Illustrating the Quantity Theory of Money in the United States and in Three Model Economies

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Abstract

We show that, between 1960 and 2009, the Quantity Theory of Money held in the United States in the long run and that it failed to hold in the short run. We ask whether standard monetary model economies from the Cash-in-Advance, the New-Keynesian, and the Search-Money frameworks can replicate these results, and we find that they do so in the long run, but that they fail in the short run because prices respond too quickly to changes in the growth rates of money.

The Quantity Theory of Money is one of the best known theories of monetary economics. It says that the rate of inflation is approximately equal to the rate of growth of money in excess of the growth rate of real output. In this article we explore whether the Quantity Theory of Money holds in the United States economy and in three standard monetary model economies, both in the short run and in the long run.

The Quantity Theory of Money was first formulated by David Hume using a thought experiment which he summarized in the following quote, "Were all the gold in England annihilated at once, and one and twenty shillings substituted in the place of every guinea, would money be more plentiful or interest lower? No surely: We should only use silver instead of gold." (Hume, 1742, *Of Interest*).

A modern version of Hume's experiment, and one which requires no imagination, occurred in the eleven European countries who decided to "annihilate" their local currencies on January 1st, 2002 and "substitute" them for the Euro. Paraphrasing Hume, was money less plentiful or interest higher in those countries after they changed their currencies? No surely: Their residents only used Euros instead of their former currencies, and went about their business as if nothing much had happened. In this and in other similar real world experiments, when currencies are redenominated

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by knocking off one or two zeros, for example, the Quantity Theory of Money relationship holds both in the short and in the long run, almost exactly.

But the words "at once" are key in Hume's quotation. When central banks conduct monetary policy, changes in the quantity of money are not introduced evenly and "at once". Instead, money is injected into one part of the economy, typically the banking system, and it spreads out gradually from there. A consequence of this relatively slow spreading out of money is that the Quantity Theory of Money fails to hold in the short-run, while the spreading out is taking place.

This short-run failure of the Quantity Theory of Money to hold, in the United States and elsewhere, can be illustrated by simply plotting the rate of inflation—plus the rate of growth of output—against the rate of growth of money. In this article we start by plotting those two series for the United States for the 1960–2009 period using quarterly data. If the Quantity Theory of Money relationship held in the short run, the data points would trace a 45 degree line. Instead, we show that their scatter plot forms a vague cloud, out of which no clear pattern emerges (see Figure 1).

Whether the Quantity Theory of Money holds in the long-run is harder to establish, because we must make operative the meaning of "long run". To do this, we replicate the method used in Lucas (1980). In that article, Lucas identifies the long-run with the low frequency fluctuations of the rate of inflation and rate of growth of money. He then uses a two-sided moving average filter to extract the low frequency movements from those series, and he shows that the plots of the filtered series move closer to the 45 degree line as he filters out the higher frequencies. Lucas considers the period between 1955 and 1975 and he does not include the rate of growth of output in his calculations. He concludes that the Quantity Theory of Money held in the long-run in the United States during that period.

Lucas' results are telling, but they are qualitative. Whiteman (1984) shows that one way to quantify Lucas' findings is to estimate the slope of the ordinary least squares linear regression of the rate of growth of prices plus the rate of growth of output on the rate of growth of money. Whiteman argues that, when the Quantity Theory of Money relationship holds, the slope of this regression will be close to one.

We replicate Whiteman's calculations, and we also compute an additional measure of closeness to the 45 degree line: the average Cartesian distance of the data points from the 45 degree line that goes through the grand mean of the sample. When the Quantity Theory of Money relationship holds, the data points will be close to the 45 degree line and this average distance will be close to zero. We compute these two measures for the United Sates in the 1960–2009 period and we conclude that the Quantity Theory of Money relationship did not hold in the short-run in the United States, but that it held in the long-run during that period. Our results confirm quantitatively both Lucas (1980)'s findings and Hume (1742a)'s thought experiment.

Next we ask whether the Quantity Theory of Money holds in three monetary model economies, which we consider to represent the standard frameworks that economists currently use to evaluate the implications of monetary policy—the Cash-in-Advance framework, the New-Keynesian framework, and the Search-Money framework. Our research should be understood as part of the search for a monetary model economy whose predictions we can trust. The methodological idea is that the more dimensions along which model economies replicate the known behavior of the monetary time series of real economies, the more we trust their predictions along other, harder to test, dimensions.

Monetary economists often test their model economies along these lines using impulse response functions, for example. But these methods do not allow them to compare model economies from different frameworks because impulse response functions depend on structural and modelling assumptions that usually differ across frameworks. Meaningful comparisons of alternative ways to model money force researchers to use evaluation methods that do not depend on the modelling assumptions of each framework, and are harder to perform. The Quantity Theory of Money, evaluated using Lucas' illustrations and our measures to quantify them, is a meaningful way to evaluate and compare the three leading monetary frameworks.

That the Quantity Theory of Money should hold in monetary model economies is both compelling and quite easily achieved formally. In the standard neoclassical models, the Quantity Theory of Money holds every period. This result is mathematically known as the "zero-degree homogeneity of real decisions in the price level". Instead, the challenge for monetary model economies is to break away from the Quantity Theory of Money relationship *in the short-run.* Or, in other words, to find a way of modelling money that makes prices respond *sluggishly* to changes in the money supply.

The three monetary frameworks that we study here represent different ideas about how to model money: the Cash-in-Advance framework focuses on the transaction role of money, the New-Keynesian framework on the role of money as a liquid asset, and the Search-Money framework on the role of money as a way to solve problems of single-coincidence of wants. To find out whether any of these three ways of modelling money succeed in making prices react sluggishly to changes in the money supply, we ask whether the Quantity Theory of Money holds in these three frameworks both in the short-run and in the long-run.

The Cash-in-Advance framework makes the use of money in exchange compulsory forcing households to buy consumption goods with money carried over from the previous period. In general, this cash-in-advance constraint is inefficient, but it solves the informational problem that would arise when trying to coordinate all the simultaneous trades; a problem that non-monetary economies ignore. In the words of Lucas (1980) the cash-in-advance framework "is an attempt to study the transaction demand for money in as simple as possible a general equilibrium setting". In this article, we use Cooley and Hansen (1989)'s cash-in-advance business cycle model as our canonical example of the cash-in-advance framework.

The New-Keynesian framework models the relationship between money and interest rates using a money demand equation. To break away from the short-run neutrality of money, New-Keynesian models assume that prices are sticky. Either they cannot be changed every period by assumption, or doing so is costly, also by assumption. The price stickiness assumption is based on empirical evidence that prices do not change often. In this article, we use the New-Keynesian monetary model economy described in Chapter 3 of Galí (2008) as our canonical example of the New-Keynesian framework.

The Search-Money framework is a successful attempt to satisfy Wallace (1998)'s dictum that "money should not be a primitive in monetary theory"; that is, that there must be an endogenous reason that justifies the existence of money. In the Search-Money framework this reason is to enable trade. Search-Money models assume that people meet in pairs and exchange goods using barter. But this means that trade only occurs when both trading partners have a good that the other one wants. This is the well-known problem of barter: trade is often limited by the absence of a double-coincidence of wants. Money solves this problem because everyone always wants money, at least as an enabler of trade. In this article, we use a stochastic extension of the Search-Money model described in Aruoba, Waller, and Wright (2011) as our canonical example of the Search-Money framework.[1]

To ensure that the comparison of the three monetary frameworks is meaningful, we choose their functional forms and parameters so that they are as similar as possible to each other. Moreover, we make the stochastic processes on the monetary shocks identical in the three model economies, and we simulate them using exactly the the same sequences of realizations of the shocks.

First, we plot Lucas' illustrations and we find that the Cash-in-Advance, the New-Keynesian, and the Search-Money frameworks all display the Quantity Theory of Money relationship in the long-run and, therefore, that they replicate the long-run behavior of the United States. In all three model economies the filtered points lie along the 45 degree line that goes through the grand mean of the sample, exactly as the Quantity Theory of Money predicts (see Figure 3.3).

Next, we simulate 100 stochastic realizations of the equilibrium processes of the three model economies and we find that Whiteman's regression coefficient is close to one, and that the Cartesian distances of the filtered points from the 45 degree line are close to zero. This confirms our qualitative results. We conclude that the differences between the three frameworks are tiny, if any, and that all three of them pass the long-run Quantity Theory of Money test with flying colors.[2]

In sharp contrast, the three monetary frameworks fail to replicate our finding that the Quantity Theory of Money relationship does not hold in the United States in the short run. While in the United States the graphs of the unfiltered data contain no suggestion of the Quantity Theory of Money, our simulations of the three model economies produce data points that lie very close to the 45 degree line. This suggests that the three model economies display a short-run Quantity Theory of Money relationship that is too tight, even though it is not exact.

Whiteman's regression coefficients and the Cartesian distances of the data points from the 45 degree lines confirm our qualitative results. We conclude that prices respond too quickly to changes in the rate of growth of money in the three frameworks that we study, and that the search for a model economy in which the response of prices to monetary innovations replicates the sluggishness found in the data still remains an important challenge for monetary economics.

1 The Quantity Theory of Money

Multiply the supply of money by m and prices will become m times larger —this is a rough but useful characterization of the Quantity Theory of Money. More precisely, the Quantity Theory of Money claims that the rate of growth of nominal prices plus the rate of growth of output is equal to the rate of growth of the money supply.

[1] Berentson, Menzio, and Wright (2011) also solve a stochastic extension of the original Search-Money model described in Lagos and Wright (2005). Their extension differs slightly from ours in the timing of the money shock. It also differs because they impose an AR(1) process on interest rates, which implies a Markov process on money, while we impose an AR(1) process on (log) money, which implies a Markov process on interest rates.

[2] Sargent and Surico (2011) study the Quantity Theory of Money in the United States between 1900 and 2005 and they argue that it is not a stable relationship. They use the Lucas Illustrations approach, they find that the slope of the Quantity Theory of Money relationship changes over time, and they attribute these changes to changes in the monetary policy regime followed by the Federal Reserve —for example, around 1980 the Fed changed from targeting monetary aggregates to targeting inflation rates. We discuss Sargent and Surico's findings in Section 2 and in Appendix B.

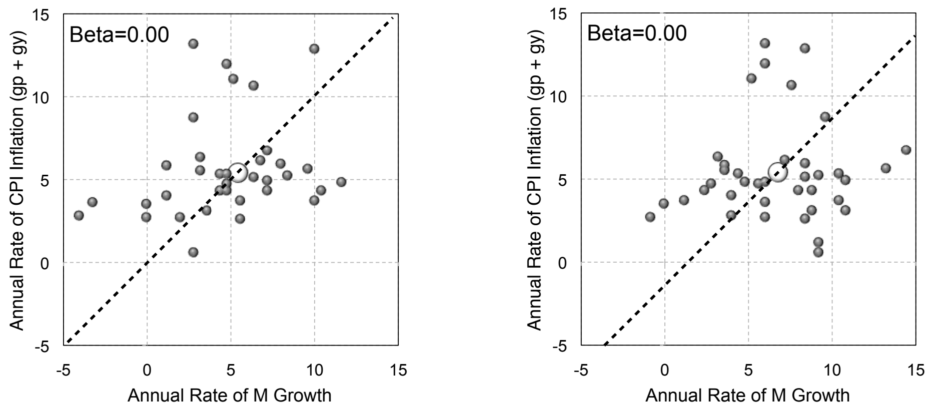
The formal expression of the Quantity Theory of Money is the following

$$MV = PY \tag{1}$$

where M is the nominal money supply, V is the velocity of circulation of money, P is the price level, and Y is real output. Let g_x be the growth rate of variable x. Then, if we assume that V is relatively constant, it follows that

$$g_M \simeq g_P + g_Y \tag{2}$$

Therefore, when the Quantity Theory of Money holds, if we graph $g_P + g_Y$ against g_M, we will get a 45 degree line. And, when the Quantity Theory of Money does not hold, we will get a meaningless bird-shot scatter plot. This is the central idea in Lucas (1980).



A: M1 in the United States $\beta = 0.0$ B: M2 in the United States $\beta = 0.0$

*The coordinates of the center of the white circle in each panel are the grand mean of the unfiltered sample.

Figure 1: The Quantity Theory in United States in the Short Run (1960:Q1–2009:Q4)

2 The Quantity Theory of Money in the United States

In this section we discuss whether the Quantity Theory of Money relationship holds in the United States both in the short run and in the long run.

2.1 The Quantity Theory of Money in the United States in the Short Run

In his 1980 article, Lucas plots the quarterly growth rate of money against the quarterly growth rate of prices for the 1955–1975 period using M1 as the monetary aggregate and he obtains a bird-shot scatter plot that shows that the Quantity Theory of Money does not hold in the short run in the United States during that period.

Here we use Lucas' idea but we make three changes: we start our sample period in 1960 and we extend it to 2009, we use both M1 and M2 as our monetary aggregates, and, while Lucas plots the rate of growth of prices against the rate of growth of money, we follow the text-book description of the Quantity Theory of Money *exactly* and plot the rate of growth of prices *plus the rate of growth* *of output* against the rate of growth of money. We implement these three changes, we plot the resulting time series, and we obtain the bird-shot scatter plots that we represent in Figure 1. Our scatter plots illustrate that the Quantity Theory of Money relationship did not hold in the United States in the short run between 1960 and 2009 either for M1 or for M2, and they confirm Lucas (1980)'s findings.[3]

2.2 The Quantity Theory of Money in the United States in the Long Run

To illustrate whether the Quantity Theory of Money holds in the long run, Lucas (1980) associates the short-run with the high-frequency fluctuations of the quantity theory time series expressed in growth rates, and the long-run with the low-frequency fluctuations of those series. To remove the high-frequency fluctuations and to obtain the low-frequency signal, Lucas transforms the original series using the following two-sided, exponentially-weighted, moving-average filter

$$x_t(\beta) = \alpha \sum_{k=1}^{T} \beta^{|t-k|} x_k \tag{3}$$

where

$$\alpha = \frac{(1-\beta)^2}{1-\beta^2 - 2\beta^{(T+1)/2}(1-\beta)} \quad 0 \leq \beta < 1 \tag{4}$$

and where T is the number of observations in the time series.[4]

A value of $\beta = 0.0$ returns the original time series, and increasingly higher values of β filter out the higher frequency fluctuations from the original time series and leave only the increasingly lower frequency fluctuations in the transformed series. Figure 4 illustrates how our version of Lucas' filter transforms the original U.S. time series as we increase the value of parameter β.[5] The filter is two-sided because the behavior of households is likely to be affected both by what happened to them in the past and by their expectations of what might happen to them in the future.[6]

One important advantage of using Lucas' methods to find out whether the Quantity Theory of Money holds in the long run is that his filter is atheoretical. This means that its results do not depend on any modelling assumptions. In contrast, other methods that are more sophisticated econometrically, such as structural VARs, require identifying assumptions that are model-dependent. Those methods are less useful to compare model economies that are fundamentally different, like those that we consider in this article.[7]

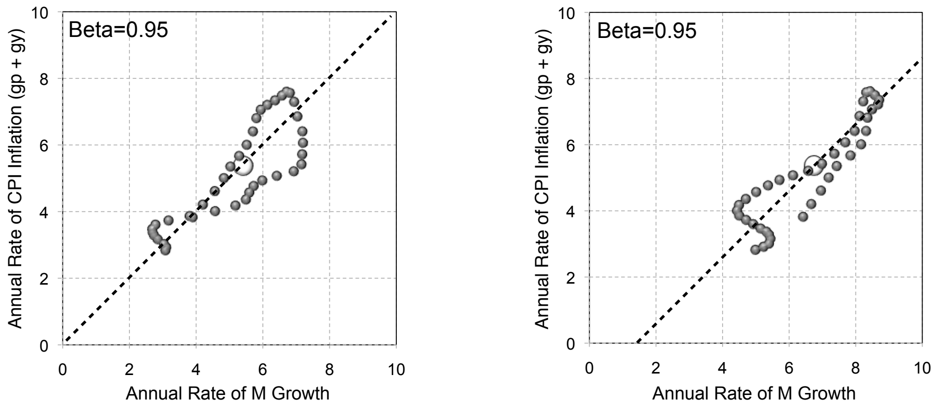
[3]We have taken all the data from FRED2 (http://research.stlouisfed.org/fred2/). The time series that we have used are GNPC96, M1SL, M2SL, and CPIAUCNS. Our results are robust to using three alternative measures for inflation: CPIAUCSL, CPILFENS, CPILFESL. We measure the growth rates as the percentage changes on the same quarter of the previous year.

[4]Parameter α guarantees that the means of the original and the filtered time series coincide. In fact, Lucas (1980) uses a slightly different definition of this parameter. He makes $\alpha = (1-\beta)/(1+\beta)$. His definition guarantees that the means coincide assuming that the lengths of the unfiltered series are infinite. Instead, we use Sargent and Surico (2011)'s small-sample correction to the value of α. This correction preserves the means of the series, but assuming that the lengths of the unfiltered series are finite.

[5]To prevent clutter, in all our figures we follow Lucas (1980) exactly and plot only the fourth quarter of every year. To prevent end-of-sample distortions, we drop the first two and last two years from each graph, even though we use them in the filter.

[6]The choice of filter is not important. For example, Benati (2005, 2009) reports similar conclusions using a band-pass filter.

[7]See Lucas (1980) for a discussion of his filter and of its frequency interpretation, and see Whiteman (1984) for further details on this discussion.



A: M1 in the United States $\beta = 0.95$ B: M2 in the United States $\beta = 0.95$

*The coordinates of the center of the white circle in each panel are the grand mean of the unfiltered sample.

Figure 2: The Quantity Theory in United States in the Long Run (1960:Q1–2009:Q4)

Higher values of β extract the higher frequency fluctuations from the original series. Therefore, if the Quantity Theory of Money relationship holds in the long-run, as we increase the value of β, the plots of the filtered time series should look increasingly like the 45 degree line that runs through the grand mean of the unfiltered series. And, if it does not hold, we have no theory to account for the relationship between those variables and we expect the filtered data to become a blob around the grand mean of the unfiltered data. In fact, Lucas (1980) shows that this is precisely what happens when he plots the unemployment rate against the rate of growth of money, for the 1955–1975 period.

In Figure 2.2 we plot the Quantity Theory of Money relationship in the United States in the long-run or, more precisely, when $\beta = 0.95$. In both panels of that figure we see that, when we filter out the high-frequency fluctuations, the original bird-shot scatters displayed in Figure 1 disappear and the observations approach the 45 degree line that runs through the grand mean of the unfiltered sample. Therefore, the scatter plots displayed in Figure 2.2 illustrate that the Quantity Theory of Money held in the United States in the long-run both for M1 and for M2 during the 1960–2009 period, and they confirm Lucas (1980)'s findings.

The long-run scatter plot for M1 is interesting from the perspective of the monetary history of the United States. In Panel A of Figure 2.2 the observations start near the bottom left-hand-side corner of the graph and they march roughly up the 45 degree line during the late 1960s and the 1970s. When they reach the top-right-hand corner of the graph, they suddenly drop down almost vertically. This period of sharply falling average growth rates of prices represents the beginning of the 1980s when the Federal Reserve, under Paul Volcker, started tightening monetary policy to fight inflation—and eventually defeat it. Then, in the 1990s and 2000s the points return to the 45 degree line as the U.S. economy transitions to a new monetary regime with a lower inflation rate and lower money growth rate.

2.3 Quantifying Lucas' Illustrations

There are two relatively straight-forward methods to quantify Lucas' illustrations. The first one is to compute the average Cartesian distance of the points in the plots from the 45 degree line that runs through the grand mean of the unfiltered observations.[8] The other one is to compute the slope of an ordinary least squares (OLS) linear regression of the growth rate of prices plus the growth rate of real output on the growth rate of money. This second method was proposed by Whiteman (1984).

The formal definition of the first statistic is the following

$$D45 = \frac{1}{\sqrt{2}T} \sum_i |x_i - y_i + (\bar{y} - \bar{x})| \tag{5}$$

where y_i is the value of the i-th observation of the growth rate of prices plus the growth rate of output, either of the original or of the filtered time series; x_i is the corresponding observation of the growth rate of money and $\bar{x}$ and $\bar{y}$ are the average values of the unfiltered x_i and y_i. Obviously, if the Quantity Theory of Money relationship holds, the value of the $D45$ statistic will be small and, if it does not hold, it will be large.

In Whiteman (1984)'s regression, the value of the OLS coefficient will be close to unity when the Quantity Theory of Money relationship holds, and it can take any value when it does not hold. Obviously, chances are that it will be different from unity in this case.

In Table 1 we report the values of these two statistics for the United States both in the short run, when $\beta = 0.00$, and in the long run, when $\beta = 0.95$, for $M1$ and for $M2$. Our numerical results confirm what we learnt from Lucas' Illustrations. The Quantity Theory of Money did not hold in the short run in the United States, between 1960 and 2009, either for M1 or for M2, but it held in the long run during the same period for both monetary aggregates. Moreover, according our two statistics, the Quantity Theory of Money relationship was tighter for M2 than for M1, both in the short run and in the long run. The distances from the 45 degree line were smaller for M2 than for M1 in both instances, and the slopes of the linear regressions were higher for M2, also in both instances (see Table 1).

Table 1: The Quantity Theory of Money Statistics in the United States

	Short Run (β=0.0)		Long Run (β=0.95)	
	M1	M2	M1	M2
Distance from 45 Degree Line ($D45$)	2.9850	2.5420	0.5003	0.3953
OLS Regression Coefficient	0.0189	0.0723	0.8179	0.9164

2.4 An Apparent Conflict with the Literature

Our finding that the Quantity Theory of Money holds in the long run in the United States is somewhat in conflict with those of Sargent and Surico (2011). They use M2 as the monetary aggregate, they divide the 1900–2005 period into four subperiods, which they identify with different

[8] The Cartesian distance of a point, (x_i, y_i), from a line, $ax + by + c = 0$, is $d = |ax_i + by_i + c|/\sqrt{a^2 + b^2}$.

monetary policy regimes, and they find that the long-run slopes of Lucas' Illustrations differ in these four regimes.

They use their results to argue that the 1984–2005 subperiod corresponds to an inflation targeting regime, and that this delivers a flatter slope. We contend that Sargent and Surico's result is not due to a breakdown in the long-run Quantity Theory of Money relationship. Instead, we think that Sargent and Surico (2011)'s slopes vary for two reasons: first, because they follow Lucas (1980) literally and leave out the growth rate of output from their calculations and, second, because of the specific subperiods in which they choose to divide their sample.

Leaving out the growth rate of output, as Lucas (1980) did when he studied the 1950–1975 period, has little effect on his illustrations because the growth rate of output was small relative to the growth rate of prices during that period—recall that in the 1970s there was a hump in the U.S. inflation rate time series. Therefore, this omission affects Lucas' original illustrations only slightly. But this is not the case for the post-1984 period, when the inflation rate was moderate and output growth was relatively high.[9]

The importance of the starting points of Sargent and Surico's subperiods is highlighted by our earlier comments that the inflation rate decreased around 1980 and that this reduction was followed by a lower growth rate of money, but with a delay. As we have already mentioned, this is illustrated by the temporary dip below the 45 degree line of the filtered points that correspond to the early 1980's in Panel A of Figure 2.2, and the subsequent return to the 45 degree line during the 1990's.

In Appendix B we provide an econometric test of our claim that the slopes of Lucas' Illustrations remained unchanged during the 1960–2009 period when we include output growth in the Quantity Theory of Money relationship. This period runs across 1984, which is the year which Sargent and Surico (2011) identify as the year when the monetary regime, and hence the slope of Lucas' illustrations, supposedly changed.

To do this, we exploit the econometric interpretation of the Quantity Theory of Money as a cointegration relationship between the logs of the price level, real output, and the money supply. We test for a structural break at an unknown point in time in the cointegrating vector—which would be the econometric interpretation of a change in the slope of Lucas' Illustrations—and we reject that such a break occurred for M2 at any time during the 1960–2009 period.

3 The Quantity Theory of Money in the Model Economies

In this section we explore the extent to which the Quantity Theory of Money relationship holds in three of the modelling frameworks most frequently used by economists to think about monetary policy: the Cash-in-Advance framework, the New-Keynesian framework, and the Search-Money framework. As we did for the United States, to answer this question we use two methods: Lucas' illustrations of the Quantity Theory of Money relationship and the two statistics that we have used in the previous section to quantify this relationship.

As we have already mentioned, for each one of these three frameworks we choose a representative model economy: for the Cash-in-Advance framework, we use the model economy described in Cooley and Hansen (1989); for the New-Keynesian framework, the model economy described in

[9]The ratio of annual inflation to GDP growth fell from roughly 2 in the 1960–1983 period, to approximately half that amount in the 1984–2009 period.

Chapter 3 of Galí (2008); and, for the Search-Money framework, a stochastic extension of the model economy described in Aruoba, Waller, and Wright (2011). Since these three model economies are standard in the literature we relegate their detailed description to Appendix A of this article.

We also describe in detail our calibration procedure in that appendix. To make our comparisons meaningful, we use the same functional forms and parameter values for the utility functions and for the processes on the technology and the monetary shocks, whenever possible.[10] We also use the same methods to characterize the equilibrium processes of our three model economies and to find their solutions.

In all three cases, we describe the equilibrium processes as systems of stochastic difference equations and we solve these systems using the default perturbation methods of Dynare that allow us to obtain quadratic laws-of-motion. Then we simulate the three model economies and we obtain samples of 204 quarterly observations to replicate the number of observations in our United States sample. To obtain these samples, we use the same seeds for the random number generators. Consequently, the sequences of realizations of the random processes are identical in the three model economies.

3.1 The Quantity Theory of Money in the Model Economies in the Short Run

Figure 3 represents Lucas' Illustrations in the short run—that is for $\beta = 0$—for M2 in the United States and for the monetary aggregates of our three model economies. We observe that the Quantity Theory of Money relationship is much stronger in the three model economies than in the United States. In the three model economies, the points lie close to the 45 degree line as predicted by Quantity Theory of Money. And in the United States data it is hard discern any pattern.

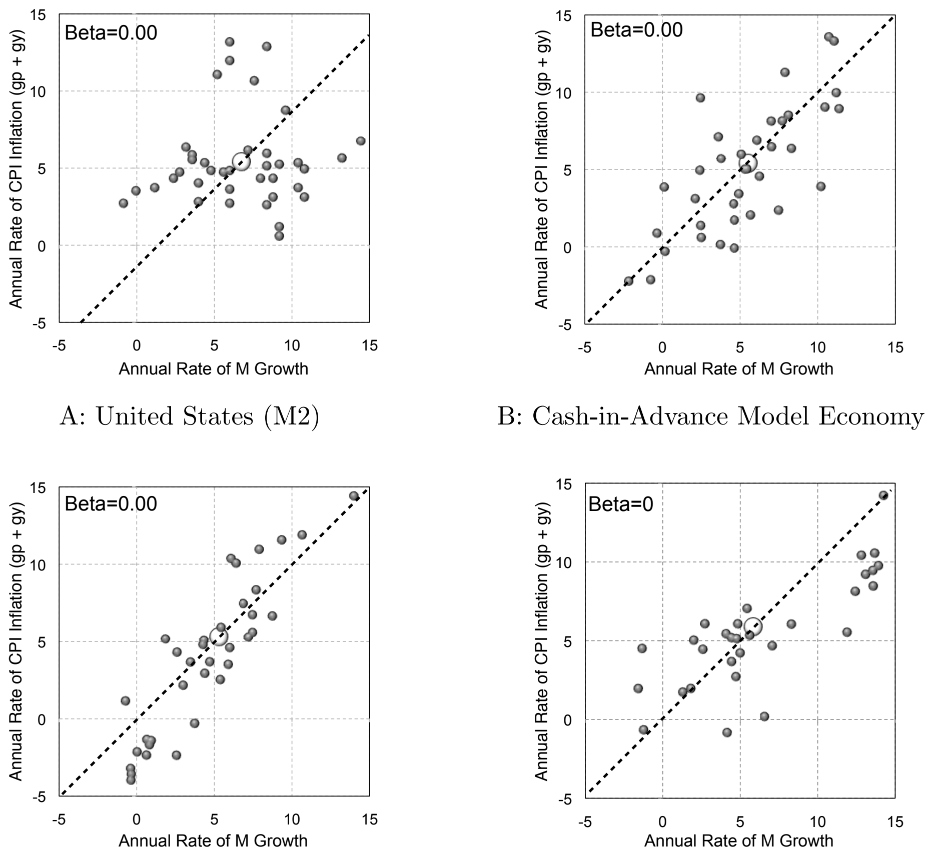
Table 2: The Quantity Theory of Money Statistics in the Short Run

	US (M2)	Cash-in-Advance	New-Keynesian	Search-Money
D45	2.5420	1.8841	1.5611	1.9278
(std dev)	–	(0.1228)	(0.1185)	(0.1408)
OLS coeff.	0.0723	1.1602	1.3366	1.4219
(std dev)	–	(0.0587)	(0.0501)	(0.0652)

We have only one United States time series from which to compute the Quantity Theory of Money statistics, but we can simulate many stochastic realizations of the equilibrium processes of our model economies. To reduce the size of the sampling error, we compute the D45 statistics and the slopes of the Quantity Theory of Money OLS linear regressions using 100 independent random samples. In Table 2 we report the sample means and the sample standard deviations of those statistics. We also reproduce the results for M2 for the United States economy to facilitate the comparisons.

Both sets of statistics confirm what we found using Lucas' Illustrations, and they establish that our graphs are not the result of a sampling oddity. The D45 statistic for the United States is 2.54,

[10]We repeated our calculations with the functional forms and the calibration targets used in the original articles, and we found that this does not change our results qualitatively. This is partly due to the fact that the original articles target similar data moments and study similar time periods.



A: United States (M2)

B: Cash-in-Advance Model Economy

C: New-Keynesian Model Economy

D: Search-Money Model Economy

*The coordinates of the center of the white circle in each panel are the grand mean of the unfiltered sample.

Figure 3: Lucas' Illustrations in the Short Run ($\beta = 0.00$)

while in all three model economies it is below 2.0. This result arises from the fact that the points in the graph for the United States form a shapeless cloud, while those in the graphs for the three model economies form clouds which are closer to the 45 degree line. Thus, the D45 statistic confirms that our three model economies display too much of the Quantity Theory of Money relationship in the short run, when compared with the United States economy.

The slopes of the Quantity Theory of Money regressions tell pretty much the same story. They are much closer to unity in the three model economies than in the United States, and the slope of the Quantity Theory of Money regression line is closest to unity in the Cash-in-Advance model economy. We interpret these results to mean that in our three model economies the rate of growth of prices responds too quickly to changes in the rate of growth of money, relative to the United States, or that our three model economies do not display enough short-run sluggishness in the response of prices.

3.2 The Departures from the Quantity Theory of Money in the Short Run

In this subsection we describe how the three model economies depart from the Quantity Theory of Money in the short run using only one equation for each one of them. Specifically, we provide an expression for the equilibrium values of the term PY/M for each model economy.[11] We provide the derivation of these equations in Appendix A, together with the full descriptions of the model economies. If the Quantity Theory of Money held exactly $(M/P)/Y$ would be constant. Therefore, these single equation expressions thus help to understand how each model economy departs from the Quantity Theory of Money in the short-run.

(a) The Cash-in-Advance Model Economy

In the Cash-in-Advance model economy we obtain that

$$\frac{PY}{M} = \frac{P(C+X)}{M} = 1 + \frac{PX}{M} \tag{6}$$

where C is consumption, and X is investment. So the Cash-in-Advance framework succeeds in departing from the Quantity Theory of Money in as far as monetary policy distorts investment decisions. These distortions take place on the cash-good (consumption) and credit-good (investment) margin.

(b) The New-Keynesian Model Economy

In the New-Keynesian model economy when we rewrite PY/M in logs we obtain that

$$p_t + y_t - m_t = \eta i_t = \eta r_t^n + \eta E_t\{f(\pi_t, \pi_{t+1}, \pi_{t+2})\} \tag{7}$$

where i_t is the nominal interest rate, η is the elasticity of money demand, r_t^n is the natural real interest rate, and $f(\pi_t, \pi_{t+1}, \pi_{t+2})$ is a linear function of current and future inflation. It is evident from expression (7) that the elasticity of money demand and the changing values of the nominal interest rates play an important role in allowing the New-Keynesian model to get away from the Quantity Theory of Money in the short run.

[11] These expressions are also related to the issue of money demand as discussed in Lucas (2000). Lucas defines money demand as the relationship between nominal interest rates and the ratio of real money holdings to real output, or $(M/P)/Y$. This ratio is the inverse of the PY/M term which we consider here.

What role do sticky prices play in this? The natural real interest rate, r_t^n, is independent of both monetary variables and the parameters that determine the degree of price stickiness. So, for sticky prices to be part of the story, they must operate through the inflation rate which evolves according to

$$\pi_t = (1-\theta)(p_t^* - p_{t-1}) \tag{8}$$

where p_t^* is the price level chosen by the firms that get to reset their prices, and $(1-\theta)$ is the share of those firms. So sticky prices affect the rate of inflation and, therefore, the nominal interest rates and they contribute to the short-run departure from the Quantity Theory of Money relationship. In practice, however, this effect is small.[12] This is because the nominal interest rate does not change immediately as predicted by Fisher's equation, $i = r + \pi$, after a monetary shock because these shocks generate a liquidity effect. But this liquidity effect is not very long-lasting and it all but disappears, when we filter out the high frequency fluctuations of the nominal time series. We conclude that the short-run behavior of the New-Keynesian model arises directly from the money demand equation, and that the role played by the degree of price stickiness is small.

(c) The Search-Money Model Economy

In the Search-Money model economy we obtain that,

$$\frac{PY}{M} = \frac{1}{z(q,K)} \frac{\gamma Y}{F_N(K,N)} \tag{9}$$

where γ is a parameter that quantifies the disutility of labor. In the simulations of this model economy total output, Y, the stock of capital, K, and the marginal product of labour, $F_N(K,N)$, are almost constant. Consequently, they do not account for the departure from the Quantity Theory of Money relationship in the short run. They are almost constant because most of the trades are non-monetary, the centralized night market is much bigger than the decentralized day market, and this market is almost unaffected by changes in the money supply. Almost all the variability in expression (9) comes from changes in $z(q,K)$, which represents the terms of trade in monetary exchanges and, more specifically, from changes in q —the amount produced and traded in the monetary exchanges that take place in the decentralized market. These changes in q are caused by the unexpected changes in the amount of money and by changes in the inflation rate, which is the cost of holding money.

In summary, the Search-Money framework succeeds in departing from the Quantity Theory of Money relationship in the short run because of the effects of changes in the money supply on the value of money. People want to hold money because it is useful for monetary trades. The value of money is jointly determined by this demand for money and by the money supply. Since the nominal price of consumption goods is the inverse of the cost of acquiring money, changes in the value of money result in changes in the price level.[13] Therefore, changes in the money supply affect the value of money, and thereby the price level. Moreover, the resulting inflation also affects output because of a holdup problem.[14] This effect is magnified because monetary trades account only for

[12] We experimented with various parameter values, and we found that the Quantity Theory of Money relationship was largely unaffected by the degree of price-stickiness, θ, but that it was very sensitive to the value of the elasticity of money demand, η.

[13] The price of consumption goods is the number of units of money that agents exchange for one unit of the consumption good. The cost of acquiring money is the number of units of the consumption good that agents give up to obtain one unit of money.

[14] In every Search-Money model inflation decreases output. Sellers in single-coincidence meetings know that they

a small fraction of total exchanges and, therefore, changes in the supply of money are large relative to the size of the total amount of monetary trades. Consequently, the effect of a given change in money supply on the value of money and, hence, on prices in the Search-Money framework, is larger than in the other two frameworks.

3.3 The Quantity Theory of Money in the Model Economies in the Long Run

Figure 3.3 represents Lucas' Illustrations in the long run, that is, for $\beta = 0.95$, for M2 in the United States and for the monetary aggregates of our three model economies. Given that the Quantity Theory of Money relationship was clearly present in the three models economies in the short run, it comes as no surprise that it also holds in all three of them in the long run. In fact, the Quantity Theory of Money relationship is so tight in every model economy that we are hard put to say in which one of them it is tightest. For that purpose, we must turn to the statistics that we report in Table 3.

The D45 statistic shows that the Quantity Theory of Money relationship is tightest in the New-Keynesian model economy, followed by the Search-Money model economy and by the Cash-in-Advance model economy. But the differences between them are small. And in all three cases the Quantity Theory of Money relationship is much tighter than in the United States. The values of the slopes of the OLS linear regressions confirm these findings.

We interpret these results to mean that in the long run the Quantity Theory of Money relationship is present both in the United States and in our three model economies. But, once again, it is sizably tighter in the model economies.

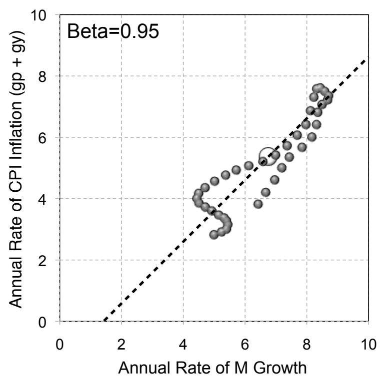
Table 3: The Quantity Theory of Money Statistics in the Long Run

	US (M2)	Cash-in-Advance	New-Keynesian	Search-Money
D45	0.3953	0.0541	0.0149	0.0188
(std dev)	–	(0.0141)	(0.0031)	(0.0039)
OLS coeff.	0.9164	0.9956	1.0015	1.0015
(std dev)	–	(0.0641)	(0.0116)	(0.0126)

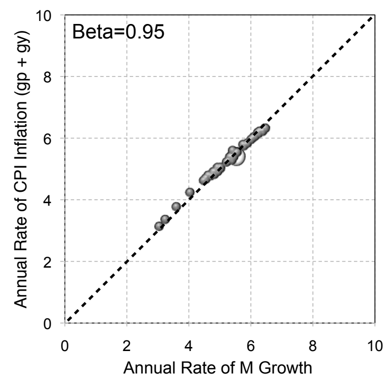
3.4 Who Wins?

We have shown that all three of our model frameworks display the Quantity Theory of Money relationship in the long run and, therefore, succeed in replicating the long-run behavior of the United States economy. But we have also shown that the Quantity Theory of Money relationship is much tighter in all three model economies than in the United States in the short run. Given the intuitive appeal of the Quantity Theory of Money, we think that this is an important shortcoming for the model economies.

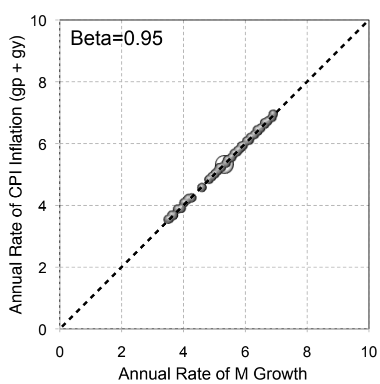
can increase the price of the consumption good because the outside option of the buyer —to hold onto the money until next period— is less attractive when the inflation rate is higher. These increased prices —known as the holdup problem— decrease economic efficiency and output. See, eg., Lagos and Wright (2005) for a discussion of the holdup problem.



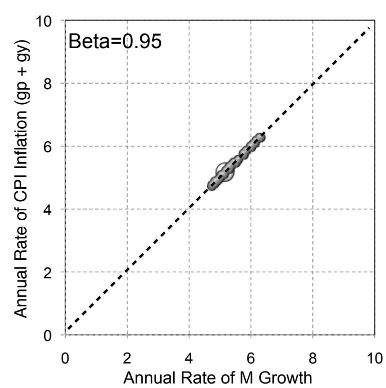
A: United States (M2)



B: Cash-in-Advance Model Economy



C: New-Keynesian Model Economy



D: Search-Money Model Economy

[*]The coordinates of the center of the white circle in each panel are the grand mean of the sample.

Figure 4: Lucas' Illustrations in the Long Run ($\beta = 0.95$)

The difficulties in capturing the short-run departures from the Quantity Theory of Money have been known to afflict the Cash-in-Advance framework since the work of Hodrick, Kocherlakota, and Lucas (1991). While the New-Keynesian and Search-Money frameworks have cast light on a number of other issues in monetary economics, they have not resolved these difficulties completely. Perhaps further research within these frameworks will succeed in enabling them to depart from the Quantity Theory of Money relationship in the short run.

Progress within the Cash-in-Advance framework in the attempt to slow down the response of prices to monetary shocks —a problem closely related to departing from the Quantity Theory of Money relationship in the short-run—appears to have stalled. Early attempts to solve this problem use constructs such as portfolio-adjustment costs (see, for instance, Christiano, 1991, and Christiano and Eichenbaum, 1995). But this line of research has trailed off since then, after being only moderately successful. Perhaps a partial exception to this rule can be found in the recent work of Alvarez, Atkeson, and Edmond (2009).

Progress within the New-Keynesian framework seems to be more promising. For example, Christiano, Eichenbaum, and Evans (2005) develop a New-Keynesian model capable of reproducing the slower reaction of the economy to monetary shocks observed in empirical studies that use impulse-response functions. This suggests that these more advanced New-Keynesian models might offer better hopes for departing from the Quantity Theory of Money relationship in the short run.

The Search-Money framework is a more recent construct, and bringing productive capital into that framework is a very recent achievement. Therefore, future refinements within this framework might enable it to depart from the Quantity Theory of Money relationship in the short run. Time will tell.

4 Conclusion

In this article we show that the Quantity Theory of Money held in the long-run in the United States between 1960 and 2009. And we also show that it failed to hold in the short-run during that period. Given the prominence of the Quantity Theory of Money in monetary theory, we argue that monetary model economies should replicate both the long-run success and short-run failure of the Quantity Theory of Money observed in the United States, if we are to trust their prescriptions for monetary policy.

Our analysis, based on the Lucas (1980) Illustrations, shows that every one of the three main frameworks that are currently used to study monetary policy —the Cash-in-Advance framework, the New-Keynesian framework, and the Search-Money framework— display the Quantity Theory of Money relationship both in the long-run and in the short-run. This failure of all three frameworks to depart from the Quantity Theory of Money in the short-run casts some doubts on their usefulness for the analysis of monetary policy —which most monetary theorists consider to be an inherently short-run phenomenon.

To break away from the Quantity Theory of Money in the short-run, the three monetary frameworks that we study here need a more sluggish response of the growth rate of prices to changes in the growth rate of money. We are not sure about what causes this sluggish response of prices to changes in money in the real world. But the generally accepted conjecture is that the way money is introduced into the economy most probably makes a difference.

When money changes are universal and simultaneous —that is, when they affect every agent at the same time— the rate of growth of prices responds immediately to changes in the rate of growth on money. But, as we mentioned in the introduction, when money enters the economy at a specific point, it has to spread around from there. The time it takes in this spreading around probably creates the sluggishness. Like the Quantity Theory of Money itself, this idea can also be traced back to David Hume; "[T]he money in its progress through the whole commonwealth...first quicken[s] the diligence of every individual before it encrease the price of labour." (Hume, 1742, *Of Money*).

In representative agent model economies there is only one point at which money can enter the economy. Once it reaches this agent, it has nowhere to spread around, and the sluggish response of prices is very hard to achieve. In this type of model economies every change in the growth rate of money is both universal and simultaneous by construction. This reasoning allows us to conjecture that agent heterogeneity may very well turn out to be a necessary condition for model economies to display the needed sluggishness.

Díaz-Giménez, Prescott, Alvarez, and Fitzgerald (1992) model the role of money as an asset in a heterogeneous household setup, and they give an early quantitative step in what could turn out to be the correct direction. The findings of Williamson (2008) and of Alvarez, Atkeson, and Edmond (2009), both of which include a small amount of agent heterogeneity, suggest that agent heterogeneity may indeed be key in replicating the sluggishness observed in the data.[15] As far as the Quantity Theory of Money relationship is concerned, the explicit modeling of agent heterogeneity is probably one of the best bets for future research.

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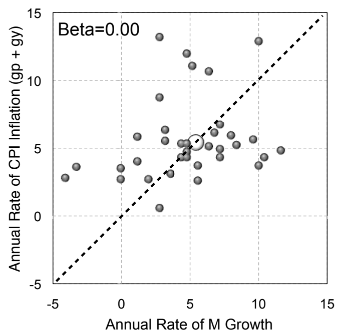
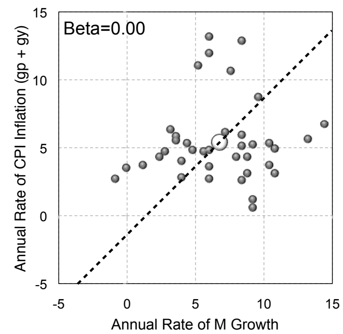
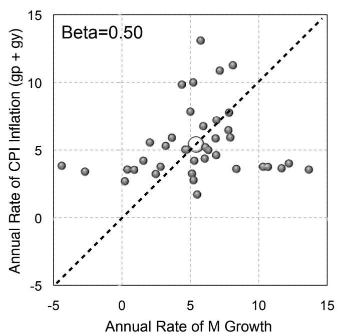


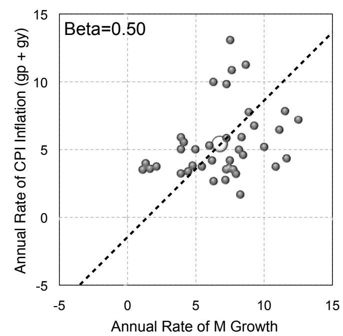
Panel A: M1 ($\beta = 0.00$)



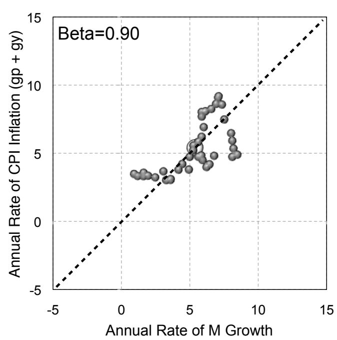
Panel B: M2 ($\beta = 0.00$)



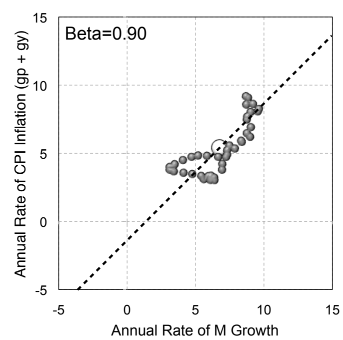
Panel C: M1 ($\beta = 0.50$)



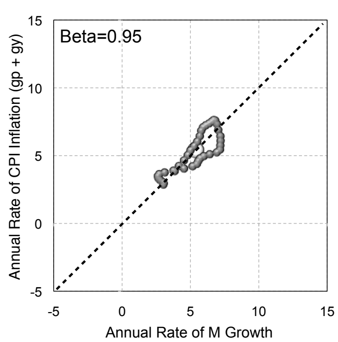
Panel D: M2 ($\beta = 0.50$)



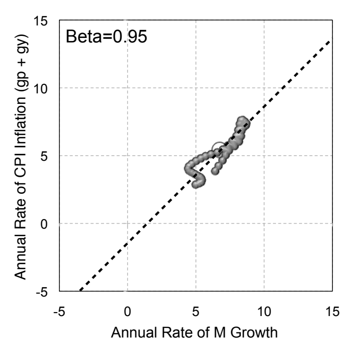
Panel E: M1 ($\beta = 0.90$)



Panel F: M2 ($\beta = 0.90$)



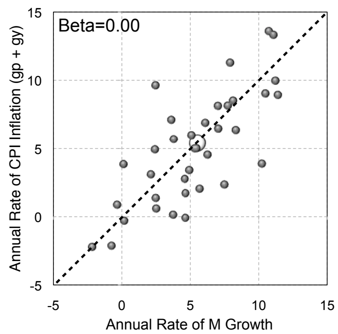
Panel G: M1 ($\beta = 0.95$)



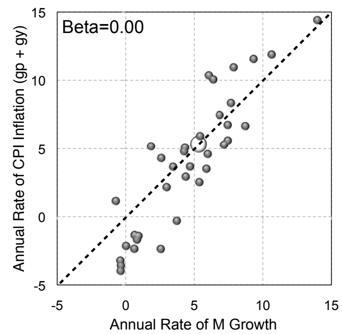
Panel H: M2 ($\beta = 0.95$)

*The coordinates of the center of the white circle in each panel are the grand mean of the sample.

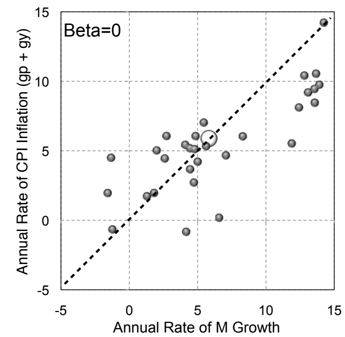
Figure 5: Lucas' Illustrations in the United States (1960:Q1–2009:Q4)



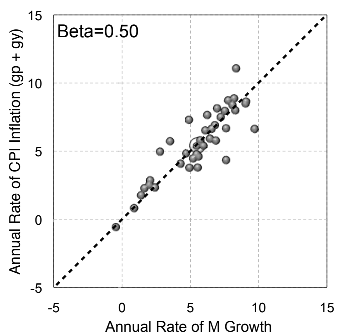
A: Cash-in-Advance ($\beta = 0.00$)



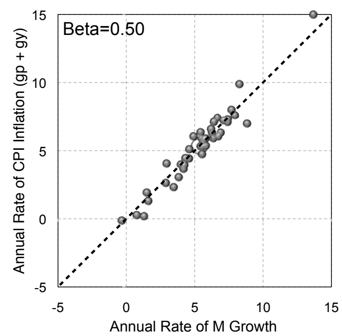
B: New-Keynesian ($\beta = 0.00$)



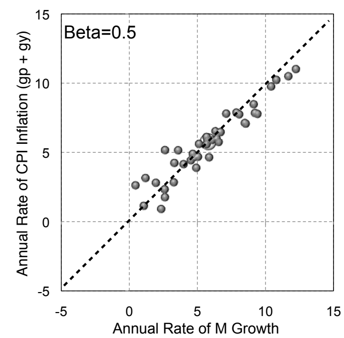
C: Search ($\beta = 0.00$)



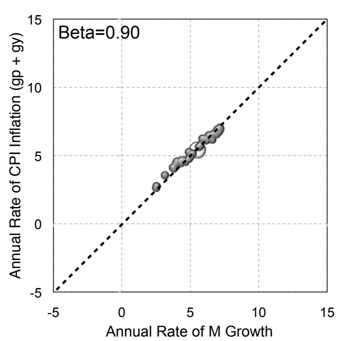
D: Cash-in-Advance ($\beta = 0.50$)



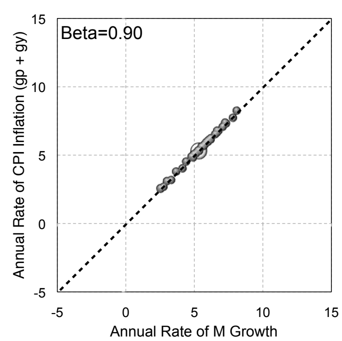
E: New-Keynesian ($\beta = 0.50$)



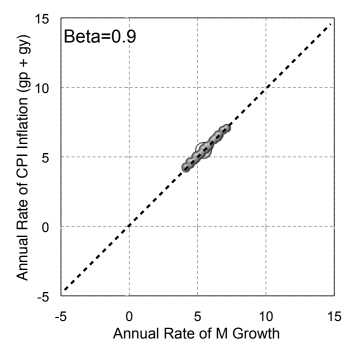
F: Search ($\beta = 0.50$)



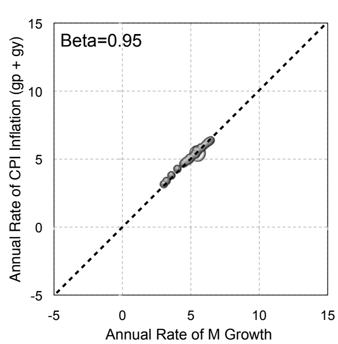
G: Cash-in-Advance ($\beta = 0.90$)



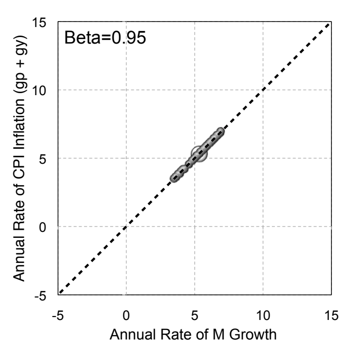
H: New-Keynesian ($\beta = 0.90$)



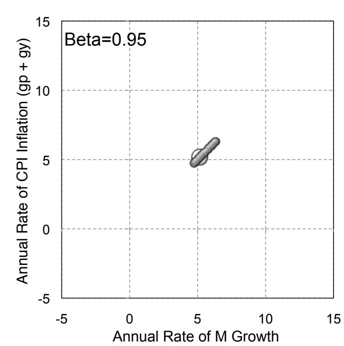
I: Search ($\beta = 0.90$)



J: Cash-in-Advance ($\beta = 0.95$)



K: New-Keynesian ($\beta = 0.95$)



L: Search ($\beta = 0.95$)

*The coordinates of the center of the white circle in each panel are the grand mean of the sample.

Figure 6: Lucas' Illustrations in the Model Economies

A The Monetary Model Economies (For Online Publication)

In this appendix we describe in detail each of the three model economies used as canonical examples for the three monetary frameworks: Cooley and Hansen (1989) for the Cash-in-Advance framework; Galí (2008) Chapter 3 for the New-Keynesian framework; and Aruoba, Waller, and Wright (2011) for the Search-Money framework. We then describe the details of the calibration and computation of all three models.

A.1 The Cash-in-Advance Model Economy

The cash-in-advance abstraction is an explicit way to model the transactions function of money by requiring that at least some goods have to be purchased with cash. This abstraction was first developed and analyzed in Lucas (1980), and more generally by Stokey and Lucas (1983, 1987), although the idea to model frictions in this way dates back to Clower (1967). Quantitative explorations of the business cycle implications of this abstraction can be found in Cooley and Hansen (1989, 1995). In this article, to represent the cash-in-advance abstraction, we use a minor variation of the model economy described in Cooley and Hansen (1989), but in its actual description we follow Nason and Cogley (1994).

In this model economy there are three goods: a consumption good, an investment good, and leisure. We assume that only the consumption good must be bought with cash carried over from the previous period, while the investment good and leisure can be purchased on credit. Model economies with this type of cash-in-advance constraint attempt to account for the distortionary effects of inflation on real activity. These distortions create an incentive for people to substitute away from activities that require cash —from consumption, in our case— towards activities that are exempt from this requirement —towards investment and leisure, in our case.

As shown by Hodrick, Kocherlakota, and Lucas (1991), one of the shortcomings of the cash-in-advance abstraction is that the model economies react too quickly to monetary shocks. Numerous extensions have attempted to deal with this shortcoming by adding liquidity effects via portfolio adjustment costs (see Lucas (1990); Fuerst (1992); Christiano and Eichenbaum (1992); and Christiano and Eichenbaum (1995), amongst others). But these extensions, while they have succeeded in addressing the issue of the liquidity effects, have had very limited success in generating a sluggish response of prices. See Christiano (1991) for an interesting discussion of the motivations, strengths, and weaknesses of the cash-in-advance approach to modelling money.

A.1.1 Households

The economy is inhabited by a continuum of identical households of measure one who order their preferences over stochastic processes of consumption and labor according to the following utility function:

$$\max E \sum_{t=0}^{\infty} \beta^t \left(\frac{C_t^{1-\sigma}}{1-\sigma} - \gamma \frac{N_t^{1+\varphi}}{1+\varphi} \right) \tag{10}$$

where $0 < \beta < 1$ is the discount factor, C_t is consumption, and N_t is labor[16]. Households in this model economy are endowed with one unit of time which they can allocate to the supply of labour services to the firm or to the enjoyment of leisure, that is $N_t \in [0, 1]$ for all t. The households face a budget constraint given by

$$P_t C_t + P_t X_t + M_t \leq P_t W_t N_t + P_t R_t K_t + M_{t-1} + T_t \tag{11}$$

where P_t is the price level, X_t is investment in capital, M_t are money holdings, W_t is the real wage, R_t is the real interest rate, K_t is capital holdings, and T_t is the lump-sum transfer of the cash injections made by monetary authorities.

The stock of capital evolves according to

$$K_{t+1} = (1 - \delta) K_t + X_t \tag{12}$$

where $0 < \delta < 1$ is the depreciation rate.

The innovation of the cash-in-advance abstraction that makes money necessary is to add a cash-in-advance constraint. This constraint requires that the consumption good must be purchased with money, in particular with money that must be 'held in advance'. That is, with money holdings that are chosen one period ahead plus the money injected into the economy in the current period. This cash-in-advance constraint is

$$P_t C_t \leq M_{t-1} + T_t \tag{13}$$

The process on money defined later, following Cooley and Hansen (1989), will make the cash-in-advance constraint always binding[17].

Therefore, the problem of the representative household is to choose C_t, N_t, M_t, X_t and K_t in order to maximize (10) subject to (11), (12), (13), and $N_t \in [0, 1]$.

A.1.2 Firms

Firms in the economy operate in competitive factor and product markets and produce output according to a constant returns-to-scale production function. These assumptions allow us to use a representative firm with a production function that takes the following form

$$Y_t = A_t K_{f,t}^{\alpha} N_{f,t}^{1-\alpha} \tag{14}$$

where Y_t is output, $K_{f,t}$ and $N_{f,t}$ are the captial and labour inputs, and A_t is a technology shock. Each period t the firms decision problem written in real terms is

$$\max_{Y_t, K_{f,t}, N_{f,t}} Y_t - W_t N_{f,t} - R_t K_{f,t} \tag{15}$$

The technology shock follows an exogenous AR(1) process in logs, given by

$$a_t = \rho_a a_{t-1} + \varsigma_t \tag{16}$$

where $a_t \equiv log(A_t)$, ς_t is an identically and independently distributed process that follows a normal distribution with zero mean and variance σ_{ς}^2.

[16]The utility function in Cooley and Hansen (1989) is $log(C_t) - \gamma N_t$, which is a subcase of ours.

[17]This assumption, that the cash-in-advance constraint always binds, was shown to be unconsequential by Hodrick, Kocherlakota, and Lucas (1991), since when it is allowed to be occasionally binding it remains the case that for quantitatively plausible calibrations it will bind almost all of the time anyway.

A.1.3 Money

The monetary authority of this economy issues non-interest bearing currency, M^s, according to the following rule

$$M^s_{t+1} = e^{\nu_t} M^s_t \tag{17}$$

where the stochastic money growth rate, ν_t, is revealed at the beginning of period t and evolves according to

$$\nu_t = (1 - \rho_m)\bar{\nu} + \rho_m \nu_{t-1} + \xi_t \tag{18}$$

where $0 < \rho_m < 1$ and where ξ is an identically and independently distributed process that follows a normal distribution with zero mean and variance σ^2_ξ.

Given the money supply rule, the government makes the required money injections to implement it each period. These injections take the following form

$$T_t = M^s_{t+1} - M^s_t \tag{19}$$

and are given as lump-sum payments to the households, adding directly to their money holdings.

A.1.4 Prices and Market Clearance

Prices in this model economy are completely flexible and they adjust instantaneously so that labor, capital and money markets always clear. That is,

$$\begin{aligned} N_t &= N_{f,t} \\ K_t &= K_{f,t} \\ M_t &= M^s_t \end{aligned} \tag{20}$$

A.1.5 Equilibrium

To solve the model it must first be made stationary. The first step to achieve this is to divide equations (11) and (13) by the price level, P_t. The second step is to replace M_t and P_t in those two equations with $\hat{M}_t = M_t/M^s_t$ and $\hat{P}_t = P_t/M^s_t$, this allows us to remove the trending variables M_t, P_t.

Once the problem is stationary, the equilibrium of the cash-in-advance model economy can be characterized by the following system of equations that combines optimality conditions, budget and technology constraints, and market clearing conditions.

$$
\begin{aligned}
K_{t+1} + \hat{M}_t/\hat{P}_t &= W_t N_t + (R_t + 1 - \delta) K_t && (21)\\
C_t &= \frac{\hat{M}_{t-1} + e^{\nu_t} - 1}{e^{\nu_t}\hat{P}_t} && (22)\\
W_t &= (1-\alpha) e^{a_t} K_t^{\alpha} N_t^{-\alpha} && (23)\\
R_t &= \alpha e^{a_t} K_t^{\alpha-1} N_t^{1-\alpha} && (24)\\
\frac{N_t^{\varphi}}{W_t} &= \beta \frac{N_{t+1}^{\varphi}}{W_{t+1}} (R_{t+1} + 1 - \delta) && (25)\\
N_t^{-\sigma} &= \frac{\gamma}{\beta} \frac{N_{t+1}^{\varphi}}{W_{t+1}} e^{\nu_{t+1}} \frac{\hat{P}_{t+1}}{\hat{P}_t} && (26)\\
\hat{M}_t &= 1 && (27)\\
a_t &= \rho_a a_{t-1} + \varsigma_t && (28)\\
\nu_t &= (1-\rho_m)\bar{\nu} + \rho_m \nu_{t-1} + \xi_t && (29)
\end{aligned}
$$

A.1.6 The Quantity Theory of Money in a Single Equation

We now describe with a single equation the quantity theory of money in way way that makes it easier to see how the Cash-in-Advance framework temporarily escape from the quantity theory of money. Specifically, we give an expression for the term PY/M Were the Quantity Theory of Money to hold exactly this term would be equal to a constant.

Using the cash-in-advance constraint, $P_t C_t = M_{t-1} + T_t$, with the aggregate resource constraint, $Y_t = C_t + X_t$, we have

$$
\frac{P_t Y_t}{M_t} = \frac{P_t (C_t + X_t)}{M_t} = \frac{M_{t-1} + T_t}{M_t} + \frac{P_t X_t}{M_t} = 1 + \frac{P_t X_t}{M_t} \qquad (30)
$$

So the Cash-in-Advance framework succeeds in breaking away from the Quantity Theory of Money in-so-far as monetary policy distorts investment decisions (distorts the cash goods vs. credit goods margin).

A.2 The New-Keynesian Model Economy

To represent New-Keynesian abstraction we use the model economy described in Chapter 3 of Galí (2008). If money were absent, both the cash-in-advance model economy described above and the New-Keynesian model economy described below would simplify to similar versions of the standard real business cycle model economy.

The main purpose of New-Keynesian model economies is to analyze monetary policy. These model economies use sticky prices, which they justify with a mixture of theoretical justifications like rational inattention with empirical evidence that prices change infrequently. Sticky prices allow money to have short-run effects, while remaining long-run neutral. The New-Keynesian approach is perhaps more interested in modelling the effects of monetary policy on the economy, than in the modelling of money itself.

In the subsections below we discuss a version of the text-book description of the basic New Keynesian model economy which we have taken from Chapter 3 of Galí (2008). Even though this model economy can be characterized fully by a system of equations obtained by log-linearization about the steady-state of an explicit model economy, we provide the details of the full model economy to highlight its similarities with the cash-in-advance economy that we have just described.

This model economy has a representative household and it assumes that prices are sticky and that they change according to a Calvo rule, (Calvo, 1983). From these micro-foundations we derive the New Keynesian Phillips curve and the dynamic Investment-Savings equation. To close the model we add a process on nominal interest rates and a money demand function that define the monetary policy rule and the relationship between the money supply and the interest rate.

A.2.1 Households

The model has a representative household who chooses consumption, labor, and savings so as to maximize an expected discounted utility function

$$\max E \sum_{t=0}^{\infty} \beta^t \left(\frac{C_t^{1-\sigma}}{1-\sigma} - \gamma \frac{N_t^{1+\varphi}}{1+\varphi} \right) \tag{31}$$

where $0 < \beta < 1$ is the discount factor, C_t is consumption, and N_t is labor. Note that this is identical to the utility function in expression (10) for the cash-in-advance model.

However, in this model economy we assume that the household consumes a continuum of goods indexed by $i \in [0, 1]$. These goods are transformed into a composite good according to the following equation

$$C_t = \left[\int_0^1 C_t(i)^{\frac{\epsilon-1}{\epsilon}} di \right]^{\frac{\epsilon}{\epsilon-1}} \tag{32}$$

In this model economy the maximization of expected discounted utility is subject to the following series of budget constraints,

$$\int_0^1 P_t(i) C_t(i) di + I_t B_t \leq B_{t-1} + P_t W_t N_t + T_t \tag{33}$$

where B_t are purchases of nominal one-period bonds which have gross rate of return I_t, W_t is the wage, T_t is a lump-sum component of income, which may include dividends from firm ownership, and P_t is the aggregate price level which is given by

$$P_t = \left[\int_0^1 P_t(i)^{1-\epsilon} di \right]^{\frac{1}{1-\epsilon}} \tag{34}$$

The representative household demands money according to a money demand function that depends on the nominal interest rates. However it is more convenient to write this demand function in logs and we provide it in expression (38) below.

A.2.2 Firms

Each differentiated consumption good is produced by a different firm. All firms have the same production technology given by $Y_t(i) = A_t N_t(i)^{1-\alpha}$, where $Y_t(i)$ is the production of firm i, A_t is a common technology level, and $N_t(i)$ is the labour used by firm i. The firms set prices a la Calvo, that is, each period firms are allowed to change prices only with probability $1-\theta$. Firms set prices to maximize their expected discounted future profits for the period in which that price is in place. Thus, problem for firm setting price in period t is

$$\max_{P_t^*} \sum_{k=0}^{\infty} \theta^k E_t\{I_{t,t+k}(P_t^* Y_{t+k|t} - \Psi_{t+k}(Y_{t+k|t}))\} \tag{35}$$

subject to a demand function

$$Y_{t+k|t} = \left(\frac{P_t^*}{P_{t+k}}\right)^{-\epsilon} C_{t+k} \tag{36}$$

which comes from the first-order conditions of the representative agents problem. Where $Y_t = \left(\int_0^1 Y_t(i)^{\frac{\epsilon-1}{\epsilon}} di\right)^{\frac{\epsilon}{\epsilon-1}}$ is production of the final (composite) good, P_t^* is the price being set, $\Psi_{t+k}(\cdot)$ is the cost function, $Y_{t+k|t}$ is the production at time $t+k$ of a firm that last changed price in period t, $I_{t,t+k}$ is the stochastic discount factor for nominal payoffs, and $P_t = [\int_0^1 P_t(i)^{1-\epsilon} di]^{\frac{1}{1-\epsilon}}$ is the aggregate price level.

The technology process, A_t, follows an AR(1) process in logs, a_t,

$$a_t = \rho_a a_{t-1} + \varsigma_t \tag{37}$$

where $\rho_a \in [0,1)$, ς_t is iid $\mathcal{N}(0, \sigma_\varsigma^2)$.

A.2.3 Money

The money demand function in logs is

$$m_t - p_t = y_t - \eta i_t \tag{38}$$

where m_t is (log) money, and p_t are (log) prices.

Monetary policy, in keeping with all of the models covered in this paper is given by an exogenous AR(1) process,

$$\nu_t = (1-\rho_m)\bar{\nu} + \rho_m \nu_{t-1} + \xi_t \tag{39}$$

where $\nu_t \equiv \Delta m_t$, $\rho_m \in [0,1)$, ξ_t is white noise. Note that the process on money in the New Keynesian model (equation (39)) in exactly the same one as was used the Cash-in-Advance model (equations (17) & (18)), just that here we write the process with m_t in logs.

A.2.4 Prices and Market Clearance

The evolution of the aggregate consumer price level is given by

$$P_t = \left[\theta P_{t-1}^{1-\epsilon} + (1-\theta)(P_t^*)^{1-\epsilon}\right]^{\frac{1}{1-\epsilon}} \tag{40}$$

Thus consumer price inflation is

$$\Pi_t^{1-\epsilon} = \theta + (1-\theta)\left(\frac{P_t^*}{P_{t-1}}\right)^{1-\epsilon} \tag{41}$$

where $\Pi_t = P_t/P_{t-1}$ is the consumer price inflation rate.

The remaining component of the model is the requirement for market clearing. The market clearing conditions are given by, $\forall$ t: that the markets for each consumption good clear, $C_t(i) = Y_t(i)$, $\forall$ $i \in [0,1]$, and that the labour market clears, $N_t = \int_0^1 N_t(i)di$.

A.2.5 Equilibrium

The system of equations that constitute the reduced form of the basic New Keynesian model are now given[18]. They are derived from the microfoundations listed previously. The difference in notation, with lowercase letters replacing the uppercase letters, is that all of the variables listed here are now in log-linear form rather than the levels represented by the uppercase letters, eg. y_t is log-deviation of output while Y_t is output. The New Keynesian Phillips curve is given by

$$\pi_t = \beta E_t\{\pi_{t+1}\} + \kappa \widetilde{y}_t \tag{42}$$

where $\kappa \equiv (\sigma + \frac{\varphi+\alpha}{1-\alpha})\frac{(1-\theta)(1-\beta\theta)}{\theta}\Theta$, and $\Theta \equiv \frac{1-\alpha}{1-\alpha+\alpha\epsilon} \leq 1$; π_t is the inflation rate, and $\widetilde{y}_t = y_t - y_t^n$ is the output gap, that is the difference between current output, y_t, and the natural level of output y_t^n which would occour if prices were flexible. The dynamic IS equation is

$$\widetilde{y}_t = -\frac{1}{\sigma}(i_t - E_t\{\pi_{t+1}\} - r_t^n) + E_t\{\widetilde{y}_{t+1}\} \tag{43}$$

where i_t is the nominal interest rate, r_t^n is the natural interest rate (again, that which would result if prices were flexible). Both of these two equations are derived from the models micro-foundations.[19]

Letting $l_t = m_t - p_t$ be real money holdings and rewritting the money market equilibrium condition as $\widetilde{y}_t - \eta i_t = l_t - y_t^n$, we can substitute out for i_t and get the following system of equations from the three above,

$$\begin{aligned}
\pi_t &= \beta E_t\{\pi_{t+1}\} + \kappa \widetilde{y}_t && (44)\\
(1+\sigma\eta)\widetilde{y}_t &= \sigma\eta E_t\{\widetilde{y}_{t+1}\} + l_t + \eta E_t\{\pi_{t+1}\} + \eta \hat{r}_t^n - y_t^n && (45)\\
l_{t-1} &= l_t + \pi_t - \Delta m_t && (46)
\end{aligned}$$

where $\hat{r}_t^n$ is the deviation from steady-state of the natural rate of interest.

The two other formulae necessary to complete the model are those for the natural level of output and the natural rate of interest expressed in terms of deviation from steady-state, both of which depend on the technology level.

$$\begin{aligned}
y_t^n &= \phi_{ya}^n a_t + \vartheta_y^n && (47)\\
\hat{r}_t^n &= -\sigma\phi_{ya}^n(1-\rho_a)a_t && (48)
\end{aligned}$$

where $\vartheta_y^n = -\frac{(1-\alpha)(\mu - log(1-\alpha))}{\sigma(1-\alpha)+\varphi+\alpha} > 0$ and $\psi_{ya}^n = \frac{1+\varphi}{\sigma(1-\alpha)+\varphi+\alpha}$. The model is thus the system of equations given by (44)-(48).

[18]This system of equations already incorporates the parametrization of $\gamma = 1$, to which the models are later calibrated.

[19]See Galí (2008) for a step-by-step derivation.

A.2.6 The Quantity Theory of Money in a Single Equation

We now describe with a single equation the Quantity Theory of Money in way way that makes it easier to see how the New-Keynesian framework temporarily escape from the Quantity Theory of MoneyṠpecifically, we give an expression for the term PY/M. Since the New-Keynesian model is log-linearized we will look at the log of this term, namely $p + y - m$. Were the Quantity Theory of Money to hold exactly this term would be equal to a constant.

First observe that simply rewriting the money demand equation, (38), we get

$$p_t + y_t - m_t = \eta i_t \tag{49}$$

where i_t is the nominal interest rate and η is the elasticity of money demand. Combining the New-Keynesian Phillips Curve, equation (44), with the dynamic IS, equation (43), we get that

$$i_t = r_t^n - \frac{\sigma}{\kappa}\pi_t + (1 + \frac{\beta}{\kappa} - \frac{\sigma}{\kappa})E_t\{\pi_{t+1}\} + \frac{\beta}{\kappa}E_t\{\pi_{t+2}\} \tag{50}$$

So the nominal interest rate depends on the natural real rate of interest r_t^n (which depends on the current technology shock) and current and future expected inflation. Thus we have that

$$p_t + y_t - m_t = \eta r_t^n + \eta E_t\{f(\pi_t, \pi_{t+1}, \pi_{t+2})\} \tag{51}$$

Now, r_t^n is independent of monetary factors and the parameters relating to sticky prices. So for sticky prices to be part of the story they must be operating through inflation. From equation, (41), we have inflation evolves as

$$\pi_t = (1 - \theta)(p_t^* p_{t-1}) \tag{52}$$

where p_t^* is the price level being chosen by those firms that get to reset their prices. So in principle, sticky prices may affect the rate of inflation, and thus the nominal interest rates — helping to break away from the Quantity Theory of Money. In practice however the effect quantitatively negligible.

A.3 The Search-Money Model Economy

The aim of Search-Money models is to provide structural reasons that justify the existence of money. This abstraction focuses on money as a facilitator of exchange based on the idea that money exists mainly to solve problems related to the presence of single-coincidence of wants. Search-Money models go a step deeper than the other two abstractions that we consider here, in which money exists simply because the modeler assumes that it does, rather than to solve an explicit problem; like the absence of a double coincidence of wants in exchange. For this reason the model is the only one of the three we consider that satisfies Wallace's Dictum for monetary economics, that "Money should not be a primitive in monetary theory — in the same way that firm should not be a primitive in industrial organization theory or bond a primitive in finance theory" (Wallace, 1998).

Search-Money models have become more popular in recent years as they have begun to overcome some teething problems that plagued them in their earlier days: for instance in Kiyotaki and Wright (1989) money holdings were restricted to being 0 or 1 units per agent. Lagos and Wright (2005) overcame these issues by introducing the concept of a centralized (Arrow-Debreu) night-market alongside the decentralized (Kiyotaki-Wright) day-market. The use of the night-market remains integral to the latest generation of Search-Money models such as Head, Liu, Menzio, and Wright

(2012) and Berentson, Menzio, and Wright (2011). To represent the Search-Money abstraction we use a stochastic extension of the model economy described in Aruoba, Waller, and Wright (2011) — the stochastic extension is necessary to allow us to use the same process on money growth as in the other models[20] The model of Aruoba, Waller, and Wright (2011) uses the same combination of decentralized day-market and competitive night-market as Lagos and Wright (2005) and incorporates physical capital.

In this model economy there are continuum of agents, a decentralized day-market, and a centralized night-market. Money is essential in the day-market because meetings are anonymous, and credit is precluded in a fraction of these meetings because there is no possibility of credibly promising to repay at a later date. As a result exchange must be quid pro quo and so without money some trades would never take place — namely, those in which there was no double-coincidence of wants. Capital investments are made during the competitive night-market, and capital is used in production during both markets[21]. The model of Aruoba, Waller, and Wright (2011) includes a government sector, we eliminate this, which requires some recalibration of the model[22,23].

A.3.1 Households

There is a continuum of households indexed by i who live forever and whose measure we normalize to 1. Time is discrete and households discount the future at rate $\beta \in (0,1)$. Each period is divided into two subperiods which are commonly referred to as "day" and "night". Households consume and supply labour in both subperiods, and their preferences over sequences of consumption and labor are ordered according to the following period utility function

$$\mathcal{U}(c,n,C,N) = u(c) - h(n) + U(C) - N \tag{53}$$

where c and C denote consumption and n and N denote labour in the day and night subperiods. Assume that u, h, and U are twice continuously differentiable with $u' > 0$,$h' > 0$,$U' > 0$,$u'' < 0$,$h'' \geq 0$ and $U'' \leq 0$. Also, $u(0) = c(0) = 0$, and suppose that there exists $q^* \in (0,\infty)$ such that $u'(q^*) = h'(q^*)$ and $C^* \in (0,\infty)$ such that $U'(C^*) = 1$ with $U(C^*) > C^*$.

Aruoba et al. (2011) propose to use the following functional form to take the model to the data

$$U(c,n,C,N) = \left\{ \left[(c+\chi)^{(1-\sigma)} - \chi^{(1-\sigma)} \right] /(1-\sigma) - \gamma n \right\} + \{\Xi \log(C) - N\} \tag{54}$$

With the exception of the inclusion of parameter χ, $u(c) = [(c+\chi)^{(1-\sigma)} - \chi^{(1-\sigma)}]/(1-\sigma)$ is the same constant elasticity of substitution utility of consumption as the ones we have used in the other two models economies; the utility of consumption is $U(C) = \Xi \log(C)$ and the disutility of labor

[20] An earlier version of this paper used the model of Lagos and Wright (2005). This model failed to break away from the Quantity Theory of Money in the short-run, performing much worse than the other models presented here.

[21] The appearance of capital in both markets is important. Earlier work by Aruoba and Wright (2003) to introduce capital, with capital appearing in only one market, led to the results that the day and night markets could be solved for seperately, and thus money had no effect on consumption, investment, or anything else in the competitive night-market.

[22] Our results are robust to leaving the government sector in the model of Aruoba, Waller, and Wright (2011).

[23] Aruoba, Waller, and Wright (2011) actually present three models. Here we follow their model 2. Their model 1 is the same model, but with a slightly different calibration. Their model 3 uses 'competitive search', setting prices in the decentralized day market by price taking, rather than Nash bargaining. For robustness we tried out using their model 3 and it makes no real difference to the results we found using their model 2.

is $h(n) = \gamma n$ in the day market. The assumption that utility is quasi-linear in labour is used by Aruoba et al. (2011) and is necessary to keep the model analyticaly tractable[24].

A.3.2 Production and Trade

The day-good, c, comes in many differentiated varieties indexed by i. Each household consumes only a subset of these goods. Each household can transform its own labour into one of these goods that the household itself does not consume by the production function $F(K_i, N_i)$, namely household i produces good i which it does not consume. Trade during the day is decentralized and anonymous and households are matched randomly in a typical search setup.

For two households i and j drawn randomly, there are three possible trading situations. The probability that one consumes what the other produces, but not vice-versa —and, therefore, there is a single coincidence of wants is ω, and we assume that it is symmetric. Then, the probability that neither one of them consumes what the other one produces is $1 - 2\omega$. In a single-coincidence meeting, if i wants the good that j produces we call i the buyer and j the seller. In a fraction ϖ of single-coincidence meetings the buyer can only pay with money, in the remaining fraction, $1 - \varpi$, the buyer has access to credit, l. By assumption captial can not be used for transaction purposes.

The night good, C, comes in a single and homogeneous variety, which is consumed by every household. Each household can transform its own labour one for one into this good. Trade, during, the night occurs in a centralized Walrasian market. Consequently, the night-good can be purchased on credit. Since money is a good, it can be traded in the night market just like any other good. Investments in capital are also made during the night market.

All the differentiated day-goods and the night-good are perfectly divisible and non-storable, with the exceptions of money and capital which are storable.

A.3.3 Money

In this model economy there is an object called *money* that is perfectly divisible and storable in any non-negative quantity. The total money stock at time t is M_t, and it evolves according to

$$M_{t+1} = e^{\nu_{t+1}} M_t \tag{55}$$

The monetary injections, $(e^{\nu_{t+1}} - 1)M_t$, are made after the night market closes and they are distributed lump-sum and equally to every household. The rate of growth of money, ν, follows an AR(1) process given by

$$\nu_t = (1 - \rho_m)\bar{\nu} + \rho_m \nu_{t-1} + \xi_t \tag{56}$$

where $0 < \rho_m < 1$ and where ξ is an identical and independently distributed process with zero mean and variance σ_ξ^2. Although for the equilibrium proofs below we only need to assume that the rate of growth of money follows a first-order Markov process. So the process on money, as given by equantions (55) and (56), is identical to that used in the New-Keynesian and Cash-in-Advance models.

[24] Quasi linearity means that there are no wealth effects in the demand for money, so all agents in the centralized night markets choose the same money holdings. As a robustness test we simulated both the New-Keynesian and Cash-in-Advance models setting the parameters so that utility was quasi-linear in labour (ie. $\varphi = 0$, $\gamma = 1$; note that $\gamma = 1$ is the value to which this parameter is calibrated in those models anyway.). The effect on the results was negligible.

A.3.4 Prices and Market Clearance

Let $1/p_t$ be the price of money in the centralized night-market, that is, p_t is the nominal price of night good C.

In the deterministic version of the model economy described in Aruoba, Waller, and Wright (2011), the only uncertainty comes from the random matching. In the stochastic extension that we use here, the rate of growth of the money supply, ν_t, is also uncertain. Consequently, in our model economy the decisions of each household at each point in time depend on its current money holdings, m, on it's capital holdings k, during the centralized night market on it's earlier borrowing during the day l, and on the aggregate state which is the rate of growth of money, ν. Therefore, the households' choices at time t can be characterized with a value function that has m, k, and ν as its arguments; as well as l in the centralized night-market.

Let $V(m, k, \nu)$ be the value function for a household when it enters the the decentralized day-market, and $W(m, k, l, \nu)$ its value function when it enters the centralized night-market. Since trade is bilateral in the day-market and the day-good is non-storable, the seller's production, n, must be equal to the buyer's consumption, c.

Let m be money holdings. The the value of trading at the day-market is

$$V_t(m, k, \nu) = \omega V_t^b(m, k\nu) + \omega V_t^s(m, k, \nu) + (1 - 2\omega)W_t(m, k, 0, \nu) \tag{57}$$

where $V_t^b(m, k, \nu)$ and $V_t^s(m, k, \nu)$ denote the values to being a buyer and being a seller, as given by

$$\begin{aligned} V_t^b(m, k, \nu) &= \varpi[u(q_b) + W_t(m - d_b, k, 0, \nu)] + (1 - \varpi)[u(\tilde{q}_b) + W_t(m, k, l_b, \nu)] && (58) \\ V_t^s(m, k, \nu) &= \varpi[-c(q_s, k) + W_t(m + d_s, k, 0, \nu)] + (1 - \varpi)[-c(\tilde{q}_s, k) + W_t(m, k, -l_s, \nu)] && (59) \end{aligned}$$

In these expressions q_b and d_b (q_s and d_s) denote the quantity of goods and money exchanged when buying (selling) for money, while $\tilde{q}_b$ and l_b ($\tilde{q}_s$ and $-l_s$) denote the quantity and the value of the loan for the buyer (seller) when trading on credit.

At the centralized night-market agents solve the following problem

$$W_t(m, k, l, \nu) = \max_{C,N,m',k'} [U(C) - N + \beta E_t\{V_{t+1}(m' + (e^{\nu'} - 1)M, \nu')|\nu\} \tag{60}$$

subject to $C = wN + (1 + r - \delta)k - k' + \frac{m-m'-l}{p}$, $C \geq 0$, $0 \leq N \leq \bar{N}$, and $m' \geq 0$, where $\bar{N}$ is the endowment of night-hours, w is the wage, and r is the interest rate on capital.[25]

It is assumed that the markets for captial and labour in the night-market are competitive, thus $w = F_N(K, N)$ and $r = F_K(K, N)$.

Now that we have defined the value functions, we consider the terms of trade in the decentralized day-market. In single-coincidence meetings, we use the generalized Nash solution in which the buyer has bargaining power $\zeta > 0$ and threat points which are given by the continuation values. In the fraction ϖ of meetings where money is used (q, d) is the consumption for money exchange pair that

[25] Notice that $(e^{\nu'} - 1)M$ is the transfer of money that is added lump-sum to the households' holdings after they exit the night-market.

maximizes the following problem

$$(q,d) = \quad \text{argmax}\ \{[u(q) + W_t(m_b - d, k_b, 0, \nu) - W_t(m_b, k_b, 0, \nu)]^{\zeta} \\ [-c(q, k_s) + W_t(m_s + d, k_s, 0, \nu) - W_t(m_s, k_s, 0, \nu)]^{1-\zeta}\} \tag{61}$$

subject to $d \leq m_b$ and $q \geq 0$. In the remaining fraction, $1 - \varpi$, of meetings where credit is available, $(\tilde{q}, l)$ is determined just like (q, d), except that the Nash bargaining problem is no longer any constraint on l, the way $d \leq m_b$ had to hold in monetary trades.

As Aruoba, Waller, and Wright (2011) observe, the solution to the bargaining problem in 61 will involve $d = m_b$. Substituting this into the bargaining problem and taking the first order condition with respect to q we have

$$\frac{m_b}{p} = \frac{z(q, k_s)w}{\gamma} \tag{62}$$

where

$$z(q,k) \equiv \frac{\zeta c(q,k)u'(q) + (1-\zeta)u(q)c_q(q,k)}{\zeta u'(q) + (1-\zeta)c_q(q,k)} \tag{63}$$

reflects the terms of trade in the bargaining meetings.

Real output, $Y = Y_D + Y_N$, is the combination of real output in the decentralized day market, $Y_D = \omega\varpi M/p + \omega\varpi\omega l/p$, and real output in the centralized night market $F(K, N)$.

Following Aruoba, Waller, and Wright (2011) we measure inflation in terms of the price level in the centralized market p_t[26].

A.3.5 Equilibrium

The system of equations that defines an equilibrium is now given. To make the model stationary we define $\hat{m}_t = m_t/M_t$ and $\hat{p}_t = p_t/M_t$; observe that in equilibrium it follows that $\hat{m}_t = 1$ for all t. The derivation of this system of equations follows almost exactly as described in Aruoba, Waller, and Wright (2011). The first three equations are related to the first-order conditions of the household.

$$z(q_t, K_t) = \beta E \left\{ \frac{z(q_{t+1}, K_{t+1})}{exp(\nu_{t+1})} \left(1 - \omega\zeta + \omega\zeta \frac{u'(q_{t+1})}{z(q_{t+1}, K_{t+1})} \right) \right\} \tag{64}$$

$$U'(C_t) = \beta E\{U'(C_{t+1})[1 + F_K(K_{t+1}, N_{t+1}) - \delta] \\ -\omega[\varpi\Gamma(q_{t+1}, K_{t+1}) + (1-\varpi)(1-\zeta)c_k(\hat{q}_{t+1}, K_{t+1})]\} \tag{65}$$

$$U'(C_t) = \frac{1}{F_N(K_t, N_t)} \tag{66}$$

The fourth equation is aggregate resource constraint

$$C_t = F(K_t, N_t) + (1-\delta)K_t - K_{t+1} \tag{67}$$

[26]We also tried using a Laspeyres measure of inflation that included prices in the decentralized markets. But since in the calibrated model the decentralized market accounts for only about 3% of total real output this made no noticable difference.

The next two equations determine the price level in the competive night market, and the real value of the credit loans made in the decentralized day market (in the fraction ϖ of meetings where credit is available)[27].

$$\hat{p}_t = \frac{\gamma}{z(q_t, K_t) F_N(K_t, N_t)} \tag{68}$$

$$l_t/p_t = F_N(K_t, N_t)[(1-\zeta)u(\tilde{q} + \zeta c(\tilde{q}, K)] \tag{69}$$

The next four equations are related to the terms of trade in the decentralized day market ($z(q,K)$, as defined in (63)), and some related derivatives and quantities.

$$z(q_t, K_t) = \frac{\zeta c(q_t, K_t) u'(q_t) + (1-\zeta) u(q_t) c_q(q_t, K_t)}{\zeta u'(q_t) + (1-\zeta) c_q(q_t, K_t)} \tag{70}$$

$$z_q(q_t, K_t) = \frac{u'(q) c_q [\zeta u'(q) + (1-\zeta) c_q] + \zeta(1-\zeta)(u(q_t) - c)(u'(q_t) c_{qq} - c_q u''(q_t))}{[\zeta u'(q_t) + (1-\zeta) c_q]^2} \tag{71}$$

$$z_K(q_t, K_t) = \frac{\zeta u'(q_t) c_k [\zeta u'(q_t) + (1-\zeta) c_q] + \zeta(1-\zeta)(u(q_t) - c) u'(q_t) c_{qK}}{[\zeta u'(q_t) + (1-\zeta) c_q]^2} \tag{72}$$

$$\Gamma(q_t, K_t) = c_K(q_t, K_t) - c_q(q_t, K_t) \frac{z_K(q_t, K_t)}{z_q(q_t, K_t)} \tag{73}$$

where c is shorthand for $c(q_t, K_t)$, c_q for $c_q(q_t, K_t)$, c_K for $c_K(q_t, K_t)$, c_{qq} for $c_{qq}(q_t, K_t)$, and c_{qK} for $c_{qK}(q_t, K_t)$. The next equation is simply the definitition of real output,

$$Y = F(K, N) + \omega\varpi M/p + \omega\varpi\omega l/p \tag{74}$$

The final equation is that defining the money growth rate,

$$\nu_t = (1-\rho_m)\bar{\nu} + \rho_m \nu_{t-1} + \xi_t \tag{75}$$

The Search-Money model with capital is thus given by the system of stochastic difference equations, (64)-(75)

A.3.6 The Quantity Theory of Money in a Single Equation

We now describe with a single equation the Quantity Theory of Money in way way that makes it easier to see how the Search-Money framework temporarily escapes from the Quantity Theory of Money. Specifically, we give an expression for the term PY/M. Were the Quantity Theory of Money to hold exactly this term would be equal to a constant.

In the Search-Money model, by equation (68), we have that

$$\frac{PY}{M} = \frac{1}{z(q,K)} \frac{\gamma Y}{F_N(K,N)} \tag{76}$$

[27] While neither l_t nor p_t are stationary, by treating l_t/p_t as a single variable the equation is stationary.

In the simulation results total output (Y), capital stock (K), and the marginal product of labour ($F_N(K,N)$) are almost constant, and thus not related to the ability of the Search-Money model to get away from the Quantity Theory of Money. They are almost constant because most of the economy is based on non-monetary trades, the centralized night market is much bigger than the decentralized day market, and so unaffected by changes in the money supply. All of the movement occours in the $z(q,K)$ term, specifically from changes in q — the amount produced/traded in the exchanges involving money in the decentralized market. The amount produced in monetary exchanges varies with the amount of money and inflation (the cost of holding money).

A.4 Calibration and Computation

For our comparisons of the three model economies to be meaningful, we choose their functional forms and parameters so that they are as similar as possible. This use of identical parameter values wherever the models coincide, of identical exogenous processes, and of identical functional forms for the utility of consumption, as well as the fact that we have solved the three model economies using identical solution methods allows us to make a genuine comparison between them. Since we have removed all other possible sources of variation, we can safely attribute any differences in their outputs with respect to the Quantity Theory of Money relationship to the different ways in which these three frameworks model money.

Parameter Choices

We have decided to use Galí (2008) as our main reference for our parameter choices, with the obvious exceptions of the parameters and functions of the Cash-in-Advance and Search-Money model economies that do not exist in the New-Keynesian framework, such as the parameters related to the search for trading partners in the Search-Money model economy.[28]

Since Galí (2008) exploits the certainty equivalence principle in his solution method, he does not define the shocks to either the technology or the money supply. Instead, we take the processes for those shocks from Cooley and Hansen (1989). We report our chosen parameter values in Table 4.[29] Our results are robust to using the original parameter calibrations of each model, that is those parameter values given in the papers from which the models are taken. Importantly, the original parameterizations of the models (in the papers from which they are taken) are all calibrated to similar postwar periods. Since the frameworks are quite different using exactly the same calibration targets for the different frameworks is not possible, although some common calibration targets, such as interest rates and capital-output ratios were used by a number of of the the original papers.

Simulation

To simulate our model economies we have used identical seeds for the random number generator so that the sequences of the realizations of the random shocks are identical in all three model economies. To obtain the model economy time series we discard the first 200 periods of each equilibrium realization to purge away the initial conditions, and then we draw a sample of 204

[28]The calibrations reported in Aruoba, Waller, and Wright (2011) are annual and so had to be adjusted. This was done using the same methodology and targets they report — some targets, such as the capital-output ratio, have to be adjusted to quarterly values.

[29]Galí (2008) pg. 52 says that $\epsilon_p = 6$, however this appears to be a type. When we use this value, we fail to replicate his results. Therefore, we use $\epsilon_p = 6/5$ instead following http://www.dynare.org/phpBB3/tviewtopic.php?f=1andt=2978. In this case we replicated Gali's results successfully.

Table 4: Parameter Values

		Cash-in-Advance	New Keynesian[a]	Search-Money
Preferences				
Time Discount factor	β	0.99	0.99	0.99
Curvature of Consumption	σ	1	1	1
Weight on Labour[b]	γ	1	1	2.1
Curvature of Labour[c]	φ	1	1	0
Technology				
Returns to Capital[d,e]	α	0.33	n.a.	0.33
Depreciation Rate[e]	δ	0.025	n.a.	0.025
Autocorrelation	ρ_a	0.9	0.9	n.a.
Variance of Shock	σ_ς	0.007	0.007	n.a.
Money				
Elasticity of Money Demand	η	n.a.	4	n.a.
Autocorrelation	ρ_m	0.5	0.5	0.5
Variance of Shock	σ_ξ	0.009	0.009	0.009
Constant Term	$\bar{\nu}$	0.014	0.014	0.014
Dist. of Shock	ξ	log-normal	log-normal	log-normal
Price Setting				
Market Power	ϵ	n.a.	6/5	n.a.
Calvo Stickiness	θ	n.a.	0.66	n.a.
Search				
Prob. of Single Coincidence[g]	ω	n.a.	n.a.	0.08
Bargaining Power	ζ	n.a.	n.a.	0.92
Night weight on consumption	Ξ	n.a.	n.a.	0.8
Make $u(0) = 0$	χ	n.a.	n.a.	0.001
Prob. of credit availability	ϖ	n.a.	n.a.	0.85

[a] Every other parameter that appears in the equations that characterize the equilibrium of the New-Keynesian model economy can be derived from the parameters that we have identified in this table using the following system of equations: $\mathcal{M} = \epsilon/(1-\epsilon)$; $\mu = log\mathcal{M}$ $\rho = -log\beta$; $\Theta = (1-\alpha)/(1-\alpha+\alpha\epsilon)$; $\lambda = \Theta(1-\theta)(1-\beta\theta)/\theta$; $\kappa = \lambda[\sigma + (\varphi+\alpha)/(1-\alpha)]$; $\vartheta_y^n = \{(1-\alpha)[\mu - log(1-\alpha)]\}/[\sigma(1-\alpha)+\varphi+\alpha]$; $\psi_{ya}^n = (1+\varphi)/[\sigma(1-\alpha)+\varphi+\alpha]$.

[b] In the Search-Money model this parameter is calibrated 2.1, as this is needed as part of the the calibration procedure of Aruoba, Waller, and Wright (2011) (setting this parameter to one in the Search-Money with capital model, while messing up the calibration, does not affect the results).

[c] In the Search-Money model the disutility of labor is linear in both the day-market and in the night-market.

[d] Abbreviation “n.a.” means “not applicable”.

[e] There is no ’returns to capital’ or ’depreciation’ in the New-Keynesian economy as there is no capital.

quarterly observations to replicate the number of observations in our United States time series. Whenever we need to obtain multiple samples, we repeat this process as necessary.

Computation

The equilibria of the three models economies that we have described above can be reduced to systems of stochastic equations. We have solved these systems using the default perturbation methods of Dynare to calculate quadratic approximations to the decision rules.[30]

[30] We have run every code with Dynare Version 4.2.1-2 using Octave 3.2.4.

B Is the Quantity Theory of Money a Stable Relationship? (For Online Publication)

Using Lucas' Illustrations, Sargent and Surico (2011) argue that the relationship between the price level, real GDP, and the money supply (measured as M2) given by the Quantity Theory of Money varies with the monetary regime. In this appendix we look at this issue using the interpretation of the long-run Quantity Theory of Money as being a cointegration relationship. Under this interpretation, a change in the Quantity Theory of Money would involve a change in the cointegrating vector that defines the long-run relationship. Using a test for a structural break in the cointegrating vector at an unknown time period developed by Seo (1998) we reject that such a break has occoured in our 1960–2009 period. The Quantity Theory of Money appears to be a stable relationship.

As we described in Section 2, Sargent and Surico (2011) look at the Quantity Theory of Money during the period 1900–2005 using the Lucas Illustrations. They divide the period into four subperiods, coinciding with different monetary policy regimes. They find that the long-run slopes of the Lucas Illustrations differ in these four regimes, using M2 as the monetary aggregate. Of particular relevance to us they identify 1984, a date in the middle of our sample, as corresponding to a change to an inflation targeting regime which delivers a flatter long-run slope. Their findings imply that a structural break in the cointegrating vector defining the Quantity Theory of Money occours around 1984 — it is this implication that we aim to test here.

The Quantity Theory of Money defines a relationship between the price level, real GDP, and the money supply — all variables that are nonstationary. This relationship holds in the long-run, but short lived departures from this relationship are common. In the language of modern econometrics the Quantity Theory of Money defines a cointegration relationship. Cointegration theory thus gives us an alternative method to the Lucas Illustrations by which to look at whether the Quantity Theory of Money holds in the US economy in the long-run. However since the models we deal with are stationary it is not appropriate for our main purpose of evaluating different approaches to modelling money.

Here we look at whether the Quantity Theory of Money is a stable relationship — as opposed to the argument of Sargent and Surico (2011) that it varies with the monetary regime. The stability of the Quantity Theory of Money can be seen as a question about the stability of the cointegrating vector. If the Quantity Theory of Money relationship changes between different monetary regimes this would appear as a structural break in the cointegrating coefficients. For the Lucas Illustrations the stability of the Quantity Theory of Money means that the slope of the Lucas Illustrations is independent of the monetary regime. Thus the validity of our use of the Lucas Illustration to analyze the period 1960–2009, which covers two different monetary regimes, requires that Quantity Theory of Money be stable.

While cointegration theory has not been applied much to the Quantity Theory of Money it has often been used to evaluate the related issue of money demand equations (Lucas, 1988; Stock and Watson, 1993; Seo, 1998).[31] The Johansen test suggests that the Quantity Theory of Money represents a cointegrating relationship.

To look at the stability of the Quantity Theory of Money we use the test for a structural break

[31] Lucas (1988) does not use any cointegration theory, but lays some theory and evidence on money demand equations which the later two papers then extend and evaluate with cointegration theory.

at an unknown date in the cointegrating vector developed by Seo (1998). We find that there has not been a break in the Quantity Theory of Money for M2[32], and we conclude that the relationship is stable. That the Quantity Theory of Money is a stable relationship means that the slope of Lucas' Illustration has remained unchanged throughout the 1960–2009 period.

The log of the Quantity Theory of Money can be expressed in Vector Error Correction Model representation as

$$\begin{pmatrix} \Delta \log Y_t \\ \Delta \log P_t \\ \Delta \log M_t \end{pmatrix} = \alpha\beta' \begin{pmatrix} \log Y_{t-1} \\ \log P_{t-1} \\ \log M_{t-1} \end{pmatrix} + \sum_{i=1}^{L-1} \Gamma_i \begin{pmatrix} \Delta \log Y_{t-1} \\ \Delta \log P_{t-1} \\ \Delta \log M_{t-1} \end{pmatrix} + u_t \tag{77}$$

where α is a 3×1 vector, β is a 3×1 vector, u_t is a 3×1 vector of independently and identically distributed shocks with mean zero and covariance matrix Σ, L is the lag-operator, and Γ_i is the matrix of i^{th}-order autocorrelation coefficients.

Vector β represents the long-run relationship between the price level, real GDP, and the money supply —the Quantity Theory of Money relationship. For the purpose of identification the first element of β is normalized to 1. Vector α captures how the system adjusts to transitory departures and returns back to the long-run relationship captured by β. We apply tests for a structural break in one or both vectors α and β.

Seo (1998)'s test is based on the Maximum Likelihood Estimate of the cointegrating vectors from the Vector Error Correction Model representation. For a structural break at a known date a simple Lagrange Multiplier test could be applied —comparing the likelihood under the null hypothesis of no break to the alternative hypothesis of a break at time t. Three different simple tests can be constructed for a break in β, in α, or in both vectors. To test for a break at an unknown date we calculate the test statistics for a break at each possible date, and the 'largest' of these individual statistics becomes itself a test statistic for a break at an unknown date.[33]

We use three definitions to determine the value of the 'largest' statistic: the average ($Avg-LM$), the exponential average ($Exp - LM$), and the supremum ($Sup - LM$). Seo (1998) derives the asymptotic properties of each of these metrics under the assumptions of *(i)* no drift, *(ii)* no trend in the data generating process, and *(iii)* a trend in the data generating process and he provides critical values for all these tests.

We apply the test for a structural break in the cointegrating vectors to our quarterly data for the 1960–2009 period[34]. We use the tests based on a trend in the data generating process, because real GDP, M2, and the price level all show clear upward trends. We use logs of all of these variables to linearize the Quantity Theory of Money relationship.

The Akaike information criterion recommends to use as many lags as possible, Hannan-Quinn suggests 13, the Bayesian information criterion suggests 3. We choose 8 because this represents 2 years of data, which keeps us in line with the literature that estimates money demand equations. In particular, Stock and Watson (1993) use cointegration to test for a relationship between real

[32] That is, we cannot reject the null hypothesis of no break in the Quantity Theory of Money for M2.

[33] The unknown date is assumed to lie in the interval $[0.15, 0.85]T$, where T is the total number of periods. That is, we assume that the break does not occur at the ends of the time period.

[34] All the data comes from FRED2 (http://research.stlouisfed.org/fred2/) and is described in full in Section 2. The regression results shown here are those based on GNPC96, CPIAUCNS, and M2SL. Our results are robust to using our other measures for inflation.

money holdings (money supply divided by the price index), real GDP, and the interest rate. Our results are robust to the use of 12 lags.[35]

As a robustness test, and to compare with the literature on money demand equations, we tried adding the interest rate to our cointegration regressions: for quarterly data the presence of the interest rate in the cointegrating relationship was consistently rejected[36]; both for 8 and for 12 lags, and using the interest rate on both 3-month and 10-year Treasury Bills (Stock and Watson (1993) also use commercial paper rates, but this series was discontinued in 1997).

All the test results that we report below are based on a 5% level of significance. The Dicky-Fuller tests confirm that all the variables contain unit roots. Johansen's test for cointegration confirms the presence of a single cointegrating relationship using both 8 and 12 lags. All the estimation and test results described up to this point were found using Gretl Version 1.9.5. The structural break test itself is performed using Gauss[37], the following results are the estimation results.

We now summarize the estimation and testing results using real GDP, Consumer Price Index, and M2 money supply,

$$\begin{pmatrix} \Delta \log Y_t \\ \Delta \log P_t \\ \Delta \log M_t \end{pmatrix} = \begin{pmatrix} 0.003_{(0.002)} \\ -0.003_{(0.002)} \\ 0.003_{(0.073)} \end{pmatrix} \begin{pmatrix} 1 \\ 5.089_{(1.302)} \\ -4.167_{(0.903)} \end{pmatrix}' \begin{pmatrix} \log Y_{t-1} \\ \log P_{t-1} \\ \log M_{t-1} \end{pmatrix} + ... \tag{78}$$

$$LR(H_0 : rank(\Pi) = 0) = 34.599*$$
$$LR(H_0 : rank(\Pi) = 1) = 17.349$$
$$LR(H_0 : rank(\Pi) = 2) = 5.204$$

$$\begin{array}{lll} Avg - LM_n^{\alpha} = 6.872*, & Exp - LM_n^{\alpha} = 4.337*, & Sup - LM_n^{\alpha} = 11.933 \\ Avg - LM_n^{\beta} = 3.737, & Exp - LM_n^{\beta} = 2.399, & Sup - LM_n^{\beta} = 9.306 \\ Avg - LM_n^{\alpha\beta} = 10.609*, & Exp - LM_n^{\alpha\beta} = 7.340*, & Sup - LM_n^{\alpha\beta} = 20.845* \end{array}$$

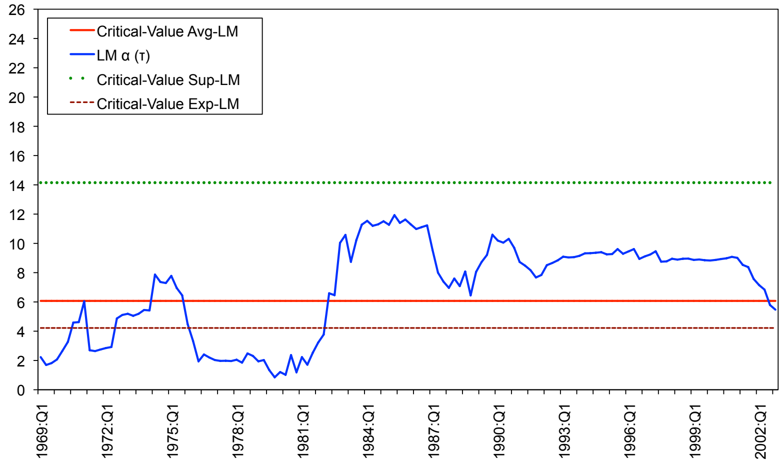
where standard errors are in parentheses and * indicates signicance at the 5% level. Figure B shows the evolution of the LM statistic over time. Using 12 lags, we further reject the possiblity of a structural break in α (or $\alpha\beta'$).

To interpret these results first recall that our main interest is in the vector, β, representing the cointegrating relationship. The tests also consider the adjustment vector, α, that represents how the system reacts to deviations from the cointegrating relationship — how the economy goes about returning to the cointegrating relationship given by the Quantity Theory of Money . For each of α and β (as well as for a joint-test of α and β) we have three test statistics ($Exp - LM$, $Avg - LM$, $Sup - LM$).

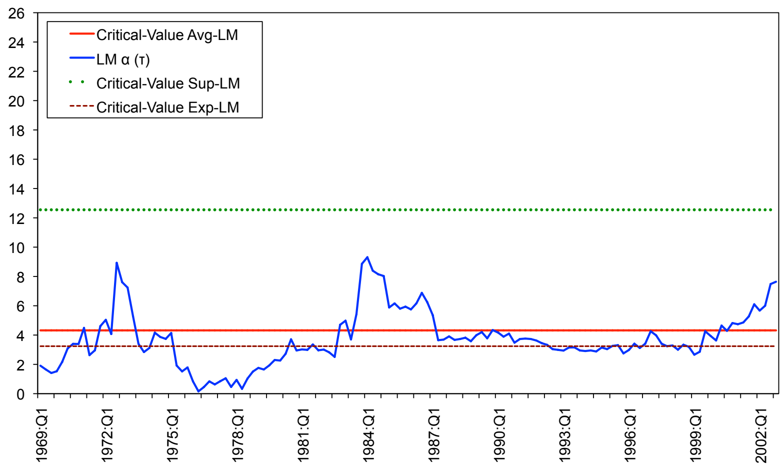
[35] When we use 12 lags we have to restrict t to be in the interval $[0.2, 0.8]$ because otherwise the $\alpha\beta$ matrix was too close to being singular. When we placed the same restriction on t with 8 lags, the results did not change.

[36] From a theoretical viewpoint whether the interest rate is a stationary or nonstationary variable is an open question in econometrics. So theoretically it is unclear whether including interest rates in a cointegration relationship makes sense. In any case our data rejected it's presence in the cointegrating relationship. We note in passing that Stock and Watson (1993) use annual data.

[37] Version 10. We thank Byeongseon Seo for a copy of his code implementing the tests.



Panel A: $LM^{\alpha}(\tau)$



Panel B: $LM^{\beta}(\tau)$

Panel C: $LM^{\alpha\beta}(\tau)$

Figure 7: Values of the LM Statistics for the Structural Break Tests in the Cointegration Vectors

In the printed estimation outputs the first part reports the estimated values for the vectors α and β, with the standard errors for the coefficients in parentheses. Then come the LR statistics of the Johansen cointegration test, which in both cases reject the first null hypothesis of no cointegrating relationship, and then accept the second null that there is a single cointegrating relationship. The third part gives the results of the tests for a structural break in the cointegrating vectors α and β (as well as for a joint-test of α and β). In the cases where these statistics are significant, indicated by $*$, we reject the null of no structural break in that vector. The results suggest that there is no structural break in β, while there is one in α.

The panels of Figure B show the evolution of the three statistics ($Exp - LM$, $Avg - LM$, $Sup - LM$) over time. The dotted horizontal lines show the critical values for the three statistics ($Exp - LM$, $Avg - LM$, $Sup - LM$), while the value of the statistic is shown by the jagged line. In the cases where we rejected the null of no structural break a sudden increase in the statistic at time t to above the critical values (the horizontal lines) suggests that this is the point in time at which it is most likely that the structural break occoured.

The results point towards the existence of a cointegrating relationship, β, which is stable over the sample period. Therefore we conclude that change of monetary policy that took place around 1984 is *not* associated with a break in the cointegrating relationship. That is, we conclude that the Quantity Theory of Money is stable over the period 1960–2009.

There is evidence of a change in the adjustment process for returning to the long-run relationship (a break in α). Figure B shows that this appears to take around 1984 (when the test statistic for α shoots up above the critical values (horizontal lines)). This might be understood as the effects of the change in the monetary policy regime on the short-run effects of monetary policy, as in Clarida, Galí, and Gertler (2000). That the estimated coefficients in the adjustment vector, α, are insignificant is likely related to the evidence of a structural break in this vector. Note that this does not represent a break in the cointegration relationship represented by the Quantity Theory of Money in the long-run, just in how the economy goes about returning to the Quantity Theory of Money when current events cause the economy to be away from the Quantity Theory of Money in the short-run.

We conclude that we do not find evidence of a change in the long-run relationship embodied by the Quantity Theory of Money, and this implies that the slope of the Lucas Illustration has not changed. There is however evidence that the adjustment of the economy back to the Quantity Theory of Money changes around 1984, when the monetary policy regime changes.