E_j)^{\alpha}}\Big]V_i - E_i \tag{17}$$

This shows that an all-pay auction with this structure of uncertainty is closely related to an unbiased Tullock contest (Tullock (1980)) where players have different valuations.[13]

6.1.1 Tullock contest success function with draws

As a second application we examine a case with n symmetric players $I = \{1, ..., n\}$ who participate in a common value all-pay auction. In this case the players' utility functions are given by equation 6 where $\gamma = 1$ and $\beta = 0$. Uncertainty is distributed on the interval $[0, \infty)$ according to the following distribution:

$$F_i(t_i|\theta) = \exp\Big[-\Big(\frac{\theta}{t_i}\Big)\Big] \quad \text{on} \quad [0, \infty) \tag{18}$$

Calculating the corresponding complete information game of this scalable game leads to the following:

$$\phi_i(e_i, e_{-i}) = \frac{e_i^2}{\Big(\sum_{j=1}^{n} e_j\Big)^2} - e_i \tag{19}$$

This complete information game is the contest success function studied by Yildizparlak (2013). The contest success function is used to model contests where ties occur with positive probabilities, such as in soccer games. The analysis provided here demonstrates that there exists a strong link between common value all pay auctions and contests with ties. This link may not seem obvious in the first place and it provides additional reasons for the importance

[13]Note that mixed strategy equilibria of the complete information Tullock contest that arise when $\beta > 2$, lead to mixed strategy equilibria in scalable strategies in the corresponding auction game. For results on mixed strategy equilibria in Tullock contests, see Baye *et al.* (1994) and Ewerhart (2014). These can be immediately accommodated by a slight change of notation in Theorem 3.2.

of studying contests with ties that goes beyond straightforward applications.

6.2 Contest success functions - 2 player case

Having proved these two specific results for N player contests to end with we prove a general result for two-player contests. We first define what it means for a two-player contest to be regular:

Definition 7. *A two-player contest success function is regular if for all* $e_i, e_j, k \in \mathbb{R}_{++}$

1. $\psi_i(e_i, e_j) = \psi_i(k.e_i, k.e_j)$
2. $\psi_i(e_i, e_j) + \psi_j(e_i, e_j) = 1$
3. *If* $e_i^H > e_i^L$*, then* $\psi_i(e_i^H, e_j) \geq \psi_i(e_i^L, e_j)$
4. $\lim_{\hat{e}_i \to 0} \psi_i(\hat{e}_i, e_j) = 0$
5. $\lim_{\hat{e}_i \to \infty} \psi_i(\hat{e}_i, e_j) = 1$

The first two conditions are the most restrictive: the first requires the contest to be homogenous of degree 0, while the second requires that the prize is always distributed. The remaining conditions are weaker: the third condition states that if a player exerts more effort then his chance of winning the prize does not decrease; finally conditions four and five ensure that if a player exerts a sufficiently small or sufficiently large amount of effort, then he will win the contest with probability close to 0 or close to 1 respectively. Many contests studied in the literature satisfy these conditions.

Before stating our result we slightly weaken one assumption made previously. Rather than being drawn from a differentiable function $F_1(t_1|\theta)$ we assume that $t_1 = \theta$. We now state the main result of this section:[14]

[14] It is clear that the proof of the main result is not affected by this change of assumption - it is only for notational convenience that $F_i(t_i|\theta)$ is assumed to be differentiable

Proposition 6.1. *Consider a contest success function ψ. The contest associated with ψ is the corresponding game of some scalable all-pay auction with private values if and only if ψ is regular.*

This result shows that in the two player case the class of regular contest success functions exactly coincides with the class of scalable private value all-pay auctions. This gives additional motivation for studying all-pay auctions and contests together, since results proved in one framework directly correspond to results proved in the other.

7 Maximal rank uncertainty

In this section we consider the property of maximal rank uncertainty, which is a key ingredient for the results. This property follows from assumption 2 which ensures that the information structure is scalable and sheds some more light on the importance of this assumption. The concept of maximal rank uncertainty generalises a similar property introduced by Morris & Shin (2007) who consider games with symmetric players in the context of global games.

When $I = \{1, ..., n\}$, let the rank of player i be defined by $r_i = \#\{j \in I | t_j \leq t_i\}$. Hence $r_i \in \{1, 2, ..., n\}$ denotes the number of players with a type $t_j \leq t_i$: if player i has the lowest type then $r_i = 1$ while if he has the highest type then $r_i = n$. Moreover let the probability that player i with type t_i has a rank r_i be given by $\Omega_i(r_i|t_i)$. We can now define maximal rank uncertainty.

Definition 8. *A game $\Gamma = \{I, \Theta, g, (F_i)_{i \in I}, (u_i)_{i \in I}\}$ is a game with maximal rank uncertainty if for all $r_i \in \{1, 2, ..., n\}$*

$$\Omega_i(r_i|t_i) = \Omega_i(r_i|t_i') \quad \textit{for all} \quad t_i, t_i' \in \Theta$$

Hence in a game with maximal rank uncertainty, a player's type does not provide him with any information about his rank. The probabilty a player assigns to being in any particular rank is the same for all types. Formally the probability $r_i \leq r^*$ - namely the probability that the rank of player i is less than or equal to r^* - does not depend on t_i. The fact that players do not gain information about their rank from observing their type is crucial for ensuring that players do not want to condition their mark-up (or mark-down) e_i on their type t_i and hence the existence of an equilibrium in G-scalable strategies. For instance in standard models of private value first price auctions, players determine their mark-down depending on their likely rank in the distribution; meanwhile in scalable games the property of maximal rank uncertainty ensures that all players choose the same percentage mark-down.

Note that in the special case where players are symmetric all possible ranks r_i are equally likely and $\Omega_i(r_i|t_i) = r_i/n$ for all r_i. The following proposition states that all scalable games are games with maximal rank uncertainty:

Proposition 7.1. *If Γ satisfies assumption 2 (and hence has a scalable information structure), then Γ is a game with maximal rank uncertainty.*

8 Conclusion

In this paper we have shown that games of incomplete information with a scalable information structure and a scalable payoff structure on the other hand are closely linked to corresponding games of complete information. In particular the equilibria of the corresponding complete information game coincide with equilibria in scalable strategies of the game of incomplete information. Understanding this link is useful for two reasons: first one can use it as a tool to model complicated situations of uncertainty, assuming that both the payoff structure and the information structure are scalable, which means that finding an equilibrium is relatively simple; secondly the link between complete information games and incomplete information games may lead to results proved in one framework to be transferred to results proved in the other. In particular this is the case for complete information games widely studied in the literature, such as the Tullock contest success function.

While the applications presented in this paper were focusing on auctions and corresponding contests, one can think of other games that may have interesting links to games in complete information. Examples include Cournot competition and certain settings of public good provision. A further complex situation with uncertainty that could be studied using the scalable game structure is the formation of asset price bubbles when players are asymmetric or risk averse.

Scalable games also shed light on games where strategy restrictions are imposed. In particular it is shown that games with certain strategy restrictions correspond to other scalable games without strategy restrictions. This helps motivate a particular choice of strategy restriction for a given problem, and potentially provides justifications for the use of certain strategy restrictions, making them less ad-hoc.

Note that mixed strategy equilibria in the complete information game, resulting in a corresponding mixed strategy equilibrium in scalable strategies in the scalable game, can immediately be accommodated by slightly modifying the notation in Theorem 3.2. Since in each complete information game, there exists an equilibrium in mixed strategies, this also ensures the existence of an equilibrium in a scalable game.

This paper has however not addressed the question of whether the equilibrium in scalable strategies is also the unique equilibrium of a scalable game. Preliminary results have shown that when actions are strategic complements or strategic complements and the complete information game is dominance solvable, then the game in incomplete information is dominance solvable and hence the equilibrium in scalable strategies is unique. The conditions are however very strong and it would be interesting to see whether one can find weaker conditions leading to a unique equilibrium.

9 Appendix A: Analysis

9.1 Additional lemma

Lemma. *If a game is scalable, then:*

$$F_i(t_i|\theta) = F_i(t_i \ominus_G \theta | 0_G)$$

Proof. Recall that $G^{-1}(0) = 0_G$. Let $\theta^{-1} = G^{-1}\Big(-G(\theta)\Big)$ and note that:

$$\theta \oplus_G \theta^{-1} = G^{-1}\Big(G(\theta) - G(\theta)\Big) = 0_G$$

Note also that:

$$t_i \oplus_G \theta^{-1} = G^{-1}\Big(G(\theta) - G(\theta)\Big) = t_i \ominus_G \theta$$

Using these two facts together with Assumption 2 we can now show the result:

$$\begin{aligned} F_i(t_i|\theta) &= F_i(t_i \oplus_G \theta^{-1}|\theta \oplus_G \theta^{-1}) \\ &= F_i(t_i \ominus_G \theta|0_G) \end{aligned}$$

□

9.2 Additional lemmas

Define $z_i = t_i \ominus_G \theta$ and $e_i = a_i \ominus_G t_i$.

Lemma 9.1.

If Γ has a scalable payoff structure, then for each player i there exist functions $C_i : \Theta \mapsto \mathbb{R}_+$ and $D_i : \Theta \mapsto \mathbb{R}_+$ such that:

$$u_i\Big(\mathbf{a};\theta;t_i\Big) = \frac{C_i(t_i)}{C_i(0)}\Big[u_i\Big(e_i, e_{-i} \oplus_G z_{-i} \ominus_G z_i; 0_G \ominus_G z_i; 0_G\Big) - D_i(0_G)\Big] + D_i(t_i)$$

Proof. This lemma follows almost immediately from the fact that Γ has a scalable payoff structure. Note that by Assumption 1 there exist suitable functions such that:

$$\frac{u_i(\mathbf{a};\theta;t_i) - D_i(t_i)}{C_i(t_i)} = \frac{u_i(\mathbf{a} \ominus_G t_i; \theta \ominus_G t_i; 0_G) - D_i(0_G)}{C_i(0_G)}$$

Noting that $a_j \ominus_G t_i = e_j \oplus_G z_j \ominus_G z_i$ and $\theta \ominus_G t_i = 0_G \ominus_G z_i$ leads to the following expression:

$$\frac{u_i(\mathbf{a};\theta;t_i) - D_i(t_i)}{C_i(t_i)} = \frac{u_i\Big(e_i, e_{-i} \oplus_G z_{-i} \ominus_G z_i; 0_G \ominus_G z_i; 0_G\Big) - D_i(0_G)}{C_i(0_G)}$$

Finally rearranging gives the result:

$$u_i\Big(\mathbf{a};\theta;t_i\Big) = \frac{C_i(t_i)}{C_i(0)}\Big[u_i\Big(e_i, e_{-i} \oplus_G z_{-i} \ominus_G z_i; 0_G \ominus_G z_i; 0_G\Big) - D_i(0_G)\Big] + D_i(t_i)$$

□

Lemma 9.2.

If Γ *has a scalable belief structure, then:*

$$g_i(\theta|t_i)\prod_{j\neq i} f_j(t_j|\theta) = -\frac{dz_i}{d\theta}\prod_{j\neq i}\frac{dz_j}{dt_j}\prod_{j=1}^{n} f_j(t_j \ominus_G \theta|0_G)$$

Proof. Note that the definition of $a \oplus_G b$ immediately implies the following : $g(\theta)\frac{d}{dt_i}\Big[t_i \ominus_G \theta\Big] = -g(t_i)\frac{d}{d\theta}\Big[t_i \ominus_G \theta\Big]$ Hence:

$$\begin{aligned} g_i(\theta|t_i) &= -\frac{d}{d\theta}\Big[t_i \ominus_G \theta\Big] f_i(t_i \ominus_G \theta|0_G) \\ &= -\frac{dz_i}{d\theta} f_i(z_i|0_G) \end{aligned}$$

Moreover, using 9.1, we get the following:

$$\begin{aligned} F_j(t_j|\theta) &= F_j(t_j \ominus_G \theta|0_G) \\ f_j(t_j|\theta) &= \frac{d}{dt_j}\Big[t_j \ominus_G \theta\Big] f_j(t_j \ominus_G \theta|0_G) \\ &= \frac{dz_j}{dt_j} f_j(z_j|0_G) \end{aligned}$$

Using these two facts we obtain:

$$g_i(\theta|t_i)\prod_{j\neq i} f_j(t_j|\theta) = -\frac{dz_i}{d\theta}\prod_{j\neq i}\frac{dz_j}{dt_j}\prod_{j=1}^{n} f_j(t_j \ominus_G \theta|0_G)$$

□

9.3 Proof of Proposition 3.1

In this subsection we provide a general proof of proposition 3.1, using the lemmas above.

Proof.

$$V_i(e_i, e_{-i}|t_i) = \int_{\Theta^n} g_i(\theta|t_i) \prod_{j\neq i} f_j(t_j|\theta) u_i\Big(e_i \oplus t_i, e^*_{-i} \oplus t_{-i}; \theta; t_i\Big) d\theta \prod_{j\neq i} dt_j$$

In order to do the substitution from $\{\theta, t_{j_1}, ...t_{j_{n-1}}\}$ to $\{z_i, z_{j_1}, ...z_{j_{n-1}}\}$ it is necessary to consider the following Jacobian matrix:

$$M = \begin{pmatrix} \frac{dz_i}{d\theta} & \frac{dz_{j_1}}{d\theta} & ... & \frac{dz_{j_{n-1}}}{d\theta} \\ \frac{dz_i}{dt_{j_1}} & \frac{dz_{j_1}}{dt_{j_1}} & ... & \frac{dz_{j_{n-1}}}{dt_{j_1}} \\ ... & ... & ... & ... \\ \frac{dz_i}{dt_{j_{n-1}}} & \frac{dz_{j_1}}{dt_{j_{n-1}}} & ... & \frac{dz_{j_{n-1}}}{dt_{j_{n-1}}} \end{pmatrix}$$

Since $z_j = t_j - \theta$, it follows that the Jacobian matrix M has only zero entries apart from in the first row and along the main diagonal. This means that the determinant is equal to the product of the main diagonal:

$$det M = \frac{dz_i}{d\theta} \prod_{j\neq i} \frac{dz_j}{dt_j}$$

Note that using Lemma 9.2 above we can see that:

$$\prod_{j\neq i} f_j(t_j|\theta) = -det(M) \prod_{j=1}^{n} f_j(z_j|0_G)$$

When changing variables we must divide by the determinant when changing variables of integration from $\{\theta, t_{j_1}, ...t_{j_{n-1}}\}$ to $\{z_1, z_2, ..., z_n\}$. Using also Lemma 9.1 and Lemma 9.2, we now perform the substitution:

$$
\begin{aligned}
V_i(e_i, e_{-i}|t_i) = & \int_{\Theta^n} g_i(\theta|t_i) \prod_{j\neq i} f_j(t_j|\theta) u_i\Big(e_i \oplus t_i, e_{-i} \oplus t_{-i}; \theta; t_i\Big) d\theta \prod_{j\neq i} dt_j \\
= & \int_{\Theta^{n-1}} \int_{D}^{-D} \frac{-detM}{detM} \prod_{j=1}^{n} f_j(z_j|0_G) \\
& \left(\frac{C_i(t_i)}{C_i(0)} \Big[u_i\Big(e_i, e_{-i} \oplus_G z_{-i} \ominus_G z_i; 0_G \ominus_G z_i; 0_G\Big) - D_i(0_G) \Big] + D_i(t_i) \right) \prod_{j=1}^{n} dz_j
\end{aligned}
$$

Reversing the domain of integration, removing the minus sign and rearranging yields:

$$
V_i(e_i|e^*, t_i) = \frac{C_i(t_i)}{C_i(0_G)} \Big[\int_{\Theta^n} \prod_{j=1}^{n} f_j(z_j|0_G) u_i\Big(e_i, e_{-i} \oplus_G z_{-i} \ominus_G z_i; 0_G \ominus_G z_i; 0_G\Big) \prod_{j=1}^{n} dz_j - D_i(0_G) \Big] + D_i(t_i)
$$

By definition $\phi(e_i, e_{-i}) = \int_{\Theta^n} \prod_{j=1}^{n} f_j(z_j|0_G) u_i\Big(e_i, e_{-i} \oplus_G z_{-i} \ominus_G z_i; 0_G \ominus_G z_i; 0_G\Big) \prod_{j=1}^{n} dz_j$. Applying this identity gives

$$
V_i(e_i|e^*, t_i) = \frac{C_i(t_i)}{C_i(0_G)} \Big[\phi_i(e_i, e_{-i}) - D_i(0_G) \Big] + D_i(t_i)
$$

Finally rearranging terms gives the result:

$$
\phi(e_i, e_{-i}) = \frac{C_i(0_G)}{C_i(t_i)} \Big[V_i(e_i|e, t_i) - D_i(t_i) \Big] + D_i(0_G)
$$

□

9.4 Proof of 3.2

Proof. Using Proposition 3.1, the proof is now straightforward. Suppose $\sigma(\mathbf{t}) = \mathbf{t} \oplus \mathbf{e}^*$. First suppose e^* is a Nash equilibrium of Γ_N. It follows that $\phi_i(e_i^*, e_{-i}^*) \geq \phi_i(\hat{e}_i, e_{-i}^*)$ for all $\hat{e}_i \in \Theta$. Using the proposition above, we can deduce that for all $\hat{e}_i \in \Theta$ and $t_i \in \Theta$:

$$
\frac{1}{C_i(t_i)} \Big[V_i(e_i^*|e_{-i}^*, t_i) - D_i(t_i) \Big] \geq \frac{1}{C_i(t_i)} \Big[V_i(\hat{e}_i|e_{-i}^*, t_i) - D_i(t_i) \Big]
$$

Multiplying by $C_i(t_i)$ and adding $D_i(t_i)$ shows that for all $\hat{e}_i \in \Theta$ and $t_i \in \Theta$:

$$V_i(e_i^*|e_{-i}^*, t_i) \geq V_i(\hat{e}_i|e_{-i}^*, t_i)$$

Hence σ^* is a Nash equilibrium of Γ.

Secondly suppose σ^* is a Nash equilibrium of Γ. Then for all $\hat{e}_i \in \Theta$ and $t_i \in \Theta$ we can deduce that $V_i(e_i^*|e_{-i}^*, t_i) \geq V_i(\hat{e}_i|e_{-i}^*, t_i)$. By reversing the steps above we can show that $\phi_i(e_i^*, e_{-i}^*) \geq \phi_i(\hat{e}_i, e_{-i}^*)$ for all $\hat{e}_i \in \Theta$. Hence e^* is a Nash equilibrium of Γ_N.

$$\begin{aligned}
\sigma \text{ is a NE of } \Gamma \quad & \text{iff} & V_i(e_i^*|e_{-i}^*, t_i) &\geq V_i(\hat{e}_i|e_{-i}^*, t_i) \\
& \text{iff} & \tfrac{1}{C_i(t_i)}\Big[V_i(e_i^*|e_{-i}^*, t_i) - D_i(t_i)\Big] &\geq \tfrac{1}{C_i(t_i)}\Big[V_i(\hat{e}_i|e_{-i}^*, t_i) - D_i(t_i)\Big] \\
& \text{iff} & \phi_i(e_i^*, e_{-i}^*) &\geq \phi_i(\hat{e}_i, e_{-i}^*) \\
& \text{iff} & e^* &\text{ is a NE of } \Gamma_N
\end{aligned}$$

□

10 Appendix B: Strategy restrictions

10.1 Proof of Theorem 4.1

Proof. For each player $i \in I$ consider an arbitrary $\sigma_i \in \Sigma_G$ such that $\sigma_i(t_i) = t_i \oplus_G e_i$. Consider also the complete information game $\{I, \phi\}$ induced by Γ. By the definition of $\phi_i(e_i, e_{-i})$:

$$\phi_i(e_i, e_{-i}) = \int_{\Theta^n} \prod_{j=1}^{n} f_j(z_j|0_G) u_i\Big(e_i, e_{-i} \oplus_G z_{-i} \ominus_G z_i; 0_G \ominus_G z_i; 0_G\Big) \prod_{j=1}^{n} dz_j$$

Since $u_i(\mathbf{a}; t_i; \theta) = \pi_i(\mathbf{a} \ominus_G \theta; t_i \ominus_G \theta)$, it follows that:

$$u_i\Big(e_i, e_{-i} \oplus_G z_{-i} \ominus_G z_i; 0_G \ominus_G z_i; 0_G\Big) = u_i\Big(e_i \oplus_G z_i, e_{-i} \oplus_G z_{-i}; 0_G; z_i\Big)$$

Using this identity and writing $t_i = z_i$ leads to:

$$\phi_i(e_i, e_{-i}) = \int_{\Theta^n} \prod_{j=1}^{n} f_j(t_j|0_G) u_i\Big(e_i \oplus_G t_i, e_{-i} \oplus_G t_{-i}; 0_G; t_i\Big) \prod_{j=1}^{n} dt_j$$

Using the fact that $\sigma_i^*(t_i) = t_i \oplus_G e_i$ and $u_i(\mathbf{a}; t_i; 0) = \pi_i(\mathbf{a}; t_i)$, it follows that:

$$\phi_i(e_i, e_{-i}) = \int_{\Theta^n} \prod_{j=1}^{n} f_j(t_j|0_G) \pi_i(\sigma_i(t_i), \sigma_{-i}(t_i); t_i) \prod_{j=1}^{n} dt_j$$

Let $\sigma_i^*(t_i) = t_i \oplus_G e_i^*$. It follows that σ^* is an equilibrium of Γ_R if and only if e^* is an equilibrium of the complete information game $\{I, \phi\}$. Moreover by theorem 3.2 e^* is an equilibrium of the complete information game $\{I, \phi\}$ if and only if σ^* is an equilibrium of Γ. Hence σ^* is an equilibrium of Γ^R if and only if it is an equilibrium of Γ.

□

11 Appendix C: Applications to Contests

11.1 Asymmetric Tullock contest

First recall that $F_i(t_i|\theta) = \exp\left[\left(\frac{V_j\theta}{t_j}\right)^{\alpha}\right]$ and note that the utility function can be rewritten as follows:

$$u_i\Big(a_i, a_{-i}; \theta; t_i\Big) = t_i \mathbf{1}_{\Big\{a_i > a_j \quad \text{for all} \quad j \neq i\Big\}} - a_i$$

This leads to the following expression for the complete information game $\phi_i(e_i, e_{-i})$:

$$\phi_i(e_i, e_{-i}) = \int_{\mathbb{R}_+^n} f_i(z_i|1) \prod_{j \neq i} f_j(z_j|1) \mathbf{1}_{\Big\{e_i z_i > e_j z_j\Big\}} d\mathbf{z} - e_i$$

Integrating this expression with respect to all $j \neq i$ gives:

$$\phi_i(e_i, e_{-i}) \quad = \quad \int_{\mathbb{R}_+} f_i(z_i|1) \prod_{j \neq i} F_j\Big(\frac{e_i z_i}{e_j}\Big|1\Big) dz_i - e_i$$

Note that $f_i(z_i|1) = \frac{\alpha V_i^\alpha}{z_i^{\alpha+1}} F(z_i|1)$. Hence:

$$\begin{aligned}
\phi_i(e_i, e_{-i}) &= \int_{\mathbb{R}_+} \frac{\alpha V_i^\alpha}{z_i^{\alpha+1}} \prod_{j \in I} F_j\Big(\frac{e_i z_i}{e_j}\Big| 1\Big) dz_i - e_i \\
&= \int_{\mathbb{R}_+} \frac{\alpha V_i^\alpha}{z_i^{\alpha+1}} \prod_{j \in I} \exp\left[\Big(\frac{v_j e_j}{z_i e_i}\Big)^\alpha\right] dz_i - e_i \\
&= \int_{\mathbb{R}_+} \frac{\alpha V_i^\alpha}{z_i^{\alpha+1}} \exp\left[\frac{1}{z_i^\alpha} \sum_{j \in I} \Big(\frac{V_j e_j}{e_i}\Big)^\alpha\right] dz_i - e_i
\end{aligned}$$

Integrating this expression gives:

$$\begin{aligned}
\phi_i(e_i, e_{-i}) &= \frac{V_i^\alpha}{\sum_{j \in I} \Big(\frac{V_j e_j}{e_i}\Big)^\alpha} \\
&= \frac{(V_i e_i)^\alpha}{\sum_{j \in I} (V_j e_j)^\alpha}
\end{aligned}$$

11.2 Tullock contest with draws

First note that $F_i(t_i|\theta) = F(t_i|\theta) = \exp\left(\frac{-\theta}{t_i}\right)$ and the utility function can be rewritten as follows:

$$u_i\Big(a_i, a_{-i}; \theta; t_i\Big) = \theta \mathbf{1}_{\Big\{a_i > a_j \text{ for all } j \neq i\Big\}} - a_i$$

This leads to the following expression for the complete information game $\phi_i(e_i, e_{-i})$:

$$\phi_i(e_i, e_{-i}) = \int_{\mathbb{R}_+^n} f(\mathbf{z}|1) \frac{1}{z_i} \mathbf{1}_{\Big\{e_i z_i > e_j z_j \text{ for all } j \neq i\Big\}} d\mathbf{z} - e_i$$

Integrating this expression with respect to all $j \neq i$ gives:

$$
\begin{aligned}
\phi_i(e_i, e_{-i}) &= \int_{\mathbb{R}_+} \frac{f(z_i|1)}{z_i} \left[\int_{\mathbb{R}_+^{n-1}} f(z_{-i}|1) \mathbf{1}_{\left\{e_i z_i > e_j z_j \text{ for all } j \neq i\right\}} dz_{-i} \right] dz_i - e_i \\
&= \int_{\mathbb{R}_+} \frac{f(z_i|1)}{z_i} \prod_{j \neq i} F\left(\frac{e_i z_i}{e_j} \middle| 1 \right) dz_i - e_i
\end{aligned}
$$

Using the fact that $F\left(\frac{e_i z_i}{e_j}\middle|1\right) = \exp\left(\frac{-e_j}{e_i z_i}\right)$:

$$
\begin{aligned}
\phi_i(e_i, e_{-i}) &= \int_{\mathbb{R}_+} \frac{\exp\left(\frac{-1}{z_i}\right)}{z_i^3} \prod_{j \neq i} \exp\left(\frac{-e_j}{e_i z_i}\right) dz_i - e_i \\
&= \int_{\mathbb{R}_+} \frac{1}{z_i^3} \exp\left(\sum_{j=1}^{n} \frac{-e_j}{e_i z_i}\right) dz_i - e_i \\
&= \int_{\mathbb{R}_+} \left[\frac{1}{z_i}\right] \left[\frac{1}{z_i^2} \exp\left(\sum_{j=1}^{n} \frac{-e_j}{e_i z_i}\right)\right] dz_i - e_i
\end{aligned}
$$

Integrating by parts gives:

$$
\begin{aligned}
\phi_i(e_i, e_{-i}) &= \left[\frac{1}{z_i} \frac{e_i}{\sum_{j=1}^{n} e_j} \exp\left(\sum_{j=1}^{n} \frac{-e_j}{e_i z_i}\right)\right]_0^\infty \\
&\quad + \int_{\mathbb{R}_+} \frac{1}{z_i^2} \frac{e_i}{\sum_{j=1}^{n} e_j} \exp\left(\sum_{j=1}^{n} \frac{-e_j}{e_i z_i}\right) dz_i - e_i
\end{aligned}
$$

Note that the first term is zero, while the second term can now be integrated directly:

$$
\begin{aligned}
\phi_i(e_i, e_{-i}) &= \left[\frac{e_i^2}{\left(\sum_{j=1}^{n} e_j\right)^2} \exp\left(\sum_{j=1}^{n} \frac{-e_j}{e_i z_i}\right)\right]_0^\infty - e_i \\
&= \frac{e_i^2}{\left(\sum_{j=1}^{n} e_j\right)^2} - e_i
\end{aligned}
$$

11.3 2-players

Proof of Proposition 6.1

Proof. In order to prove this result, take some regular contest success function ϕ. Suppose t_i is drawn from the distribution $F_i(t_i|\theta) = 0$ if $t_i < \theta$ and $F_i(t_i|\theta) = 1$ if $t_i \geq \theta$. Hence $t_i = \theta$ and the complete information game is now given as follows:

$$\begin{aligned}\phi_i(e_i, e_j) &= \int_{\mathbb{R}_{++}} \mathbf{1}_{\{e_i \geq e_j z_j\}} dF_j(z_j|1) - e_i \\ \phi_j(e_i, e_j) &= \int_{\mathbb{R}_{++}} \mathbf{1}_{\{e_j z_j \geq e_i\}} dF_j(z_j|1) - e_j\end{aligned}$$

Integrating these expressions gives:

$$\begin{aligned}\phi_i(e_i, e_j) &= F_j(\frac{e_i}{e_j}|1) - e_i \\ \phi_j(e_i, e_j) &= 1 - F_j(\frac{e_i}{e_j}|1) - e_i\end{aligned}$$

Now define $F_j(t_j|\theta) := \psi_i(t_j/\theta, 1)$. Note that since ψ is regular (i) $\psi_i(t_j/\theta, 1)$ is weakly increasing in t_i, (ii) $\lim_{t_j \to 0} \psi_i(t_j/\theta, 1) = 0$ and (iii) $\lim_{t_j \to \infty} \psi_i(t_j/\theta, 1) = 1$. It follows that $F_j(t_j|\theta)$ inherits these properties and so $F_j(t_j|\theta)$ is a cumulative probability distribution. Moreover it follows directly from the definition that $F_j(t_j|\theta)$ is homogenous of degree 0. Hence $F_j(t_j|\theta)$ is an admissible cumulative probability distribution to be used in a scalable private value auction. Now using the fact that $F_j(t_j|\theta) := \psi_i(t_j/\theta, 1)$ we can re-write the complete information game as follows:

$$\begin{aligned}\phi_i(e_i, e_j) &= \psi_i(\frac{e_i}{e_j}|1) - e_i \\ \phi_j(e_i, e_j) &= 1 - \psi_i(\frac{e_i}{e_j}|1) - e_i\end{aligned}$$

Finally using the fact that (i) $\psi_i(e_i, e_j)$ is homogenous of degree 0 and (ii) $\psi_i(e_i, e_j) + \psi_j(e_i, e_j) = 1$ we obtain:

$$\begin{aligned}\phi_i(e_i, e_j) &= \psi_i(e_i|e_j) - e_i \\ \phi_j(e_i, e_j) &= \psi_j(e_i|e_j) - e_i\end{aligned}$$

This shows that the contest function ψ is the representation of some scalable private value all-pay auction and completes the proof.

$\square$

12 Appendix D: Maximal rank uncertainty

12.1 Proof of Proposition 7.1

Proof. Define $\mathbb{J}_i^K \subseteq \mathcal{P}(I)$ such that $J \in \mathbb{J}_i^K$ whenever $i \in J$ and $|J| \leq k$. Using this notation we can write the probability $\Omega_i(k|t_i)$ as follows:

$$\Omega_i(k|t_i) = \sum_{J \in \mathbb{J}_i^K} \int_\Theta g_i(\theta|t_i) \prod_{j \in J} F_j(t_i|\theta) \prod_{j \notin J} \Big(1 - F_j(t_i|\theta)\Big) d\theta$$

Each integral captures the probability that exactly the players $j \in J$ have a type $t_j \leq t_i$. To calculate $\Omega_i(k|t_i)$ it is then necessary to sum over all $J \in \mathbb{J}_i^K$. Using Proposition 9.2 it follows that $g_i(\theta|t_i) = -\frac{d[t_i \ominus_G \theta]}{d\theta} f(t_i \ominus_G \theta|0)$. Moreover by Assumption 2 it follows that $F_j(t_i|\theta) = F_j(t_i \ominus_G \theta|0)$. Applying these transformations we reach:

$$\Omega_i(k|t_i) = \sum_{J\in\mathbb{I}_K} \int_\Theta -\frac{d[t_i \ominus_G \theta]}{d\theta} f_i(t_i \ominus_G \theta|0_G) \prod_{j\in J} F_j(t_i \ominus_G \theta|0_G) \prod_{j\notin J} \Big(1 - F_j(t_i \ominus_G \theta|0_G)\Big) d\theta$$

Now substituting $\theta' = t_i \ominus_G \theta$:

$$\Omega_i(k|t_i) = \sum_{J\in\mathbb{I}_K} \int_\Theta f_i(\theta'|0_G) \prod_{j\in J} F_j(\theta'|0_G) \prod_{j\notin J} \Big(1 - F_j(\theta'|0_G)\Big) d\theta'$$

This expression does not depend on t_i and hence $\Omega_i(k|t_i) = \Omega_i(k|t_i')$ $\square$

References

Abreu, D., & Brunnermeier, M. 2003. Bubbles and Crashes. *Econometrica*, **71**, 173–204.

Angeletos, G. M., & Pavan, A. 2007. Effiicient use of information and the social value of information. *Econometrica*, **75**, 1103 – 1142.

Baye, M., & Hoppe, H. 2003. The strategic equivalence of rent-seeking, innovation, and patent-race games. *Games and Economic Behaviour*, **44**, 217–226.

Baye, M., Kovenock, D., & De Vries, C. 1994. The solution to the Tullock rent-seeking game when R¿2: Mixed strategy equilibria and mean dissipation rates. *Public Choice*, **81**, 363–380.

Brunnermeier, M., & Morgan, J. 2010. Clock games: Theory and Experiments. *Games and Economic Behaviour*, **68**(2), 532–550.

Carlsson, H. 1991. A Bargaining model where Parties Make Errors. *Econometrica*, **59**, 1487–96.

Carlsson, H., & Van Damme, E. 1993. Global Games and Equilibrium Selection. *Econometrica*, **61**, 989–1018.

Clark, D., & Riis, C. 1998. Contest succes functions; an extension. *Economic Theory*, **11**, 201–204.

Colombo, L., Femminis G., & Pavan, A. 2014. Information Acquisition and Welfare. *Review of Economic Studies*. Forthcoming.

Compte, O., & Postlewaite, A. 2013. *Simple Auctions*. PIER Working Paper 13-017.

Ewerhart, C. 2014. Mixed Equilibria in Tullock Contests. *Economic Theory*. Forthcoming.

Hartigan, J. E. 1983. *Bayes Theory*. Springer-Verlag.

Klemperer, P. 1999. Auction Theory: A guide to the literature. *Journal of Economic Surveys*, **13**, 227–286.

Lazear, E., & Rosen, S. 1981. Rank-Order Tournaments as Optimum Labour Contracts. *Journal of Political Economy*, **89**, 851–864.

Liu, H. 2014. Equilibrium selection in common-value second-price auctions. *Games and Economic Behaviour*, **84**, 1–6.

Maskin, E., & Riley, J. 2000. Assymetric Auctions. *Review of Economic Studies*, **67**.

Milgrom, P. 1981. Rational expectations, information acquisition, and competitive bidding. *Econometrica*, **49**, 921–943.

Milgrom, P., & Weber, R. 1982. A Theory of Auctions and Competitive Bidding. *Econometrica*, **50**, 1089–1122.

Morris, S., & Shin. 2002. The Social Value of Public Information. *American Economic Review*, **95**, 1521–1534.

Morris, S., & Shin, H. 2007. *Common Belief Foundations of Global Game*. Working Paper.

Myatt, D., & Wallace, C. 2012. Endogenous Information Acquisition in Coordination Game. *Review of Economic Studies*, **79**, 340–375.

Myatt, D., & Wallace, C. 2013. *Cournot Competition and the Social Value of Information*. Working paper.

Pavan, A. 2014. *Attention, Coordination, and Bounded Recal*. Working paper.

Satterthwaite, M. A., Williams, S. R., & Zachariadis, K. E. 2014. Optimality versus Practicality in Market Design: A comparison of Two Double Auctions. *Games and Economic Behaviour*.

Tullock, G. 1980. *Toward a theory of the rentseeking society*. A&M University Press, College Station. Chap. Efficient rent seeking, pages 97–112.

Ui, T., & Yoshizawa, Y. 2014. Characterizing Social Value of Information. Working Paper.

Vives, X. 1988. Aggregation of Information in Large Cournot Market. *Econometrica*, **56**, 851–876.

Vives, X. 2011. Strategic supply function competition with private information. *Econometrica*, **79**(6), 1919–1966.

Yildizparlak, A. 2013. *Contests with ties and an application to soccer*. Unpublished Working Paper 2013.