"Engineering Electoral Systems to Increase Turnout"

Orestis Troumpounis *
International Doctorate in Economic Analysis
Universitat Autònoma de Barcelona
Orestis.Troumpounis@uab.cat
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Abstract

We study two alternatives to a PR system: first, a varying size of the parliament (VSP) depending on participation and second, the introduction of a participation quota. We analyze a two party competition model where parties exert some costly effort that mobilizes voters to participate. We characterize and compare equilibria levels of effort exerted by parties and level of turnout. We show that there always exists an effective quota that increases turnout. The effect of a varying size parliament on turnout depends on the benefit parties obtain for being in parliament. Finally, turnout is always increasing in the benefit parties obtain.

1 Introduction

"Voice and exit are often alternative ways of exerting influence, but with regard to voting the exit option spells no influence; only voice can have an effect" Lijphart (1997)

Lijphart refers to abstention as the exit option of voters in elections. Indeed in most electoral systems used in parliamentary elections voters abstaining do not affect the constitution of the parliament. In this paper we analyze two ways in

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which the exit option (i.e. abstention) may influence the result of an election. In both alternatives we suggest the target is to study the level of participation and the level of effort put by politicians in order to convince voters to participate.

Arguing in line with Lijphart we believe that low turnout and unequal participation may harm democracy. The basic reason is that low turnout often implies unequal and socioeconomically biased turnout and as a result biased elected policies. As a mean to prevent low turnout Lijphart endorses compulsory voting and claims that "it is the only institutional mechanism that can assure high turnout virtually by itself". We disagree with the compulsory voting endorsement both for normative and positive reasons. The typical normative argument against compulsory voting applies. In the effort to enhance democracy the tool is highly "undemocratic" since it restricts individual’s right not to vote. Our positive argument and as we show in the paper is that compulsory voting is not the only institutional mechanism that boosts turnout by itself.

Following Lijphart’s work the debate about compulsory voting has revived mainly among political scientists. Very little attention had been paid by economists till recently. The first contribution focusing on compulsory versus voluntarily voting is by Börgers (2004). He analyzes a costly voting model and he shows that compulsory voting is never desirable. After this result work by Ghosal and Lockwood (2009) and Krasa and Polborn (2009) keep the costly voting setup but change some assumptions. They show conditions under which turnout is too low and hence increasing participation would be desirable. In our paper we focus in a costly voting setup but rather than adding in the compulsory versus voluntarily voting debate we provide two alternative ways of increasing turnout.

The first alternative we study is a varying size parliament (VSP) where the size of the parliament depends on the level of participation. The size of the parliament is determined after the election takes place and abstention is translated into empty seats in the parliament. This electoral system in contrast with a fixed size parliament (FSP) is the first alternative that gives "voice" to the exit option.

To the best of our knowledge a system of a varying size parliament has not been implemented in any election. There exist some movements supporting such

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1A well established fact is that the better educated and richer voters tend to participate with higher probability.

2Lijphart claims that compulsory voting is not an obligation to vote which would be undemocratic rather than an obligation to show up in the election, since voters have the right to cast a blank, invalid ballot etc. Even though this argument is true the problem of the non-influence of the "exit" choice is still present.
a system. For example, in 2009 ILP\(^3\) collected ninety thousand signatures during an initiative for an electoral change in Catalunya. The suggested electoral system was including a varying size parliament. Moreover, the "European citizens for a None Of The Above option” network\(^4\) is organized in nine European countries and their target is a varying size parliament depending on the level of blank votes in national and European elections.

Second, we study the effect of introducing a quota of participation in order the election to be valid. In case of too low participation then a new election takes place. We assume that this is too costly for political parties since they have to compete in a new election. The alternative of introducing a quota in an election has been practiced mainly in presidential elections in ex-USSR countries. In some cases like in Serbia 2002 and Ukraine 1995 the non-fulfillment of a fifty percent quota result in repeated elections. In the next election in both countries the quota had been abolished. A fifty percent quota is still present in the cases of Moldova and Hungary. In Macedonia the president decided to decrease from a 50\% to a 40\% (January 2009).

The model we adapt in order to study the effect of the two above alternatives is a group-based model of turnout as used by Snyder (1989), Shachar and Nalebuff (1999) and Herrera and Mattozzi (2009).

Snyder uses a group-based model in order to study campaign resource allocation among districts in legislative elections under two different assumptions for parties objectives: maximize the probability of winning and maximize the number of seats in the parliament. He shows that allocation differs depending on the objective of the party.

Sachar & Nalebuff divide the world in leaders and followers. Leaders perform a strategic calculation and decide the level of effort to exert. Voters follow a nonstrategic voting rule and their participation decision depends on the level of effort put by the leaders. In this setup the authors study the positive correlation between turnout and closeness of the race in US elections.

The paper by Mattozzi & Herrera is the one closest to ours since they study the effect of a participation quota in referenda. In contrast to our model the quota in case of referenda may result into lower participation levels. This is the case when the status quo ”party” rather than mobilizing it’s supporters to vote against the change, has incentives to stay passive, not mobilize it’s voters and hence ”win” the referendum by not satisfying the quota.

\(^3\)http://www.lleielectoral.cat/web/
\(^4\)http://www.cevb.org/index.php?lng=en
In our model there are two exogenously given parties and the society is divided into two equal pools of potential voters for each party. The assumption of the equally divided society that implies full symmetry may seem too restricting but has an interesting feature. It allows us to study the pure effect of the instruments we introduce in order to increase turnout given that we consider the simplest structure of the society. Each party exerts some costly effort that is observable and desirable by individuals. Parties utility depends on the number of seats obtained in the parliament. Voters are not strategic and support their preferred party once it’s effort is high enough to compensate the cost of participating. We assume that there are no partizan voters in the sense that no supporters would be willing to vote their preferred party in case it doesn’t exert any effort at all.

Effort may be interpreted in several ways. The most obvious interpretation is campaign spending (Herrera et al. (2008) : Snyder (1989)). Alternatively, it can be thought as quality of the party (or the inverse of corruption (Polo, 1998)), an endogenous valence characteristic (Meirowitz (2008) : Ashworth and de Mesquita (2005)), or simply effort of politicians to persuade voters to participate through rhetoric (Dickson and Scheve, 2006). In any of the above interpretations it always has to be the case that effort is desirable voters while it is costly for politicians.

The paper is organized as follows. In section two we present the model. In section three we analyze the benchmark case without a quota. We characterize and compare equilibria for both the FSP and the VSP. In section four we introduce a quota and again we characterize and compare equilibria under both a FSP and a VSP. In section five we conclude. Proofs can be found in the Appendix.

2 The Model

There are two exogenously given parties \( j, \neg j \) and the society is divided into two equal pools of potential supporters of each party. Each individual has a personal cost of voting \( c \in [0, 1] \) that is drawn from a uniform distribution. Parties decide simultaneously the amount of effort to exert in order to persuade their potential supporters to vote for them. We assume that voters receive a benefit from voting their preferred party that is strictly concave in parties efforts. More specifically, if party \( j \) exerts effort \( e_j \), the benefit of voters supporting party \( j \) is captured
by the function $\rho(e_j)$. Function $\rho(e_j) : \mathbb{R}_+ \to [0, 1]$ , is continuous for $e_j > 0$, strictly increasing, strictly concave and takes value $\rho(0) = 0$. The voting rule followed by the individuals is as follows: For a given level of effort made by party $j$, a voter that supports party $j$ and has a voting cost equal to $c$ votes for the party if and only if $\rho(e_j) \geq c$. Because of the uniformly distributed cost of voting the probability that an individual participates is $\Pr(\rho(e_j) \geq c) = \rho(e_j)$, which implies that the vote shares for each alternative are: $v_j = \frac{1}{2} \rho(e_j)$.

Abstention is given by: $v_A = 1 - v_j - v_{-j} = 1 - \frac{1}{2} [\rho(e_j) + \rho(e_{-j})]$. Notice that abstention is always decreasing in $e_j$ and $e_{-j}$.

Parties profit function is defined as:

$$\pi_j = Bs_j - e_j$$

where $B$ is the payoff to parties if they get all the seats in the parliament and $s_j$ is the percentage of seats the party obtains. We assume that the cost of effort is linear and equal to one.

3 The Benchmark Case

3.1 FSP - Fixed Size Parliament

We begin our analysis with the case of the FSP system. Party $j$ exerting effort $e_j$ obtains vote share $v_j = \frac{1}{2} \rho(e_j)$ and after the seat allocation seat share $s_j^F = \frac{\rho(e_j)}{\rho(e_j) + \rho(e_{-j})}$.

**Proposition 1.** Under FSP there exists a unique equilibrium $e^F_j(B) = e^F_{-j}(B) = e^F(B)$, where $e^F(B)$ is strictly increasing in $B$ and is the unique solution of

$$\frac{\rho(e_j)}{\rho(e_j) + \rho(e_{-j})} = \frac{B}{4}$$

**Proof.** Parties maximize profits. The optimization problem of party $j$ is:

$$\max \{ Bs_j - e_j \}$$

which is equivalent to:

$$\max \{ B \frac{\rho(e_j)}{\rho(e_j) + \rho(e_{-j})} - e_j \}$$

Taking F.O.C. with respect to effort for party $j$ we obtain that it has to hold:

$$\frac{\rho(e_{-j})\rho'(e_j)}{[\rho(e_{-j}) + \rho(e_j)]^2} = \frac{1}{B}$$

(1)
Solving the same problem for party \(-j\) we get:

\[
\frac{\rho(e_j)\rho'(e_{-j})}{\rho(e_{-j}) + \rho(e_j)^2} = \frac{1}{B}
\]  
(2)

From the above two conditions in equilibrium it has to hold that:

\[
\frac{\rho(e_{-j})\rho'(e_j)}{\rho(e_j)\rho'(e_{-j})} = 1
\]

which implies that in equilibrium:

\[
\frac{\rho'(e_j)}{\rho(e_j)} = \frac{\rho'(e_{-j})}{\rho(e_{-j})}
\]  
(3)

Notice that \(\frac{\rho'(e_j)}{\rho(e_j)}\) is strictly decreasing in \(e_j\) for all \(e_j > 0\). This is because of the monotonicity and concavity of function \(\rho(e_j)\). More specifically we have:

\[
\frac{d}{de_j} \frac{\rho'(e_j)}{\rho(e_j)} = \frac{\rho''(e_j)\rho(e_j) - \rho'(e_j)^2}{[\rho(e_j)]^2} < 0
\]

Given the above it must be that in equilibrium effort level \(e^V_{-j} = e^V_j\) and substituting in equation 1 or 2 we obtain that both parties exert the same amount of effort that satisfies the equilibrium condition:

\[
\frac{\rho'(e^V_j)}{\rho(e^V_j)} = \frac{\rho'(e^V_{-j})}{\rho(e^V_{-j})} = \frac{4}{B}
\]  
(4)

Notice that from the equilibrium condition if \(B\) increases then \(\frac{\rho'(e^V_j)}{\rho(e^V_j)}\) is decreasing. Given that this is strictly decreasing in \(e_j\), we can conclude that \(e^V_j\) is strictly increasing in \(B\).

3.2 VSP - Varying Size Parliament

Now we move to the case of the varying size parliament. The difference is the seat allocation. In case of the VSP we have \(s^V_j = v_j = \frac{1}{2}\rho(e_j)\).

**Proposition 2.** Under VSP there exists a unique equilibrium \(e^V_j(B) = e^V_{-j}(B) = e^V(B)\) that is strictly increasing in \(B\) and is the unique solution of \(\rho'(e^V_j) = \frac{2}{B}\).

The proof is straightforward. Party j solves: Max.\(\{\frac{1}{2}B\rho(e_j) - e_j\}\). Taking
F.O.C. we obtain a unique solution that has to satisfy:

\[ \rho'(e^V_j) = \frac{2}{B} \]  

(5)

Because of symmetry we would get the same solution out of the maximization problem of party \( -j \). Notice that because of the strict concavity of \( \rho(e_j) \) effort level in equilibrium is strictly increasing in \( B \).

Before comparing the two alternatives we should mention the first tool to increase turnout. This can be done by offering to parties higher benefits. So far we have seen that in our model independent of the electoral rule, effort and hence turnout is increasing in parties’ benefit.

3.3 Comparing VSP and FSP

Having analyzed separately the two electoral systems and having proved the uniqueness of equilibria in both cases, the following proposition summarizes the comparison between the two.

**Proposition 3.** \( e^V(B) > e^F(B) \) if and only if \( B > B^* \), where \( B^* \) is the unique solution of \( \rho(e^F_j(B)) = \frac{1}{2} \).

**Proof.** From equilibrium conditions 4 and 5 we get that for party \( j \) it holds that \( \rho'(e^V_j) = \frac{2}{B} \) and \( \frac{\rho'(e^F_j)}{\rho(e^F_j)} = \frac{4}{B} \). From the above two we obtain:

\[ 2\rho'(e^V_j) = \frac{\rho'(e^F_j)}{\rho(e^F_j)} \Rightarrow \rho'(e^V_j) = \frac{1}{2\rho'(e^F_j)} \]  

(6)

In order \( e^V_j > e^F_j \) and because of the concavity of \( \rho(e_j) \) it has to be that \( \frac{\rho'(e^V_j)}{\rho(e^V_j)} < 1 \Rightarrow \frac{1}{2\rho'(e^F_j)} < 1 \Rightarrow \rho(e^F_j) > \frac{1}{2} \). In the same way \( e^V_j = e^F_j \) if \( \rho(e^F_j) = \frac{1}{2} \) and \( e^V_j < e^F_j \) if \( \rho(e^F_j) < \frac{1}{2} \). As we have shown above \( e^F_j \) (and hence \( \rho(e^F_j) \)) is strictly increasing in \( B \). Hence, there will exist a unique \( B^* \) such that \( \rho(e^F_j) = \frac{1}{2} \) and \( e^V_j = e^F_j \). If \( B > B^* \) \( \Rightarrow \rho(e^F_j) > \frac{1}{2} \Rightarrow e^V_j > e^F_j \) while if \( B < B^* \) \( \Rightarrow \rho(e^F_j) < \frac{1}{2} \) and \( e^V_j < e^F_j \).

From the above proposition we conclude that which electoral system will imply higher effort depends on the level of \( B \). For our model the threshold value of \( B^* \) is the one that implies a fifty percent participation in a FSP election. Hence, according to our results in elections that turnout is higher than half of
the population the introduction of a VSP would boost turnout even further. On
the other hand, a VSP should not be used as an instrument to increase turnout
in cases that participation under the existing system is lower than fifty percent.
In such case the VSP would decrease turnout even further.

4 Introducing a quota \( q > 0 \)

4.1 FSP

So far we have studied the effect of introducing the VSP in terms of participation
and effort. As we have shown the VSP may result in lower levels of effort and
participation if the benefit parties obtain by being in parliament is low. In this
section we study which is the effect of introducing a quota in the standard fixed
size parliament system. In case that the quota is not satisfied the election is
not valid and candidates obtain zero profits. On the other hand voters follow
as before a simple expressive voting rule not taking in consideration possible
future costs of an invalid election. Before the equilibrium characterization a
formal definition of upper and lower thresholds of quota level follows.

Definition 1. Let \( q^F(B) \) be the unique solution of \( q^F = \rho(e^F(B)) \) and \( \overline{q}^F(B) \)
the unique solution of \( \overline{q}^F = \rho\left(\frac{B}{2}\right) \). Moreover, let \( e^{F,q}(q) \) be the unique solution
of \( \rho(e^{F,q}(q)) = q \).

Notice that the lower threshold of the quota \( q^F(B) \) is the equilibrium level of
participation in the case without a quota. The upper threshold \( \overline{q}^F(B) \) is defined
as the level of participation when each party exerts effort equal to the benefit
of obtaining half of the seats in the parliament and zero profits. Finally, \( e^{F,q} \) is
the effort level required by each party in order to fulfill half of the quota.

Proposition 4. For all \( B \) and \( q \) there exists a unique symmetric pure strategy
Nash equilibrium characterized as follows:

1. \( e^{F,q}_j(B, q) = e^{F,q}_{-j}(B, q) = e^F(B) \) if \( q < q^F \)
2. \( e^{F,q}_j(B, q) = e^{F,q}_{-j}(B, q) = e^{F,q}(q) \) if \( q^F \leq q < \overline{q}^F \)
3. \( e^{F,q}_j(B, q) = e^{F,q}_{-j}(B, q) = 0 \) if \( q \geq \overline{q}^F \)

Moreover there exist multiple asymmetric equilibria if the following condi-
tions hold:
\( e_j > e_{-j} \) such that \( v_j(e_j) + v_{-j}(e_{-j}) = q \) is an equilibrium if \( e_j < \frac{B\rho(e_j)}{2q} \) and \( \rho'(e_{-j}) \leq \frac{q^2}{B\rho(e_j)} \).

(Proof can be found in the Appendix)

The above proposition suggests that introducing a quota that is lower than the non-quota equilibrium participation level clearly has no effect on effort and participation. In case that the quota introduced is not satisfied by the non-quota equilibrium level of turnout (i.e. the quota is effective), parties exert higher effort and participation will be higher such that the quota in equilibrium is binding. Notice that an effective quota (i.e. \( q \geq \rho(e) \)) always results in higher levels of effort and hence participation unless the quota is too high (i.e. \( q \geq q^F \)). If the later is the case parties do not exert any effort since this would imply negative profits.

Claim 1. The set \([q^F(B), \overline{q}^F(B)]\) is non empty.

The proof of the above claim is straightforward. As we have shown in the proof of proposition 4, \( e^F(B) \leq \frac{B}{4} \) which implies that the set \( [\rho(e^F(B)), \rho(\frac{B}{2})] \) is non-empty. This result tells us that if we are interested in increasing effort and participation this can always be done by introducing a quota. The quota has to be chosen accordingly so that it is not too high. Hence, in contrast with the discussion of VSP versus FSP now we obtain that a quota can always increase effort and participation.

Claim 2. Both \( q^F(B) \) and \( \overline{q}^F(B) \) are increasing in \( B \).

As we have shown participation in FSP is increasing in the benefit parties obtain. This implies that the lower threshold will be larger in cases of higher benefit. Moreover the higher is the benefit parties obtain the larger is the potential effect of a quota.

4.2 VSP

Before analyzing the case of a VSP and a quota a similar definition as in the case of a FSP follows.

Definition 2. Let \( q^V(B) \) be the unique solution of \( q^V = \rho(e^V(B)) \) and \( \overline{q}^V(B) \) the unique solution of \( \overline{q}^V = \rho(\frac{B}{2}) \). Moreover, let \( e^{V,q}(q) \) be the unique solution of \( \rho(e^{V,q}(q)) = q \).
Again, the lower threshold is defined as the non-quota VSP participation level. The upper threshold is the case that parties exert effort equal to $B/2$ and make zero profits. Finally, $e_{V,q}(q)$ is the effort level required by each party competing in a VSP in order to fulfill half of the quota.

**Proposition 5.** For all $B$ and $q$ there exists a unique symmetric equilibrium characterized as follows:

1. $e_{V,q}^{V,q}(B,q) = e_{V,q}^{V,q}(B,q) = e^V(B)$ if $q < q^V$
2. $e_{V,q}^{V,q}(B,q) = e_{V,q}^{V,q}(B,q) = e^V(q)$ if $q^V \leq q < q^V$
3. $e_{V,q}^{V,q}(B,q) = e_{V,q}^{V,q}(B,q) = 0$ if $q \geq q^V$

Moreover there will exist multiple asymmetric equilibria satisfying the following: $e_j > e_{-j} \geq e^V(B)$ such that $v_j(e_j) + v_{-j}(e_{-j}) = q$ and $e_j < \frac{B}{2}$.

(Proof can be found in the Appendix)

**Claim 3.** Both $q^V(B)$ and $q^V(B)$ are increasing in $B$.

As we see from proposition 5 and claim 3 the equilibria under both a VSP and a FSP are of exactly the same kind. The intuition of our results is the same as in the case of a FSP.

**Claim 4.** In case of a VSP in all asymmetric equilibria party $-j$ makes higher profits than party $j$ (free riding)

In case of a VSP there exist multiple equilibria as described in proposition 5. This equilibria describe a situation in which an effective quota is introduced and one of the parties exerts higher effort than the other. The lower party’s effort is equal or higher than in case of a non-quota VSP. The party performing higher effort has to fulfill the rest of the quota in order to avoid a new election. The quota is always binding and the party exerting lower effort free rides and makes higher profits.

4.3 Comparing FSP and VSP with quota

As we have seen so far the quota-equilibria structure is the same both in case of a FSP and a VSP. In both cases if the quota is lower than the non-quota participation level then the equilibrium is the same as in the case of no quota. If the quota is higher than the non-quota participation level (i.e effective) then parties increase their effort level such that in equilibrium the quota is binding. An effective quota is never slack in equilibrium. In this way independent of $B$
the designer can actually decide and impose the participation level in equilibrium. The only problem arises in case that the quota is higher than the upper threshold. This is because if parties try to fulfill such a quota this implies negative profits and hence they rather not exert any effort at all. In both case of a VSP or a FSP the upper threshold is increasing in $B$. This implies that the designer can impose higher levels of turnout in elections that parties obtain high benefits for being in office.

**Claim 5.** $q^V(B) > q^F(B)$ if and only if $B > B^*$. Moreover, $\bar{q}^F = \bar{q}^V$ for all $B$’s.

The above claim compares the lower and upper thresholds of the set of efficient quotas. The upper threshold is the same under both electoral systems. On the other hand, when comparing between the lower thresholds, since they are defined as the non-quota equilibrium turnout level which one is higher depends on $B$. In case that $B$ is higher than $B^*$ as we have shown in proposition 3 this implies higher participation in case of a VSP and hence a higher lower threshold for the VSP case.

## 5 Conclusion

In this paper we studied the effect of different instruments in order to increase turnout in PR elections. We didn’t enter into an analysis of whether low or high turnout is more efficient in terms of information aggregation, welfare etc. Neither we targeted in comparing existing electoral systems in terms of turnout. What we hope to have shown are two different and very simple ways of increasing turnout in existing proportional representation systems.

First we studied the case of a varying size parliament. This system increases turnout in case that parties obtain high benefit for being in parliament. This tool could be used in cases that participation is high in order to boost it even further. Under no means should be introduced as a solution to low participation since it would decrease it even further.

The second instrument we introduced is a quota of participation. As we have shown there always exists the possibility of increasing turnout by introducing a

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5Iaryczower and Mattozzi (2009) show that campaign spending in PR system is lower than in majoritarian systems. & Myerson (1999) offers a game theoretic comparison among electoral systems.
quota. In contrast to a VSP the designer can introduce a quota and determine turnout independent of the benefit parties obtain.

The quota system has a more drastic effect than the VSP. This comes from the fact that in the fear of a repeated election parties are forced to fulfill the quota introduced by the designer. In both tools studied in case of higher turnout the common element is that parties’ profits are lower.

Finally, we have shown that participation is increasing in the benefits parties obtain for being in parliament. Hence, this is a third alternative to increase turnout but the mechanism through which participation is higher is very different.

All the above results of course are related to the assumptions of our model and the symmetric setup we have analyzed. For example, the society is divided into two equal pools of supporters, there exists no party with an advantage in terms of the effect of their mobilization effort etc. This may seem as a drawback but we believe that it has a nice feature. It shows the pure effect of the tools suggested above even in the simplest structures of the society. We believe that even in more complicated cases the results would be similar and this may be the first step for extending the existing work.
Appendix

Proof of Proposition 4

We analyze all possible Nash equilibria. Those can belong in the following categories:

Symmetric: \( e_j = e_{-j} \) such that:

1. \( v_j + v_{-j} = q \Rightarrow \rho(e_j) + \rho(e_{-j}) = 2q \)
2. \( v_j + v_{-j} > q \Rightarrow \rho(e_j) + \rho(e_{-j}) > 2q \)
3. \( v_j + v_{-j} < q \Rightarrow \rho(e_j) + \rho(e_{-j}) < 2q \)

Asymmetric: \( e_j > e_{-j} \) such that:

4. \( v_j + v_{-j} = q \Rightarrow \rho(e_j) + \rho(e_{-j}) = 2q \)
5. \( v_j + v_{-j} > q \Rightarrow \rho(e_j) + \rho(e_{-j}) > 2q \)
6. \( v_j + v_{-j} < q \Rightarrow \rho(e_j) + \rho(e_{-j}) < 2q \)

In general our approach is the following. For each pair of levels of effort \((e_j, e_{-j})\) we have to guarantee that there are no incentives for any of the parties to deviate. Hence, if effort level \(e_j\) is part of a Nash equilibrium it has to be the solution of the (constrained) maximization problem of party \(j\) given the effort of the opponent \(e_{-j}\). Moreover, we have to guarantee that each effort level being part of a Nash equilibrium implies positive profits given that the deviation \(e_j = 0\) implying zero profits is always available.

Cases that the quota is slack

Case 2 - Symmetric - \( e_j = e_{-j} = e \)

Given \(e_{-j} = e\) the maximization problem of party \(j\) is:

\[
\text{Max.}\{ B \frac{\rho(e_j)}{\rho(e_j) + \rho(e)} - e_j \} \text{ subject to } \rho(e_j) + \rho(e) \geq 2q
\]

The Lagrangian of this problem is:

\[
L = B \frac{\rho(e_j)}{\rho(e_j) + \rho(e)} - e_j + \lambda[\rho(e_j) + \rho(e) - 2q]
\]

Taking F.O.C. with respect to effort we obtain:

\[
L_{e_j} = B \frac{\rho'(e_j)\rho(e)}{[\rho(e_j) + \rho(e)]^2} - 1 + \lambda\rho'(e_j)
\]

Given the quota is slack \(\lambda = 0\). Hence, \(e_j = e\) must satisfy:

\[
B \frac{\rho'(e)\rho(e)}{[\rho(e) + \rho(e)]^2} - 1 = 0 \implies \frac{\rho'(e)}{\rho(e)} = \frac{4}{B} \implies e = e^F
\]

In order the constraint to be slack it must be that \(\rho(e^F) > q \implies q^F > q\).

In order to guarantee positive profits it must be that \(e^F < \frac{B}{2}\). Notice that
because of the concavity of $\rho$ this is always true since it holds that $\frac{\rho'(e^F)}{\rho(e^F)} \leq \frac{1}{e^r} \implies \frac{4}{\rho} \leq \frac{1}{e^r} \implies e^F \leq \frac{B}{\rho}$. 

The same would hold for party $-j$.

Case 5 - Asymmetric

e_j > e_{-j}$ and the condition slack can not be an equilibrium. As before for $\lambda = 0$ we would get that for party $j$ it should hold that: $B \frac{\rho'(e_j)\rho(e_{-j})}{\rho(e_j)+\rho(e_{-j})} - 1 = 0$

in the same way for party $-j$ it should hold that: $B \frac{\rho'(e_{-j})\rho(e_j)}{\rho(e_{-j})+\rho(e_j)} - 1 = 0$

combining the above two it has to be true that $\frac{\rho'(e_j)}{\rho(e_j)} = \frac{\rho'(e_{-j})}{\rho(e_{-j})}$ which as we have shown is true only for $e_j = e_{-j}$.

Cases that the quota is binding

Case 1 - Symmetric - $e_j = e_{-j} = e^{F,q}$

Given effort $e_{-j} = e^{F,q}$ the maximization problem of party $j$ is:

$\text{Max.} \{B \frac{\rho(e_j)}{\rho(e_j)+q} - e_j\} \text{ subject to } \rho(e_j) \geq q$

The Lagrangian of this problem is:

$L = B \frac{\rho(e_j)}{\rho(e_j)+q} - e_j + \lambda \left[ \rho(e_j) - q \right]$

Taking F.O.C. with respect to effort we obtain:

$L_{e_j} = B \frac{\rho'(e_j)q}{[\rho(e_j)+q]^2} - 1 + \lambda \rho'(e_j)$

If $e_j = e^{F,q} > 0$ is the solution it must hold that $B \frac{\rho'(e^{F,q})q}{4q} - 1 + \lambda \rho'(e^{F,q}) = 0$.

Moreover given that the constraint is binding it has to be that $\lambda \geq 0$ which is true if $\rho'(e^{F,q}) \leq \frac{4q}{B} \implies \frac{\rho'(e^{F,q})}{\rho(e^{F,q})} \leq \frac{4}{B} \implies \frac{\rho'(e^{F,q})}{\rho(e^{F,q})} \leq \frac{\rho'(e^F)}{\rho(e^F)} \implies e^{F,q} \geq e^F \implies \rho(e^{F,q}) \geq \rho(e^F) \implies q \geq q^F$. Finally it has to be that profits are positive which holds if $e^{F,q} < \frac{B}{2} \implies q < \bar{q}^F$. The same would apply for party $-j$.

Case 4 - Asymmetric

Given effort $e_{-j}$ the maximization problem of party $j$ is:

$\text{Max.} \{B \frac{\rho(e_j)}{\rho(e_j)+\rho(e_{-j})} - e_j\} \text{ subject to } \rho(e_j) + \rho(e_{-j}) \geq 2q$

The Lagrangian of this problem is:

$L = B \frac{\rho(e_j)}{\rho(e_j)+\rho(e_{-j})} - e_j + \lambda_j \left[ \rho(e_j) + \rho(e_{-j}) - 2q \right]$

Taking F.O.C. with respect to effort we obtain:

$L_{e_j} = B \frac{\rho'(e_j)\rho(e_{-j})}{\rho(e_j)+\rho(e_{-j})^2} - 1 + \lambda_j \rho'(e_j)$

Given the quota is binding it must be that $\lambda_j \geq 0 \implies \rho'(e_j) \leq \frac{4q^2}{B\rho(e_{-j})}$

Doing the same for party $-j$ we obtain that $\lambda_{-j} \geq 0$ if $\rho'(e_{-j}) \leq \frac{4q^2}{B\rho(e_j)}$.

Notice that since $e_j > e_{-j}$ it holds that $\rho'(e_j) < \rho'(e_{-j})$ and $\frac{4q^2}{B\rho(e_j)} < \frac{4q^2}{B\rho(e_{-j})}$ hence $\rho'(e_{-j}) \leq \frac{4q^2}{B\rho(e_{-j})}$ is sufficient to guarantee that both $\lambda_j$ and $\lambda_{-j}$ are positive.

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In order to guarantee positive profits for both parties it must hold that
\[ e_j < \frac{B \rho(e_j)}{2q} \] and \[ e_{-j} < \frac{B \rho(e_{-j})}{2q} \]. Notice that since \( e_j > e_{-j} \) it holds that \( \frac{\rho(e_j)}{e_j} < \frac{\rho(e_{-j})}{e_{-j}} \). Hence, \( e_j < \frac{B \rho(e_j)}{2q} \) is sufficient to guarantee that both parties make positive profits.

Cases not satisfying the quota

Case 3 - Symmetric

\( e_j = e_{-j} = e > 0 \) can not be an equilibrium. The quota is not satisfied and parties exerting positive effort implies negative profits \( \pi_j = -e < 0 \). Hence \( (e, e) \) can not be an equilibrium given that party \( j \) has incentive to deviate to zero effort and zero profits respectively.

\[ e_j = e_{-j} = 0 \]

Given \( e_{-j} = 0 \) for whatever positive effort party \( j \) obtains all the seats in the parliament if the quota is fulfilled. The constrained maximization problem of party \( j \) is: \( \text{Max.} \{ B - e_j \} \) subject to \( \rho(e_j) \geq 2q \). The solution of this problem is \( e_j^{2q} \) such that \( \rho(e_j^{2q}) = 2q \). Finally we have to guarantee that profits are positive which is the case if \( e_j^{2q} < B \). If the later inequality doesn’t hold then \( e_j = 0 \).

Case 6 - Asymmetric

\( e_j > e_{-j} \) and the quota not satisfied can not be an equilibrium given that \( e_j > 0 \) and quota not satisfied implies negative profits for party \( j \) and hence incentive to deviate to \( e_j = 0 \)

Summing up the cases regarding symmetric equilibria we have obtained:

1. \( e_j = e_{-j} = e \) such that \( \rho(e) = q \) is an equilibrium if \( q^F \leq q < q^F \)
2. \( e_j = e_{-j} = e \) such that \( \frac{\rho(e)}{\rho(e + e)} = \frac{4}{B} \) is an equilibrium if \( q < q^F \)
3. \( e_j = e_{-j} = 0 \) is an equilibrium if \( e_j^{2q} \geq B \) where \( e_j^{2q} \) is the unique solution of \( \rho(e_j^{2q}) = 2q \)

Finally if \( q \geq q^F \) then \( e_j^{2q} > B \) which is the condition for case 3. This is because
\[
q \geq q^F \implies e_j^{2q} \geq \frac{q}{2} \implies e_j^{2q} > B \quad \text{[since \( e_j^{2q} > e_j^{2q} \implies \frac{\rho(e_j^{2q})}{\rho(e_j^{2q})} < \frac{\rho(e_j^{2q})}{\rho(e_j^{2q})} \implies \frac{2q}{e_j^{2q}} < \frac{q}{e_j^{2q}} \implies e_j^{2q} < \frac{q}{2} \text{ and combining with } e_j^{2q} > B \text{ we obtain that } e_j^{2q} > B]}.
\]

Proof. of Proposition 5

We follow the same approach as in the proof of proposition 4.

Cases that the quota is slack
Case 2 - Symmetric - $e_j = e_{-j} = e$

Given effort $e_{-j} = e$ the maximization problem of party $j$ is:

$$\text{Max.} \{ \frac{1}{2} B \rho(e_j) - e_j \} \text{ subject to } \rho(e_j) + \rho(e) \geq 2q$$

The Lagrangian of this problem is:

$$L = \frac{1}{2} B \rho(e_j) - e_j + \lambda [\rho(e_j) + \rho(e) - 2q]$$

Taking F.O.C. with respect to effort we obtain:

$$L_{e_j} = \frac{1}{2} B \rho'(e_j) - 1 + \lambda \rho'(e_j)$$

Given the quota is slack $\lambda = 0$. Hence, $e_j = e$ must satisfy:

$$\rho'(e_j) = \frac{2}{B} \implies e_j = e^V$$

In order for the constraint to be slack it must be that $\rho(e^V) > q \implies q^V > q$. In order to guarantee positive profits it must be that $e^V < \frac{B}{2}$. Notice that because of the concavity of $\rho$ it holds that $\rho'(e^V) \leq \frac{\rho(e^V)}{e^V} \implies \rho'(e^V) \leq \frac{1}{e^V} \implies e^V \leq \frac{B}{2}$.

The same applies for party $-j$.

Case 5 - Asymmetric

$e_j > e_{-j}$ and the condition slack can not be an equilibrium. As before for $\lambda = 0$ we would get that for party $j$ it should hold that: $\rho(e_j) = \frac{2}{B}$ in the same way for party $-j$ it should hold that: $\rho(e_{-j}) = \frac{2}{B}$. Combining the two it has to be true that $\rho(e_{-j}) = \rho(e_{-j})$ which because of the strict concavity of $\rho(e_j)$ is true only for $e_j = e_{-j}$.

Cases that the quota is binding

Case 1 - Symmetric $e_j = e_{-j} = e^{V,q}$

Given effort $e_{-j} = e^{V,q}$ the Lagrangian of this problem is:

$$L = \frac{1}{2} B \rho(e_j) - e_j + \lambda [\rho(e_j) + \rho(e^V) - 2q]$$

Taking F.O.C. with respect to effort we obtain:

$$L_{e_j} = \frac{1}{2} B \rho'(e_j) - 1 + \lambda \rho'(e_j)$$

If $e_j = e^{V,q} > 0$ is the solution it must hold that $\frac{1}{2} B \rho'(e^{V,q}) - 1 + \lambda \rho'(e^{V,q}) = 0$

Moreover given that the constraint is binding it has to be that $\lambda \geq 0$ which is true if $\rho'(e^{V,q}) \leq \frac{2}{B} \implies \rho'(e^{V,q}) \leq \rho'(e^V) \implies \rho(e^{V,q}) \geq \rho(e^V) \implies q \geq q^V$.

Finally it has to be that profits are positive which holds if $e^{V,q} < \frac{B}{2} \implies q < q^V$. The same applies for party $-j$.

Case 4 - Asymmetric - $e_j > e_{-j}$

Solving as in case 1 we obtain for party $j$ that it has to hold $\frac{1}{2} B \rho'(e_j) - 1 + \lambda_j \rho'(e_j) = 0$. Given the quota is binding it must be that $\lambda_j \geq 0 \implies \rho'(e_j) \leq \frac{2}{B} \implies e_j \geq e^V$. Doing the same for party $-j$ we obtain that $e_{-j} \geq e^V$. In
order to guarantee positive profits for both parties given that $e_j > e_{-j}$ it is sufficient that $e_j < \frac{B}{2}$.

Cases not satisfying the quota

Case 3 - Symmetric

$e_j = e_{-j} = e > 0$ can not be an equilibrium as already shown in proof of proposition 4.

$e_j = e_{-j} = 0$

As in the proof of proposition 4 in order party $j$ not to have incentives to deviate to $e_j \geq e^{2q}$ it has to be that $e_j^{2q} \geq B$.

Case 6 - Asymmetric

$e_j > e_{-j}$ and the quota not satisfied can not be an equilibrium given that $e_j > 0$ and quota not satisfied implies negative profits for party $j$ and hence incentive to deviate to $e_j = 0$

Summing up the cases regarding symmetric equilibria we have obtained:

1. $e_j = e_{-j} = e^{V,q}$ such that $\rho(e^{V,q}) = q$ is an equilibrium if $\frac{q}{2} \leq q < \overline{q}$

2. $e_j = e_{-j} = e^{V}$ such that $\rho(e^{V}) = \frac{2}{B}$ is an equilibrium if $q < \frac{q}{2}$

3. $e_j = e_{-j} = 0$ is an equilibrium if $e_j^{2q} \geq B$ where $e_j^{2q}$ is the unique solution of $\rho(e_j^{2q}) = 2q$

Finally, in the same as shown in case of proposition 4 if $q \geq \overline{q}$ then $e_j^{2q} > B$ and the condition for case 3 holds. \qed
References


