# Competition and Networks of Collaboration*

*Job Market Paper*

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**Abstract**

I develop a model of collaboration between competitors. In this model, agents collaborate in pairs, and a structure of collaboration is represented by a network. Agents' payoffs consist of two terms. The first term, which I call "performance", increases in the number of agent's connections. The second term, a "prize" in a competition, increases in agent's relative position in the distribution of performance. In contrast with most of the network formation literature, I assume that the agents are forward looking, and I use von Neumann-Morgenstern stable sets as a solution for the network formation process. First, I find a necessary and sufficient condition for the stability of the efficient network. Second, I find a set of networks which are stable whenever the efficient network is not. These networks consist of two mutually disconnected complete components. I interpret this pattern as a group competition between "insiders" and "outsiders". The larger of the two groups, namely the group of the insiders, gains an advantage in the competition by staying disconnected from the outsiders. Finally, I extend my model to accommodate for heterogeneity in agents' abilities, and show that a high ability does not necessarily transform into a strong position in the competition

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# 1 Introduction

We often observe collaboration between direct competitors. For instance, firms, that compete in the market for a final product, often collaborate at the R&D stage. Similarly, co-workers, that compete for a promotion, usually collaborate with their rivals in the course of the competition. Agents in these situation face a dilemma: if they collaborate, they themselves become stronger competitors, but they also strengthen the positions of their rivals. Nevertheless, we often see agents choosing extensive collaboration over autarkic competition.

This paper studies the following questions about such collaboration. Under what conditions do competitors collaborate at the efficient level? If those conditions fail to hold, what are stable patterns of collaboration? How do these patterns depend on the exogenous strength of the competitors?

I address these questions with a model in which an endogenous structure of collaboration is represented by a network. I start with a basic setup that features homogeneous agents and simple bilateral collaboration. Any two agents in this model can collaborate with each other, and each agent can collaborate with as many agents as he wants. Naturally, in this setup, a link represents a collaboration between two agents. A link between two agents exists so long as both of them approve it. If one of the two agents decides to break a link, the collaboration ceases.

Agents in my model produce an output, and seek collaboration in order to be more productive. Each agent's output increases in the number of direct partners (or equivalently links) he has. In addition, agents compete with each other for a prize. I model the competition as a tournament in which the prize is split equally between agents with maximal output. An agent's payoff consists of two terms. The first term is a portion of the prize an agent wins, and the second term is the output an agent produces.

I assume that agents are forward looking and capable of cooperation. Whenever an agent decides which action to take, he examines the ultimate outcomes of these actions rather than their immediate effects. I also assume that agents are immune to collective action problems and failure of coordination: if a coalition of agents can credibly coordinate their actions for their mutual benefit, they do so. A vast majority of papers on network formation assume that agents are myopic (see Jackson and Wolinsky (1996), Dutta and Mutuswami (1997), Jackson (2008), and many others). In this paper I depart from this tradition. Forward looking agents fit better for my purposes, since I study small networks with extensive communication between agents. It is natural, for example, to assume that, prior to signing a bilateral agreement, firms carefully consider the long-run effect it will have on their current partners.[1]

[1]Such coordination of actions between allied firms is very common. For example, on August 16, 2011, HTC, a Google's partner on the market for handsets, filed a lawsuit against Apple, accusing the latter of infingement of three HTC's patents. This happened a day after Google announced plans for an acquisition of Motorola Mobility. Both Google and HTC foresaw that alliance of Apple

I use von Neumann-Morgenstern stable sets (see von Neumann and Morgenstern (1944)) to define a solution for this model. A von Neumann-Morgenstern stable set of networks must satisfy two properties: external and internal stability. Intuitively, internal stability guarantees that agents that are connected in a stable networks do not attempt to switch to another stable network, and external stability guarantees that agents that are connected in an unstable network would rearrange their links into a stable network. The notion of von Neumann-Morgenstern stable sets is contextual: agents consider a particular network to be stable because it satisfies external and internal stability with respect to itself and other elements of a stable set.[2]

This setup speaks to a variety of applications, such as networks of bilateral agreements between firms, collaboration between employees, networks of organized crime, racial and ethnic discrimination in small societies, and others. Although these applications are very different from each other, they all have several common features that play a central role in my model. Agents in these applications compete with each other, and their performance in the competition depends on their collaboration with rivals.

For instance, let us consider some firms that operate in the same market for a final product. They have an opportunity to collaborate at the R&D stage, and as data shows firms often use this opportunity extensively.[3] Under what conditions do firms collaborate at the efficient level? How does competitiveness of the market influence collaboration? How does collaboration affect the market structure? My model can be used to tackle all these questions.

Although there is an extensive literature on firms' collaboration, my model is able to offer new insights. It differs from the existing literature in two respects. First, we characterize conditions under which efficient outcome can *not* be stable. Second, my model makes sharper predictions about the structure of stable collaboration networks. A vast majority of papers on R&D collaboration consider firms to be myopic (see Goyal and Moraga-Gonzalez (2001), Goyal and Joshi (2003), and Marinucci and Vergote (2011)). These papers find that efficient outcomes are *always* stable, but there are some inefficient outcomes that can be stable as well. In an efficient outcome, a firm has to collaborate with many others. If a firm is myopic, it can be trapped in the situation in which, for example, it has no connections. If there are increasing returns to connections, a myopic firm can not escape this trap by creating new connections one by one. Put differently, these models are driven by the lack of

and Microsoft would try to hinder the acquisition of Motorola Mobility, and HTC's lawsuits are seen by some as an attempt to protect Google's acquisition (see Robin Kwong and Joseph Menn, (2011, August 16), "HTC files fresh lawsuit against Apple." *Financial Times*).

[2]Von Neumann-Morgenstern stable set is self-referential and can be formally represented as a fixed point of some function.

[3]For instance, Hagedoorn (2002) reports that the amount of inter-firm collaboration in R&D almost tripled in the period from 1980 to 1998. He also reports that joint ventures constituted roughly half of R&D partnerships in 1980, but by 1998 this share dropped to roughly 10%. He argues, that firms preferences shifted from joint ventures towards contractual partnerships.

firms' foresight: a firm fails to coordinate its actions with its "future self". Dutta et al. (1998) avoid this problem by looking at CPNE of network formation game. They find results similar to mine, but their framework is very limited — they consider a game with only three firms.

The main results of my paper are as follows. I find necessary and sufficient conditions for efficient outcomes to be stable. To obtain this condition, I look at the collection of networks in which winners are only connected to other winners. Let us call this collection $\mathcal{C}$.[4] An efficient network is stable if and only if it maximizes the utility of winners in the set $\mathcal{C}$. This criterion resembles a *union mentality*. Winners are concerned only with their own payoffs, just as unions are concerned only with well-being of union members. Winners do not internalize the effect they impose on others, just as unions do not internalize the effect they impose on nonunion employees.

I also find stable networks, when this condition does not hold. A set of networks that maximizes winners' payoffs in the set $\mathcal{C}$ is always stable. These networks exhibit a special property: each network consists of two mutually disconnected groups both of which feature full collaboration. One group necessarily constitutes a majority and hence is a group of winners. Members of the larger group protect their winners' positions by refusing to collaborate with members of the smaller group. I call the larger group "insiders" and the smaller one "outsiders". We can interpret this rule as a norm, that favors the majority and is enforced by it.

These results are coherent with properties of networks of collaboration that we observe in practice. To illustrate this, let us go back to the example with R&D collaboration. Theory suggests, that if the stakes in the competition are high enough, the efficient network of collaboration, in which firms sign all available collaboration agreements, is not stable. Moreover, there are stable networks in which firms that dominate the market, i.e. "insider" firms, refuse to collaborate with "outsider" firms. This strategy allows the "insiders" to maintain their dominant position in the market. Such configurations are observed in practice. For example, Bekkers et al. (2002) provide an overview of the market for GSM infrastructure and terminals. They report, that an alliance of five firms led by Motorola dominated this market in the 1990s. Their total market share in 1996 was above 85%. Bekkers et al. (2002) argue, that bilateral cross-licensing agreements played a central role in this alliance. Its participants made their portfolios of GSM-essential patents available to their allies and unavailable to the rest of the firms. This strategy is predicted by our results, and it has proven to be very successful in capturing almost all of the GSM market.

More generally, our model provides two important insights on patent wars. First, if stakes in the competition are high, patent wars are inevitable. Second, the possibility of patent war forces firms to form alliances.

To obtain more insights, I extend this model to the case of both heterogeneous agents and more complex collaboration relationships. I assume, that agents differ in their ability to convert collaboration into output. Using this setup, I find stable net-

[4]Notice, that an efficient network belongs to this collection of networks.

works with empirically relevant properties. As in basic model, these stable networks feature two mutually disconnected groups of unequal size. This time, however, I find that, first, the smaller of the two groups features tighter collaboration, and, second, a high ability of an agent does not necessarily transform into a strong position in the competition.

Both results stem from the following observation. A high ability member of the larger group can be easily substituted by an outsider with a low ability. Indeed, links they can both generate are the same from the others' point of view. The high ability agent is a strong competitor and a potential threat for members of the larger group. Hence, he is less welcome in the larger group unless he pretends to have a low ability by restricting his output. In the stable networks that I find, high ability agents follow this strategy of restricting their own output in exchange for the membership in a larger group.

I do not study collaboration in competition in its full generality. Instead, I focus on one aspect of it, that was *not* examined previously in the literature. Cooperative and forward looking agents may collectively manipulate their connections to gain an advantage in a competition. Such behavior results in structural holes in networks of collaboration. Such networks are inefficient since they favor some agents more than others. This aspect is best explained with an example.

Consider three agents, $A$, $B$ and $C$. In our model, the efficient network for these three agents is a triangle (complete network). Imagine agent $A$ deviates from the efficient network by dropping his link with agent $C$. In the resulting network, agent $B$ has two links, and agents $A$ and $C$ each have one link. The absence of a link between agent $A$ and agent $C$ imposes a positive externality on agent $B$, since he becomes the only winner in the competition. Now imagine that agent $B$ reciprocates to agent $A$'s action and drops his link with agent $C$ as well. Let us compare the resulting network with only one link $AB$ to the efficient network $\{AB, AC, BC\}$. On one hand both $A$ and $B$ bear some losses when they delete their links with $C$. On the other hand, in the network with one link both $A$ and $B$ get half of the prize whereas in the efficient network they get only one third each. For some parameters, the benefits outweigh the losses, and therefore it is profitable for the coalition $\{A, B\}$ to drop their links with $C$.

One may object that such a deviation by agents $A$ and $B$ does not lead to a stable network because there might be further deviations from it. For instance agents $A$ and $C$ both find it profitable to revive their missing link: agent $A$ becomes the only winner, and agent $C$ gains a link. There are, indeed, deviations from the network with only one link $AB$, but I argue that they are not credible. Suppose, that all three agents perceive networks $\{AB\}$, $\{AC\}$, and $\{BC\}$ as stable.[5] Then, the deviation by agents $A$ and $C$ will be followed by agents $B$ and $C$, who will drop their links with $A$ and connect with each other. This counter-deviation will lead us back into a stable

[5]Recall, that our stability notion is self-referential. Agents' beliefs about a stable set of networks are supported by internal and external stability of this set.

set of networks. This observation supports agents' beliefs about stability of networks $\{AB\}$, $\{AC\}$, and $\{BC\}$.

This idea may be extended to a setup with an arbitrary number of agents. Let us start with a complete network. A group of agents, that constitutes a majority, can unilaterally break their connections with the rest of the agents. This action leads to losses in output caused by the absence of some links. Notice, that this loss is asymmetric across members of the larger and the smaller groups. Suppose the sizes of the groups are $k_1$ and $k_2$ such that $k_1 > k_2$. The loss that an agent incurs is determined by the number of links he loses, which is $k_2$ for a typical member of the larger group and $k_1$ for a member of the smaller one. The difference of losses in output gives members of the larger group an advantage in the competition and allows them to keep the prize within their group. Notice, that members of the smaller group are powerless against such strategy used by the larger group, since the links that connect the two groups can be ceased unilaterally by one side.

The rest of the papers is organized as follows. I set up the model in Section 2. In Section 3 I introduce the solution concept for my model and discuss its properties. The main results are presented in Section 4. I extend the model to accommodate for heterogeneity in agents' talents in Sections 5. Related literature is discussed in Section 6. Finally, Section 7 concludes. All proofs are at the end of the paper in the Appendix.

# 2 Framework

There are $N$ identical agents in my model. Each agent produces an output and competes against the others in a tournament. The agents, who produced the highest output, win the tournament.

A key ingredient of my model is collaboration between competitors. The more the agents collaborate, the higher their outputs are. I assume that collaboration takes place in pairs. Although, collaboration in groups is not allowed as a primitive of the model, it can be represented as a set of the collaborations in pairs. I do not put any restriction on the number of partners agents can have.

I represent a structure of collaboration relationships by a network. Naturally, nodes represent agents and links — collaboration. Some additional definitions related to networks are introduced and discussed in Section 2.1.

The agents set up their links with each other prior to producing the output and competing in the tournament. I assume that the network of collaboration is fully observable by all agents at each step of this process. An agent is free to choose his partners, but each link requires a consent from the counterpart. An existing link, however, can be unilaterally removed by any of the two participants.

I do not assume any particular protocol, that agents have to follow in setting up their links. Instead, I use a cooperative approach. I define which deviations agents can pursue from a current network. After that, I define stable networks, as the ones that are immune to such deviations. I discuss this approach in detail in Section 3

Once all links have been established, each agent produces an output. I assume that an agent's output is increasing in the number of his links: an output of agent, that collaborates with $x$ partners, is given by $f(x)$. In addition, I assume that each new partner strictly improves the performance of the agent, i.e. $f(x)$ is strictly increasing.

Finally, once all agents have produced their output, they compete with each other in the tournament. The agents with the maximal output are announced as winners of this tournament. The winners are awarded a prize, that they split equally between themselves. I denote the size of the prize by $g$ and the number of winners by $w$. Thus, each winner gets $\frac{g}{w}$.

I assume, that the collaboration is beneficial for the agents even after I control for its effect on the competition outcome. This means that even the agent that has no chance of winning prefers to have as many links with others as possible. In particular, I assume that agents receives his output as a part of his payoff.[6] This assumption holds in many applications. For instance, the compensation schemes for employees usually take into account both their absolute and relative performance. In this example, the part of the compensation, that depends on the employee's absolute performance is determined, among many other things, by the extent of the employee's collaboration with others.

Notice, that both the division of the prize and the agents' outputs are determined by a current network. Suppose an agent $i$ has $x_i$ links in the network. By $x_{-i}$ I denote the vector that contains the numbers of links of all other agents. Then, the agent $i$'s payoff is given by

$$u_i(x_i, x_{-i}) = \begin{cases} f(x_i) + \frac{g}{w} & \text{, if } x_i \geq x_j \text{ for all } j \in N \\ f(x_i) & \text{, otherwise.} \end{cases}$$

This payoff of consists of two parts discussed above: one is the share of the prize $\frac{g}{w}$ that the agent receives in case he wins the tournament, and the other is the agent's output $f(x_i)$.

I do not allow monetary transfers between the agents in this model. In general, the possibility of transfers has a stark effect on the outcome of the network formation process (see, for instance, Dutta and Mutuswami (1997)).

## 2.1 Networks

As I mentioned already, the structure of the collaboration is modeled using a network, in which nodes represent the agents, and links — the collaborations.

[6] This assumption is without loss of generality, as long as this portion of agent's payoff is strictly increasing in the number of agent's connections

The set $N$ of agents is the set of nodes (or vertexes). Let $\Lambda$ be a set of all subsets of $N$ of size two. An element $ab \in \Lambda$ represents the link between agents $a$ and $b$. Nodes $a$ and $b$ are called the ends of the edge $ab$. A network $\psi$ is a collection of links between agents, i.e. a subset of $\Lambda$. Therefore the set $2^\Lambda$ is the collection of all networks on $N$.

Let $\psi \in 2^\Lambda$. With some abuse of notations, I say that $\psi(a,b) = 1$ if $ab \in \psi$, and $\psi(a,b) = 0$ otherwise. Also, let $M \subset N$. By $E_M(\psi)$ I denote the set of edges that have the elements of the set $M$ as their ends (if $M = N$ I drop the index and write $E(\psi)$). Also, by $I_M(\psi)$ I denote the set of edges that have *only* the elements of the set $M$ as their ends. Finally, $O_M(\psi) \equiv E_M(\psi)\backslash I_M(\psi)$.

Any set operation on networks are taken with respect to the set of edges. For example, for $\psi, \gamma \in 2^\Lambda$, $\psi \cap \gamma$ is the network that satisfies $E(\psi \cap \gamma) = E(\psi) \cap E(\gamma)$.

A path between two nodes, $a$ and $b$, is a collection of edges of the form $\{a_0a_1, a_1a_2, \ldots, a_{k-1}a_k\}$ such that $a_0 = a, a_k = b$ and $a_i \neq a_j$ for any $i \neq j$. If such a path exist in $\psi$ I say that $a$ and $b$ are connected in $\psi$. The set of nodes $M$ is connected in $\psi$ if any $a, b \in M$ are connected in $\psi$. The maximal (in the set inclusion sense) connected set of nodes $M$ is called a component. In this case, with a slight abuse of notations, I call $E_M(\psi)$ a component as well. Component $M$ is regular if for any $a, b \in M : |E_a(\Psi)| = |E_b(M)|$, and is complete if for any $a \in M : |E_a(\psi) = |M| - 1$.

By $M(\psi)$ I denote the set of nodes with maximal number of edges in $\psi$:

$$M(\psi) = \arg\max_{k \in N}\{|E_k(\psi)|\}.$$

Given these notations, we can rewrite agent $i$'s utility in network $\psi$ as

$$U_i(\psi) = f(|E_i(\psi)|) + \mathbf{1}_{i \in M(\psi)} \frac{g}{|M(\psi)|}$$

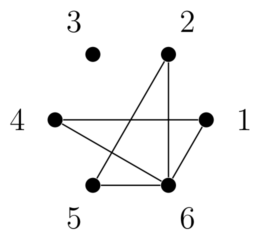
Throughout the paper, I refer to the winners of the tournament as "insiders" and to the rest as "outsiders". The meaning of these terms will become more clear once I present the results.

# 3 Stability

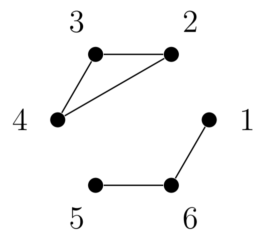
In this section I define the solution concept that I use through out the paper. Before proceeding to the formal definitions, I must say, that I follow the approach of the cooperative game theory. I use the stable sets of von Neumann and Morgenstern (1944) combined with the notion of farsighted agents first introduced by Harsanyi (1974). Similar approach is used in the literature on networks, two-sided matching, and coalition formation (see Ray and Vohra (1997), Diamantoudi and Xue (2007), Herings et al. (2009) and others). Although this modeling choice might seem to be ad hoc at this stage, I invite the reader to follow me to the end of this section. After introducing the definitions I discuss the strong sides and limitations of this solution in detail.

The solution that I use for my model is a stable set of networks. Such set is defined to be immune to *credible* deviations by coalitions of agents. Before introducing the formal definition of a stable set, I must describe what these coalitions *can* do, and what they *would like* to do.

Previously, we mentioned that a link between two agents can appear only if they both agree to this. At the same time the existing link can disappear even if only one of the two partners drops it. This rule naturally extends to the coalitions of agents. Let us look, for example, on the networks $\psi_1$ and $\psi_2$ on the Fig. 1. What coalition of




(a) Network $\psi_1$;




(b) Network $\psi_2$.

Figure 1

agents can enforce the transition from the network $\psi_1$ to the network $\psi_2$? Observe, that links 23, 34, and 24 do not exist in network $\psi_1$, but do exist in the network $\psi_2$. This means, that, participation of agents 2, 3, and 4 is necessary for this transition. Also, observe that links 14, 46, 25, and 26 must be removed in order to reach the network $\psi_2$. Agents 2 and 4 can do that as well, so we can conclude, that coalition $\{2, 3, 4\}$ can enforce the transition from the network $\psi_1$ to the network $\psi_2$.

The following definition generalizes this principle.

**Definition 1.** *A coalition $S$ can enforce a transition from a network $\gamma$ to $\psi$ or*

$$\psi \xrightarrow{S} \gamma$$

*if for all $i, j \in N, i \neq j$ the following holds:*

*(i)* $\gamma(i,j) > \psi(i,j) \implies \{i,j\} \subset S$;

*(ii)* $\gamma(i,j) < \psi(i,j) \implies \{i,j\} \cap S \neq \emptyset$.

Observe, that there is an asymmetry in the process of creation and removal of links. In particular, $\psi \xrightarrow{S} \gamma$ does *not* imply that coalition $S$ can reverse this process, i.e. $\gamma \xrightarrow{S} \psi$. If we go back to the example on the Fig. 1, we can see that $\psi_1 \overset{\{2,3,4\}}{\rightarrow} \psi_2$ and $\psi_2 \overset{\{1,2,4,5,6\}}{\rightarrow} \psi_1$.[7]

The previous definition only states what agents can do, but it does not take into account their incentives. The traditional approach says that the transition will take place if all active participants are better off. It is captured by the notion of *blocking*: if transition from $\gamma$ to $\psi$ can be enforced by some coalition of agents that benefit from the transition, network $\psi$ block network $\gamma$. Depending on which coalitions are allowed to act, there are notions of setwise or pairwise (myopic) blocking (see Jackson (2008), Goyal (2007), Roth and Sotomayor (1992)).

**Definition 2.** *A network $\psi$ setwise blocks $\gamma$ or*

$$\psi \rhd \gamma$$

*if there exists a coalition $S$ such, that*

*(i)* $\gamma \xrightarrow{S} \psi$

*(ii)* $\mathbf{U}_S(\psi) > \mathbf{U}_S(\psi)$.[8]

*In addition, $\psi$ pairwise blocks $\gamma$ (or $\psi \rhd_p \gamma$) if $|S| = 2$.*

Usually, a stable network is defined as one that is not blocked by any other network. I depart from this approach for several reasons. First, according to this definition, agents can not plan their actions ahead. For instance, there can be a situation in which an agent prefers to have a certain number of links or no links at all.[9] If he is unconnected, it follows from this definition, that he does not want to create the first link because he can not foresee that he will have a chance to create the second one.

Second, agents evaluate their actions as if a network that results from a transition is always stable. This requirement is excessively strong and often stable outcomes simply do not exist. In particular, in my model I can always find reasonable parameters for which the set of unblocked networks (or an abstract core defined with respect to $\rhd$) is empty.

**Proposition 1.** *Suppose $N > 2$. There is always $g$ and $f(\cdot)$ such that no setwise unblocked networks exist.*

This issue is not just a technical detail. The fact that all possible networks are blocked raises a concern that some blocking networks are not credible (because they themselves are blocked as well). It is possible to weaken the definition of stable network ruling out non-credible blocks and still get reasonable predictions.

[7] In this case, the coalitions that can enforce these two transitions are unique.
[8] For two vectors $\mathbf{x}$ and $\mathbf{y}$, $\mathbf{x} > \mathbf{y}$ if $x_i > y_i$ for all $i$.
[9] It happens, for example, when links are complementary and costly.

Suppose, there is a sequence of transitions that is enforced by a sequence of coalitions:

$$\psi_1 \overset{S_1}{\to} \psi_2 \overset{S_2}{\to} ... \overset{S_K}{\to} \psi_{K+1}.$$

I do not require that each step is an immediate improvement for the acting coalition. Instead, all acting coalitions must prefer the ultimate outcome of this transition to the networks in which they are asked to act. For example, the coalition $S_i$ must prefer the network $\psi_K$ to the network $\psi_i$. Otherwise, if this property is violated for some coalition $S_i$, the transition will stop at step $i$ and the network $\psi_K$ will never be reached. One might argue, that if coalition $S_i$ refuses to act, there might be some other coalition $\widehat{S}_i$ that can step in and proceed with the transition. It is indeed true, and I am going to account for it by looking at *all* possible transition paths.

Let us formally define the blocking for farsighted agents.

**Definition 3.** *A network $\psi$ setwise farsightedly blocks $\gamma$ or*

$$\psi \succ \gamma$$

*if there exists a sequence $\{(S_i, \phi_i)\}_{i=1}^K$, $\forall i = 1, ..., K : S_i \subset N$ and $\phi_i \in 2^\Lambda$ such, that*

*(i)* $\gamma = \phi_1 \overset{S_1}{\to} \phi_2 \overset{S_2}{\to} \dots \overset{S_K}{\to} \psi$

*(ii)* $\mathbf{U}_{S_k}(\psi) > \mathbf{U}_{S_k}(\phi_k)$ *for all $k \leq K$.*

*A sequence $\{(S_i, \phi_i)\}_{i=1}^K$ that satisfies properties (i) and (ii) is called $\gamma$-to-$\psi$ path.*

This definition is widely used in the literature on the coalition formation with forward looking agents. This notion is discussed in Ray (2007) in detail. The main difference is that I use the notion of enforceability that is tailored for network formation problems.

Notice, that setwise farsighted blocking is weaker[10] than setwise blocking, hence Proposition 1 applies to it as well. Therefore, instead of looking at unblocked networks, I look at networks that are unblocked only by stable networks, and I define the former to be members of the latter. Put differently, a stable set of networks consists of *all* networks that are unblocked by other networks in this set.

Such sets are fully characterized by internal and external stability.

**Definition 4.** *A set of networks $\mathcal{R}$ is von Neumann-Morgenstern farsightedly stable, if it satisfies the following two conditions:*

*(IS) for any $\psi, \gamma \in \mathcal{R} : \psi \not\succ \gamma$;*

*(ES) for any $\gamma \notin \mathcal{R}$ there exist $\psi \in \mathcal{R} : \psi \succ \gamma$.*

[10] Formally, $\rhd \subset \succ$.

Let $A(\cdot)$ be a function that for a given set of networks $\mathcal{X}$ returns a set $A(\mathcal{X})$ of all networks that are unblocked by any network in $\mathcal{X}$. Then $\mathcal{R}$ is von Neumann-Morgenstern farsightedly stable if and only if

$$\mathcal{R} = A(\mathcal{R}).$$

It is implicitly assumed in the definition that the agents believe that a set $\mathcal{R}$ consists of stable networks. This belief, however, is reinforced by the two properties of internal and external stability. In contrast with the core, the stability of an element of the stable set relies on other elements of the same set. It is also implicitly assumed that agents are optimistic: if they find themselves in the network outside of a stable set, it is enough for them to have only one path back into the set to believe that this set is actually stable. All agents that are active in the transition believe that there is going to be no deviations along the path and that they will definitely reach the stable set.

In general, there exist numerous myopically profitable deviations from any stable set $\mathcal{R}$. The definition of a stable set requires, that for any deviation there exists a path back to the stable set. However, there is no explicit requirement that the original deviators are worse off once the new stable networks is reached. One might think, that agents could use deviations to switch from a less desirable stable network to a more desirable one. It turns out that it is not the case. Such property of stable sets follows from the combination of internal and external stability. It is called *consistency* and it was introduced by Chwe (1994).

Suppose, a coalition of players deviates from a stable network to some network outside of the stable set. Then, there always exists a path back to the stable set such, that at least one of the deviators is worse off compared to the original stable network. Since there is always at least one agents who is punished, the deviations will not take place.

Chwe (1994) states, that stable sets are immune to one-step deviations by a coalition of players. This result can be easily extended to a sequential deviation as well.

**Proposition 2.** *Let $\mathcal{R}$ be a stable set, and $\rho \in \mathcal{R}$ be a stable network. Take any $\psi \succ \rho$, and any $\rho$-to-$\psi$ path $\{(S_k, \gamma_k)\}_{k=1}^{K}$. For any stable network $\widehat{\rho} \in \mathcal{R}$ such that $\widehat{\rho} \succ \psi$, there exist an agent $i \in \bigcup_{k=1}^{K} S_k$ such that*

$$U_i(\rho) \geq U_i(\widehat{\rho}).$$

As I already mentioned, the literature on network formation in general and on collaboration in particular mostly uses pairwise stability notion. Although pairwise stability is a reasonable solution for large networks in which effect of a single agent on the network is negligible, it is unequal to the task of understanding the environments in which a single agent has much control over the network structure. This is the main reason why my approach is very different from the traditional one.

Herings et al. (2009) study similar stability notion. They, however, look at the pairwise farsighted blocking and they substitute internal stability with the weaker condition. Although this weakening guarantees existence, it also makes their stability less discriminative. Some networks that can never be stable in my framework are stable in theirs. Similarly, one can use the adaptation of Chwe's largest consistent set, which is always non-empty, to solve a network formation problem, but as in case of Herings et al. (2009), the largest consistent set has little predictive power.

For the rest of the paper, I often write "block" instead of "setwise farsightedly block" and "stable" instead of "von Neumann-Morgenstern farsightedly stable" to save space. Also, I refer to networks from a given stable set as stable. However, one should keep in mind that these networks are stable *with respect to* the corresponding stable set. Finally, one should not confuse stable networks with elements of an abstract core.

# 4 Main Results

There are two main questions to be answered. Under what conditions are the efficient networks stable? If the efficient networks are not stable then what networks are?

To answer these questions, first, I have to characterize efficient networks. In my setup, the complete network $\Lambda$ is the unique efficient network. Intuitively it is easy to see: the distribution of the prize in the tournament does not have an effect on the aggregate welfare, since the sum of the prizes is constant and equal to $g$. Therefore, an efficient network is the one that has the maximal aggregate output. Recall that any link strictly improves the outputs of the two agents that are connected by it, hence the complete network is the unique efficient network.

**Proposition 3.** *The complete network $\Lambda$ is the unique efficient network.*

Let us start with the first questions. What are the necessary and sufficient conditions for the unique efficient network $\Lambda$ to belong to a stable set? The following theorem provides these conditions.

**Theorem 1.** *Take a stable set $\mathcal{R}$, and a network from this stable set $\rho \in \mathcal{R}$. The set of winners in this network $M(\rho) = N$ if and only if*

$$N \in \arg\max_{m > \frac{N}{2}} \left\{ f(m-1) + \frac{g}{m} \right\} \tag{1}$$

Theorem 1 does not mention the efficient network $\Lambda$ explicitly. Notice, however, that the efficient network $\Lambda$ is symmetric, hence $M(\Lambda) = N$. To obtain the desired characterization, one has to combine this observation with Theorem 1.

**Corollary 2.** *The efficient network $\Lambda$ is stable if and only if*

$$N \in \arg\max_{m > \frac{N}{2}} \left\{ f(m-1) + \frac{g}{m} \right\}.$$

This result is very different from the analogs obtained in the literature (see for example Goyal (2007) or Marinucci and Vergote (2011)). Previously, it has been argued that the efficient outcome is *always* stable, and in addition, some inefficient outcomes can be stable as well. In contrast with that, The Corollary 2 says that the efficient network *can not* be stable if the condition (1) is not satisfied.

This difference in results is due to the assumption that the agents are forward looking and are able to coordinate with their peers when creating their connections. For a moment I focus on forces that drive the result in Corollary 2, since they have not been discussed in the existing literature.

In my model, any agent in any network benefits from creating an extra link. Let us look at two arbitrary agents who are not collaborating with each other. A link between them is mutually profitable since each of them gets an extra payoff of $f(x+1) - f(x) > 0$, and their chances of winning the tournament weakly increase.

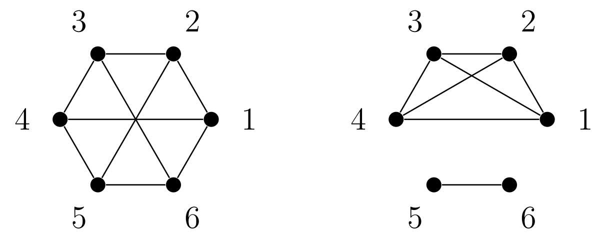
Each of these two agents discussed above have two options: to create the missing link, or leave things as they are. I already mentioned, that if the agents choose the first option, their payoffs strictly increase. On the other hand, if one of them chooses the second option, he imposes a *positive externality* on the rest of his competitors. Indeed, if these two agents do not collaborate, their performance in the tournament is lower than it could be. As a result, the chances of winning for their competitors are higher. The price that the agent who refused to collaborate pays for imposing this externality is the opportunity cost.

The benefits of such externality, if it is properly exploited, can outweigh the total opportunity costs of all agents involved. One way to exploit the externality is for agents to coordinate their missing links.

To illustrate this point, let us look at two networks on Fig. 2. In the regular network $\psi_r$, all agents have three links each, hence they each get a payoff of $f(3) + \frac{g}{6}$. Any agent in this network misses two links, but the externality imposed by this agent on others cancels out by the externality others impose on him.

Observe, however, that the agents 1,2,3, and 4 can coordinate their missing links and use the externality they impose on each other for their own good. In particular, they can delete their links to agents 5 and 6, and create missing links with each other. In the resulting network, $\psi_n$, each of them still has three links, but their payoff, $f(3) + \frac{g}{4}$, is higher than in $\psi_r$ since agents 5 and 6 do not get their share of the prize anymore.

In this case the agents who coordinate their actions to exploit the externality have the same number of link in the beginning and at the end. It does not have to be the case. A group of agents might be willing to lose some links if in return they get a




(a) Regular network $\psi_r$; (b) Nonregular network $\psi_n$;

Figure 2: Externality in regular and nonregular networks.

high enough share of the prize.

This effect drives the results in Theorem 1 and Corollary 2. In particular, if the condition (1) does not hold, it means that there exists $m \in (\frac{N}{2}, N)$, such that

$$f(m-1) + \frac{g}{m} > f(N-1) + \frac{g}{N}.$$

This inequality guarantees that one can find a group of $m$ agents who are willing to disconnect themselves from the rest in order to be the only winners in the competition.

Needless to say that the existence of such deviation from the efficient network alone does not guarantee that the efficient network can not be stable. One need to show that such deviation is credible, i.e. is not blocked by other credible networks. The following example makes it clear.

**Example 1.** Suppose there are three agents: 1,2, and 3. Also, suppose that the prize in the tournament, $g$, satisfies

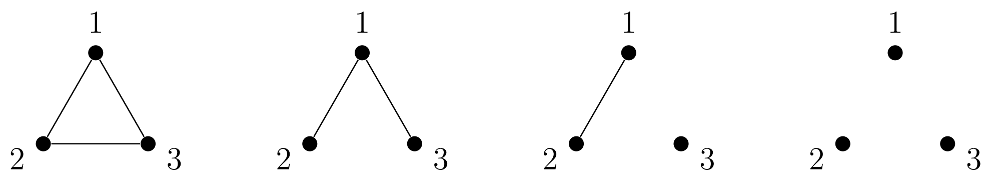
$$g > 6(f(2) - f(1)).$$

Clearly, in this case, the condition (1) is violated, hence Corollary 2 tell us that efficient network can not be stable. Let us establish it without using the corollary. I will show that in this case the unique stable set consists of three networks each consisting of a single link: $\mathcal{R} = \{\{12\}, \{23\}, \{13\}\}$.

Let us look at the Fig. 3. It depicts all possible networks up to a permutation of the agents. First, notice, that for any network except $\psi_2$, there are two agents who strictly prefer some network in $\mathcal{R}$ to the original network. In particular agents 1 and 2 prefer $\{12\}$ over both $\Lambda$ and $\psi_3$, and agents 2 and 3 prefer $\{23\}$ over $\psi_1$. Clearly, these pairs are sufficient to enforce these transitions, hence $\mathcal{R}$ is externally stable.

Internal stability of $\mathcal{R}$ is also easy to see. All connected agents are indifferent between their partners, hence they will not participate in any transition between the networks in $\mathcal{R}$. So far I established that the set $\mathcal{R}$ is stable. The question is if there are any other stable sets. The answer to this question is "no", and I prove it by contradiction.

Assume that there is another stable set $\mathcal{P}$. This stable set must contain the network $\psi_1$, since it is the only network (up to a permutation of agents) that blocks the network $\psi_2$. However, the network $\psi_1$ does not block the efficient network $\Lambda$, hence the latter should also belong to the stable set. The desired contradiction arises because agents 2 and 3 can enforce the transition from $\psi_1$ to $\Lambda$ and moreover they both strictly prefer $\Lambda$ over $\psi$. This means that $\Lambda \succ \psi_1$ and the internal stability of the set $\mathcal{P}$ is violated.




(a) Network $\Lambda$ (b) Network $\psi_1$ (c) Network $\psi_2$ (d) Network $\psi_3$

Figure 3: Example 1

Intuitively, although the network $\psi_1$ blocks the network $\psi_2$, this block is not credible. One of the two agents who enforce the transition, namely agent 1, foresees that after he collaborates with the agent 3, the latter will also collaborate with the agent 2. Moreover, agent 1 thinks of the collaboration between agents 2 and 3 (i.e. of $\Lambda \succ \psi_1$) as being credible, since the other network $\psi_1$, which is (wrongly) assumed to be stable, does not block $\Lambda$.

The novel tension between stability and efficiency presented in Corollary 2 is particularly striking, if one tries to approach this model with the traditional stability concept, i.e. pairwise stability. If agents are myopic, the pairwise stability unambiguously predicts that the efficient network is the unique stable one.

Theorem 1 is more than just a necessary and sufficient conditions for the stability of the efficient outcome. It also establishes, that if the efficient network is not stable, then any stable network features ex post inequality in agents' payoffs. The surprising part of this result is that agents are identical, and yet, if the condition (1) does not hold, it is impossible that they all have the same payoff in a stable outcome.

What are the implications of Theorem 1 for the examples we discussed in the Introduction? Let us look at high-tech firms that compete with each other in the market and can collaborate by means of cross-licensing agreements. These agreements allow firms to use each other's patent portfolios without paying royalties. The firms which have an access to their rivals' patents gain a competitive advantage on the market. This advantage, however, does not come for free: a typical cross-licensing agreement also allows firm's counterparts to access its patents and, hence, makes the latter stronger rivals.

Applied to this environment, Theorem 1 predicts that if the stakes in the competition are high (i.e. the market is thick) patent wars between alliances of firms are unavoidable. In the course of patent war groups (or alliances) of firms manipulate the structure of the collaboration network in order to hinder the performance of their competitors. The resulting network of collaboration is nonregular, and firms that sign more cross-licensing agreements gain a larger share of the market. [11]

Another application of Theorem 1 is a discrimination. McAdams (1995) argues that a race discrimination in the U.S. is there to maintain the gap in the social status between whites and blacks. People, who, in addition to their wealth, also value being wealthier than others, may sacrifice some mutually beneficial connections in order to maintain the gap in wealth between discriminating and discriminated groups.

Theorem 1 in this case states that if the concern for the social status is significant, discrimination necessarily arises in the stable network of collaboration. It is important to notice, that this theory predicts discrimination that arises in the society of identical agents. This prediction is consistent with the observation that discrimination happens along economically irrelevant markers such as skin color or ethnicity.

Of course one has to be cautious when applying Theorem 1 to the environments discussed above because my framework is extremely stylized. In order to obtain more accurate predictions, the framework has to be modified to accommodate for the specifics of a particular environment.

Although Theorem 1 hints some properties of stable sets, it does not tell us what networks are stable. This brings me to the second question of this section: if the efficient network is not stable, then what networks are? The next theorem gives the answer to this question.

**Theorem 3.** *Let,*

$$\mathcal{M}^* = \operatorname*{arg\,max}_{\frac{N}{2}<m\leq N}\left\{f(m-1)+\frac{g}{m}\right\}, \text{ and} \tag{2}$$

$$\mathcal{G}(m) = \{\psi \in 2^{\Lambda} \mid \exists M \subset N : |M| = m, E(\psi) = I_M(\Lambda) \cup I_{N\setminus M}(\Lambda)\}. \tag{3}$$

*The set of networks* $\mathcal{G}^* = \bigcup_{m\in\mathcal{M}^*} \mathcal{G}(m)$ *is stable.*

I construct a stable set of networks for any set of the parameters. This construction consists of several steps. First, I find the maximizer for the problem that was introduced in Theorem 1. For simplicity, lets assume that the maximizer is unique, i.e.

$$m^* = \operatorname*{arg\,max}_{\frac{N}{2}<m\leq N}\left\{f(m-1)+\frac{g}{m}\right\}.$$

[11] A recent example of such patent war is a struggle between the alliance of Microsoft and Apple on one side and Google on the other side. Throughout 2011, these companies were engaged in the patent war in the market for smartphones.

Take the set $M$ that consists of any $m^*$ agents, and look at the network in which the set $M$ and its complement $N \backslash M$ form two complete components, i.e. the agents in $M$ are connected to all other agents in $M$ and are not connected to the agents outside of $M$ (similarly the agents in $N \backslash M$ are only connected to all other agents in $N \backslash M$). Let us denote this network by $\gamma$. The set of networks $\mathcal{G}^*$, obtained from $\gamma$ by all permutation of the agents is stable. The maximizer $m^*$ determines the size of the group of winners in any network that belongs to the constructed stable set.

Notice, such set always exists since the maximization problem (2) is finite. The example of the set $\mathcal{G}^*$ for $N = 4$ and $m^* = 3$ is given on Fig. 4.

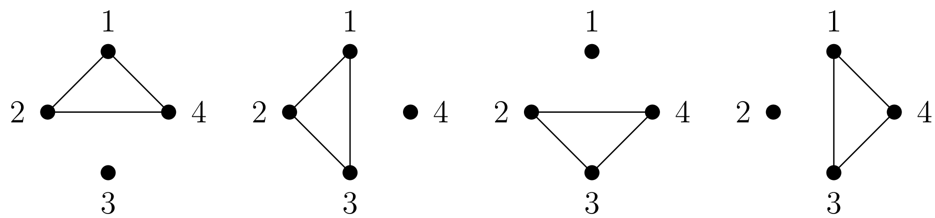



Figure 4: Set of networks $\mathcal{G}^*$ for $N = 4$ and $m^* = 3$

Observe, that in any network in the stable set $\mathcal{G}^*$ the set of winners (or insiders) does not have any links with the set of losers (or outsiders). The insiders do not connect with the outsiders because otherwise the former expose themselves to the danger of being overthrown by the latter. In this set of stable outcomes, the competition is shifted to the group level. It is particularly interesting, because I do not allow for the group competition in the fundamentals of the model. And yet, the agents organize groups and act as if these groups compete against each other.

The criterion for the size of the insiders' group resembles so called "union mentality". The size of the insiders' group is set to maximize the utility of its typical member, and at the same time neither the well-being of the outsiders not any efficiency considerations are taken into account.

As I discussed above, a group of insiders coordinates on which links they sacrifice to eliminate their competitors. I notice, that the number of links sacrificed is in general excessive. For example, for $N = 6$ and $m^* = 4$, the stable network is shown on Fig. 5a. There exist a network in which the same set of agents wins the tournament and at the same time, each agent has strictly higher payoff (see Fig. 5d) and yet this network is not stable. The winners can maintain their status only if they isolate themselves completely from the outsiders. In such case, there is nothing the outsiders can do to destabilize the current network.

The subgroup of winners, in principle, may attempt to collaborate with outsiders and obtain an extra payoff. The stable set is constructed in such a way, that any attempt of this kind will be followed by exclusion of the deviators and substituting them with outsiders. This threat is credible, since any deviation from the stable network perturbs the payoffs of the agents that did participate in the deviation.

Let us go back to the example with firms signing cross-licensing agreements. Theorem 3 describes the structure of a particular stable set. In this stable set a group of agents (or in this case firms) forms a component that dominates the competition. Insiders' dominant position is enforces through absence of links to the outsiders.

Bekkers et al. (2002) provide an empirical evidence from the GSM industry, that fits the predictions of Theorem 3. According to their overview, by the time the GSM standard was established, several companies held patents that are essential in developing the products that comply with the standard. Five of them, which are Ericsson, Nokia, Siemens, Motorola, and Alcatel, actively participated in cross licensing the patents. The same five companies later dominated the market for GSM infrastructure and terminals. Their total market share in 1996 was above 85%. At the same time, three other companies, Phillips, Bull, and Telia, held roughly as many patents as Alcatel and were not able to convert them into a significant market share. Moreover, they performed worse than Ericsson and Siemens, which did not have large patent portfolios and yet were ranked the first and the third largest GSM companies in 1996.

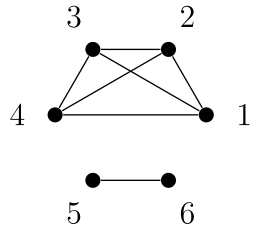
Bekkers et al. (2002) argue that the companies, that dominated the GSM market in the 1990s, could maintain the control over the market because they where able to shift the competition from the individual level to the group level. On the individual level, none of them had a competitive advantage over their rivals. However, they could gain an advantage on the group level by tailoring the size of their alliance.

The set of networks $\mathcal{G}^*$ is not a unique stable set. However, this set is not arbitrary: it is crucial for understanding the driving forces behind the Theorem 1. Let me look at the example, that provides an intuition for Theorem 1 and Theorem 3.

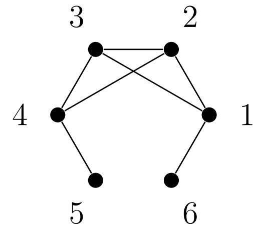
**Example 2.** Assume, that $N = 6$ and $m^* = 4$. Theorem 3 tells us, that there exists a stable set $\mathcal{G}^*$, that consists of the networks with two complete components of sizes 4 and 2. One of such networks is shown on Fig. 5a.

The set $\mathcal{G}^*$ has several properties that are worth illustrating. First, it is internally stable (which is one of the requirements for stability). Second, in any network in set $\mathcal{G}^*$ the insiders are not connected to the outsiders. Third, the set $\mathcal{G}^*$ is consistent. I discuss the intuition behind these properties in detail in the context of this example.

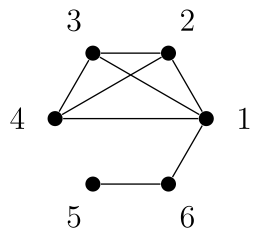
Let us start with internal stability. The set $\mathcal{G}^*$ is internally stable, because any network in $\mathcal{G}^*$ consists of two complete component. Take any agent, say agent 1. In the network on Fig. 5a, the agent 1's payoff is $f(3) + \frac{g}{4}$. If agent 1 were in the smaller component, like agent 6 is in the current network, his payoff would be $f(1)$. These are the only two payoffs any agent receives in any network in $\mathcal{G}^*$. Since agents 1, 2, 3, and 4 already receive the maximum possible payoff, they would not want to switch to any other network in $\mathcal{G}^*$. The only agents that would prefer to switch to some other network in $\mathcal{G}^*$ are 5 and 6. However, the only action they can take is dropping the one link they have and switching to the network on Fig. 6b. As the result of this action, the payoffs of agents 1, 2, 3, and 4 do not change, hence agents 5 and 6 are stuck in



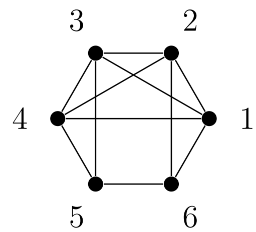

(a) Stable network;




(b) A network, that is payoff equivalent to a stable one;




(c) Deviation by agents 1 and 6;




(d) The network, that strictly Pareto dominates the stable one.

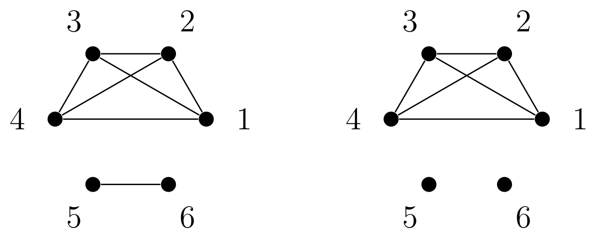
Figure 5: Stable network and deviations from it.

one of the two networks on Fig. 6. They both strictly prefer the stable network on Fig. 6a to the one on Fig. 6b.

Each network in $\mathcal{G}^*$ features the group of insiders who control the distribution of the prize through enforcing a particular structure of the collaboration. For each agent there are many sets of partners with whom he can take over the tournament. Intuitively, for internal stability to hold, agents must be indifferent between their partners.

The second property of $\mathcal{G}^*$ is that in any network in this set the insiders do not collaborate with the outsiders. There is only one reason why some links are missing in any stable network, not only in the networks in $\mathcal{G}^*$. The irregularity of a network favors some agents who create it by assigning a share of the prize that is larger than $\frac{g}{N}$ to these agents. Put differently, a group of agents weaken their rivals by not collaborating with them.

Clearly, if the the prize is assigned to the agents with the highest performance, a minimal difference in performance is sufficient for the purpose of dominating the tournament. This minimal difference can be achieved by the absence of less than $\frac{N}{4}$ links. Notice, that for every network in the set $\mathcal{G}^*$, there are $m^*(N-m^*) \geq \frac{N}{4}$ missing links. For instance, in our current example it is enough to drop 1 link to ensure that $m^* = 4$ agents win the tournament. However in each network in the set $\mathcal{G}^*$, there are




(a) Stable network;

(b) The only possible deviation by agents, who do not win the tournament.

Figure 6: Internal stability

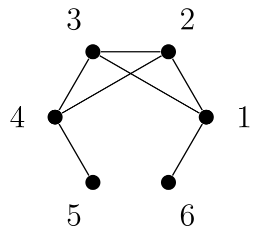
8 missing links.

This extra inefficiency which comes from deleting more links than necessary for ensuring the dominant position in the tournament, is a price for (internal) stability. To illustrate it, let me look at the network on Fig. 5b. This network is payoff equivalent to the stable one on Fig. 5a, since agents 1, 2, 3, and 4 have three links each and agents 5 and 6 have one link each in both networks. The only difference between the two is that agents that win the tournament are connected to ones that do not. The links between insiders and outsiders create an opportunity for the the latter to become winers in the tournament. For example, agent 6 can start a transition to a stable network, in which he is one of the tournament winners. In order to do this, agent 6 has to drop his link to agent 1 (see Fig. 7b). Once it is done, agent 1 no longer wins the tournament, hence his payoff is low enough for him to participate in the transition. If the agent 1 drops all his link, agent 2 also drops out of the set of winners. At this point (see Fig. 7c), there are five agents whose payoff is below $f(3) + \frac{g}{4}$. Four of them (agents 5, 6, 1, and 2) can form a new set of winners in network on Fig. 7d.

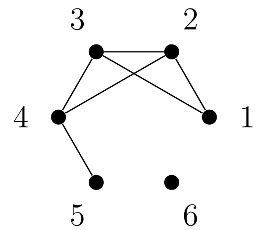
The same result holds even for the networks, that strictly Pareto dominate stable ones. Even though all 6 agents would myopically agree to switch from the stable network to the one on Fig. 5d, they could not keep this deviation stable in the long run, because agents 5 and 6 would be tempted to act further and to force a network that favors them the most.

Intuitively, links between insiders and outsiders stand in the way of the stability for large sets (the set $\mathcal{G}^*$ consists of $C_N^{m^*}$ networks) because these links give the outsiders the leverage against the insiders. In particular, the outsiders may want the membership in the insiders' group in return for their collaboration with some insiders.

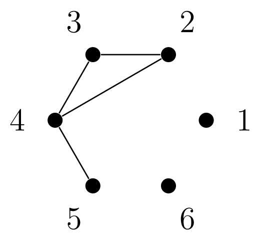
As the Proposition 2 states, any stable set is consistent, i.e. for any deviation from the stable set, there exists a credible counterdeviation back to the stable set, such that some of the agents who originally deviated prefer the initial network to



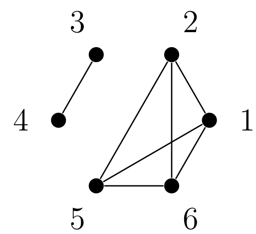
(a) Initial network



(b) A move by agent 6 after which agent 1 falls out of the set of winners;



(c) Agent 1 takes an action after which agent 4 becomes the only winner;



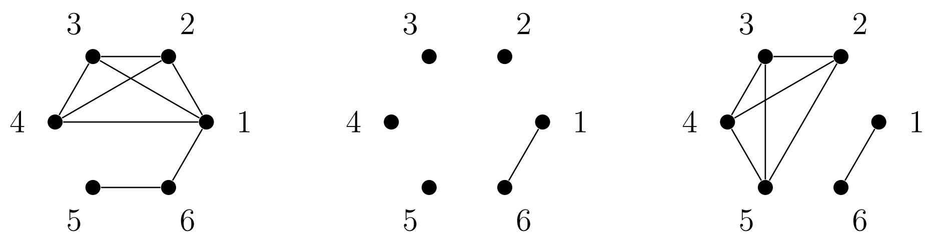
(d) Stable network.

Figure 7: Links between insiders and outsiders.

the current one. This property makes sure, that deviations can be punished in a consistent way.

To illustrate this, suppose agents 1 and 6 deviate from the stable network on the Fig. 5a to the network on Fig. 5c. If the latter network was stable, both agents would be better off: both agents would get an extra link, and agent 1 would still be the winner in the tournament. It is clear however that the resulting network is not stable. Moreover, there is a counterdeviation such, that agent 1 loses all his links but one and drops out of the set of winners. Since, in the network on Fig. 8a, agent 1 is the only winner in the tournament, agents 2, 3, 4, and 5 have low enough payoffs to act on switching to a stable network. These agents could drop all their links, leaving agents 1 and 6 in isolation (see Fig. 8b), and relink with each other later on. As a result, agents 2, 3, 4, and 5 would be the new set of insiders (see Fig. 8c). Agent 1 clearly prefers the initial network on Fig. 8a to the on Fig. 8c, hence he would never attempt the considered deviation in the first place. Since all agents are the same, any insider that deviates from the stable network and "hurts" other insiders can be effectively substituted with some outsider. Such punishment is credible, because any outsider is happy to take a place of an agent, that participated in the deviation.

The punishment can not be guaranteed to all agents, that deviated at the same time (since, for example, some deviations involve all agents), but it is not necessary for credibility. Even if one of the participants of the deviation is credibly threatened, this particular deviation can not take place. Since this result holds for *all* deviations, none of them are credible.




(a) Deviation by agents 1 and 6; (b) Agents 1 and 6 in isolation. (c) Stable network.

Figure 8: A deviation by agents 1 and 6 and a path back into the stable set.

Although I do not provide the full characterization of stable sets, I point to some interesting properties that are common to all of them. Theorem 1 shows, that under a certain conditions, in any stable network some agents are discriminated. No network with perfect equality of payoffs across all agents can be a part of a stable set.

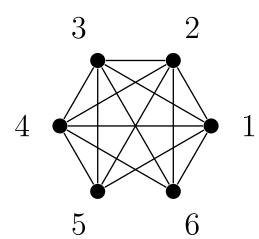



Figure 9: A stable singleton set.

For example, the singleton set on the Fig. 9 is always stable. In this network agents 5 and 6 do not win the tournament and their payoffs are strictly smaller than the ones of agents 1, 2, 3, and 4.

# 5 Extensions

When agents are identical the group of winners in the stable networks that I find is selected arbitrarily. A natural question, that arises, is how the agents split into insiders and outsiders if some of them become more productive than the others. Is it the case, that most (or least) productive agents become the most likely to be discriminated? What happens to the payoff inequality across agents? How does it compare to the payoff inequality in the efficient outcome?

To answer these questions I study the extension of the model, that features the heterogeneous agents and various intensities of collaboration between them. In particular, I study the case in which the agents are heterogeneous in their ability to transform collaboration into the output. I start with introducing various intensity of collaboration. After that, in Section 5.1, I add agents' heterogeneity to this extended model.

I assume, that an agreement between two agents $i$ and $j$ is characterized by two numbers $x_{ij}$ and $x_{ji}$ between 0 and 1. These numbers represent the intensity of collaboration. In particular $x_{ij}$ represents the intensity of "help" that agent $j$ provides to agent $i$. An agent $i$, after all agreements are finalized, receives the amount of "help" that is equal to

$$x_i = \sum_{j \neq i} x_{ij}.$$

An aggregate output (or performance) of an agent $i$ in this case is $f(x_i)$.

The agents' payoffs consist of two parts. As in the symmetric model, all agents get their outputs as a payoffs. In addition to it, agents benefit from outperforming their competitors. An agent $i$ gets a prize $g$ with probability

$$p_i = \frac{f(x_i)}{\sum_j f(x_j)}.$$

Agent $i$'s aggregate payoff in this game is

$$u_i(x_i, x_{-i}) = f(x_i) + g \frac{f(x_i)}{\sum_j f(x_j)}.$$

The agreements in this model are more complex, than in the symmetric one. In particular, it is possible, that agent $i$ lowers the performance of agent $j$ without necessarily changing its own performance: switching from the an agreement $(x_{ij}, x_{ji})$ to an agreement $(x_{ij}, \widehat{x}_{ji})$, where $\widehat{x}_{ji} < x_{ji}$ will achieve this effect.

A network $\psi$ in this model is not just a collection of pairs from the set $N$. Instead, a network is a function that for each *ordered* pair of agents returns the intensity of the *directed* link: $\phi : N \times N \to [0, 1]$. Using our previous notation, if two agents, $i$ and $j$, are connected by a link $(x_{ij}, x_{ji})$ in $\phi$, then $\phi(i, j) = x_{ij}$ and $\phi(j, i) = x_{ji}$.

The stability concept that was used for the basic model carries over to the extended model without any modifications.

I find that the result of Theorem 3 holds in this framework as well. The only difference is that the size of the group of insiders is defined using the new payoff function. Similarly to the basic model, I can define

$$\mathcal{M}^* = \arg \max_{\frac{N}{2} < m \leq N} \left\{ f(m-1) + \frac{g/m}{1 + \frac{(N-m)f(N-m-1)}{mf(m-1)}} \right\} \tag{4}$$

and

$$\mathcal{G}^* = \bigcup_{m \in \mathcal{M}^*} \left\{ \gamma \mid \exists M \subset N : |M| = m, \gamma(i,j) = \begin{cases} 0, & \text{if } |\{i,j\} \cap M| = 1 \\ 1, & \text{otherwise} \end{cases} \right\}. \tag{5}$$

The set $\mathcal{G}^*$ is stable. Formally, this result follows from Proposition 4 presented in the following section.

In the stable set $\mathcal{G}^*$ the internal collaboration within groups of winners and outsiders is at the maximal level, and the collaboration between groups is absent. This observation reassures the statement that the absence of links between the two groups of agents is crucial for the stability of this configuration. Indeed, in the current extension of the model agents can tune their collaboration much more fine than in the basic model and yet they choose not to collaborate with some of their potential partners.

## 5.1 Heterogeneity

I assume, that agents are heterogeneous in productivity. By $\alpha_i$ I denote agent $i$'s ability to convert the incoming links into output. In addition I assume that agent's technology is linear, i.e. $f_i(x_i) = \alpha_i x_i$.

In this section I find stable sets of networks that resemble ones described in Theorem 3. In particular, if all agents have the same abilities, the set that I find is defined by the equation (5).

Networks in the new stable sets feature two disconnected tightly intralinked groups of agents. Notice, that the term "group of winners" is not applicable in this framework, because any agent with nonzero output gets certain share of the prize. Moreover, since agents are heterogeneous in abilities, an agents with just a few links could in principle win a larger share of the prize than an agents with the large number of links. Nevertheless, in the sets that I find the larger group still can be called the group of insiders based on the following observation. Any agent strictly prefers to be in the larger group than in the smaller group because the former one has more opportunities for collaboration.

In the model with homogeneous agents, the stable set is chosen among the sets that consist of networks with two complete components. In the maximization problem (4), $m$ parametrizes the collection of such sets.

My current setup requires me to look at all sets of networks that consist of two components. I can not restrict my attention on only complete components, because in this model preferences of participants are much less aligned, than in the case of homogeneous agents. To find the stable set, I have to take two steps: first, I characterize the collection of sets that are potential candidate sets, and then I provide the conditions under which a candidate set is stable.

I define a candidate set for each size $m \in \left(\frac{N}{2}, N\right)$ of the majority group. I denote the candidate set by $\mathcal{R}_m$. Such set are constructed to satisfy the following criteria.

Let us look at the set of all networks that have a complete component of size $(N-m)$:

$$\mathcal{D}_m = \left\{ \psi \mid \exists M \subset N : |M| = m \text{ and } \forall i \in N \backslash M : \psi(i,j) = \begin{cases} 1, \text{ if } j \in N \backslash M \\ 0, \text{ otherwise} \end{cases} \right\}$$

Define a set of functions that point to a complete component (CC) in each of these networks:

$$\mathcal{C}_m = \left\{ c : \mathcal{D}_m \to 2^N \mid \forall \psi \in \mathcal{D}_m : c(\psi) \text{ form a CC in } \psi \text{ and } |c(\psi)| = N - m \right\}$$

**Definition 5.** *Fix $\frac{N}{2} \leq m < N$. A candidate pair for $m$ is a set of networks $\mathcal{R}_m \subset \mathcal{D}_m$ and a function $c_m \in \mathcal{C}_m$ such that*

*(i) for any set $M \subset N : |M| = m$ there exists at least one network $\rho \in \mathcal{R}_m$ such that $c_m(\rho) = N \backslash M$;*

*(ii) for any agent $i \in N$ and any two networks $\psi, \phi \in \mathcal{R}_m$ such, that $i \notin c_m(\psi) \cup c_m(\phi)$,*

$$U_i(\psi) = U_i(\phi)$$

*(iii) for any network $\rho \in \mathcal{R}_m$ there exist no network $\psi$ such that*

$$\psi \in \{\phi \in \mathcal{D}_m \mid \exists \widehat{c}_m \in \mathcal{C}_m : c_m(\rho) = \widehat{c}_m(\phi)\}$$

*and*

$$\mathbf{U}_{N \backslash c_m(\rho)}(\psi) > \mathbf{U}_{N \backslash c_m(\rho)}(\rho).$$

**Lemma 1.** *A candidate pair $(\mathcal{R}_m, c_m)$ exists for any $m : \frac{N}{2} < m < N$.*

For each candidate pair $(\mathcal{R}_m, c_m)$ I define a vector of the maximum payoffs that agents receive in networks inside $\mathcal{R}_m$: $\overline{\mathbf{U}}(\mathcal{R}_m)$. In particular agent $i$ receives his maximum payoff in $\mathcal{R}_m$ when he belongs to the larger of the two groups. Take some $\rho \in \mathcal{R}_m : i \notin c_m(\rho)$, then $\overline{U}_i(\mathcal{R}_m) = U_i(\rho)$.

Now, I can proceed to finding the stable sets. The properties above are enough to show that any candidate set is internally stable. All I have to do now is to find a candidate set that is also externally stable. Take a network from the outside of the candidate set. If there are agents that are well connected and at the same time are unhappy (relatively to the networks in the candidate set), they can initiate the transition in to the candidate set. If it is true for any network outside of the candidate set, the latter is externally stable.

**Proposition 4.** *Take a candidate pair $(\mathcal{R}_k, c_k)$. If there exists no network $\psi \notin \mathcal{R}_k$ such, that*

*(i)* $M = \{i \in N \mid U_i(\psi) \geq \overline{U}_i(\mathcal{R}_k)\} \neq \emptyset$*;*

*(ii)* $N \backslash M \subset K$*, where* $K$ *is a complete component,*

*then* $\mathcal{R}_k$ *is stable.*

Let us obtain more intuition about this proposition by looking at the situation when it is violated. Take some candidate set $\mathcal{R}_k$. Suppose there is a network $\psi \notin \mathcal{R}_k$, such that there is a set of agents $M$, and every agent in that set prefers $\psi$ over networks in $\mathcal{R}_k$. Also, the rest of the agents form a complete component. The latter play a role of outsiders in this situation. It is so, because whatever they do — and they can not do much, since their aggregate performance is already at the maximum level — they can not threaten the agents in $M$. At the same time agents in $M$ like the current network more than any potentially stable network, hence they will never make a first step in switching to a network in $\mathcal{R}_k$. As a result, the set $\mathcal{R}_k$ fails to be externally consistent.

**Proposition 5.** *Take two agents, $i$ and $j$, such that $i, j < N$ and $\alpha_i > \alpha_j$. Let $\psi \in \mathcal{R}_m$ be the network in which both agents are in the majority group: $i, j \in M(\psi)$. Then,*

$$\frac{U_i(\Lambda)}{U_j(\Lambda)} > \frac{U_i(\psi)}{U_j(\psi)}$$

The inequality of payoffs within the dominant group is lower in any candidate set, than in the efficient network. The payoffs get "squeezed" because talented agents represent a higher threat to the other participants, and can be easily substituted with less talented ones. If an agent with the high ability wants to stay with the dominant group, he must compete modestly. Since this proposition applies to all candidate sets, it also applies to the stable one.

## 5.2 Three-Agents Example

Suppose, there are three agents with abilities $\alpha_1 > \alpha_2 > \alpha_3$. Under the efficient set of agreements , i.e. when $\Lambda(i, j) = 1$ for all $i, j \in \{1, 2, 3\}$, the payoff of the agent $i$ is

$$u_i^\Lambda = 2\alpha_i + g \frac{\alpha_i}{\sum\limits_j \alpha_j}$$

Any two agents can exclude the third one from the competition for the prize, by unilaterally breaking the agreements. Moreover, if the size of the prize $g$ is large enough, the agents will benefit from it. The stable set inherits several properties from the symmetric model. If there are two stable networks (or in this case sets of agreements) in which an agent belongs to the discriminating group, he is indifferent between these two networks. Also, for each agent there exists the stable set of agreements, in which he is included into a discriminating group and a networks in which he is excluded from one. Finally, in the symmetric model, a discriminating group always forms a complete component. In the extended setting, this property transforms into the following: for any stable set of agreements, there should be no set of agreements that generates the same or smaller (in the set inclusion sense) network and that is preferred by all agents in the discriminating group.

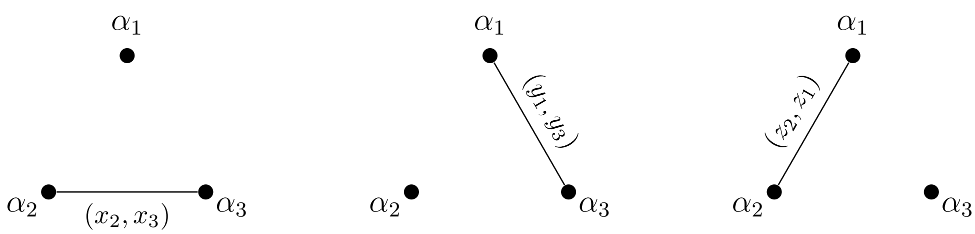



Figure 10: Networks $\psi_1, \psi_2$ and $\psi_3$ (from the left to the right)

I am going to use these properties to characterize the stable set of agreements for the three-agents example. I start with three networks on the Fig. 10. Agent 1 should obtain the same payoff in networks $\psi_2$ and $\psi_3$, agent 2 — in $\psi_1$ and $\psi_3$, and agent 3 — in $\psi_1$ and $\psi_2$. This gives me three conditions:

$$\begin{aligned}
x_3\left(1+\frac{g}{x_2\alpha_2+x_3\alpha_3}\right) &= y_3\left(1+\frac{g}{y_1\alpha_1+y_3\alpha_3}\right)\\
x_2\left(1+\frac{g}{x_2\alpha_2+x_3\alpha_3}\right) &= z_2\left(1+\frac{g}{z_1\alpha_1+z_2\alpha_2}\right)\\
y_1\left(1+\frac{g}{y_1\alpha_1+y_3\alpha_3}\right) &= z_1\left(1+\frac{g}{z_1\alpha_1+z_2\alpha_2}\right)
\end{aligned}$$

Take an arbitrary solution to this system of equations $(x_2, x_3, y_1, y_3, z_1, z_2)$. Suppose for example, that both $x_2 < 1$ and $x_3 < 1$. Then, I can find a set of agreements $\widehat{\psi}_1$, that induces the same network, i.e. $\mathcal{N}(\psi_1) = \mathcal{N}(\widehat{\psi}_1)$[12] such that $\widehat{x}_2 = \frac{x_2}{\max\{x_2,x_3\}}$ and $\widehat{x}_3 = \frac{x_3}{\max\{x_2,x_3\}}$. Notice, that both $\widehat{x}_2$ and $\widehat{x}_3$ are weakly less then 1, hence feasible, and both agents 2 and 3 strictly prefer $\widehat{\psi}_1$ to $\psi_1$. This example violates one of the properties of stable set, so I am going to pick a solution such that one of the agents in each of three discriminating pairs is assigned 1 in the agreement. In particular $x_3 = y_3 = z_2 = 1$. The rest of the solution is $z_1 = \frac{\alpha_2}{\alpha_1} \leq 1$, $y_1 = \frac{\alpha_2}{\alpha_1} x_2 \leq 1$ if $x_2 \leq 1$

[12]I define $\mathcal{N}(\cdot)$ in the following way: $\mathcal{N} \circ \psi(i,j) = 1$ if $\max\{\psi(i,j), \psi(j,i)\} > 0$.

and

$$x_2 \left( 1 + \frac{g}{x_2\alpha_2 + \alpha_3} \right) = 1 + \frac{g}{2\alpha_2}. \tag{6}$$

Notice, that $\frac{d}{dx_2}F(x_2) \geq 0$, $F(0) = 0$ and $F(1) = 1 + \frac{g}{\alpha_2+\alpha_3} > 1 + \frac{g}{2\alpha_2}$, where

$$F(x_2) = x_2 + \frac{gx_2}{x_2\alpha_2 + \alpha_3},$$

is a continouos function on $[0, 1]$, hence the solution to the equation (6) exists and is less than 1.

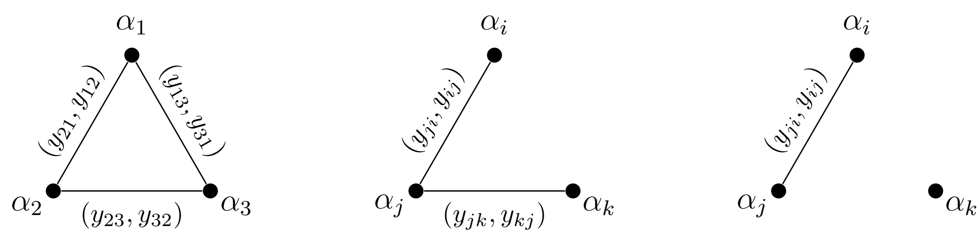



Figure 11: Networks $\phi_3, \phi_2$ and $\phi_1$ (from the left to the right)

I will show that if

$$g > \frac{2\alpha_2(2x_2 - 1)(2\alpha_2 x_2 + \alpha_3)}{\alpha_3} \tag{7}$$

any set of agreements is blocked by one of the elements in $\mathcal{R} = \{\psi_1, \psi_2, \psi_3\}$.

I start with agreements that induce a network with only one link. Take an agreement $\phi_1 \notin \mathcal{R}$ such that $\mathcal{N}(\phi_1)$ has only one link (see Fig. 11). There is a set of agreements $\psi \in \mathcal{R}$ such that $\mathcal{N}(\phi_1) = \mathcal{N}(\psi)$. Either agent $i$ or $j$ prefer $\psi$ to $\phi_1$: suppose it is $i$. Then, agent $i$ can break the current arrangement in $\phi_1$, and after that agent $j$ will be willing to create a new agreement $\psi$. To summarize, for any $\phi \notin \mathcal{R}$ such that $\mathcal{N}(\phi)$ has only one link, there always exist $\psi \in \mathcal{R} : \psi \succ \phi$.

Take a set of agreements $\phi_3$ (Fig. 11) and ask the following question: under what condition on $g$, for any triple $(y_1, y_2, y_3) \in (0, 2]^3$ there exist an $i \in \{1, 2, 3\}$ such that agent $i$'s payoff in at least one of the three networks $\psi_1, \psi_2$ or $\psi_3$ is larger than in $\phi_3$? Formally, I need to find a condition on $g$ such that at least one of the three following inequalities holds for any triple $(y_1, y_2, y_3) \in (0, 2]^3$:

$$\begin{aligned}
\alpha_1 y_1 \left(1 + \frac{g}{\sum_i \alpha_i y_i}\right) &< \alpha_2 \left(1 + \frac{g}{2\alpha_2}\right) \\
\alpha_2 y_2 \left(1 + \frac{g}{\sum_i \alpha_i y_i}\right) &< \alpha_2 \left(1 + \frac{g}{2\alpha_2}\right) \\
\alpha_3 y_3 \left(1 + \frac{g}{\sum_i \alpha_i y_i}\right) &< \frac{\alpha_3}{x_2} \left(1 + \frac{g}{2\alpha_2}\right).
\end{aligned}$$

From these three inequalities it follows that the condition on $g$ is

$$g > \max_{(y_1,y_2,y_3)\in(0,2]^3} \left\{ \frac{\min\left\{\frac{\alpha_1}{\alpha_2} y_1, y_2, x_2 y_3\right\} - 1}{\frac{1}{2\alpha_2} - \frac{\min\left\{\frac{\alpha_1}{\alpha_2} y_1, y_2, x_2 y_3\right\}}{\sum_i \alpha_i y_i}} \right\} = \frac{2\alpha_2(2x_2 - 1)(2\alpha_2 x_2 + \alpha_3)}{\alpha_3}.$$

Under the condition (7), in any set of agreements $\phi$ that induce either the network $\mathcal{N}(\phi_3)$ or $\mathcal{N}(\phi_2)$, I can find an agent, that prefers one of the three sets of agreements in $\mathcal{R}$ to $\phi$.

Finally I show that for any set of agreements $\phi \notin \mathcal{R}$ I can find $\psi \in \mathcal{R}$ such, that $\psi \succ \phi$. If $\mathcal{N}(\phi)$ has only one link, I am done. Suppose that $\mathcal{N}(\phi)$ has more than one link. By the argument, under condition (7), I can find an agent $i$ that prefers some element of $\mathcal{R}$ to $\phi$. Let me look at the link between the other two agents, $j$ and $k$ in $\phi$. For that I delete all links of agent $i$ in $\phi$ obtaining a new set of agreements $\phi_{jk}$. Now there can be two cases: either $\phi_{jk}$ belongs to $\mathcal{R}$ or not. Suppose, $\phi_{jk} \notin \mathcal{R}$. Then either agent $j$ or $k$ prefers one of the elements in $\mathcal{R}$ to $\phi_{jk}$. Without loss of generality suppose it is $j$. Then $j$ can break the agreement with $k$ in $\phi_{jk}$, and after that arrange the agreement $\psi$ with $i$ such that $\psi \in \mathcal{R}$, which means that $\psi \succ \phi$. Now, suppose that $\phi_{jk} \in \mathcal{R}$. Then, either there exist a second agent in $\phi$, say $j$ that prefers an element in $\mathcal{R}$ to $\phi$, or agent $i$ has non-trivial links to both agent $j$ and $k$. In the first scenario, agents $i$ and $j$ simultaneously break all existing agreements and then link with each other in $\psi \in \mathcal{R}$. In the second scenario agent $i$ brakes just one link to agent $j$, switching to network $\phi_{ij}$, which both agents $i$ and $j$ enjoy less than an element in $\mathcal{R}$. After that, agents $i$ and $j$ simultaneously brake links with agent $k$ and then relink with each other in $\psi \in \mathcal{R}$. In both scenarios $\psi \succ \phi$.

If the condition (7) holds, the set $\mathcal{R}$ satisfies both internal and external stability conditions, and hence is von Neumann-Morgenstern stable. There are several new properties of this set, that were absent in the symmetric model. The condition (7) only depends on $\alpha_2$ and $\alpha_3$. The closed form solution for the threshold for $g$ is

$$\frac{g}{\alpha_3} > \frac{12 - \left(1 + 2\frac{\alpha_2}{\alpha_3}\right) + 3\sqrt{\left(1 + 2\frac{\alpha_2}{\alpha_3}\right)^2 + 16}}{4}$$

I can set $\alpha_3 = 1$ as a normalization. Observe, that the threshold is an increasing function of $\alpha_2$. Indeed, when the inequality increases (in this case the measure of inequality is $\alpha_2/\alpha_1$), it becomes more costly in terms of absolute output to sustain equal payoffs across $\psi_1, \psi_2$ and $\psi_3$, hence the collusive outcome can be sustained only if the size of the tournament prize also goes up.

Also, note that the threshold does not depend on $\alpha_1$. The coalition of agents 2 and 3 have enough power to exclude agent 1 from the competition for the prize. Agent 1 has no choice but to keep his own performance down to appear not too strong of a competitor to agents 2 and 3. When linked for example to agent 3, agent 1 mimics the productivity of agent 2. No matter how high his true productivity is, he can compensate it by low intensity of his links, i.e. low $x_{1i}$.

# 6 Related Literature

My paper is closely related to Marinucci and Vergote (2011). They develop a model of competition in R&D. In their model firms also can use research joint ventures to save on costs of R&D. They find, that under certain conditions, asymmetric networks of cooperation is possible in the equilibrium. The main difference between my results and Marinucci and Vergote (2011) is that they can not exclude the complete network from the equilibrium: there is always an equilibrium that admits efficient outcome. Also, their results hold only under very strong condition for the payoff functions, that are due to the way they model the competition in R&D.

Also, my paper contributes to the program announced in Salop and Scheffman (1983). In their paper, Salop and Scheffman (1983) state, that firms can capture the market by increasing the costs of production for their rivals. In the companion paper, Salop and Scheffman (1987) describe various strategies that are available to firms to raise the costs for their competitors. They notice that some of those strategies can be more effective than predatory pricing. The mechanism, that is described in my model, can be used by a coalition of firms to gain control over the market. The coalition of the firms does not need to engage in predatory pricing to implement this mechanism. In fact, prices are likely to rise, since some of the firms will be forced to leave the market, or to include the royalty fees in their prices.

Dutta et al. (2005) study the dynamic network formation process with farsighted agents. They establish the conditions, under which the efficient outcome is stable. They model a network formation process as a dynamic game endowed with Markov Perfect Equilibrium. Agents are farsighted, therefore each agent, when evaluating certain strategy, takes into account the effect of his actions on the future play. In my model I also assume that agents are farsighted, but I allow the network structure to settle before the payoffs are realized.

My stability notion is related to one introduced by Ray and Vohra (1997) and later by Diamantoudi and Xue (2007). Both Ray and Vohra (1997) and Diamantoudi and Xue (2007) study the model of coalition formation with farsighted agents. I modify their blocking relation to make it applicable to my network formation model. Ray and Vohra (1997) and Greenberg (1990) provide a discussion on the beliefs of agents that can support the stable outcome.

Solution concept, that I use is also closely related to one developed by Herings et al. (2009). The difference is twofold. When they define the blocking relation for networks, they allow weak improvement for all but one agent. I exclude this possibility, because it can lead to large number of cycles, in particular between networks that are in my set $\mathcal{G}^*$. As a result the dominance relation becomes to rich, which in its turn can lead to existence problems. Also, instead of using von Neumann-Morgenstern stable sets, they use what they call pairwise farsighted stability. Their notion is defined in such a way, that existence is automatically guaranteed[13]. Their solution in general lacks the internal stability property. The stable set that is defined in my paper is always pairwise farsightedly stable. The reverse is not true.

Chwe (1994) introduces a solution, which he calls largest consistent set, that is related to ours. Any stable set of mine is a subset of Chwe's largest consistent set. In particular, the consistency of stable set (see Proposition 2) follows from this observation.

# 7 Conclusion

This paper studies networks of collaboration between competitors. Prior to competition, agents set up collaboration relationships with each other. Once a network of the collaboration is settled, each agent produces an output. This output depends on the number of agent's partners in the network. The competition is modeled as a tournament, therefore the agents with the highest output win the competition and split the prize between each other. In our model, we assume that the agents value their connections not only because the latter improve their chances of winning the tournament, but also because the agents value their output per se. This implies that the efficient outcome is the complete network. Moreover, each link in the network is bilaterally optimal, i.e. any two unconnected agents find it optimal to connect with each other.

[13] They define two properties that a pairwise farsightedly stable set must satisfy and then say that the solution is the minimal (in the set inclusion sense) set that satisfies these two properties. Also they note that grand set satisfies these two properties, hence the existence follows from the finite number of sets.

I assume that the agents are farsighted and can coordinate their actions with each other. I use von Neumann-Morgenstern stable sets to characterize stable outcomes in my model. My framework works best for small networks with abundance of communication between agents. It can applied to such problems as patent wars between alliances of firms, collaboration networks in promotion tournaments, internal structure of organized crime, etc.

Although my model does not capture all the the details of these applications, it provides the important insights that are common for all of these environments. First, it characterizes the necessary and sufficient conditions for the stability of the efficient outcome. Second, it highlights stable networks that have empirically relevant properties. These networks feature two complete components. Such a structure can be interpreted as a competitions between two groups: a group of "insiders" that dominates the competition and the group of "outsiders" that is forced to lose. These groups are represented by the two components.

This result is particularly interesting, because group competition is not a primitive of the model. Nevertheless, the stable networks that I find are organized in a fashion that resembles a competition on the group level.

I depart from the traditional approach of the network formation literature: I assume that agents are farsighted. This allows me to obtain the sharper predictions about the stable outcomes. In particular, in contrast with existing literature, I find the conditions under which the efficient outcome *can not* be stable.

Finally, I discuss the directions for the future research. First, it would be interesting to investigate if there can be more then two components in a stable network. The stable networks that I find feature two complete components. The smaller component does not receive any of the prize, so its members do not have any incentives to split further into two components. However, if we assume that the agents in the smaller component get a significant share of the prize, the smaller component may split into two.

Second, it would be interesting to study extension of the model that accommodates the agents that are heterogeneous in their effect on others. In the extension that I study the agents differ from each other in their ability to convert the collaboration into the output. Put differently, the agents are either good or bad "performers". A natural question is what if the agents are either good or bad "collaborators".

# Proofs

## Proof of Proposition 1

First, observe that $\psi \cup \{ij\} \rhd \psi$ for all $i \neq j$ and $\psi : \psi(i,j) = 0$. This means that any network, except for $\Lambda$ is blocked. Suppose that $f(N-1) + \frac{g}{N} < f(m-1) + \frac{g}{m}$ for some $m \in \left(\frac{N}{2}, N\right)$. Take a coalition $S = \{1, 2, ..., m\}$, and let us look at a network $\psi$ such that

$$\psi(i,j) = \begin{cases} 0 & \text{, if } |\{i,j\} \cap S| = 1, \\ 1 & \text{, otherwise.} \end{cases}$$

Then, $\psi \rhd \Lambda$, and hence the core of $(2^\Lambda, \rhd)$ is empty.

## Lemma 2

**Lemma 2.** *If $\psi \succ \gamma$, then there exist the sequences $\{(S_i, \gamma_i)\}_{i=1}^K$, $\forall i = 1, ..., K : S_i \subset N$ and $\gamma_i \in 2^\Lambda$ such, that $\gamma = \gamma_1 \xrightarrow{S_1} \gamma_2 \xrightarrow{S_2} \dots \xrightarrow{S_K} \psi$ and $\psi \succ \gamma_k$ for all $k \leq K$*

*Proof.* By definition there exist a b-sequence of networks such, that $U_{S_k}(\psi) \gg U_{S_k}(\gamma_k)$ for all $k \leq K$. If we take the tail of that sequence starting from some $k$ we get a new sequence that is a b-sequence for pair $\psi$ and $\gamma_k$. $\square$

## Lemma 3

**Lemma 3.** *If*

$$\arg\max_{m > \frac{N}{2}} \{f(m-1) + g(m)\} \cap N = \emptyset$$

*complete network can not be stable: $\{\Lambda\}$ is never stable.*

*Proof.* Let $m^* = \arg\max_{m > \frac{N}{2}} \{f(m-1) + g(m)\}$. By contradiction, assume, that $\{\Lambda\}$ is a stable set. By external stability $\Lambda \succ \psi$ for all $\psi \in 2^\Lambda$. Let $\mathcal{G}$ be a set of networks, in which agents $1, ..., m^*$ form a complete component. Take an arbitrary $\psi \in \mathcal{G}$. By Lemma 2, there exists a b-sequence $\psi = \psi_1 \xrightarrow{S_1} \dots \xrightarrow{S_K} \Lambda$. Let $\psi_i$ be the latest element in the sequence that lies in $\mathcal{G}$. Then, $\psi_{i+1} \cap \mathcal{G} = \emptyset$ and hence $S_i \cap \{1, ..., m^*\} \neq \emptyset$. Also, for any $i \in \{1, ..., m^*\}$

$$U_i(\Lambda) = f(N-1) + g(N) < f(m^* - 1) + g(m^*) = U_i(\psi).$$

However, if $(\psi_i, S_i)$ is a part of a b-sequence, that ends in $\Lambda$, $U_{S_i}(\Lambda) \gg U_{S_i}(\psi)$, hence the contradiction. $\square$

## Proof of Proposition 3

Take some network $\psi$. The aggregate utilitarian welfare of this network is

$$W(\psi) = \sum_{i \in N} U_i(\psi) = g + \sum_{i \in N} f(x_i)$$

Since $f(\cdot)$ is strictly increasing, $W(\Lambda) > W(\psi)$ for any $\psi \neq \Lambda$.

## Proof of Theorem 1

Assume by contradiction that $\Lambda \in \mathcal{R}$, where $\mathcal{R}$ is a stable set. Take $\psi \in \mathcal{G}^*$. I know, that $\Lambda \not\succ \psi$ (see the proof of Lemma 3). Also, it is easy to see, that $\psi \succ \Lambda$. Since $\mathcal{R}$ is stable, there exist $\gamma \in \mathcal{R}$ such that $\gamma \succ \psi$.

I am going to show that $\Lambda \succ \gamma$, which leads to the contradiction, since by assumption both of those networks are in $\mathcal{R}$.

Define the functions $\underline{U}(x) = \min_{i \in N}\{U_i(x)\}$ and $\overline{U}(x) = \max_{i \in N}\{U_i(x)\}$. Note, that $\overline{U}(\Lambda) = \underline{U}(\Lambda)$.

Since I am proving that $\Lambda \succ \gamma$, I need to find a sequence $\gamma \to \gamma_1 \to ... \to \Lambda$ that satisfies the definition of dominance relation $\succ$. I am going to come up with recursive definition of this sequence, i.e. $\gamma_{i+1} = D(\gamma_i)$. Before introducing function $D(\cdot)$, I have to define the domain of it.

By $M(\psi) = \{i \mid i \in \arg\max_{k \in N} |E_k(\psi)|\}$ I denote the set of nodes in $\psi$ that have maximum number of edges. Let

$$\Omega = \{\psi \in 2^\Lambda \mid \psi \neq \psi' \sqcup \psi'', \text{ such that } N(\psi') \subset M(\psi) \text{ and } \psi' \text{ is a component}\}$$

With $\Omega$ in hands, I define $D : \Omega \to 2^\Lambda$. Take $\psi \in \Omega$. $D(\psi)$ is obtained from $\psi$ by performing two operations:

(i) delete all edges in $\psi \backslash E_{M(\psi)}(\psi)$;

(ii) delete one arbitrary edge in $O_{M(\psi)}(\psi)$.

Note that $(\psi \backslash E_{M(\psi)}(\psi)) \cap O_{M(\psi)}(\psi) = \emptyset$, hence operation (ii) is well defined.

The following lemma makes sure that if I start from an element in $\Omega$, the sequence that is defined by recursive rule $\gamma_{i+1} = D(\gamma_i)$ does not go outside of the domain of $D(\cdot)$.

**Lemma 4.** $D(\Omega) \subset \Omega \cup \{\chi \in 2^\Lambda \mid |E(\chi)| = 1\}$

*Proof.* Take $\psi \in \Omega : |E_{M(\psi)}| = 2$. By definition of $\Omega$ I know that $M(\psi)$ must be a singleton in this case. After applying procedure (i) of $D(\cdot)$ I obtain $\hat{\psi}$, that has only two edges: $|E(\hat{\psi})| = 2$, hence $D(\psi) \in \{\chi \in 2^\Lambda \mid |E(\chi)| = 1\}$.

Now, take $\psi \in \Omega : |E_{M(\psi)}| > 2$. After applying procedure (i) of $D(\cdot)$ I obtain $\hat{\psi} \in \Omega$, that satisfies $\hat{\psi} \backslash E_{M(\hat{\psi})}(\hat{\psi}) = \emptyset$.

If $|M(\psi)| = 1$, it is clear that $D(\psi) \in \Omega$. Suppose, that $|M(\psi)| > 1$. By definition of $\Omega$ there exist $i \in M(\psi) : E_i(\psi) \cap O_{M(\psi)}(\psi) \neq \emptyset$. Take $j \in M(\psi) \backslash i$. Again by definition of $\Omega$, either $i$ and $j$ belong to the same component of $\psi$ or there exist $k \in N \backslash M(\psi)$, such that $j$ and $k$ belong to the same component. Procedure (ii) of $D(\cdot)$ deletes edge that belongs to $E_i(\psi) \cap O_{M(\psi)}(\psi)$. Now, the components that were disjoint with $i$ in $\psi$ are untouched by procedure (ii), hence they satisfy the definition of $\Omega$. Also note that $M(D(\psi)) = M(\psi) \backslash i$, and $i$ is still connected to elements in $M(\psi)$ that shared the same component with $i$ in $\psi$. Hence I can not find a component in $D(\psi)$, that consist of only elements of $M(D(\psi))$, i.e. $D(\psi) \in \Omega$. $\square$

Finally, I can define the sequence $\{\gamma_i\}_{i \in K}$, that satisfies the definition of dominance. Start with $\gamma_0 \in \Omega$, and apply the recursive rule $\gamma_{i+1} = D(\gamma_i)$ until $D(\gamma_{K-2}) \in \{\chi \in 2^\Lambda \mid |E(\chi)| = 1\}$. Observe, that $|E(\gamma_{K-2})| = 2$ and $|M(\gamma_{K-2})| = 1$. Pick an arbitrary regular network $\gamma_{K-1} \supset \gamma_{K-2}$ of degree 2[14]. The last element of the sequence is $\gamma_K = \Lambda$.

If $\gamma \in \Omega$, then I set $\gamma_0 = \gamma$. Suppose $\gamma \notin \Omega$. Then, there exist $\hat{M} \subset M(\gamma)$ that forms a component. I observe that $|M(\gamma)| \leq \frac{N}{2}$, since by original assumption $\gamma \succ \psi$. Take an arbitrary agent $i \in N \backslash M(\gamma)$ and obtain network $\gamma_0$ by connecting this agent to all nodes in $N \backslash M(\gamma)$. Obviously, $M(\gamma_0) = i$ and $\gamma_0 \in \Omega$.

I observe, that under this algorithm, when switching from $\gamma_i$ to $\gamma_{i+1}$ a link is deleted only if one of the nodes is in $N \backslash M(\gamma_i)$, and link is created if both of the nodes are in $N \backslash M(\gamma_i)$. Also, for any network $\psi \neq \Lambda$, $U_{N \backslash M(\psi)}(\psi) < \overline{U}(\Lambda)$.

I have shown that $\Lambda \succ \gamma$ by constructing the sequence $\gamma \rightarrow \gamma_0 \rightarrow \ldots \rightarrow \gamma_K = \Lambda$, that satisfies definition of setwise dominance, which contradicts my assumption that $\gamma \in \mathcal{R}$.

Finally, as I just showed, $\Lambda \notin \mathcal{R}$, hence $\exists \psi \in \mathcal{R} : \psi \succ \Lambda$. Take a network $\lambda : M(\lambda) = N$. Then for any $\psi \succ \Lambda$, I have $\psi \succ \lambda$, hence $\lambda \notin \mathcal{R}$.

## Proof of Theorem 3

I have to prove that $\mathcal{G}^*$ is both internally and externally stable.

I start with internal stability. Take two networks $\psi, \gamma \in \mathcal{G}^*$. Internal stability requires that $\psi \not\succ \gamma$. Suppose, by contradiction that $\psi \succ \gamma$. Then, there exist a sequence $\gamma = \gamma_1 \rightarrow \ldots \rightarrow \gamma_K = \psi$, that satisfies the definition of dominance. Suppose, that all networks $\gamma_i$ with $i \leq k$ are such, that $E_{M(\gamma)}(\gamma_i) = E_{M(\gamma)}(\gamma)$, and $\gamma_{k+1}$ is the first network in the sequence that violates this property. Then, $\gamma_k \rightarrow \gamma_{k+1}$

[14] It is easy to see that it always exist. For example I can use the following procedure: two remaining edges in $\gamma_{K-2}$ are between agent $i$ and say $l$ and $m$. Starting with agent $l$ connect all agents but agent $i$ in one line that ends with agent $m$. The resulting network $\gamma_{K-1}$ is a cycle.

violates the definition of dominance since $U_{M(\gamma)}(\gamma_k) = U_{M(\gamma)}(\gamma) \leq U_{M(\gamma)}(\psi)$, hence the contradiction.

For external stability, I need to show, that for any $\psi \notin \mathcal{G}^*$ there exist $\gamma \in \mathcal{G}^*$, such that $\gamma \succ \psi$. Since the set $\mathcal{G}^*$ consist of networks that are equivalent up to permutation of agents, for any $\gamma, \gamma' \in \mathcal{G}^*$, I have $\overline{U}(\gamma) = \overline{U}(\gamma')$. With some abuse of notation I will say that $\overline{U}(\mathcal{G}^*) = \overline{U}(\gamma)$ for some $\gamma \in \mathcal{G}^*$.

Take an arbitrary $\psi \notin \mathcal{G}^*$. Suppose that $U_{M(\psi)}(\psi) \leq \overline{U}(\mathcal{G}^*)$. Pick an arbitrary set $M$ of agents such that $|M| = m^*$ and modify $\psi$ by dropping all edges in $E_M(\psi)$. Let us call the resulting network $\psi_2$. Network $\psi_3$ is obtained from network $\psi_2$ by connecting agents in the set $M$ into a complete component. Finally network $\psi_4 \in \mathcal{G}^*$ is obtained from $\psi_3$ by connecting agents in the set $N\backslash M$ into a complete component. It is easy to see that in transitions $\psi_1 \to \psi_2 \to \psi_3$ only agents from the set $M$ are involved and their payoff in the final network is larger than along the way: $U_M(\psi_4) = \overline{U}(\mathcal{G}^*)$. The rest of the agents are only involved in the last step of transition $\psi_3 \to \psi_4$ and since $N\backslash M \cap M(\psi_3) = \emptyset$ and each agent in $N\backslash M$ gets weakly more links in $\psi_4$ than in $\psi_3$, the definition of dominance is met by the sequence $\psi = \psi_1 \to ... \to \psi_4$.

Now suppose, that $U_{M(\psi)}(\psi) > \overline{U}(\mathcal{G}^*)$. I have to go through three cases: (i) $|M(\psi)| < \frac{N}{2}$, (ii) $|M(\psi)| = \frac{N}{2}$ and (iii) $|M(\psi)| > \frac{N}{2}$.

In case (i) when $|M(\psi)| < \frac{N}{2}$, pick an arbitrary set $M \subset (N\backslash M(\psi)) : |M(\psi)| < |M| \leq m^*$. Obtain network $\psi_2$ by simultaneously dropping all edges in $O_M(\psi)$ and adding all missing edges between elements in $M$. In the resulting network $M(\psi_2) = M$. Also, $U_M(\psi_2) \leq \overline{U}(\mathcal{G}^*)$ with equality only in case of $|M| = m^*$. If $|M| < m^*$, pick an arbitrary set $\hat{M} \supset M : |\hat{M}| = m^*$. Obtain network $\psi_3$ by creating a complete component that consists of $\hat{M}$ and finally obtain $\psi_4$ by adding all missing links between agents in $N\backslash\hat{M}$. As before, I can see, that $\mathcal{G}^* \ni \psi_4 \succ \psi$, since we've constructed proper sequence $\psi = \psi_1 \to ... \to \psi_4$.

In case of $|M(\psi)| = \frac{N}{2}$, if $M(\psi)$ forms a union of components, take agents $N\backslash M(\psi)$ and obtain network $\psi_2$ by duplicating components formed by $M(\psi)$, using agents $N\backslash M(\psi)$. The resulting network has a property that $M(\psi_2) = N$ and that $\overline{U}(\psi_2) = \underline{U}(\psi_2) < \overline{U}(\mathcal{G}^*)$. In the next step, obtain network $\psi_3$ from network $\psi_2$ by selecting the set $M \supset (N\backslash M(\psi)) : |M| = m^*$, simultaneously dropping all edges in $O_M(\psi_2)$ and adding all missing edges between elements in $M$. The resulting network $\psi_3$ consists of complete component of size $m^*$ and some other components. Finally obtain network $\psi_4 \in \mathcal{G}^*$ by creating all missing links between agents in $N\backslash M$. Obtained sequence $\psi = \psi_1 \to ... \to \psi_4$ satisfies the definition of dominance, hence $\mathcal{G}^* \ni \psi_4 \succ \psi$.

If $|M(\psi)| = \frac{N}{2}$ and there exist an element in $M(\psi)$ that is connected to $N\backslash M(\psi)$. Drop one link between $M(\psi)$ and some $i \in N\backslash M(\psi)$ and obtain network $\psi'$. Observe that $|M(\psi')| = |M(\psi)| - 1 < \frac{N}{2}$, hence I can use the argument above (see case (i)) to construct the sequence $\psi' = \psi_1 \to ... \to \psi_4$ with one modification, that when set $M$ is picked it must be that $i \in M$. This will guarantee that I can add $\psi$ to the beginning of this sequence and still satisfy the definition of dominance.

Finally, if $|M(\psi)| > \frac{N}{2}$ and $U_{M(\psi)}(\psi) > \overline{U}(\mathcal{G}^*)$, it must be the case that each element in $M(\psi)$ has weakly more links then $|M(\psi)|$, i.e. each agent in $M(\psi)$ is connected to some agent in $N\backslash M(\psi)$. I already discussed the case when $|M(\psi)| = N$, so I only need to go through the case when $|M(\psi)| < N$. Pick $\hat{M} \subset M(\psi) : |\hat{M}| = \frac{|M(\psi)|}{2}$ if $|M(\psi)|$ is even and $|\hat{M}| = \frac{|M(\psi)|+1}{2}$ if $|M(\psi)|$ is odd. Break the links that connect $\hat{M}$ to $N\backslash M(\psi)$ and obtain network $\psi'$. Clearly $M(\psi') = M(\psi)\backslash\hat{M}$ and $|M(\psi')| < \frac{N}{2}$. Now I can use the argument for case (i) with the modification that the set $M$ must be such that $N\backslash M(\psi) \subset M$. This condition will guarantee that I can add network $\psi$ to the beginning of the sequence $\psi' \to ... \to \psi_4 \in \mathcal{G}^*$ in a way that goes along with the definition of dominance.

## Proof of Proposition 2

The external stability of $\mathcal{R}$ guarantees the existence of $\widehat{\rho} \in \mathcal{R}$. Fix a b-sequence $\{(S_k, \gamma_k)\}_{k=1}^{K}$, that supports $\psi \succ \rho$. By contradiction assume, that for any agent $i \in \bigcup\limits_{k=1}^{K} S_k$, I have $U_i(\rho) < U_i(\widehat{\rho})$. Then, $\widehat{\rho} \succ \rho$, which contradicts an assumption that $\mathcal{R}$ is stable.

## Proof of Lemma 1

Let us look at the set of networks $\mathcal{R}_m$ each of which consists of two groups, that are not connected between each other. The sizes of the groups are $m$ and $N - m$, where $m > \frac{N}{2}$. For a network $\psi \in \mathcal{R}_m$, I am going to denote a larger group by $M(\psi)$.

Suppose I have $N$ agents ordered by their productivity: $\alpha_1 > \alpha_2 > ... > \alpha_N$.

For the internal stability of $\mathcal{R}_m$, I require that for any two networks $\psi, \phi \in \mathcal{R}_m$, and for any agent $i \in M(\psi) \cap M(\phi)$ I have, that $U_i(\psi) = U_i(\phi)$. Also, I require, that in any network $\psi \in \mathcal{R}_m$, the agent $i_{(1)}(M(\psi)) = \arg \min\limits_{i \in M(\psi)} \{\alpha_i\}$, should have the highest possible intensity of connections, i.e. $M - 1$.

Given these two properties, I conclude that for any two networks $\psi, \phi \in \mathcal{R}_M$ such that $i_{(1)}(M(\psi)) = i_{(1)}(M(\phi))$, the aggregate output in $\psi$ is the same as in $\phi$. The set

$$\Psi_k = \{\psi \in \mathcal{R}_m \mid i_{(1)}(M(\psi)) = k\}$$

is the set of networks in which the agent $k$ is the one with the lowest productivity within the majority group. Then, I am going to denote the aggregate output in those networks by $Y_k$. It is easy to see, that any time agent $i$ participates in the majority in some networks in $\Psi_k$, the intensity of his links is always the same. I am going to denote it by $x_i^k$.

Take two networks $\psi, \phi \in \Psi_k$, such that $k \notin M(\psi)\Delta M(\phi) = \{i, j\}$. Since the aggregate output in $\psi$ is the same as in $\phi$, I have that

$$\alpha_i \left(x_i^k - (N - M - 1)\right) = \alpha_j \left(x_j^k - (N - M - 1)\right) \tag{8}$$

Take a network $\psi \in \Psi_k$, such that agent 1 belongs to the majority in $\psi : 1 \in M(\psi)$. The aggregate output in $\psi$ is

$$Y_k = \sum_{i \in M(\psi) \setminus k} \alpha_i \left( x_i^k - (N - M - 1) \right) + \alpha_k (M - 1) + \sum_{i \in N \setminus k} \alpha_i (N - M - 1),$$

but as I noted, $\alpha_i \left( x_i^k - (N - M - 1) \right)$ is constant across $i$, so

$$Y_k = (M - 1) \left( \alpha_1 \left( x_1^k - (N - M - 1) \right) + \alpha_k \right) + \sum_{i \in N \setminus k} \alpha_i (N - M - 1), \tag{9}$$

If I know the aggregate outputs for all networks, i.e. $Y_k$, for $k = m, ..., N$, I can solve for $x_1^k$ using the equations (9), and after that I can solve for the rest of $x_i^k$ using equations (8).

All I need to find is $Y_k$ for $k = m, ..., N$. First I reduce this problem to the problem of two unknowns. Take a network $\psi \in \Psi_k$ such, that agent $m \in M(\psi)$. Then it must be that

$$\alpha_m x_m^k \left( 1 + \frac{g}{Y_k} \right) = \alpha_m (m - 1) \left( 1 + \frac{g}{Y_m} \right), \tag{10}$$

so once I know $Y_m$ I can solve for all $Y_k, k > M$.

To come up with the equation for $Y_m$, observe that

$$\alpha_i x_i^N \left( 1 + \frac{g}{Y_N} \right) = \alpha_i x_i^m \left( 1 + \frac{g}{Y_m} \right).$$

The aggregate output for networks in $\Psi_m$ is

$$Y_m = \sum_{1 \leq i \leq m-1} \alpha_i x_i^m + \alpha_m (m - 1) + \sum_{m+1 \leq i \leq N} \alpha_i (N - m - 1).$$

By substituting $x_i^m$ with $x_i^N$ from the previous equation I obtain

$$\frac{\left( Y_m - \alpha_m (2m - N) - \sum_{i \in N} \alpha_i (N - m - 1) \right)}{\left( Y_N - \alpha_N (2m - N) - \sum_{i \in N} \alpha_i (N - m - 1) \right)} = \frac{\left( 1 + \frac{g}{Y_N} \right)}{\left( 1 + \frac{g}{Y_m} \right)}$$

This is an extra equation, that solves for $Y_m$.

## Proof of Proposition 5

First, note that

$$\frac{U_i(\Lambda)}{U_j(\Lambda)} = \frac{\alpha_i}{\alpha_j},$$

and

$$\frac{U_i(\psi)}{U_j(\psi)} = \frac{\alpha_i x_i^N}{\alpha_j x_j^N}.$$

By equation (8) I obtain that

$$\alpha_i x_i^N = \alpha_j x_j^N - (\alpha_j - \alpha_i)(N - M - 1),$$

hence,

$$\frac{U_i(\psi)}{U_j(\psi)} - \frac{U_i(\Lambda)}{U_j(\Lambda)} = \left(\frac{x_j^N - (N - M - 1)}{x_j^N}\right)\left(1 - \frac{\alpha_i}{\alpha_j}\right) < 0.$$

## Proof of Proposition 4

First observe that $\mathcal{R}_k$ is internally stable by construction. I only need to prove that $\mathcal{R}_k$ satisfies external stability. With some abuse of notation, by $\overline{U}_i$ I denote a maximum payoff of agent $i$ in the set $\mathcal{R}_k$:

$$\overline{U}_i = \max_{\psi \in \mathcal{R}_k} \{U_i(\psi)\}.$$

Take an arbitrary $\psi \notin \mathcal{R}_k$ and look at the set of agents that would prefer to switch to one of the networks in $\mathcal{R}_k$ rather than stay in $\psi$:

$$A(\psi) = \{i \in N \mid U_i(\psi) < \overline{U}_i\}$$

If $|A(\psi)| \geq k$, then clearly I can find $\rho \in \mathcal{R}_k$, such that $\rho \succ \psi$. Suppose, $|A(\psi)| < k$.

I am going to set up an induction argument at this point. Assume, that there is a network $\psi_l$ such, that $|A(\psi_l)| < k$. I will show, that there is a network $\psi_{l+1}$, such that:

(i) $A(\psi_l) \cup j \subset A(\psi_{l+1})$, for some $j \in N$,

(ii) $\psi_l = \psi^0 \overset{S_0}{\to} \psi^1 \overset{S_1}{\to} ... \overset{S_n}{\to} \psi^{n+1} = \psi_{l+1}$ such, that

  (a) $\forall i = 1, ..., n : \overline{U}_{S_i} > U_{S_i}(\psi^i)$, and

  (b) $\bigcup_{i=1}^{n} S_i \subset A(\psi_{l+1})$.

If this statement holds, then I can construct a strictly increasing sequence $A(\psi) \subset A(\psi_1) \subset ... \subset A(\psi_L)$, such that $|A(\psi_L)| = k$. Note that the conditions that I used guarantee, that the sequence of corresponding networks is a b-sequence for a $\rho \succ \psi$, where $\rho \in \mathcal{R}_k$, and $U_{A(\psi_L)}(\rho) = \overline{U}_{A(\psi_L)}$.

Let us get back to proving the induction. I claim that I can find $\psi_{l+1}$ that satisfies the conditions above. Let us look at $\widehat{\psi}_l$ such, that

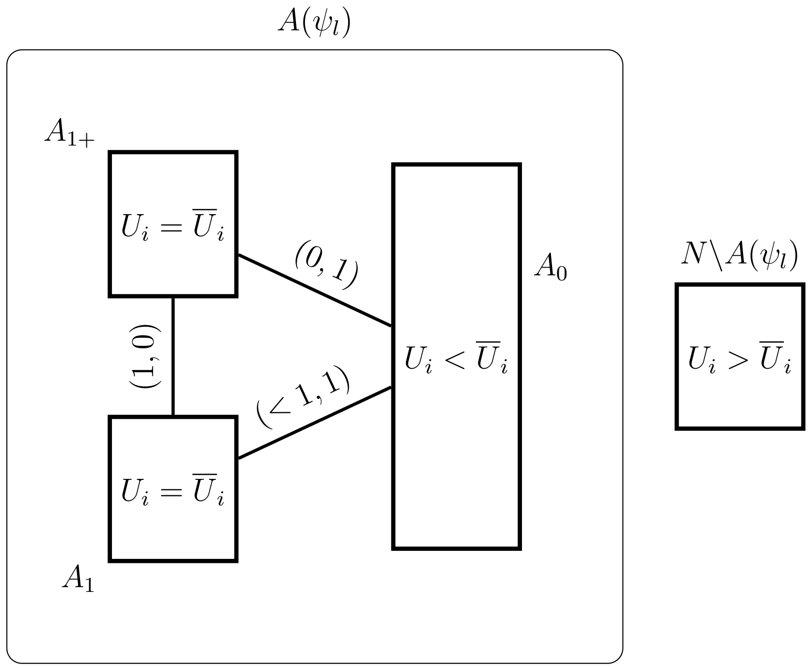



Figure 12: The structure of network $\phi_l^0$.

(a) $\psi_l \overset{A(\psi)}{\to} \widehat{\psi}_l$,

(b) $A(\psi_l)$ form a component in $\widehat{\psi}_l$,

(c) $x_{A(\psi_l)}(\widehat{\psi}_l) = \alpha x_{A(\psi_l)}(\rho)$, where $\rho \in \mathcal{R}_k : U_{A(\psi)}(\rho) = \overline{U}_{A(\psi)}$.

Observe, that the condition (c) guarantees, that for any two agents $i, j \in A(\psi)$

$$\frac{U_i(\widehat{\psi}_l)}{U_j(\widehat{\psi}_l)} = \frac{\overline{U}_i}{\overline{U}_j}$$

and hence there exists $\alpha_0 \in \left[0, \frac{|A(\psi_l)|-1}{k-1}\right]$ such, that $A(\psi_l) = A(\widehat{\psi}_l(\alpha_0))$[15]. Starting from $\alpha_0$, increase $\alpha$ gradually until $A(\widehat{\psi}_l(\alpha)) \cap (N \backslash A(\psi_l)) \neq \emptyset$ or one of the agents in $A(\psi_l)$ reaches his capacity in links. In principle, it could also be the case, that $U_{A(\psi_l)}(\widehat{\psi}_l(\alpha)) = \overline{U}_{A(\psi_l)}$, but that would be a violation of conditions of the proposition.

If $A(\widehat{\psi}_l(\alpha)) \cap (N \backslash A(\psi_l)) \neq \emptyset$, I am done. If not, then I keep increasing $\alpha$ for all agents that did not reach the capacity constraint yet. Again I stop either when $A(\widehat{\psi}_l(\alpha)) \cap (N \backslash A(\psi_l)) \neq \emptyset$, in which case I am done, or when the agents that did not

[15] I abuse notatations slightly. The function $\widehat{\psi}_l(\alpha)$ maps $\alpha \in \left[0, \frac{|A(\psi_l)|-1}{k-1}\right]$ into network described by conditions above

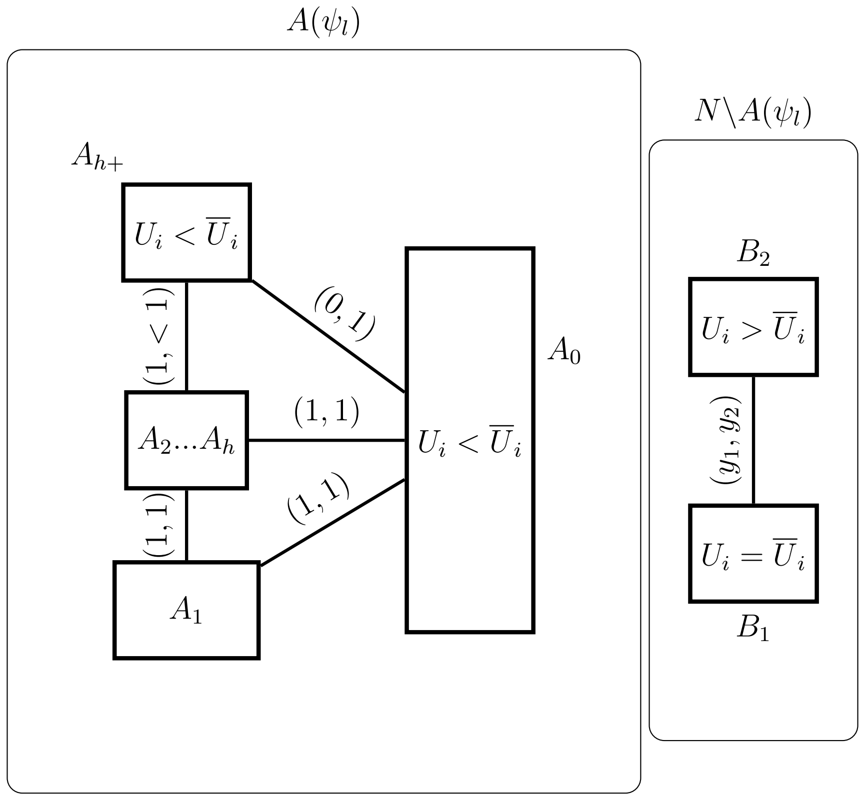



Figure 13: The structure of network $\psi_l^h$.

yet reach the capacity constraint have a payoff $\overline{U}_i$. Suppose the latter happens at network $\phi_l^o$.

I am going to show that it is possible to raise the aggregate output of the set of agents $A(\psi_l)$ up to a point, when some agent $i \in N\backslash A(\psi_l)$ have a payoff equal to $\overline{U}_i$. Moreover, it is possible to raise it in a reversible way, i.e. such that the coalition $A(\psi_l)$ can go back to a network $\phi_l^0$ only utilizing agents whose current payoff is lower than $\overline{U}$.

First, let us look at the properties of $\phi_l^0$ (see Fig. 12). Notice, that agents in $A_0$ have $|A(\psi_l)|$ incoming connections each. Also, Agents in $A_1$ have strictly more than $|A_0|$ connections.

If I set $x_i = |A(\psi_l)|$ for all $i \in A_1$. In this case, the payoffs of all agents in $A_{2+}$ become strictly less than $\overline{U}$. I can proportionally increase the intensity for agents in $A_{1+}$ until either their payoffs become equal to $\overline{U}$ again, or one of the agents in $N\backslash A(\psi_l)$ gets payoff $\underline{U}$. Suppose the former happens. If the former happens I repeat the same procedure again, i.e. set $x_i = |A(\psi_l)|$ for all $i \in A_1$.

I continue until I hit the network in which there exists $i \in N\backslash A(\psi_l)$ such that $U_i = \overline{U}_i$. Suppose it happens at step $h$, i.e. the network $\phi_l^h$.

Since $|A_0| > |N \backslash A(\psi_l)|$, there exists network $\widehat{\phi}_l^h$ such, that agents in $B_1$ have the same intensity of incoming links, but all of them come from $A_0$, and the rest of the links are exactly the same as in $\phi_l^h$.

There exists a sequence of networks, that starts in $\phi_l^h$ and leads back to $\phi_l^0$, and only agents whose payoffs are lower than $\overline{U}$ are active along this sequence. Indeed, agents in $A_0$ can break link to agents in $A_0$, which guarantees, that latter have payoffs below $\overline{U}$. After that, agents in $A_1 \cup A_0$ break their links with agents in $A_2$, and simultaneously link with each other. After this step, agents in $A_2$ are the ones with the low payoff. Agents proceed in the same fashion, working their way up along the sequence of $A_i$ until they reach the top. This algorithm applies for any transition from $\phi_l^j$ to $\phi_l^{j-1}$.

Notice now, that for any network $\psi_l^j$ I can find a corresponding network $\widehat{\phi}_l^j$, in which agents in $B_1$ have their links coming from $A_0$ instead of $B_2$.

Once I reach $\widehat{\phi}_l^0$, all agents in $A(\psi_l)$ delete all their link. As a result all agents in $A(\psi_l) \cup B_1$ have no links.

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