# Ranking Multidimensional Alternatives and Uncertain Prospects*

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**Abstract**

We introduce a two-stage ranking of multidimensional alternatives, including uncertain prospects as particular case, when these objects can be given a suitable matrix form. The first stage defines a ranking of rows and a ranking of columns, and the second stage ranks matrices by applying natural monotonicity conditions to these auxiliary rankings. Owing to the theory of additive separability developed here, this framework is sufficient to generate very precise numerical representations. We apply them to four types of multidimensional objects: (1) streams of commodity baskets through time, (2) uncertain social prospects, (3) uncertain individual prospects, and (4) monetary input-output matrices. Application (1) enters the paper mostly as an illustration, and the main results of the paper concern the other three. In application (2), we prove the strongest existing form of Harsanyi's (1955) Aggregation Theorem and cast light on the comparison between the ex ante and ex post Pareto principle, when expected utility assumptions do not hold. In application (3), we provide a novel derivation of subjective probability similar to Anscombe and Aumann (1963). Lastly, application (4) delivers a numerical measure of economic integration.

## 1 Introduction and overview

Consider the classic problem in consumer theory, i.e., to define a preference over intertemporal consumption plans ranging over several goods. A convenient way to tackle this problem is to begin with two sets of simpler preferences, the first set comprising of preferences defined on time sequences of consumption for each given good, and the second set comprising of preferences defined on goods baskets for each given time period. Then, an overall ranking of consumption plans

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will follow from aggregating the information contained in these two auxiliary rankings.

Now suppose that a social observer wants to compare social prospects, which allocate money across both individuals and states of the world. This can be dealt with as before, by first supposing two sets of simpler preferences, and then aggregating them. Here, one set is obtained by fixing the individual and letting the states vary, and the other, by fixing the state and letting the individuals vary. Put differently, the observer will judge the social prospects both from an *ex ante* individual perspective and from an *ex post* social perspective, and his final preferences will take these two sets of judgments into account.

Changing the model again, suppose that an individual decision-maker wants to compare uncertain prospects, which are defined by their consequences in each state of the world, while these consequences themselves are risky lotteries on a set of final outcomes. The same technique leads one to introduce two distinct sets of preferences over the probabilities of outcomes - the first set being obtained by fixing the state, and the second one, by fixing the outcome - and then define preferences over uncertain prospects by aggregating these two sets.

In the previous cases, the objects are practical alternatives, and preferences must be defined between them, but in related settings, multidimensional quantities are compared more abstractly. Suppose that a statistician aims at comparing national economies in terms of the extent to which they integrate their different sectors, and for this purpose, relies on their monetary input-output tables. (We assume the tables involve the same list of sectors for all economies, and the transactions of a larger economy are made commensurable with those of a smaller one by dividing the raw figures by a size factor.) Taking the problem as before, the statistician would define two sets of integration rankings, one in which sectors stand as sellers, and the other in which they stand as purchasers, and then synthesize these partial comparisons into a final ranking. Thus, the framework of this paper applies to information-processing, as well as preference construction.

In this article, we develop an aggregative theory to formalize the method and account for all four examples at one go. The first example is mentioned primarily for didactic purposes, and our real stake lies with the remaining three. Initially, the theory was meant only for the second example, having been motivated by earlier work by the first author on social choice under uncertainty,[1] but it proved easy and rewarding to state it in fuller generality, so as to cover more applications. This said, social choice under uncertainty looms large in what follows, being an area to which the theory applies especially well.

Before proceeding, we briefly sketch the main technical ideas of the paper. In general, the objects of preference or ranking take the form of *matrices of real numbers*, with the indexes of rows and columns representing two qualitatively different attributes of these objects. To take more than two attributes into consideration, it is enough to increase the number of rows, columns, or both; thus, states of nature may be introduced into the intertemporal choice application,

[1] See the unpublished paper by Blackorby et al. (2004). A comparison follows Theorem 3.

and multiple commodities or time periods into the applications to uncertain prospects. More subtly, it could happen that the attributes exhibit some kind of logical interdependency. In this case, the matricial form of the alternatives would be inappropriate.[2]

We assume that objects are ranked as follows. Each row index generates a ranking of those rows which are feasible given that index. Likewise, each column index generates a ranking of the feasible columns for that index. The overall ranking of feasible matrices takes these auxiliary rankings into account by monotonically increasing with them, i.e., if two matrices differ only in one row, and one matrix has this row ranked above the corresponding row of the other, then the first matrix is higher than the second in the overall ranking. The same holds for columns instead of rows. By a further monotonicity condition, two matrices that differ in only one coordinate (ie. row-column pair), are ranked as the numbers in that coordinate; this fixes the direction of the overall ranking in another way. These three axioms — called Row Preferences, Column Preferences and Coordinate Monotonicity —often become familiar and plausible conditions once the application context is fixed. In the intertemporal choice problem, with the matrix components representing dated quantities of goods, the axioms are very standard dominance or monotonicity conditions. In the uncertain social choice problem, with the matrix components representing state-dependent utility values, they translate into statewise dominance conditions at the individual level and unanimity-preservation (Pareto) conditions at the social level. In the individual decision making problem, they boil down to dominance again. We also impose Continuity on the overall ranking.

Under technical assumptions to be spelled out below, the four conditions together deliver a representation theorem of a classic format: the overall ranking of matrices can be represented by a *fully additively separable* value function, i.e., a sum of value functions defined for each coordinate (Proposition 2). This functional form was axiomatized by Debreu (1960) and Gorman (1968b), and it has since then pervaded microeconomic theory (see Blackorby, Primont, and Russell, 1978) and multiattribute decision theory (see Fishburn, 1970, Keeney and Raiffa, 1976, Wakker, 1989). However, it is not obtained here in exactly the same way as in these works. They assume that the ranking of vector-valued alternatives is totally separable —roughly speaking, defined componentwise — but we *deduce* this property from the aforementioned monotonicity axioms. Also, we take account of feasibility constraints by relaxing the assumption, made by both Debreu and Gorman, that the set of alternatives is a full Cartesian product.

Another important connection is with the early microeconomic literature on *consistent aggregation* (see Green (1964), and van Daal and Merkies (1984, 1988) for surveys; the pioneering result is due to Nataf (1948)). Once translated into numerical representations, our four axioms are seen to entail consistent aggregation, which is known from this literature to entail additive separability. However, our theory goes farther by dispensing with two unpalatable assump-

[2]However, it is possible that a finer description of the attributes make them suitably independent. For a discussion, see Keeney (1981).

tions of these earlier works: first, the Cartesian product structure, and second, the differentiability of numerical representations. The latter assumption precludes one from stating a proper axiomatic basis since it has no counterpart at the preference level.

As it reexpresses the two-stage analysis of the four examples, Proposition 2 shows that, for all its naturalness, this analysis is constraining and sometimes undesirable. Depending on the applications, it can be seen to deliver either a positive characterization or an impossibility theorem. The same ambivalence underlies the main results of the paper, Theorem 3, Theorem 5, and Proposition 6, to be described now.

These results need more axioms and structure, and in particular, require the overall ranking to be invariant between rows (Row Invariance), or between columns (Column Invariance), or both at the same time. With these additional assumptions, Theorem 3 strengthens the additively separable representation of Proposition 2 into a *weighted sum of value functions*, where the value functions may differ only across columns, or only among columns, or not at all, depending on the chosen invariance condition(s). We apply Theorem 3 to uncertain social choice, taking the numbers in the matrices to be utility values rather than physical quantities; this *welfarist* interpretation makes our formalism very effective to handle issues in normative economics. As is well-known, when social alternatives are uncertain, the Pareto principle can have two forms, either *ex ante* or *ex post*, and the question arises whether they can be made compatible. This has been debated in welfare economics (Hammond (1981)), in moral philosophy by (Broome (1991)), and in axiomatic decision theory (Mongin (1995)). The widespread answer is that the two forms of the Pareto principle are compatible only if the individuals' and the social observer's *ex ante* preferences obey stringent restrictions. However, this conclusion depends on the prior assumption that the individuals and the social observer satisfy the axioms of expected utility theory, and little is known on the compatibility problem when this major assumption is relaxed. Because the decision-theoretic properties encapsulated in our axioms are so weak, Theorem 3 shows what happens in this case. Somewhat shockingly, the conclusion remains negative: the same stringent conditions are necessary to achieve compatibility between *ex ante* and *ex post* Pareto.

A related connection is with Harsanyi's (1955) Aggregation Theorem, which states that a Paretian and von Neumann-Morgenstern aggregate of individual von Neumann-Morgenstern utility functions is a weighted sum of these utility functions. Viewed in this light, Theorem 3 is a generalization that replaces Harsanyi's von Neumann-Morgenstern assumptions by mere dominance conditions. In the end, von Neumann-Morgenstern theory turns out to indispensable, because our theorem *deduces* it at the same time as the weighted sum rule, so this is another ambivalent finding. On the one hand, we reinforce Harsanyi's intriguing argument for utilitarianism; on the other, we establish once and for all that his argument cannot live outside of the narrow framework of some form of expected utility decision theory.

Theorem 5 relies on a different trade-off in assumptions. It weakens the domain assumptions of Proposition 2 and Theorem 3, and in exchange, it reinforces the ranking conditions by combining dominance with *betweenness*. This condition emerged in discussions of non-expected utility theory as an attractive stopping place, because, like von Neumann-Morgenstern independence, it entails linear indifference curves, but unlike it, permits these curves not to be parallel (see Chew, 1983, and Dekel, 1986). In the conclusions of Theorem 5, the ranking of matrices is represented by a *twice weighted sum* of numbers, with one set of weights holding for rows and the other for columns. At this stage, straight linearity has replaced the additively separable representations. Theorem 5 completes the discussion of *ex ante* and *ex post* forms of the Pareto principle by reconciling them at a still very stringent price, and when compared with Harsanyi's Aggregation Theorem, it provides another generalization, in which the full von Neumann-Morgenstern theory is now replaced with dominance plus betweenness (and this axiomatic choice licences the relatively weak domain assumptions).

Theorem 5 also casts light on individual decision theory, through yet another interpretation of the rows and columns of alternatives; this is the third example above. Matrices now become *mixed prospects* in the sense of Anscombe and Aumann (1963) —i.e., prospects that associate states of nature with von Neumann-Morgenstern lotteries. Theorem 5 then provides a derivation of subjective probability from preferences under uncertainty. Its novelty lies with the weak assumptions. We require the induced preference over lotteries to satisfy only dominance and betweenness, not the whole of von Neumann-Morgenstern theory, as Anscombe and Aumann do, and we take feasibility constraints into account, which they did not do, since they only consider what is in effect a Cartesian product of prospects.

Our final result, Proposition 6, extends Proposition 2 to tackle the problem of measuring economic integration, as in the fourth example above. Since the statistician is likely to work with *normalized* monetary input-output data, we face a constraint that was not present in the other applications. As we however show, the problem can be circumvented, and the additively separable representation that emerges clarifies the sense in which our axioms, when interpreted in terms of economic integration, entail a numerical measurement for this concept, unlike in current production analysis, where it is captured by discrete criteria. This is a possibility result, and it confirms the wide expressive power of the framework proposed here.

# 2 Basic framework, with an application to intertemporal choice

We fix two sets of indexes, $N := \{1, \ldots, i, ..., n\}$ and $M := \{1, \ldots, j, ..., m\}$, with $n, m \geq 2$, in order to represent the relevant attributes of the objects to be ranked. These are identified with bundles of quantities $x^i_j$ for all $(i, j) \in N \times M)$, which we analyze as follows. First, we define an *alternative* $\mathbf{X}$ to be an element of the Cartesian product $\mathbb{R}^{N\times M}$. We will usually write $\mathbf{X}$ in matrix form, i.e., $\mathbf{X} = [x^i_j]^{i\in N}_{j\in M}$ , but sometimes also as a vector of rows or as a vector columns, i.e.,

$$\mathbf{X} = (\mathbf{x}^1, \mathbf{x}^2, \ldots, \mathbf{x}^n) \text{ and } \mathbf{X} = (\mathbf{x}_1, \mathbf{x}_2, \ldots,\ \mathbf{x}_m),$$

where, for each $j \in M$, $\mathbf{x}_j := [x^i_j]^{i\in N}$, an element of $\mathbb{R}^N$, and for each $i \in N$, $\mathbf{x}^i := [x^i_j]_{j\in M}$, an element of $\mathbb{R}^M$.

Second, we assume that feasibility constraints restrict the set of alternatives. For technological reasons, it may be impossible to realize all and every distribution of goods through time periods or amongst individuals; for economic reasons, some distributions of money among individuals may be excluded in some states of the world, and so on. To cover many cases at once, we take the *set of feasible alternatives* to be an open, connected subset $\mathcal{X} \subseteq \mathbb{R}^{N\times M}$. This is in line with some advanced utility-theoretic literature (Segal, 1992; Chateauneuf and Wakker, 1993). The next sections will introduce more restrictions on $\mathcal{X}$. We assume that only feasible alternatives can be compared or need comparing, and thus introduce an order $\succeq$ on $\mathcal{X}$ rather than $\mathbb{R}^{N\times M}$. Define $\mathcal{X}^i := \{\mathbf{x}^i; \mathbf{X} \in \mathcal{X}\}$, for all $i \in N$. Define $\mathcal{X}_j := \{\mathbf{x}_j;\ \ \mathbf{X} \in \mathcal{X}\}$, for all $j \in M$. The following axioms will be maintained throughout on $\succeq$.

CONTINUITY**:** The order $\succeq$ is *continuous*, i.e., its upper and lower contour sets are closed subsets of $\mathcal{X}$.

ROW PREFERENCES**:** For all $i \in N$, there is an order $\succeq^i$ on $\mathcal{X}^i$ such that, for all $\mathbf{X},\ \mathbf{Y} \in \mathcal{X}$, and all $i \in N$, if $\mathbf{x}^h \approx^h\ \ \mathbf{y}^h$ for all $h \in N \setminus \{i\}$, then $\mathbf{X} \succeq\ \mathbf{Y}$ if and only if $\mathbf{x}^i \succeq^i\ \mathbf{y}^i$.

COLUMN PREFERENCES**:** For all $j \in M$, there is an order $\succeq_j$ on $\mathcal{X}_j$ such that, for all $\mathbf{X},\ \mathbf{Y} \in \mathcal{X}$, and all $j \in M$, if $\mathbf{x}_k \approx_k\ \ \mathbf{y}_k$ for all $k \in M \setminus \{j\}$, then $\mathbf{X} \succeq\ \mathbf{Y}$ if and only if $\mathbf{x}_j \succeq_j\ \mathbf{y}_j$.

COORDINATE MONOTONICITY: For all $i \in N$ and $j \in M$, and all $\mathbf{X}, \mathbf{Y} \in \mathcal{X}$ with $x^h_k = y^h_k$ for all $(h, k) \in N \times M \setminus \{(i, j)\}$, we have $\mathbf{X} \succeq \mathbf{Y}$ if and only if $x^i_j \geq y^i_j$.

The last of these axioms is best understood in terms of two sufficient conditions stated in the following lemma. Here and below, vector inequalities have the usual componentwise definition.[3]

**Lemma 1** *Let $\mathcal{X} \subseteq \mathbb{R}^{N\times M}$ be an open set, and let $\succeq$ be an order on $\mathcal{X}$ that has* COLUMN PREFERENCES *and* ROW PREFERENCES*. If $\succeq$ satisfies either of the following conditions, then $\succeq$ satisfies* COORDINATE MONOTONICITY*.*

ROW MONOTONICITY: *For all $i \in N$ and $j \in M$, and any $\mathbf{x}, \mathbf{y} \in \mathcal{X}^i$ with $x_k = y_k$ for all $k \in M \setminus \{j\}$, we have $\mathbf{x} \succeq^i \mathbf{y}$ if and only if $x_j \geq y_j$.*

[3]If $\mathbf{v} = (v_1, ..., v_q)$ and $\mathbf{v}' = (v'_1, ..., v'_q)$, we write $\mathbf{v} \geq\ \mathbf{v}'$ if $v_p \geq v'_p$, for all $p \in \{1, ..., q\}$, and $\mathbf{v} > \mathbf{v}'$ if the same holds with $\mathbf{v} \neq \mathbf{v}'$. We say that $\mathbf{v}$ is *non-negative* (*strictly positive*) if $\mathbf{v} \geq 0$ (resp. $\mathbf{v} > 0$).

Column Monotonicity: *For all $j \in M$ and $i \in N$, and any $\mathbf{x}, \mathbf{y} \in \mathcal{X}_j$, with $x^h = y^h$ for all $h \in N \setminus \{i\}$, we have $\mathbf{x} \succeq_j \mathbf{y}$ if and only if $x^i \geq y^i$.*

*Conversely, if $\mathcal{X}$ is convex, then* Coordinate Monotonicity *is equivalent to each of* Row Monotonicity *and* Column Monotonicity.

The proofs of Lemma 1 and all other results are in the Appendix.

In the intertemporal choice problem, $N$ and $M$ will conventionally represent time periods and goods, respectively. Thus, with the numbers $x^i_j$ measuring physical quantities, Row Preferences says that, for each given time, $\succeq$ is increasing with respect to the instantaneous preferences over baskets of goods, and Column Preferences says that, for each given good, the overall preference $\succeq$ is increasing with respect to the preferences over consumption streams. These are dominance properties in the sense considered by multiattribute preference theory (see, e.g., Keeney and Raiffa, 1976, ch. 3). Coordinate Monotonicity, Row Monotonicity and Column Monotonicity are familiar monotonicity conditions from consumer theory, saying in effect that all the goods, at all times, are valuable.

In the uncertain social choice problem, $N$ and $M$ will conventionally represent individuals and states of nature, respectively. We can take the $x^i_j$ to be physical quantities, as in the previous case, or to be utility values, which conceptually amounts to endorsing a *welfarist* position in normative economics.[4] We consider the latter interpretation, both because it illustrates another use of the formalism, and because it connects with the theoretical issues highlighted in the introduction. Thus, what the social preference $\succeq$ ranks are *ex ante* social allocations viewed in utility terms, and Row Preferences has two implications: (a) if all individuals are indifferent between two social prospects, then so is the social preference; (b) if an individual ranks a social prospect above another, and all others are indifferent, then the social preference ranks the former above the latter. Statement (a) is the *ex ante* Pareto Indifference condition. Statement (b) is not quite the *ex ante* Strict Pareto condition, since it must be applied iteratively to deliver this condition, and the domain must be rich enough for the iteration to take place.[5] Thus, the *ex ante* Pareto Principle holds in a somewhat weakened way.

Now, Column Preferences means that the *ex ante* social preference $\succeq$ is increasing with respect to each of its *ex ante* preferences conditional on states. Since the $x^i_j$ are utility numbers, Row Monotonicity makes the same claim for the $\succeq^i$ vis-à-vis their own conditionals. This is a classic dominance property in the theory of decision under uncertainty; it is satisfied not only by expected utility, but also by rank-dependent utility and most received non-expected utility

[4] In normative economics, *welfarism* is the claim that individual utility values capture all the information on alternatives that may be relevant to the social evaluation.

[5] Given our basic domain assumption, we can only conclude that *ex ante* Strict Pareto holds *locally*. (That is: for any $\mathbf{X} \in \mathcal{X}$, there is an open neighbourhood $\mathcal{Y}_\mathbf{X} \subseteq \mathcal{X}$ with $\mathbf{X} \in \mathcal{Y}_\mathbf{X}$ such that, for any $\mathbf{Y} \in \mathcal{Y}_\mathbf{X}$ with $\mathbf{x}^i \succeq^i \mathbf{y}^i$ for all $i \in N$, and $\mathbf{x}^i \succ^i \mathbf{y}^i$ for some $i \in N$, it is the case that $\mathbf{X} \succ \mathbf{Y}$.) If $\mathcal{X}$ is convex, one can take $\mathcal{Y}_\mathbf{X} = \mathcal{X}$ for all $\mathbf{X} \in \mathcal{X}$.

construals.[6] Column Monotonicity means that in every realized state, the *ex post* social preference satisfies both Pareto Indifference (trivially) and an individual-by-individual version of Strict Pareto (nontrivially). This is the *ex post* Pareto Principle, though in the same weaker form as the *ex ante* principle. As before, this interpretation takes the $x^i_j$ to be utility numbers.

In the individual decision-making example, a suitable reinterpretation of rows and columns delivers the dominance property just stated, as well as a dual form of it, where conditionals are taken on prizes instead of outcomes. The full meaning of the axioms for this case is spelled out in Section 4.

In the assessment of economic integration, Coordinate Monotonicity is natural, and Row and Column Preferences express an equally natural division of the initial problem. Presumably, the statistician will have more feeling on which of two economies is more integrated if she compares them sector by sector, or commodity by commodity; this is what these two axioms say.

We now move to more technical assumptions, which are essential to the proofs. For all $\mathbf{Y} \in \mathcal{X}$, and all $i \in N$ and $j \in M$, the $(i,j)$-*section* of $\mathcal{X}$ through $\mathbf{Y}$ is the set $\{\mathbf{X} \in \mathcal{X};\, x^i_j = y^i_j\}$, an $(N \cdot M - 1)$-dimensional subset of $\mathbb{R}^{N\times M}$. We say $\mathcal{X}$ is *sectionally connected* if each $(i,j)$-section is connected. This condition is neither stronger nor weaker than ordinary connectedness; see the examples by Segal (1992), Wakker (1993), and Chateauneuf and Wakker (1993), which also illustrate why this is an important restriction. In words, to say that $\mathcal{X}$ is (path-)connected means that, given any two feasible alternatives $\mathbf{X}$ and $\mathbf{Y}$, it is possible continuously to transform $\mathbf{X}$ into $\mathbf{Y}$ by moving along a continuous path of feasible alternatives.[7] Sectional connectedness resembles connectedness, except that it requires one to transform $\mathbf{X}$ into $\mathbf{Y}$ while holding constant the value of one coordinate. The set $\mathcal{X} \subseteq \mathbb{R}^{N\times M}$ is both connected and sectionally connected if it is convex or (an even more restrictive condition) it is a *box* —i.e. $\mathcal{X} = \prod_{i\in N}\prod_{j\in M}\mathcal{B}^i_j$, where $\mathcal{B}^i_j$ is a real interval for all $i \in N$ and $j \in M$.

Finally, we say that $\mathcal{X}$ is $\succeq$-*indifference connected* if, for all $\mathbf{Y} \in \mathcal{X}$, the indifference set $\{\, \mathbf{X} \in \mathcal{X};\, \mathbf{Y} \approx \mathbf{X}\}$ is a connected subset of $\mathcal{X}$. The above papers also illustrate why this restriction matters to additive separability. Here are two cases in which it holds.

**(a)** If $\mathcal{X}$ is an open box in $\mathbb{R}^{N\times M}$, then $\mathcal{X}$ is $\succeq$-indifference connected. (See Appendix for proof.)

**(b)** Suppose $\mathcal{X}$ is a convex and comprehensive subset of $\mathbb{R}^{N\times M}_+$. If $\succeq$ is quasiconcave, then $\mathcal{X}$ is $\succeq$-indifference connected.[8]

For all $i \in N$ and $j \in M$, let $\mathcal{X}^i_j := \{x^i_j;\, \mathbf{X} \in \mathcal{X}\} \subseteq \mathbb{R}$. Now to our first result.

[6] As in Savage (1972) and elsewhere in decision theory, this interpretation identifies *ex ante* preferences conditional on $j$ with *ex post* preferences occurring when $j$ is realized.

[7] Any open subset of a Euclidean space is connected if and only if it is path-connected, so that we may identify the two notions here.

[8] The set $\mathcal{X} \subseteq \mathbb{R}^{N\times M}$ is *comprehensive* if for all $\mathbf{X} \in \mathcal{X}$, and all $\mathbf{X}' \in \mathbb{R}^{N\times M}$, if $\mathbf{X}' \leq \mathbf{X}$ then $\mathbf{X}' \in \mathcal{X}$. The order $\succeq$ is *quasi-concave* if all of its upper contour sets are convex.

**Proposition 2** *Let $\mathcal{X} \subseteq \mathbb{R}^{N \times M}$ be open. Let $\succeq$ be an order on $\mathcal{X}$ that has* Row Preferences *and* Column Preferences, *and which satisfies* Continuity *and* Coordinate Monotonicity. *Then:*

**(a)** *For all $\mathbf{X} \in \mathcal{X}$, there is an open neighbourhood $\mathcal{Y} \subseteq \mathcal{X}$ with $\mathbf{X} \in \mathcal{Y}$, and for all $i \in N$ and $j \in M$, there are continuous increasing functions $u_j^i : \mathcal{X}_j^i \longrightarrow \mathbb{R}$ such that $\succeq$ is represented on $\mathcal{Y}$ by the additive function $U : \mathcal{Y} \longrightarrow \mathbb{R}$ defined by*

$$U(\mathbf{Y}) \;:=\; \sum_{i \in N} \sum_{j \in M} u_j^i(y_j^i), \quad \textit{for all } \mathbf{Y} \in \mathcal{Y}.$$

*Furthermore, in this representation, the $u_j^i$ are unique up to positive affine transformations with a common multiplier.*[9]

**(b)** *Suppose $\mathcal{X}$ is also connected, sectionally connected, and $\succeq$-indifference connected. Then we can take $\mathcal{Y} = \mathcal{X}$ in part (a).*

**(c)** *In this case, for all $i \in N$, the order $\succeq^i$ is represented by the function $U^i : \mathcal{X}^i \longrightarrow \mathbb{R}$ defined by*

$$U^i(\mathbf{x}) \quad := \quad \sum_{j \in M} u_j^i(x_j), \qquad \textit{for all } \mathbf{x} \in \mathcal{X}^i.$$

**(d)** *Likewise, for all $j \in M$, the order $\succeq_j$ is represented by the function $U_j : \mathcal{X}_j \longrightarrow \mathbb{R}$ defined by*

$$U_j(\mathbf{x}) \quad := \quad \sum_{i \in N} u_j^i(x^i), \qquad \textit{for all } \mathbf{x} \in \mathcal{X}_j.$$

Proposition 2 is related to Debreu (1960)'s theorem on additively separable representations, but unlike this classic result, it does not explicitly assume that the preference order is totally separable. Indeed, the proof first establishes total separability via the theory of overlapping separability developed in Gorman (1968b), and only then, using Debreu (1960), it concludes that there exists a local additively separable representation around any given alternative. Part (b) consists in gluing these local representations together, via the special connectedness conditions. We leave it for the reader to check that consistent aggregation, in Green's (1964) or van Daal and Merkies's (1984) sense, holds of the numerical functions representing the orders defined here. Had we retained

[9] That is, if the functions $\widetilde{u}_j^i : \mathcal{X}_j^i \longrightarrow \mathbb{R}$ are such that $\succeq$ is represented on $\mathcal{Y}$ by the function $\widetilde{U}$ defined by

$$\widetilde{U}(\mathbf{Y}) \;:=\; \sum_{i \in N} \sum_{j \in M} \widetilde{u}_j^i(y_j^i), \quad \text{for all } \mathbf{Y} \in \mathcal{Y},$$

then there exist $a > 0$ and $b_j^i \in \mathbb{R}$ such that, for all $i \in N$ and $j \in M$, $\widetilde{u}_j^i(y_j^i) = a\, u_j^i(y_j^i) + b_j^i$ for all $\mathbf{Y} \in \mathcal{Y}$.

Nataf's (1948) strong Cartesian product and differentiability assumptions, we could have applied his theorem and obtained Proposition 2(b) at once.[10]

In general, the functions $u_j^i$ are all different, and to obtain a relationship between them is the object of the following sections and their more advanced results. Our applications to uncertain social choice, individual decision-making, and economic integration, require these later results, but Proposition 2 offers a perspective on the application to intertemporal choice, as we now discuss. In this case, $u_j^i$ is a utility function for consumption of good $j$ at time $i$, $U^i$ is a utility function over consumption bundles at time $i$, $U_j$ is a utility function over streams of good $j$, and $U$ is a utility function for consumption plans. There is a classical stock of arguments for rejecting additive separability with respect to *goods*, and be suspicious of it when it applies to *time periods*.

Jevons and Walras discussed the "equation of exchange" —today's textbook equality between marginal utility ratios and marginal rates of substitution —in terms of separable, and even additively separable, utility functions for consumption goods, and they also stated their demand theory in this way. Edgeworth pointed out that this was unnecessary for the purpose, still a mild point, but later neo-classicals found more distressing objections. Implying as it does that the marginal rate of substitution of $a$ for $b$ only depends on the quantities of $a$ and $b$, separability (more generally than additive separability) makes the law of demand automatic under diminishing marginal utilities, thus wiping out the possibility of a prevailing income effect. Moreover, separability makes it impossible to classify consumer goods into complements and substitutes. These critical messages were taken aboard long ago by demand theory, and it comes to no surprise that postwar theorist Gorman[11] expressed doubts about the very assumptions that he was exploring mathematically.

Additively separable representations have on the whole been more successful when they concern time preferences. Ramsey may have been the first to employ such a functional form in his saving model, and it has persisted in the neoclassical literature on intertemporal choices of consumption, investment or money balances. This can be explained by analytical convenience, but no doubt also by the fact that the objections from demand theory lose their force here. However, there are worrying specific objections, in particular that for some goods, the quantity of today's consumption influences the utility of tomorrow's consumption through habit formation.[12]

Given this controversial pedigree, Proposition 2 sounds like a mixed blessing. Some might use to axiomatize old style neo-classical economics, but many others will rather argue from the strong functional forms against the applicability of the axioms to the case. The ambivalence is typical of our results; it will transpire throughout Sections 3 and 4.

[10] Though Nataf's (1948) theorem is correct, its proof is rather obscure. The curious reader may consult the clarifications and improvements adduced by van Daal and Merkies (1988).

[11] More obviously in Gorman (1968a) than in the other papers.

[12] This by now classic objection is discussed in detail by Browning (1991). Other problems raised by temporal separability are discussed in the theoretical management literature (see, e.g., Keeney and Raiffa, 1976), as well as in health economics (see, e.g., Gold et al. (1996).

## 3 Social choice under uncertainty

Although too strong in one sense, the conclusion of Proposition 2 is too weak in another, because the additively separable representation does not impose any relation between the utility functions defined coordinatewise. Sections 3 and 4 make it more informative by introducing both more axiomatic conditions and more structural assumptions. In the former group, we will require that there be a single preference order on rows, or a single preference order on columns, or both. Formally, define

$$\mathcal{X}_M := \bigcup_{j\in M} \mathcal{X}_j \quad \text{and} \quad \mathcal{X}^N := \bigcup_{i\in N} \mathcal{X}^i.$$

We will require at least one of the following axioms.

ROW INVARIANCE: There is a single preference order $\succeq^N$ defined on $\mathcal{X}^N$, such that for all $i \in N$, the order $\succeq^i$ is the restriction of $\succeq^N$ to $\mathcal{X}^i$.

COLUMN INVARIANCE: There is a single preference order $\succeq_M$ defined on $\mathcal{X}_M$, such that for all $j \in M$, the order $\succeq_j$ is the restriction of $\succeq_M$ to $\mathcal{X}_j$.

Since our framework treats rows and columns symmetrically, and their meaning can be fixed at will, there is no point in considering both conditions unless they apply at the same time. When only one of them applies, we will conventionally select COLUMN INVARIANCE.

In the group of structural conditions, we will require that there be a single set of feasible rows, or a single set feasible columns, or both. Formally, we will require the domain $\mathcal{X}$ to satisfy at least one of the following structural conditions:

***Identical Row Spaces*:** $\mathcal{X}^1 = \mathcal{X}^2 = \cdots = \mathcal{X}^n = \mathcal{X}^N$.

***Identical Column Spaces*:** $\mathcal{X}_1 = \mathcal{X}_2 = \cdots = \mathcal{X}_m = \mathcal{X}_M$.

Under the first condition, there is a common projection $\mathcal{X}_j^*$ of the $\mathcal{X}^i$ on $j \in M$ , and $\mathcal{X}^N \subseteq \prod_{j\in M} \mathcal{X}_j^*$. Under the second condition, there is a common projection $\mathcal{X}_*^i$ of the $\mathcal{X}_j$ on $i \in N$, and $\mathcal{X}_M \subseteq \prod_{i\in N} \mathcal{X}_*^i$. Here are formal cases where they hold.

**Examples.** (a) If $\mathcal{X}$ is an open box in $\mathbb{R}^{N\times M}$, then $\mathcal{X}$ satisfies both *Identical Row Spaces* and *Identical Column Spaces*.

(b) Suppose that, for all $\mathbf{y} \in \mathcal{X}_M$, there exists $\mathbf{X} \in \mathcal{X}$ such that $\mathbf{x}_j = \mathbf{y}$ for all $j \in M$. Then $\mathcal{X}$ satisfies *Identical Column Spaces.* $\diamondsuit$

Note that ROW and COLUMN INVARIANCE are so formulated that no logical implication holds between them and *Identical Row Spaces* and *Identical Column Spaces*, respectively. However, the two sets of restrictions are often acceptable or rejectable together. In the intertemporal choice problem, ROW INVARIANCE and

*Identical Row Spaces* are implausible, while Column Invariance and *Identical Column Spaces* are stringent without being absurd. (The former says that one time ranks commodity baskets like another when they are available at both times; this excludes habit formation. The latter adds that exactly the same baskets are available at both times; this excludes technical interdependencies between periods. Existing time-separable representations in consumer theory often make these assumptions.)

In the uncertain social choice problem, with $x_j^i$ representing utility, Row Invariance becomes the implausible claim that the individuals have the same preferences. But *Identical Row Spaces* is not so easy to discard. It says that the set of utility vectors is common to all individuals, which makes sense if some interpersonal utility comparisons have already taken place. Meanwhile, Column Invariance says that *ex post* social preferences are state-independent, while *Identical Column Spaces* says that the same social outcomes exist in each state. These two state-independence assumptions are made by Savage (1972) and Anscombe and Aumann (1963) when they derive a subjective probability from preferences under uncertainty, and they have prevailed in the theoretical discussion of *ex ante* versus *ex post* Paretianism that concern us.[13] On their part, the individuals may have state-dependent preferences; this is explained below.

Now to our first main result. Given a set $L = \{1, 2, \ldots, \ell\}$ and a vector $\mathbf{p} = (p_1, ..., p_\ell) \in \mathbb{R}^L$, we say that $\mathbf{p}$ is a *weight vector* on $L$ if $p_k \geq 0$ for all $k \in L$, and $\sum_{k \in L} p_k = 1$. The expression *probability vector* is mathematically appropriate, but we reserve it for those cases in which elements of $L$ represent states of nature. The set of weight vectors on $L$ is denoted by $\Delta_L$.

**Theorem 3** *Suppose $\mathcal{X} \subseteq \mathbb{R}^{N \times M}$ is open, connected, sectionally connected, $\succeq$-indifference connected, and satisfies Identical Column Spaces. Then $\succeq$ has* Row Preferences *and* Column Preferences *and satisfies* Coordinate Monotonicity, Continuity, *and* Column invariance *if and only if:*

**(a)** *For all $i \in N$, there is an increasing, continuous function $u^i : \mathcal{X}_*^i \longrightarrow \mathbb{R}$, such that the order $\succeq_M$ is represented by the function $W_M : \mathcal{X}_M \longrightarrow \mathbb{R}$ defined by*

$$W_M(\mathbf{x}) \quad := \quad \sum_{i \in N} u^i(x^i), \qquad \text{for all } \mathbf{x} \in \mathcal{X}_M. \tag{1}$$

**(b)** *There is a strictly positive weight vector $\mathbf{p} \in \Delta_M$, such that for all $i \in N$, the order $\succeq^i$ is represented by the function $U_{\mathbf{p}}^i : \mathcal{X}^i \longrightarrow \mathbb{R}$ yielding the $\mathbf{p}$-weighted value of $u^i$. That is:*

$$U_{\mathbf{p}}^i(\mathbf{x}) \quad := \quad \sum_{j \in M} p_j \, u^i(x_j), \qquad \text{for all } \mathbf{x} \in \mathcal{X}^i. \tag{2}$$

[13] The papers by Mongin (1998), Chambers and Hayashi (2006), and Gajdos et al. (2008) are exceptions.

**(c)** *The order $\succeq$ is represented by the function $W : \mathcal{X} \longrightarrow \mathbb{R}$ which computes the* $\mathbf{p}$*-weighted value of the function $W_M$ from part (a). That is:*

$$W(\mathbf{X}) \ := \ \sum_{j \in M} p_j \, W_M(\mathbf{x}_j) \ = \ \sum_{j \in M} \sum_{i \in N} p_j \, u^i(x^i_j) \ = \ \sum_{i \in N} U^i_{\mathbf{p}}(\mathbf{x}^i), \ \textit{for all } \mathbf{X} \in \mathcal{X}. \tag{3}$$

**(d)** *In this representation, the weight vector $\mathbf{p}$ is unique, and the functions $u^1, \ldots, u^n$ are unique up to positive affine transformations with a common multiplier.*

In terms of intertemporal choice, Theorem 3 says that time $j$ does not influence the shape of the utility functions $u^i$ defined for each commodity $i$, its role being channelled through the weights $p_j$, which should be viewed as discounting factors.

In terms of uncertain social choice, the functions $U^i_{\mathbf{p}}$ and $W$ of Theorem 3(b,c) are the individuals' and the social observer's *ex ante* utility functions. If $\mathbf{p}$ is regarded as a probability vector, then these functions are shown to be of the *expected utility type*. This is a striking result if one thinks of the non-committal decision theory that we assumed at the start. We required only two things: first, that both the individuals and social observer satisfy dominance (a property that most non-expected utility models fulfil), and second, that the social observer has Paretian and state-independent preferences. Theorem 3(b) does *not* impose state-independent preferences on the individual agents, because the $x^i_j$ are taken to be preexisting utility values which may very well come from some *state-dependent* utility functions, exogenous to our modelling.

Theorem 3(a,c) gives another description of the social observer's preferences, this time in terms of social welfare functions. The *ex post* welfare functions $W_M$ and the *ex ante* welfare function $W$ are sums of the corresponding individual utility functions, i.e., have the mathematical form of a weighted utilitarian rule. This is another striking result in view of the purely ordinal form of the axioms. Whether the derived representation bears more than a formal analogy with classical utilitarianism is a complex question that we do not discuss here.

Finally, Theorem 3(d) confers uniqueness to the functional representations, under the usual proviso that the mathematical pattern in which they appear must be respected.[14] Without such uniqueness, the representations would have no conceptual bearing; for instance, it would not be sensible to view $\mathbf{p}$ as representing a probability.

With these interpretations at hand, Theorem 3 states that the *ex ante* and *ex post* Pareto principles are compatible only if (1) the individuals and the social observer are all expected utility maximizers, and (2) they compute their expected utilities by using the *same* subjective probabilities. Hammond's (1981) welfare economics paper is the classic source for both the compatibility problem and the answer that (2) is necessary for its solution. When investigating consistent ways of aggregating Savage preferences, Mongin (1995) implicitly raised

[14]Non-affine monotonic transforms of the $u^i$ would represent the $\succeq^i$ equally well, but destroy the expected utility form of the representations in Theorem 3(b,c).

the compatibility problem. His axiomatic treatment enlarges the set of possibilities somewhat. If the individuals' and the social observer's utility functions are all alike up to positive affine transformations, then the *ex ante* and *ex post* principles are compatible, and more subtly, they can be so when weaker Pareto conditions than the Pareto principle apply. These other possibilities lie outside the present framework, so it comes as no surprise that only condition (2) survives. The main news concerns the necessary condition (1). The above papers (and others as well) *assume* that both the individuals and the social observer satisfy the axioms of subjective expected utility, whereas we now *prove* this in the representation theorem. To appreciate the step forward, take *probabilistically sophisticated agents*, i.e., agents who have well-defined subjective probabilities despite satisfying not subjective expected utility, but some generalization of it. They would satisfy our weak decision-theoretic conditions; thus, if the observer insisted on respecting both the *ex ante* and *ex post* Pareto principle, they would inexorably turn into subjective expected utility maximizers!

It is unclear whether (2) signals an impossibility or only a severe, though implementable restriction. Among the interpreters, Broome (1991) seems to take the latter view, whereas Mongin and d'Aspremont (1998) favour the former. The choice of answer depends on one's underlying philosophy of probability, and on the further issue of when probabilities are computed: is it at the completely *ex ante* stage, or rather at some *interim* stage? On one interpretation, probabilities are subjective in the sense promoted by Savage, and moreover, they are pure priors, i.e., embody no outside information at all; this would make their interpersonal agreement very unlikely. On another interpretation, they are still subjective in the same sense, but count as imperfect priors, thus in effect as posteriors, because they embody some outside information; this would make their interpersonal agreement less unlikely. (Some will argue that a pure prior is a fiction and that this is the only appropriate alternative of the two.) Finally, probabilities could be objective in one of the senses that philosophers of probability have argued for.[15] This last interpretation would make (2) unproblematic, but it does not fit in with the present frame of analysis, which is exclusively preference-based, like Savage's. For the weight function to represent an objective probability, at least *some* probabilistic information would have to be included into the assumptions.

Several solutions have been proposed to escape from (2) when it is interpreted as an impossibility, many of which prioritize the *ex post* form of the Pareto principle over the *ex ante* form,[16] while a few others defend the opposite priority, and still others reach compatibility by relaxing some decision-theoretic component of the framework. We will not evaluate these theoretical possibilities here, but Theorem 3 has a clear bearing on them, especially on the last group.[17]

[15] An interesting recent option is *objective Bayesianism* (see Williamson, 2010)

[16] This is the classic solution since Hammond, (1981). For a more refined treatment, see Fleurbaey (2011).

[17] Mongin and d'Aspremont (1998) evaluate the solutions proposed at the time. More recently, Gilboa et al. (2004), Chambers and Hayashi (2006), and Keeney and Nau (2011) have taken up the challenge.

By the same token, Theorem 3 is closely related in spirit to Harsanyi's (1955) Aggregation Theorem. According to this classic result, if the individuals have von Neumann-Morgenstern preferences on a lottery set, and if the social observer satisfies the Pareto principle and herself entertains von Neumann-Morgenstern preferences on the lottery set, then her preferences can be represented by a positively weighted sum of the von Neumann-Morgenstern representations of the individual preferences. Harsanyi interpreted this piece of reasoning as constituting an argument for utilitarian ethics. Our framework does not contain lotteries, so in order to bridge the gap with Harsanyi, we should replace his theorem by one of the variants that were devised for state-contingent prospects instead of lotteries.[18] When this is done, Theorem 3 appears to be a *stronger* form of the classic result: expected utility theory now belongs to the conclusions, and the utilitarian-looking social welfare functions follow from weaker assumptions than before.

Two previous works suppressed the expected utility assumptions in Harsanyi's theorem, and they call for a comparison. In the unpublished paper that the present one supersedes, Blackorby et al. (2004) started from a Cartesian product set of state-contingent prospects, expressed conditions related to the present ones but stated in utility terms directly, and eventually derived an additively separable representation for social preference. At a closer look, this representation boils down to expected utility, so that this early result can be swept under Theorem 3 as a particular case. Not so for the theorem by Gajdos et al. (2008), which requires a specialized framework in the style of Anscombe and Aumann (1963). The individual and social preferences there obey weaker forms of von Neumann-Morgenstern independence and the sure-thing principle, and they can be state-dependent. Under an appropriate Pareto condition, the stringent conclusion (2) of a unique subjective probability emerges in more general form, and the social utility representation can be expressed as a weighted sum the individual ones. This result is closer to Harsanyi's original than ours by its choice of framework and assumptions.[19]

When $M$ is interpreted as a set of time periods, Theorem 3 becomes a statement about intertemporal social choice. COORDINATE MONOTONICITY, ROW PREFERENCES and COLUMN PREFERENCES express Pareto or dominance conditions, while ROW INVARIANCE says that the social observer's preferences are unchanging over time. The weight vector $\mathbf{p}$ now describes a sequence of discount factors, which are *common to all agents*. This conclusion reveals a tension between applying the Pareto principle at each moment of time, and applying it to entire social histories, granting the mild decision-theoretic conditions. As before, it may be interpreted as either a sheer impossibility or only a severe

[18] Mongin (1995) provides a state-contingent version for Savage's framework, and Blackorby et al. (1999) provides another for Anscombe and Aumann's.

[19] The aggregative results of Crès et al. (2011) and Nascimento (2012) are also non-expected utility variants of Harsanyi's Aggregation Theorem. Unlike ours and those of Gajdos et al. (2008), they rely on identical utility functions. This explains the difference in conclusions - in particular, the initial non-expected utility component is not destroyed by the aggregation process as it is here.

restriction; we lean towards the former view.

It remains to investigate the case in which the four conditions introduced by this section jointly apply. If $\mathcal{X}$ has both *Identical Column Spaces* and *Identical Row Spaces*, there is a single open subset $\mathcal{X}_*^*$ such that $\mathcal{X}_j^i = \mathcal{X}_*^*$ for all $(i,j) \in N \times M$.

**Corollary 4** *Suppose $\mathcal{X} \subseteq \mathbb{R}^{N\times M}$ is open, connected, sectionally connected, $\succeq$-indifference connected, and has both Identical Row Spaces and Identical Column Spaces. Then $\succeq$ has* Row Preferences *and* Column Preferences *and satisfies* Coordinate Monotonicity, Continuity, Row Invariance *and* Column Invariance *if and only if there is a single increasing, continuous function $u : \mathcal{X}_*^* \longrightarrow \mathbb{R}$, and two strictly positive weight vectors $\mathbf{q} = (q^1, \ldots, q^n) \in \Delta_N$ and $\mathbf{p} = (p_1, \ldots, p_m) \in \Delta_M$, such that:*

**(a)** *The order $\succeq_M$ is represented by the function $W_M : \mathcal{X}_M \longrightarrow \mathbb{R}$ defined by*

$$W_M(\mathbf{x}) \quad := \quad \sum_{i \in N} q^i\, u(x^i), \qquad \text{for all } \mathbf{x} \in \mathcal{X}_M.$$

**(b)** *The order $\succeq^N$ is represented by the function $W^N : \mathcal{X}^N \longrightarrow \mathbb{R}$ defined by*

$$W^N(\mathbf{x}) \quad := \quad \sum_{j \in M} p_j\, u(x_j), \qquad \text{for all } \mathbf{x} \in \mathcal{X}^N.$$

**(c)** *The order $\succeq$ is represented by the function $W : \mathcal{X} \longrightarrow \mathbb{R}$ defined by*

$$W(\mathbf{X}) \;:=\; \sum_{j \in M} \sum_{i \in N} q^i\, p_j\, u(x_j^i), \qquad \text{for all } \mathbf{X} \in \mathcal{X}.$$

**(d)** *In this representation, the weight vectors $\mathbf{q}$ and $\mathbf{p}$ are unique, and the function $u$ is unique up to a positive affine transformation.*

Since Row (Column) Invariance is unacceptable when the rows (columns) refer to individuals, we must shift away from collective interpretations. Here is one from individual decision theory. Take $N$ to be a set of time periods, while keeping $M$ to be a set of states of nature. Thus, $\succeq$ represents intertemporal preferences under uncertainty. Elements of $\mathcal{X}^N$ represent *instantaneous prospects* (which by *Identical Row Spaces* could be realized at any moment in time), while elements of $\mathcal{X}_M$ represent *ex post* consumption streams (which by *Identical Column Spaces* could be realized in any state of nature). Now, by Row Invariance and Column Invariance, respectively, preferences are state-independent over *ex post* consumption streams, and time-independent over instantaneous prospects. The conclusion is that the agent maximizes the expected value of a discounted utility sum.

# 4 Individual choice and subjective probability

We will now consider a variation of Theorem 3, which drops the structural conditions that $\mathcal{X}$ be sectionally connected and indifference connected, and have identical column spaces. In exchange, we will need to impose a stronger axiom on $\succeq$.

Let $\mathcal{Y} \subseteq \mathbb{R}^M$ be an open set. A subset $\mathcal{Z} \subset \mathcal{Y}$ will be said to be *flat* if $\mathcal{Z} = \mathcal{Y} \cap \mathcal{H}$, where $\mathcal{H}$ is an affine hyperplane in $\mathbb{R}^M$. We also call *flat* a preference order $\succeq$ on $\mathcal{Y}$ all indifference sets of which are flat. This is obviously the case if $\succeq$ is represented by a linear utility function $u(\mathbf{y}) = \sum_{j \in M} c_j \, y_j$, but the converse is false, because flatness does not force the indifference hyperplanes to be parallel. If $\mathcal{Y}$ is convex, then $\succeq$ is flat only if its indifference sets are convex. More specifically, if $\mathcal{Y}$ is a convex set of probability vectors, then $\succeq$ is flat if and only if it satisfies the *betweenness* property. The latter is a restriction of von Neumann-Morgenstern independence to indifferent lotteries, and it implies linear, but not necessarily parallel indifference sets. The derived representation replaces the expected utility form by a *weighted utility* form (see Chew, 1983, and Dekel, 1986). It has sometimes been suggested that betweenness offers a plausible middle ground between empirical and normative validity.[20]

**Theorem 5** *Suppose $\mathcal{X} \subseteq \mathbb{R}^{N \times M}$ is open and connected, and $\mathcal{X}_M$ is also connected. Suppose that either $\succeq_M$ is flat, or $\succeq^i$ is flat for every $i \in N$. Then $\succeq$ has* Row Preferences *and* Column Preferences *and satisfies* Continuity, Coordinate Monotonicity, *and* Column Invariance *if and only if there is a strictly positive weight vector $\mathbf{q} \in \Delta_N$, and a strictly positive weight vector $\mathbf{p} \in \Delta_M$, such that:*

**(a)** *$\succeq_M$ is represented by the linear function $W_M : \mathcal{X}_M \longrightarrow \mathbb{R}$ defined by*

$$W_M(\mathbf{x}) \quad := \quad \sum_{i \in N} q^i \, x^i, \qquad \text{for all } \mathbf{x} \in \mathcal{X}_M.$$

**(b)** *For all $i \in N$, the order $\succeq^i$ is represented by the linear function $W^N : \mathcal{X}^i \longrightarrow \mathbb{R}$ defined by*

$$W^N(\mathbf{x}) \quad := \quad \sum_{j \in M} p_j \, x_j, \qquad \text{for all } \mathbf{x} \in \mathcal{X}^i.$$

*(Thus, $\succeq$ also satisfies* Row Invariance.*)*

**(c)** *$\succeq$ is represented by the linear function $W : \mathcal{X} \longrightarrow \mathbb{R}$ defined by*

$$W(\mathbf{X}) \; := \; \sum_{i \in N} \sum_{j \in M} q^i \, p_j \, x^i_j, \qquad \text{for all } \mathbf{X} \in \mathcal{X}.$$

**(d)** *Furthermore, $\mathbf{q}$ and $\mathbf{p}$ are unique in this representation.*

[20]See Epstein (1992) and Sarin and Wakker (1998) for more discussion along this line.

The flatness restriction, hence Theorem 5, are relevant to the uncertain social choice problem. Here, $\mathbf{x} \in \mathcal{X}^i$ is a personal prospect for $i$, $x_j$ is the utility this individual receives if state $j$ is realized, and flatness of $\succeq^i$ is a variant form of betweenness, as applied to state-contingent alternatives $X$ instead of lotteries (and furthermore without $\mathcal{X}$ being necessarily convex). The two weight vectors obtained in the conclusions have a clear meaning: $\mathbf{q}$ compares the individuals and $\mathbf{p}$ (a probability vector) compares the states. Thus, Theorem 5 reinforces the message of Theorem 3. As part (b) indicates, the individual *ex ante* preferences obey the expected utility form with the same subjective probability $\mathbf{p}$, and moreover —this is the new implication —these preferences have something in common. Indeed, they transform utility amounts $x_j$ in the same way —by taking their $\mathbf{p}$-weighted sum. In parts (a) and (c), $W_M$ is an *ex post* social welfare function, and $W$ is an *ex ante* social welfare function, and both are of a classical utilitarian form, while $W$ is also of the expected utility type.[21] Thus, the conclusions together express the reconciliation of *ex ante* with *ex post* Paretianism and the high price that this imposes on the diversity of individual characteristics; this time, a price is payed also on the preference side.[22]

Like Theorem 3, this result can be likened to Harsanyi's, or rather, to its state-contingent variations. Suppose that the individuals satisfy the betweenness property on top of dominance, and that the observer similarly satisfies dominance, and is *ex ante* and *ex post* Paretian as far as his social welfare criteria are concerned. Weak as they are compared with Harsanyi's, these assumptions suffice to entail his sum-of-utility formulas. Again, it is interesting to compare Theorem 5 with the earlier results of Blackorby et al. (2004) and Gajdos et al. (2008).

We now change directions, and give an interpretation of Theorem 5 in terms of the Anscombe and Aumann (1963) axiomatization of subjective probability. Famously, these authors modify Savage's axiomatization by allowing some probabilistic information to enter their primitives. They define prospects as associating states of nature with consequences taken in a set of lotteries, instead of Savage's nondescript set of consequences. Technically, this change was motivated by the need to derive subjective probability for *finite* sets of states, like those considered in this paper. Let us interpret $M$ as being the set of states and $N$ as being the (also finite) set of final outcomes on which the set of lotteries is constructed. If we take $x_j^i$ to represent the probability of getting outcome $i$ in state $j$, then alternatives in $\mathcal{X}$ become Anscombe-Aumann prospects, with $\mathbf{x}^j$ being the lottery associated with state $j$, and $\mathbf{x}_i$ stating the probabilities of

[21] Unlike those of Theorem 5, the social welfare functions delivered by Theorem 3 were unweighted. However, this is a purely apparent difference, since the initial utility amounts $x_j^i$ could be subjected to increasing transformations, and then. a weighted sum of individual utilities would also result in Theorem 3.

[22] Bearing in mind that the $x_j^i$ represent utility values, we should refrain from claiming that individual preferences are alike. The conclusion in (b) is only that *if* individuals attribute the same utility vector to a prospect, they will end up with the same utility for this prospect.

outcome $i$ conditional on the states in $M$. This sketch must however be refined, because it would make $\mathcal{X}_M$ —a lottery set —only $(N-1)$-dimensional, and thus violate our full-dimensionality requirement on $\mathcal{X}$.

We will single out an outcome, to be denoted by 0, redefine $N$ to be the set of all outcomes *except* the 0 element, and rewrite the set of lotteries as

$$\Delta_N^0 \quad := \quad \{\mathbf{x} \in \mathbb{R}_+^N ; \sum_{i=1}^n x^i \leq 1\},$$

where for all $\mathbf{x}$, the probability that outcome 0 occurs is $1 - \sum_{i=1}^n x^i$. This is an $N$-dimensional set, so the framework applies if we take prospects to be elements of $(\Delta_N^0)^M$. Supposing as before that feasibility restrictions hold, we take $\mathcal{X} \subseteq (\Delta_N^0)^M$ to be open and connected in $\mathbb{R}^{N \times M}$. Because 0 does not explicitly enter the definition of $\mathcal{X}$, it can be restricted only through the application of the axioms to the other outcomes, and this happens only through COORDINATE MONOTONICITY. This axiom now says that it is better, *ceteris paribus*, to shift probability mass *away* from 0 to any other outcome in $N$. (If $x_k^h = y_k^h$ for all $(h,k) \in N \times M \setminus \{(i,j)\}$ as the antecedent requires, then $x_j^i > y_j^i$ only if $x_j^0 < y_j^0$.) Thus, COORDINATE MONOTONICITY means that 0 is the *worst possible* outcome. There is a single decision-maker in the present application, so this can be assumed without loss of generality.

As for the other conditions, COLUMN PREFERENCES is entailed by the dominance (or "monotonicity") axiom in Anscombe and Aumann (1963), to the effect that the agent's preference for prospects increases with respect to each of her preferences conditional on states. Meanwhile, COLUMN INVARIANCE makes the agent's preferences conditional on states effectively state-independent; this is also entailed by their axiom. Implicitly, they have *Identical Column Spaces*, but we can avoid this loss of generality. An addition to their system is made by ROW PREFERENCES, which extends dominance from conditionals on *states* to conditionals on *outcomes*.

Now to the conclusions of Theorem 5. The vectors $\mathbf{q}$ and $\mathbf{p}$ define a normalized utility function over $N$ and a subjective probability vector on $M$, respectively, and they together serve to compute expected utility values in $W_M(\mathbf{x})$ (part (a), this is in effect the von Neumann-Morgenstern representation theorem for lotteries) and in $W(\mathbf{X})$ (part (c), this is the important one). With the uniqueness statement (d), (c) reproduces the conclusions of the Anscombe and Aumann (1963) subjective probability representation theorem.

The added value lies in the derivation from weak assumptions. Our driving condition is betweenness, and if we choose to apply it to $\succeq_M$, we see that it exactly plays the role of Anscombe and Aumann's assumption that von Neumann-Morgenstern independence regulates preferences over lotteries.[23] In other words, only *part* of von Neumann-Morgenstern theory is needed to derive a subjective probability from the agent's preferences under uncertainty (and the full

[23]If we applied betweenness to the $\succeq^i$ preferences, an alternative derivation would result. Then, the betweenness condition would evoke Savage's sure-thing principle in weaker form.

von Neumann-Morgenstern theory will be obtained as a bonus). We also generalize by allowing the set of uncertain prospects not to be a Cartesian product. Subjective probability theorists have often been taken to task for the large sets of prospects they rely one, the problem being that states put feasibility constraints on outcomes and that there may be dependencies, typically complementarities, holding between different states. Our domain assumption does some justice to this classic objection. The price for these improvements is that we need the unconventional ROW PREFERENCES condition. However, it is justified after the fact by conclusion (b), which holds true in Anscombe and Aumann (1963) even if they do not mention it.

# 5 Economic integration

We now return to the framework of Section 2, and apply it to our last example, i.e., the assessment of economic integration across a set of national economies. In this application, the sectors define both rows and columns, so $N = M$, and $\mathcal{X}$ is a space of square $(n \times n)$ matrices. We suppose that *row* $i$ records *the inputs* to sector $i$ from other sectors, while *column* $j$ records *the outputs* of sector $j$ to the other sectors. If monetary input-output matrices are to be compared in terms of the degree of economic integration prevailing in the economy, and in terms of no other property, such as the size of the economy, some normalization of the data is clearly in order. However, this will make the set $\mathcal{X}$ less than full dimensional in $\mathbb{R}^{n\times n}$, a problem not unlike that encountered in the decision-theoretic application.

One possibility is to reexpress the $x^i_j$ as fractions of total GDP, which amounts to setting $\sum_{i=1}^n \sum_{j=1}^n x^i_j = 1$ in the original matrices, hence confining $\mathcal{X}$ within an $(n^2-1)$-dimensional subspace of $\mathbb{R}^{n\times n}$. Alternatively, for all $i \in N$, the entry $x^i_j$ could be the fraction of total input into sector $i$ coming from sector $j$; this would imply that $\sum_{j=1}^n x^i_j = 1$ for all $i \in N$. A third possibility is to stipulate, for all $j \in N$, that $x^i_j$ is the fraction of the total output from sector $j$ which goes to sector $i$; this would imply that $\sum_{i=1}^n x^i_j = 1$ for all $j \in N$. Either way, $\mathcal{X}$ would be confined within an $(n^2-n)$-dimensional subspace of $\mathbb{R}^{n\times n}$. Any of these three normalizations secures a fixed total sum of $x^i_j$, which resolves the problem addressed by different sizes of the economy. However, with any of these normalizations, the domain $\mathcal{X}$ is no longer an open subset of $\mathbb{R}^{n\times n}$, so one key structural condition of Proposition 2 fails. Furthermore, some of our axioms become vacuous. Specifically, COORDINATE MONOTONICITY is vacuous if matrices are normalized by total GDP, and on top of this, ROW PREFERENCES (or COLUMN PREFERENCES) becomes vacuous if the rows (or columns) are normalized by the total input (or output) of each sector.

These technical difficulties can be superseded by making use of a feature that any economic integration ranking should arguably possess. Diagonal elements of the matrix represent a flow *within* one sector, rather than between sectors, and as such, they are irrelevant to the comparison. Formally, let $L := \{(i, j) \in N \times N;$

$i \neq j\}$ be the set of *off-diagonal* elements in the square $N \times N$. Instead of assuming that $\mathcal{X}$ is a subset of $\mathbb{R}^{N\times N}$, we will define it to be a subset of $\mathbb{R}^{L}$. When $x^i_j$ are fractions of either GDP, or total inputs, or total outputs, the hidden flows of goods from each sector to itself act as slack variables, so $\mathcal{X}$ is open in $\mathbb{R}^L$ and incurs no loss of dimensionality.

It is easy to adapt the axioms to this modified framework. Coordinate Monotonicity becomes:

> *For all $(i,j) \in L$, and all $\mathbf{X}, \mathbf{Y} \in \mathcal{X}$ with $x^h_k = y^h_k$ for all $(h,k) \in L \setminus \{(i,j)\}$, we have $\mathbf{X} \succeq \mathbf{Y}$ if and only if $x^i_j \geq y^i_j$.*

For all $i \in N$, let $L^i := \{(i,j);\ \ j \in N \text{ and } j \neq i\}$, and let $\mathcal{X}^i$ be the projection of $\mathcal{X}$ onto $\mathbb{R}^{L^i}$. For all $j \in N$, let $L_j := \{(i,j);\ \ i \in N \text{ and } i \neq j\}$, and let $\mathcal{X}_j$ be the projection of $\mathcal{X}$ onto $\mathbb{R}^{L_j}$. Row and Column Preferences are now understood to apply with these new definitions of $\mathcal{X}^i$ and $\mathcal{X}_j$.

Arguably, there is still another feature to the ranking problem. For the purpose of assessing economic integration, the off-diagonal entries should not be treated differently from one another. If they are equal in value, the flow of commodities from sector $i$ to sector $j$ contributes just as much to the overall index as the flow of commodities from sector $h$ to sector $k$. But nothing yet forces the ranking to satisfy this property.

Let $\Pi_N$ be the group of all permutations of $N$. For any $\pi \in \Pi_N$, and any $\mathbf{X} \in \mathbb{R}^L$, we define a new matrix $\pi(\mathbf{X}) \in \mathbb{R}^L$ by permuting both the rows and the columns of $\mathbf{X}$ simultaneously. Formally, $\pi(\mathbf{X}) := \mathbf{Y}$, where $y^i_j := x^{\pi(i)}_{\pi(j)}$ for all $(i,j) \in L$. This is a well-defined operation, because $(i,j) \in L$ if and only if $(\pi(i), \pi(j)) \in L$.[24] A subset $\mathcal{X} \subseteq \mathbb{R}^L$ will be said to be *permutation-invariant* if $\pi(\mathbf{X}) \in \mathcal{X}$ for all $\mathbf{X} \in \mathcal{X}$ and all $\pi \in \Pi_N$. Given one such $\mathcal{X}$, the following axiom prohibits discriminating between sectors.

Impartiality: For all $\pi \in \Pi_N$ and all $\mathbf{X}, \mathbf{Y} \in \mathcal{X}$, $\mathbf{X} \succeq \mathbf{Y}$ if and only if $\pi(\mathbf{X}) \succeq \pi(\mathbf{Y})$.

Here is the final result of the paper.

**Proposition 6** *Let $\mathcal{X}$ be a connected, sectionally connected, relatively open subset of $\mathbb{R}^L$, and let $\succeq$ be an order on $\mathcal{X}$ such that $\mathcal{X}$ is $\succeq$-indifference connected.*

[24] For example, take $n = m = 3$, let

$$\mathbf{X} = \begin{bmatrix} \bullet & 0.05 & 0.08 \\ 0.12 & \bullet & 0.17 \\ 0.15 & 0.13 & \bullet \end{bmatrix},$$

and suppose $\pi(1) = 2$, $\pi(2) = 3$ and $\pi(3) = 1$. Then

$$\sigma(\mathbf{X}) = \begin{bmatrix} \bullet & 0.17 & 0.12 \\ 0.13 & \bullet & 0.15 \\ 0.05 & 0.08 & \bullet \end{bmatrix}.$$

**(a)** *The order* $\succeq$ *has* Row Preferences *and* Column Preferences*, and satisfies* Coordinate Monotonicity *and* Continuity *if and only if for all* $(i,j) \in L$*, there exist continuous, increasing functions* $v_j^i : \mathcal{X}_j^i \longrightarrow \mathbb{R}$ *such that* $\succeq$ *is represented by the function* $V$ *defined by*

$$V(\mathbf{X}) := \sum_{(i,j)\in L} v_j^i(x_j^i) \text{ for all } \mathbf{X} \in \mathcal{X}.$$

*In this case, for all* $i \in N$*, the order* $\succeq^i$ *is represented by the function* $V^i$ *defined by*

$$V^i(\mathbf{x}^i) := \sum_{(i,j)\in L^i} v_j^i(x_j^i) \text{ for all } \mathbf{x}^i \in \mathcal{X}^i,$$

*and for all* $j \in N$*, the order* $\succeq_j$ *is represented by the function* $V_j$ *defined by*

$$V_j(\mathbf{x}_j) := \sum_{(i,j)\in L_j} v_j^i(x_j^i) \text{ for all } \mathbf{x}_j \in \mathcal{X}_j.$$

*In these representations, the functions* $\{v_j^i\}_{i,j\in N}$ *are unique up to positive affine transformations with a common multiplier.*

**(b)** *If* $\mathcal{X}$ *is permutation-invariant, then there is a single open interval* $\mathcal{X}_*^* \subseteq \mathbb{R}$ *such that* $\mathcal{X}_j^i = \mathcal{X}_*^*$ *for all* $(i,j) \in L$*. If the order* $\succeq$ *is as in part (a), then it also satisfies* Impartiality *if and only if there is a single continuous increasing function* $v : \mathcal{X}_*^* \longrightarrow \mathbb{R}$ *such that* $v_j^i = v$ *for all* $(i,j) \in L$*. Thus, the representations in part (a) simplify to* $\sum_{(i,j)\in L'} v(x_j^i)$ *(where* $L'$ *is either* $L^i$*, or* $L_i$*, or* $L$*, as appropriate).*

In words, under the revised monotonicity conditions, the ranking of economic integration takes the form of an additively separable function, which sums up quantities evaluating the flows between every pair of distinct sectors in the economy. Furthermore, if the ranking treats all sectors the same, then these basic quantities are obtained from a single function. While the existing theory of input-output analysis approaches economic integration in terms of *discrete* concepts, the main of which is the algebraic decomposability of matrices, we propose a new ranking that is *amenable to a numerical index.*[25]

# 6 Conclusion

The paper has developed a new theory for ranking multiattribute alternatives, which permits multiple applications. The applications covered here are sufficient to illustrate its power, but other applications will be developed elsewhere. Even in the field of normative economics broadly conceived, where the theory originates, there seems to be more room for concrete work. We may put

[25] Compare with Miller and Blair (2009) or Kurz and Salvadori (1995).

GDP time-series, wealth distributions, or systems of interpersonal utility comparison into suitable matrix forms, and each time check whether or not the axioms introduced here meaningfully apply. Some of these cases will raise the loss of dimensionality problem that complicated our application to input-output tables. Not all them will accommodate the special invariant preference and identical spaces axioms that enhanced our treatment of uncertain prospects. So the forthcoming applications are likely to range all the way down from the generic additive separability result in Section 2 to the specific ones in Sections 3, 4 and 5.

## Appendix: Proofs

*Proof of Lemma 1.* Clearly, ROW MONOTONICITY or COLUMN MONOTONICITY imply COORDINATE MONOTONICITY. We show the nontrivial converse. Suppose $\mathcal{X}$ is convex, and satisfies COORDINATE MONOTONICITY; we will show that it satisfies COLUMN MONOTONICITY. Let $j \in M$ and $i \in N$ , and let $\mathbf{x}, \mathbf{y} \in \mathcal{X}_j$. Suppose $x^h = y^h$ for all $h \in M \setminus \{i\}$; we must show that $\mathbf{x} \succeq_j \ \mathbf{y}$ if and only if $x^i \geq y^i$.

**Case 1.** First suppose $\mathcal{X}$ is a box. Then we can find $\widetilde{\mathbf{X}}, \widetilde{\mathbf{Y}} \in \mathcal{X}$ such that $\widetilde{\mathbf{x}}_j = \mathbf{x}$ and $\widetilde{\mathbf{y}}_j = \mathbf{y}$, while $\widetilde{\mathbf{y}}_k = \widetilde{\mathbf{x}}_k$ for all $k \in M \setminus \{j\}$. Thus, we have:

$$(\mathbf{x} \succeq_j \mathbf{y}) \iff \left( \widetilde{\mathbf{X}} \succeq \widetilde{\mathbf{Y}} \right) \iff (\widetilde{x}_j^i \geq \widetilde{y}_j^i) \iff (x^i \geq y^i),$$

as desired, by applying first COLUMN PREFERENCES, then COORDINATE MONOTONICITY, and finally the definition of $\widetilde{\mathbf{X}}$ and $\widetilde{\mathbf{Y}}$.

**Case 2.** Now let $\mathcal{X}$ be any open convex set. Then the coordinate projection $\mathcal{X}_j$ is also open and convex, so the line segment $\mathcal{L}$ between $\mathbf{x}$ and $\mathbf{y}$ is in $\mathcal{X}_j$. For each $\mathbf{z} \in \mathcal{L}$, we can find an open box $\mathcal{B}_\mathbf{z} \subseteq \mathcal{X}_j$ that contains $\mathbf{z}$, and an open box $\widetilde{\mathcal{B}_\mathbf{z}} \subseteq \mathcal{X}$ that projects onto $\mathcal{B}_\mathbf{z}$. Apply the argument from *Case 1* to $\widetilde{\mathcal{B}_\mathbf{z}}$ to show that $\succeq_j$ satisfies COLUMN MONOTONICITY when restricted to $\mathcal{B}_\mathbf{z}$. Since $\mathcal{L}$ is compact, it can be covered with a finite collection of boxes like $\mathcal{B}_\mathbf{z}$, and $\succeq_j$ satisfies COLUMN MONOTONICITY on each one. An inductive argument leads one to conclude that $\mathbf{x} \succeq_j \ \mathbf{y}$ if and only if $x^i \geq y^i$.

The proof of ROW MONOTONICITY is similar, only using ROW PREFERENCES instead of COLUMN PREFERENCES.□

*Proof of Example (a) just above Proposition 2.* Without loss of generality, we can take $\mathcal{X} = (0,1)^{N \times M}$. Fix $\mathbf{X} \in \mathcal{X}$, letting $\mathcal{Y} := \{\mathbf{Y} \in \mathcal{X};\ \mathbf{Y} \approx \mathbf{X}\}$. Given$\mathbf{Y}_1, \mathbf{Y}_2 \in \mathcal{Y}$, we must find a path in $\mathcal{Y}$ connecting$\mathbf{Y}_1$ to $\mathbf{Y}_2$.

Define $\mathbf{1} \in \mathbb{R}^{N \times M}$ by setting $\mathbf{1}_j^i := 1$ for all $i \in N$ and $j \in M$. By CONTINUITY and COORDINATE MONOTONICITY, there exists $r_\mathbf{1} \in (0,1)$ such that $r_\mathbf{1}\, \mathbf{1} \in \mathcal{Y}$. Let $\mathcal{Z}_1 \subset \mathcal{X}$ be the open line segment from $\mathbf{Y}_1$ to $\mathbf{1}$. For all $\mathbf{Z} \in \mathcal{Z}_1$, COORDINATE MONOTONICITY implies that $\mathbf{Z} \succeq \mathbf{Y}_1$. Again by CONTINUITY and COORDINATE MONOTONICITY, there exists $r_{\mathbf{Z}} \in (0,1]$ such that $r_{\mathbf{Z}}\,\mathbf{Z} \in \mathcal{Y}$. The set $\mathcal{L}_1 := \{r_{\mathbf{Z}}\,\mathbf{Z};\ \mathbf{Z} \in \mathcal{Z}_1\}$ is a continuous path in $\mathcal{Y}$ from $\mathbf{Y}_1$ to $r_{\mathbf{1}}\,\mathbf{1}$.

Likewise, a continuous path $\mathcal{L}_2$ can be found in $\mathcal{Y}$ from $\mathbf{Y}_2$ to $r_{\mathbf{1}}\,\mathbf{1}$. A path in $\mathcal{Y}$ from $\mathbf{Y}_1$ to $\mathbf{Y}_2$ results from joining it to $\mathcal{L}_1$. □

The proof of Proposition 2 is based on the Debreu-Gorman theory of additive representations for separable preference orders, which requires some background. Let $I$ be an indexing set (e.g. $I = N \times M$), let $\mathcal{Y}$ be an open subset of $\mathbb{R}^I$, and for all $i \in I$, let $\mathcal{Y}_i$ be the projection of $\mathcal{Y}$ onto the $i$-th coordinate. A preference order $\succeq$ on $\mathcal{Y}$ has a *fully additive representation* if there exist functions $u_i : \ \mathcal{Y}_i \longrightarrow \mathbb{R}$, for all $i \in I$, such that if we define $U : \ \mathcal{Y} \longrightarrow \mathbb{R}$ by

$$U(\mathbf{y}) \quad := \quad \sum_{i \in I} u_i(y_i),$$

then $U$ represents $\succeq$.

For any $\mathbf{y} \in \mathcal{Y}$, we say that $\succeq$ admits a *fully additive representation near $y$* if there is an open neighbourhood $\mathcal{Y}' \subseteq \mathcal{Y}$ around $\mathbf{y}$, such that $\succeq$ admits a fully additive representation when restricted to $\mathcal{Y}'$. We will use the following result.

**Lemma A1** *Let $\mathcal{Y}$ be an open, connected, sectionally connected subset of $\mathbb{R}^I$, and let $\succeq$ be a continuous, indifference-connected preference order on $\mathcal{Y}$, which is strictly increasing in every coordinate. If $\succeq$ admits a fully additive representation near every $y \in \mathcal{Y}$, then $\succeq$ admits a fully additive representation on $\mathcal{Y}$. Furthermore, this global additive representation is unique up to a positive affine transformation.*

*Proof.* See Theorem 2.2 of Chateauneuf and Wakker (1993). □

Let $J \subseteq I$, and let $K := I \setminus J$. For any $\mathbf{y} \in \mathcal{Y}$, define $\mathbf{y}_J := [y_j]_{j \in J}$ (an element of $\mathbb{R}^J$) and $\mathbf{y}_K := [y_k]_{k \in K}$ (an element of $\mathbb{R}^K$). We say that $\succeq$ is *$J$-separable* (or that $J$ is a *$\succeq$-separable* subset of $I$) if the following holds. For all $\mathbf{x}, \mathbf{y}, \mathbf{x}', \mathbf{y}' \in \mathcal{Y}$, if

$$\begin{array}{rclcrcl} \mathbf{x}_K & = & \mathbf{y}_K, & & \mathbf{x}_J & = & \mathbf{x}'_J, \\ \mathbf{x}'_K & = & \mathbf{y}'_K, & \text{and} & \mathbf{y}_J & = & \mathbf{y}'_J, \end{array}$$

then $(\mathbf{x} \succeq \mathbf{y}) \iff (\mathbf{x}' \succeq \mathbf{y}')$. We say that $\succeq$ is *totally separable* if every subset $J \subseteq I$ is $\succeq$-separable. A well-known result applies these concepts to the case where $\mathcal{Y}$ is an open box.

**Lemma A2** *If $\succeq$ is a continuous, totally separable preference order on an open box $\mathcal{B} \subseteq \mathbb{R}^I$, and $\succeq$ is increasing in every coordinate, then $\succeq$ has a fully additive utility representation.*

*Proof.* See Theorem 3 in Debreu (1960). □

Let $J \subseteq I$ and $K := I \setminus J$. We say that $J$ is *strictly $\succeq$-essential* if, for any $\mathbf{y} \in \mathcal{Y}$, there exist $\mathbf{x}, \mathbf{x}' \in \mathcal{Y}$ such that $\mathbf{x}_K = \mathbf{x}'_K = \mathbf{y}_K$, but $\mathbf{x} \succ \mathbf{x}'$. (In words, it is possible to create a strict preference by only manipulating the $J$ coordinates, while keeping the $K$ coordinates fixed at any stipulated values.)

**Lemma A3** *Let $\succeq$ be a continuous preference order on an open box $\mathcal{B} \subseteq \mathbb{R}^I$. Let $J, K \subseteq I$ be two $\succeq$-separable subsets, such that $J \cap K \neq \emptyset$. Suppose that $J$, $K$, and $J \cap K$ are all strictly $\succeq$-essential. Then:*

**(a)** *$J \cup K$ is $\succeq$-separable.*

**(b)** *$J \cap K$ is $\succeq$-separable.*

*Proof.* See Theorem 1 by Gorman (1968b) for the original result, Theorem 4.7 of Blackorby et al. (1978) for a restatement, and Theorem 11 and Proposition 16 of von Stengel (1993) for the most general treatment. □

Now, for any $i \in N$, define $M_i := \{(i,j);\ j \in M\}$. We can write $\mathbb{R}^{N\times M} = \mathbb{R}^{M_1} \times \mathbb{R}^{M_2} \times \cdots \times \mathbb{R}^{M_n}$. For any $j \in M$, define $N_j := \{(i,j);\ i \in N\}$. Similarly, we can write $\mathbb{R}^{N\times M} = \mathbb{R}^{N_1} \times \mathbb{R}^{N_2} \times \cdots \times\ \mathbb{R}^{N_m}$.

**Lemma A4** *As in Lemma A3, let $\succeq$ be a continuous preference order on an open box $\mathcal{B} \subseteq \mathbb{R}^{N\times M}$. For all $i \in N$ and $j \in M$, suppose the sets $N_j$ and $M_i$ are $\succeq$-separable, and the set $\{(i,j)\}$ is $\succeq$-strictly essential. Then $\succeq$ is totally separable.*

*Proof.* Clearly, the union of two strictly $\succeq$ -essential subsets of $N \times M$ is strictly essential. Since every singleton subset of $N \times M$ is strictly $\succeq$-essential, it follows that every subset of $N \times M$ is strictly $\succeq$-essential.

To show from the assumptions that $\succeq$ is totally separable, consider the cases of singleton and doubleton subsets of $N \times M$. Singletons $\{(i,j)\}$ are intersections of the $\succeq$-separable subsets $M_i$ and $N_j$, hence $\succeq$-separable by Lemma A3(b). A slightly more roundabout application of Lemma A3 shows that doubletons are $\succeq$-separable. Finally, prove that *any* subset $J \subseteq N \times M$ is $\succeq$-separable, by induction on $|J|$, doubleton separability, and Lemma A3(a). (See also Corollary to Theorem 3.7 in Keeney and Raiffa, 1976.) □

*Remark.* To show that doubletons are separable in the proof of Lemma A4, we need $n \geq 2$ and $m \geq 2$. This is the key place in the proofs where this assumption is necessary.

*Proof of Proposition 2.* (a) Given $\mathbf{X} \in \mathcal{X}$, there is an open box $\mathcal{B}$ of $\mathbb{R}^{N\times M}$ such that $\mathbf{X} \in \mathcal{B} \subseteq \mathcal{X}$. We first show that if $\succeq$ is restricted to $\mathcal{B}$, then it is $M_i$-separable for all $i \in N$. Let $\mathbf{Y}, \mathbf{Z}, \overline{\mathbf{Y}}, \overline{\mathbf{Z}} \in \mathcal{B}$, and suppose that (a) $\mathbf{y}^h = \mathbf{z}^h$ for all $h \in N \setminus \{i\}$, (b) $\mathbf{y}^i = \overline{\mathbf{y}}^i$, (c) $\overline{\mathbf{y}}^h = \overline{\mathbf{z}}^h$ for all $h \in N \setminus \{i\}$, and (d) $\mathbf{z}^i = \overline{\mathbf{z}}^i$. Then

$$(\mathbf{Y} \succeq \mathbf{Z}) \iff \left(\mathbf{y}^i \succeq^i \mathbf{z}^i\right) \iff \left(\overline{\mathbf{y}}^i \succeq^i \overline{\mathbf{z}}^i\right) \iff \left(\overline{\mathbf{Y}} \succeq \overline{\mathbf{Z}}\right),$$

showing that $\succeq$ is $M_i$-separable. (The first equivalence is by (a) and Row Preferences, the second by (b) and (d), and the last one by (c) and Row Preferences.)

By a similar argument based on Column Preferences, if $\succeq$ is restricted to $\mathcal{B}$, then it is $N_j$-separable for all $j \in M$.

It remains to show that $\succeq$ has a fully additive representation on $\mathcal{B}$. By CONTINUITY, $\succeq$ is continuous on $\mathcal{B}$. COORDINATE MONOTONICITY implies that every coordinate is strictly essential. We have just shown that $M_i$ and $N_j$ are separable for all $i$ and $j$; thus Lemma A4 implies that $\succeq$ is totally separable on $\mathcal{B}$. Finally, Lemma A2 and COORDINATE MONOTONICITY yield an additive representation of $\succeq$ on $\mathcal{B}$. This proves part (a) with $\mathcal{Y} = \mathcal{B}$.

**Proof of (b).** This follows from part (a), along with COORDINATE MONOTONICITY, CONTINUITY and Lemma A1.[26]. (Alternately, we could have directly proved (b) by applying Theorem 1 of Segal (1992).)

**Proof of (d).** Fix $\mathbf{X} \in \mathcal{X}$, and consider the section of $\mathcal{X}$ in the $j$th dimension through $\mathbf{X}$, as defined by:

$$\mathcal{S}_j(\mathbf{X}) \quad := \quad \{\mathbf{Y} \in \mathcal{X}\ ;\ \mathbf{y}_k = \mathbf{x}_k,\ \text{for all } k \in M \setminus \{j\}\}.$$

Let $\mathcal{X}_j(\mathbf{X}) := \{\mathbf{y}_j;\ \mathbf{Y} \in \mathcal{S}_j(\mathbf{X})\} \subseteq \mathcal{X}_j$. COLUMN PREFERENCES implies that $\succeq$, when restricted to $\mathcal{S}_j(\mathbf{X})$, is equivalent to $\succeq_j$ on $\mathcal{X}_j(\mathbf{X})$. Thus, part (b) implies that the order $\succeq_j$ on $\mathcal{X}_j(\mathbf{X})$ is represented by the function $U_j^{\mathbf{X}}$ defined by

$$U_j^{\mathbf{X}}(\mathbf{y}) \quad := \quad \overbrace{\sum_{k \in M \setminus \{j\}} \sum_{i \in N} u_k^i(x_k^i)}^{\text{a constant}} + \sum_{i \in N} u_j^i(y^i),$$

for all $\mathbf{y} \in \mathcal{X}_j(\mathbf{X})$. Thus, the function $U_j := \sum_{i \in N} u_j^i(y^i)$ also represents $\succeq_j$ on $\mathcal{X}_j(\mathbf{X})$. This holds for all $\mathbf{X} \in \mathcal{X}$; thus $U_j$ represents $\succeq_j$ on $\mathcal{X}_j = \bigcup_{\mathbf{X} \in \mathcal{X}} \mathcal{X}_j(\mathbf{X})$.

**Proof of (c).** Similar to the proof of (d), only using ROW PREFERENCES instead of COLUMN PREFERENCES. □

To prove Theorem 3, we must solve a Pexider functional equation on a general domain. The solution is provided by the following result.

**Lemma A5** *Let $\mathcal{Y} \subseteq \mathbb{R}^J$ be an open, connected set. For all $j \in [1 \ldots J]$, let $\mathcal{Y}_j$ be the projection of $\mathcal{Y}$ onto the $j$th coordinate, and let $\mathcal{Y}_0 := \{\sum_{j=1}^J y_j;\ \mathbf{y} \in \mathcal{Y}\}$.[27] For all $j \in [0 \ldots J]$, let $f_j : \mathcal{Y}_j \longrightarrow \mathbb{R}$ be functions, at least one of which is continuous, and suppose they satisfy the Pexider equation:*

$$f_0\left(\sum_{j=1}^{J} y_j\right) \quad = \quad \sum_{j=1}^{j} f_j(y_j), \qquad \text{for all } \mathbf{y} \in \mathcal{Y}.$$

*Then there exist (unique) constants $a, b_0, b_1, b_2, \ldots, b_J \in \mathbb{R}$ such that $b_0 = \sum_{j=1}^J b_j$, and such that, for all $j \in [0 \ldots J]$, $f_j(y) = a\,y + b_j$ for all $y \in \mathcal{Y}_j$.*

[26] This is the one place in the proof that makes use of sectional connectedness and indifference connectedness.

[27] Thus, $\mathcal{Y}_0, \mathcal{Y}_1, \ldots, \mathcal{Y}_J$ are all open intervals in $\mathbb{R}$.

*Proof.* See Theorem 1 and Corollary 2 in Radó and Baker (1987) . □

*Proof of Theorem 3.* The "if" direction is obvious; we will prove the "only if" direction.

**Proof of (a).** This follows from adapting the representations in Proposition 2(d) to the fact that $\mathcal{X}$ now satisfies *Identical Column Spaces* and $\succeq$ now satisfies COLUMN INVARIANCE. (Specifically, for all $i \in N$, and all $\mathbf{x} \in \mathcal{X}_M$, define $u^i(x^i) := u^i_1(x^i)$, and put $W_M = U_1$, where $U_1$ is defined by setting $j = 1$ in Proposition 2(d).)

To prove parts (b)-(d), we we need the following claim.

**Claim 1:** *For any $j \in M$, there exist constants $a_j > 0$ and $b^i_j \in \mathbb{R}$ such that $u^i_j(x^i) = a_j\, u^i_1(x^i) + b^i_j$ for all $\mathbf{x} \in \mathcal{X}_M$ and $i \in N$.*

*Proof.* By *Identical Column Spaces*, $\mathcal{X}_M$ is the same as $\mathcal{X}_j$ for any $j \in M$, so it is an open and connected set of $\mathbb{R}^n$ by the usual properties of the projection map. Let $j \in M$, and let $U_1$ and $U_j$ be as in Proposition 2(d). By COLUMN INVARIANCE, both $U_1$ and $U_j$, represent $\succeq_M$ on $\mathcal{X}_M$. Thus, there are continuous, increasing transformations $g_j : \mathbb{R} \longrightarrow \mathbb{R}$ such that $U_j = g_j \circ U_1$, or

$$\sum_{i \in N} u^i_j(x^i) \quad = \quad g_j\left(\sum_{i \in N} u^i_1(x^i)\right), \quad \text{for all } \mathbf{x} \in \mathcal{X}_M. \tag{A1}$$

The image set $\mathcal{Z} := \{(u^1_1(x^1), \ldots, u^n_1(x^n))\ ;\ \mathbf{x} \in \mathcal{X}_M\}$ is also open and connected in $\mathbb{R}^N$, because the $u^i_1$ are continuous and increasing, hence open.[28] If we make the change of variables $z^i := u^i_1(x^i)$ for all $i \in N$, then (A1) becomes the Pexider equation:

$$\sum_{i \in N} u^i_j \circ (u^i_1)^{-1}(z^i) \quad = \quad g_j\left(\sum_{i \in N} z_i\right), \quad \text{for all } \mathbf{z} \in \mathcal{Z}.$$

Lemma A5 applied to $\mathcal{Z}$ yields constants $a_j$ and $b^1_j, \ldots, b^n_j \in \mathbb{R}$ such that $u^i_j \circ (u^i_1)^{-1}(z^i) = a_j\, z^i + b^i_j$ for all $\mathbf{z} \in \mathcal{Z}$ and all $i \in N$, hence such that $u^i_j(x^i) = a_j\, u^i_1(x^i) + b^i_j$ for any $\mathbf{x} \in \mathcal{X}_M$. Finally, $a_j > 0$ because $u^i_j$ and $u^i_1$ are both increasing. ♢ `Claim 1`

**Proof of (c).** Let $A := \sum_{j \in M} a_j$ and $p_j := a_j/A$ for all $j \in M$, so that $\mathbf{p} = (p_1, \ldots, p_m)$ is a strictly positive weight vector on $M$ . Claim 1 implies that, for all $i \in N$ and $j \in M$, and all $\mathbf{X} \in \mathcal{X}$,

$$u^i_j(x^i_j) \quad = \quad A\, p_j\, u^i(x^i_j) + b^i_j. \tag{A2}$$

[28] Any function $\phi$ from an open subset of $\mathbb{R}$ to $\mathbb{R}$ that is continuous and increasing is also open. We will make repeated use of this property.

If we let $U : \mathcal{X} \longrightarrow \mathbb{R}$ be as in Proposition 2(a,b), and define $B := \sum_{i\in N}\sum_{j\in M} b^i_j$, then for all $\mathbf{X} \in \mathcal{X}$,

$$\begin{aligned} U(\mathbf{X}) &= \sum_{i\in N}\sum_{j\in M} u^i_j(x^i_j) \quad = \quad A\cdot\sum_{i\in N}\sum_{j\in M} p_j\, u^i(x^i_j) + \sum_{i\in N}\sum_{j\in M} b^i_j \\ &= A\cdot\sum_{j\in M} p_j \left(\sum_{i\in N} u^i(x^i_j)\right) + B \quad = \quad A\cdot W(\mathbf{X}) + B, \end{aligned}$$

where $W$ is defined as in equation (3). Thus, $W$ is an increasing transform of $U$, so it represents $\succeq$ on $\mathcal{X}$.

**Proof of (b).** Let $U^i$ be as in Proposition 2(c). Then for all $\mathbf{x} \in \mathcal{X}^i$, we have

$$U^i(\mathbf{x}) \;=\; \sum_{j\in M} u^i_j(x_j) \;=\; A\sum_{j\in M} p_j\, u^i(x_j) + \sum_{j\in M} b^i_j \;=\; A\, U^i_{\mathbf{p}}(\mathbf{x}) + [\text{a constant}],$$

where the second equality is by (A2). Thus, $U^i_{\mathbf{p}}$ represents $\succeq^i$.

**Proof of (d).** For all $i \in N$, let $\widetilde{u}^i : \mathbb{R}\longrightarrow\mathbb{R}$ be a continuous and increasing function, and let $\widetilde{\mathbf{p}} \in \Delta_M$ be a strictly positive weight vector. Suppose that $\succeq_M$ is represented by the function $\widetilde{W}_M : \mathcal{X}_M \longrightarrow \mathbb{R}$ defined by

$$\widetilde{W}_M(\mathbf{x}) \quad := \quad \sum_{i\in N} \widetilde{u}^i(x^i), \qquad \text{for all } \mathbf{x} \in \mathcal{X}_M.$$

and that $\succeq$ is also represented by the function $\widetilde{W} : \mathcal{X}\longrightarrow\mathbb{R}$ defined by

$$\widetilde{W}(\mathbf{X}) \quad := \quad \sum_{j\in M}\sum_{i\in N} \widetilde{p}_j\, \widetilde{u}^i(x^i_j), \qquad \text{for all } \mathbf{X} \in \mathcal{X}.$$

Now, $\sum_{i\in N} \widetilde{u}^i(x^i) = g(\sum_{i\in N} u^i(x^i))$ for some increasing and continuous transformation $g$.[29] Thus carrying the same functional equation argument as for Claim 1, we conclude that there are constants $a > 0$ and $b^1, \ldots, b^n \in \mathbb{R}$ such that

$$\widetilde{u}^i(x^i) \quad = \quad a\, u^i(x^i) + b^i, \tag{A3}$$

for all $i \in I$ and $\mathbf{x} \in \mathcal{X}_M$. Thus, $u^1, \ldots, u^n$ are unique up to a common affine transformation, as was to be proved.

Meanwhile, the uniqueness part of Proposition 2(a) yields constants $A > 0$ and $b^i_j \in \mathbb{R}$, for all $i \in N$ and $j \in M$, such that

$$\widetilde{p}_j\, \widetilde{u}^i(x^i_j) \quad = \quad A\, p_j u^i(x^i_j) + b^i_j, \tag{A4}$$

[29] If $f$ and $h$ are continuous real-valued functions on some connected subset $B \subseteq \mathbb{R}$, and $g$ is an increasing real-valued function such that $h = g \circ f$, then $g$ is continuous on $f(B)$. We will make repeated use of this fact.

for all $\mathbf{X} \in \mathcal{X}$, $i \in N$ and $j \in M$. Let $\mathbf{x} \in \mathcal{X}^1$. The set $\mathcal{X}^1$ is open, and $u^1$ is continuous and increasing; thus, there exist some $\varepsilon > 0$ and some $\mathbf{y} \in \mathcal{X}^1$ such that $u^1(x_j) - u^1(y_j) = \varepsilon$ for all $j \in M$. But then, for all $j \in M$,

$$\begin{aligned} a\varepsilon\widetilde{p}_j &= a\widetilde{p}_j\, u^1(x_j) - a\, \widetilde{p}_j\, u^1(y_j) = \widetilde{p}_j\, \widetilde{u}^1(x_j) - \widetilde{p}_j\, \widetilde{u}^1(y_j) && \text{(by Eq.(A3))} \\ &= A\, p_j\, u^1(x_j) - A\, p_j\, u^1(y_j) \quad = \quad A\, \varepsilon\, p_j, && \text{(by Eq.(A4)).} \end{aligned}$$

It follows that $a\,\varepsilon\widetilde{\mathbf{p}} = A\,\varepsilon\mathbf{p}$, and thus $A = a$, since $\mathbf{p}$ and $\widetilde{\mathbf{p}}$ are weight vectors. Thus, $\mathbf{p} = \widetilde{\mathbf{p}}$, which completes the proof of (d). □

*Proof of Corollary 4.* Again, we prove the "only if" direction. Theorem 3(c) says that $\succeq$ is represented by the function $W : \mathcal{X} \longrightarrow \mathbb{R}$ defined by equation ( 3). Now, by the variant of this theorem using Row-independent Preferences and Identical row spaces, there is a weight vector $\mathbf{q} = (q^1, q^2, \ldots, q^n) \in \Delta_N$, and, for all $j \in M$, there is an increasing, continuous function $v_j : \mathcal{X}^*_j \longrightarrow \mathbb{R}$, such that $\succeq^N$ is represented by the function $W^N : \mathcal{X} \longrightarrow \mathbb{R}$ defined by

$$W^N(\mathbf{x}) \quad := \quad \sum_{j \in M} v_j(x_j), \quad \text{for all } \mathbf{x} \in \mathcal{X}^N, \tag{A5}$$

while $\succeq$ is represented by the function $\widetilde{W} : \mathcal{X} \longrightarrow \mathbb{R}$ defined by

$$\widetilde{W}(\mathbf{X}) \quad := \quad \sum_{j \in M} \sum_{i \in N} q^i\, v_j(x^i_j), \quad \text{for all } \mathbf{X} \in \mathcal{X}. \tag{A6}$$

Now fix $x_0 \in \mathcal{X}^*_*$. By Theorem 3(d) and its variant, we can subtract relevant constants from the functions $\{v_j\}_{j \in M}$ and $\{u^i\}_{i \in N}$, to ensure that

$$v_j(x_0) = 0 \text{ for all } j \in M, \text{ and } u^i(x_0) = 0 \text{ for all } i \in N. \tag{A7}$$

Since $\succeq$ is represented by both $W$ and $\widetilde{W}$, there is some continuous, increasing function $f : \mathbb{R} \longrightarrow \mathbb{R}$ such that:

$$f\left(\sum_{j \in M} \sum_{i \in N} p_j\, u^i(x^i_j)\right) \quad = \quad \sum_{j \in M} \sum_{i \in N} q^i\, v_j(x^i_j), \quad \text{for all } \mathbf{X} \in \mathcal{X}. \tag{A8}$$

For all $i \in N$ and $j \in M$, define $g^i_j(\zeta) := q^i\, v_j \circ (u^i)^{-1}(\zeta/p_j)$ for all $\zeta \in \mathbb{R}$ where this definition makes sense. Define $\Xi := \{[p_j u^i(x^i_j)]^{i \in N}_{j \in M};\ \mathbf{X} \in \mathcal{X}\}$, an open, connected subset of $\mathbb{R}^{N \times M}$. Then substituting $\xi^i_j := p_j u^i(x^i_j)$ into both sides of equation (A8) yields

$$f\left(\sum_{j \in M} \sum_{i \in N} \xi^i_j\right) \quad = \quad \sum_{j \in M} \sum_{i \in N} g^i_j(\xi^i_j), \quad \text{for all } \boldsymbol{\xi} \in \Xi.$$

Now Lemma A5 implies that there exists a constant $a > 0$ such that $f(\zeta) = a\,\zeta = g^i_j(\zeta)$ for all $i \in N$ and $j \in M$. (Equation (A7) implies that the added constants of Lemma A5 are all 0.) By rescaling $\{v_j\}_{j\in M}$ if necessary, we can assume that $a = 1$; hence $g^i_j(\zeta) = \zeta$. But $g^i_j(\zeta) = q^i\, v_j \circ (u^i)^{-1}(\zeta/p_j)$, so this implies that $p_j\, u^i = q^i\, v_j$, for all $(i,j) \in N \times M$. Dividing these equations by $q^i\, p_j$ (which are nonzero), we obtain

$$u^i/q^i \quad = \quad v_j/p^j, \quad \text{for all} \quad (i,j) \in N \times M.$$

It follows that there is a single increasing continuous function $u : \mathcal{X}^*_* \to \mathbb{R}$ such that

$$\textbf{(a)}\ \ u^i/q^i = u \ \text{ for all } i \in N \quad \text{and} \quad \textbf{(b)}\ \ v_j/p_j = u \ \text{ for all } j \in M. \qquad \text{(A9)}$$

Substituting equation (A9)(a) into equation (1) yields part (a) of the result. Substituting (A9)(b) into (A5) yields part (b), while substituting (A9)(b) into (A6) yields part (c). Part (d) is straightforward. □

The proof of Theorem 5 relies on the following Lemma.

**Lemma A6** *Let $\mathcal{Z} \subseteq \mathbb{R}^M$ be an open set. For all $j \in M$, let $\mathcal{Z}_j$ be the projection of $\mathcal{Z}$ onto the $j$th coordinate, and let $u_j : \mathcal{Z}_j \longrightarrow \mathbb{R}$ be a continuous increasing function. Define $U(\mathbf{z}) = \sum_{j=1}^m u_j(z_j)$ for all $\mathbf{z} \in \mathcal{Z}$, and let $\succeq$ be the preference order on $\mathcal{Z}$ represented by $U$. Then $\succeq$ is flat if and only if the functions $u_1, \ldots, u_m$ are affine.*

*Proof.* If $u_1, \ldots, u_m$ are affine, then clearly $\succeq$ is flat. To prove the converse, fix $\mathbf{z} \in \mathcal{Z}$, and let $\mathcal{Y}(\mathbf{z}) := \{\mathbf{y} \in \mathcal{Z};\ \mathbf{z} \approx \mathbf{y}\}$ be its indifference surface. If $\succeq$ is flat, then there is some hyperplane $\mathcal{H} \subset \mathbb{R}^M$ such that $\mathcal{Y}(\mathbf{z}) = \mathcal{H} \cap \mathcal{Z}$. The equation of this hyperplane is $\sum_{j=1}^m a_j\, y_j = b$, with all the $a_j$ being non-zero because $u_j$ is increasing and $\mathcal{Z}$ is open in $\mathbb{R}^M$.[30] Without loss of generality, suppose $a_1 = 1$. Then for all $\mathbf{y} \in \mathbb{R}^{[2\ldots m]}$,

$$\left( y_1 \ = \ b - \sum_{j=2}^m a_j\, y_j \right) \implies \Big( (y_1, \mathbf{y}) \in \mathcal{H} \Big).$$

Let $\mathcal{Y}'$ be a connected component of $\mathcal{Y}(\mathbf{z})$; then $\mathcal{Y}'$ is a relatively open subset of $\mathcal{H}$. If $C := U(\mathbf{z})$, then $U(\mathbf{y}) = C$ for all $\mathbf{y} \in \mathcal{Y}'$. Define $\mathcal{Y}'_j$ to be the projection of $\mathcal{Y}'$ onto the $j$th coordinate, and $\mathcal{Y}'_{[2\ldots m]}$ to be the projection of $\mathcal{Y}'$ onto $\mathbb{R}^{[2\ldots m]}$. The set $\mathcal{Y}'_{[2\ldots m]}$ is an open and connected in $\mathbb{R}^{[2\ldots m]}$ by the usual properties of the projection map.

For all $\mathbf{y} \in \mathcal{Y}'_{[2\ldots m]}$, if $y_1 = b - \sum_{j=2}^m a_j\, y_j$, then $(y_1, \mathbf{y}) \in \mathcal{Y}'$ and $U(y_1, \mathbf{y}) = C$. In other words,

$$u_1\left( b - \sum_{j=2}^m a_j\, y_j \right) + \sum_{j=2}^m u_j(y_j) \ = \ C, \qquad \text{for all } \mathbf{y} \in \mathcal{Y}'_{[2\ldots m]}.$$

[30] For example, suppose $a_1 = 0$; then there exists $\epsilon > 0$ such that $\mathbf{y} = (z_1 + \epsilon, z_2, z_3, \ldots, z_M)$ is in $\mathcal{H} \cap \mathcal{Z} = \mathcal{Y}(\mathbf{z})$, and thus, $U(\mathbf{y}) = U(\mathbf{z})$, which contradicts the fact that $u_1$ is increasing.

and this can be rewritten as a Pexider equation:

$$u_1\left(\sum_{j=2}^{m}\widetilde{y_j}\right) \quad = \quad \sum_{j=2}^{m}\widetilde{u_j}\left(\frac{\widetilde{y_j}-b/m}{a_j}\right), \qquad \text{for all } \mathbf{y}\in\mathcal{Y}'_{[2...m]}.$$

by putting $\widetilde{y_j} := \frac{b}{m} - a_j\, y_j$ and $\widetilde{u_j} := \frac{C}{m} - u_j$ for all $j \in [2...m]$. Lemma A5 implies that, for all $j \in M$, the function $\widetilde{u}_j$ is affine on $\mathcal{Y}'_j$. Thus, $u_j$ is affine when restricted to $\mathcal{Y}'_j$.

By repeating this argument for all connected components of $\mathcal{Y}(\mathbf{z})$, and for all $\mathbf{z} \in \mathcal{Z}$, we can cover $\mathcal{Z}_j$ with open subsets such that $u_j$ is affine on each subset. But $\mathcal{Z}$ is connected, so $\mathcal{Z}_j$ also is, and by a standard argument based on path-connectedness, we can conclude that for all $j \in M$, $u_j$ is an affine function on $\mathcal{Z}_j$. $\square$

*Proof of Theorem 5.* We prove the "only if" direction.

**Claim 1:** *For any $\mathbf{X} \in \mathcal{X}$, there exists an open neighbourhood $\mathcal{B}_{\mathbf{X}} \subseteq \mathcal{X}$ and constants $c^i_{j,\mathbf{X}} > 0$ for all $i \in N$ and $j \in M$, such that $\succeq$ is represented on $\mathcal{B}_{\mathbf{X}}$ by the function $U_{\mathbf{X}} : \mathcal{B}_{\mathbf{X}} \longrightarrow \mathbb{R}$ defined by $U_{\mathbf{X}}(\mathbf{B}) := \sum_{i\in N}\sum_{j\in M} c^i_{j,\mathbf{X}} \cdot b^i_j$ for all $\mathbf{B} \in \mathcal{B}_{\mathbf{X}}$. We can assume* $\max\{c^i_{j,\mathbf{X}};\, i \in N \text{ and } j \in M\} = 1$.

*Proof.* Proposition 2(a) yields an open rectangular neighbourhood

$$\mathcal{B}_{\mathbf{X}} = \prod_{i\in N}\prod_{j\in M}\mathcal{B}^i_{j,\mathbf{X}} \subset \mathcal{X},$$

as well as continuous, increasing functions $u^i_{j,\mathbf{X}} : \mathcal{B}^i_{j,\mathbf{X}} \longrightarrow \mathbb{R}$, for all $i \in N$ and $j \in M$, such that $\succeq$ is represented on $\mathcal{B}_{\mathbf{X}}$ by the function $U_{\mathbf{X}}$ defined by $U_{\mathbf{X}}(\mathbf{B}) := \sum_{i\in N}\sum_{j\in M} u^i_{j,\mathbf{X}}(b^i_j)$ for all $\mathbf{B} \in \mathcal{B}_{\mathbf{X}}$ . We consider the two flatness assumptions of the theorem in turn.

**Case 1.** Suppose $\succeq^i$ is flat for all $i \in N$. Let $\mathcal{B}^i_{\mathbf{X}} := \prod_{j\in M} \mathcal{B}^i_{j,\mathbf{X}}$; then $\mathcal{B}^i_{\mathbf{X}} \subseteq \mathcal{X}^i$. Proposition 2(c) says that $\succeq^i$ is represented on $\mathcal{B}^i_{\mathbf{X}}$ by the function $U^i_{\mathbf{X}}$ defined by $U^i_{\mathbf{X}}(\mathbf{b}) := \sum_{j\in M} u^i_{j,\mathbf{X}}(b^j)$ for all $\mathbf{b} \in \mathcal{B}^i{}_{\mathbf{X}}$. Lemma A6 implies that for all $j \in M$, $u^i_{j,\mathbf{X}}$ is affine on $\mathcal{B}^i_{j,\mathbf{X}}$; i.e., that for all $j \in M$ , there exist constants $c^i_{j,\mathbf{X}} > 0$ and $d^i_{j,\mathbf{X}} \in \mathbb{R}$ such that $u^i_{j,\mathbf{X}}(b) = c^i_{j,\mathbf{X}}\, b + d^i_{j,\mathbf{X}}$ for all $b \in \mathcal{B}^i_{j,\mathbf{X}}$. Without loss of generality, we can set $d^i_{j,\mathbf{X}} = 0$ in these equations. By the first assumption of the theorem, they hold for all $i \in N$. Noting that $c^i_{j,\mathbf{X}} > 0$ by Coordinate Monotonicity, we can multiply the coefficients $\{c^i_{j,\mathbf{X}}\}^{i\in N}_{j\in M}$ by a positive constant without changing the representations, and thus ensure that

$$\max\{c^i_{j,\mathbf{X}}; i \in N, j \in M\} = 1.$$

**Case 2.** Suppose $\succeq_M$ is flat. Fix $j \in M$, and let $\mathcal{B}^{\mathbf{X}}_j := \prod_{i\in N} \mathcal{B}^i_{j,\mathbf{X}}$; then $\mathcal{B}^{\mathbf{X}}_j \subseteq \mathcal{X}_j$. Proposition 2(d) and Column Invariance imply that $\succeq_M$ is represented on $\mathcal{B}^{\mathbf{X}}_j$ by the function $U^{\mathbf{X}}_j$ defined by $U^{\mathbf{X}}_j(\mathbf{b}) := \sum_{i\in N} u^i_{j,\mathbf{X}}(b_i)$ for all $\mathbf{b} \in \mathcal{B}^{\mathbf{X}}_j$.

Lemma A6 implies that for all $j \in M$, $u^i_{j,\mathbf{X}}$ is affine on $\mathcal{B}^i_{j,\mathbf{X}}$. This holds for all $j \in M$. Now proceed as in *Case 1.* $\diamondsuit$ `Claim 1`

**Claim 2:** *There exist constants $c^i_j > 0$ for all $i \in N$ and $j \in M$, such that $\succeq$ is represented on $\mathcal{X}$ by the function $U : \mathcal{X} \longrightarrow \mathbb{R}$ defined by $U(\mathbf{X}) := \sum_{i \in N} \sum_{j \in M} c^i_j \cdot x^i_j$ for all $\mathbf{X} \in \mathcal{X}$.*

*Proof.* Take $\mathbf{X}, \mathbf{X}' \in \mathcal{X}$ and the associated rectangular neighbourhoods $\mathcal{B}_{\mathbf{X}}$ and $\mathcal{B}_{\mathbf{X}'}$ of Claim 1, supposing that $\mathcal{B}' = \mathcal{B}_{\mathbf{X}} \cap \mathcal{B}_{\mathbf{X}'} \neq \emptyset$. Then Claim 1 implies that the functions $U_{\mathbf{X}}$ and $U_{\mathbf{X}'}$, defined by $U_{\mathbf{X}}(\mathbf{B}) = \sum_{i \in N} \sum_{j \in M} c^i_{j,\mathbf{X}} \cdot b^i_j$ and $U_{\mathbf{X}'}(\mathbf{B}) = \sum_{i \in N} \sum_{j \in M} c^i_{j,\mathbf{X}'} \cdot b^i_j$ for all $\mathbf{B} \in \mathcal{B}'$, both represent $\succeq$ on $\mathcal{B}'$, so they are ordinally equivalent on $\mathcal{B}'$. Thus, there is some continuous, increasing function $g : \mathbb{R} \longrightarrow \mathbb{R}$ such that:

$$g\left(\sum_{i \in N} \sum_{j \in M} c^i_{j,\mathbf{X}} \cdot b^i_j\right) \quad = \quad \sum_{i \in N} \sum_{j \in M} c^i_{j,\mathbf{X}'} \cdot b^i_j, \quad \text{for all } \mathbf{B} \in \mathcal{B}'.$$

Lemma A5 for this Pexider equation yields $A > 0$ such that $c^i_{j,\mathbf{X}'} b^i_j = A\, c^i_{j,\mathbf{X}} b^i_j$ for all $i \in N$ and $j \in M$ and $\mathbf{B} \in \mathcal{B}'$. Since $\mathcal{B}'$ is open in $\mathbb{R}^{N \times M}$, we may divide by $b^i_j$ in each of these equations, and conclude that $c^i_{j,\mathbf{X}'} = A\, c^i_{j,\mathbf{X}}$ for all $i \in N$ and $j \in M$. However, $\max\{c^i_{j,\mathbf{X}};\ i \in N \text{ and } j \in M\} = 1 = \max\{c^i_{j,\mathbf{X}'};\ i \in N \text{ and } j \in M\}$, hence $A = 1$, and $c^i_{j,\mathbf{X}} = c^i_{j,\mathbf{X}'}$ for all $i \in N$ and $j \in M$.

Now, by another argument based on the path-connectedness of $\mathcal{X}$, we cancel the dependence on $\mathbf{X}$ in the $c^i_{j,\mathbf{X}}$coefficients and conclude that $\succeq$ is represented on all of $\mathcal{X}$ by the function $U$ defined as above. $\diamondsuit$ `Claim 2`

**Claim 3:** *For all $j \in M$, the order $\succeq_M$ is represented on $\mathcal{X}_j$ by the function $U_j$ defined by $U_j(\mathbf{x}) := \sum_{i \in N} c^i_j\, x^i$, for all $\mathbf{x} \in \mathcal{X}_j$.*

*Proof.* The proof is the same as for Proposition 2(d), but with Column Invariance. $\diamondsuit$ `Claim 3`

**Claim 4:** *For all $j, k \in M$, if $\mathcal{X}_j \cap \mathcal{X}_k \neq \emptyset$, then there exists a constant $a_{jk} > 0$ such that $c^i_j = a_{jk}\ c^i_k$ for all $i \in N$.*

*Proof.* Let $\mathcal{X}_{jk}$ be any connected component of $\mathcal{X}_j \cap \mathcal{X}_k$. Claim 3 implies that the functions $U_j$ and $U_k$ both represent $\succeq_M$ on $\mathcal{X}_{jk}$. Thus, they are ordinally equivalent, yielding another Pexider equation. Just as in the proof of Claim 2, we can use Lemma A5 to find $a_{jk} > 0$ such that $c^i_j\, x^i = a_{jk}\, c^i_k\, x^i$ for all $\mathbf{x} \in \mathcal{X}_{jk}$ and $i \in N$. Since $\mathcal{X}_{jk}$ is an open subset of $\mathbb{R}^M$, this implies that $c^i_j = a_{jk}\, c^i_k$ for all $i \in N$. $\diamondsuit$ `Claim 4`

**Claim 5:** *For all $j, k \in M$, then there exists a constant $a_{jk} > 0$ such that $c^i_j = a_{jk}\ c^i_k$ for all $i \in N$.*

*Proof.* Fix $j$ and $k$, and observe that there are a subset of indexes $\Lambda = \{\lambda_1, ..., \lambda_L\} \subseteq M$ with $\lambda_1 = j$, $\lambda_L = k$, such $\mathcal{X}_{\lambda_\ell} \cap \mathcal{X}_{\lambda_{\ell+1}} \neq \varnothing$ for all $\ell \in [1 \ldots L]$.

(This follows from the fact that $\mathcal{X}_M$ is a connected set; we skip the easy topological argument.) Let $a_{j\lambda_2}, a_{\lambda_2\lambda_3}, \ldots, a_{\lambda_{L-1}k} > 0$ be the constants obtained in Claim 4. Then define $a_{jk} := a_{j\lambda_2} \cdot a_{\lambda_2\lambda_3} \cdots a_{\lambda_{L-1}k}$. Then iterated application of Claim 4 yields the result. $\diamondsuit$ `Claim 5`

**Proof of (a).** For all $i \in N$, define $q^i := c_1^i$, and for all $j \in N$, define $p_j := a_{j1}$. Then for all $i \in N$ and $j \in M$, Claim 5 implies that

$$c_j^i \quad = \quad a_{j1}\, c_1^i \quad = \quad p_j\, q^i. \tag{A10}$$

Thus, fixing $j$ and using the definition of $U_j(\mathbf{x})$ in Claim 3, we have that for all $\mathbf{x} \in \mathcal{X}_j$,

$$U_j(\mathbf{x}) \quad = \quad \sum_{i \in N} c_j^i\, x^i \quad = \quad \sum_{i \in N} p_j\, q^i\, x^i \quad = \quad p_j \sum_{i \in N} q^i\, x^i \quad = \quad p_j\, W_M(\mathbf{x}),$$

Then Claim 3 implies that $W_M$ represents $\succeq_M$ on $\mathcal{X}_j$. Since this holds for all $j \in M$, $W_M$ represents $\succeq_M$ on $\mathcal{X}_M = \bigcup_{j \in M} \mathcal{X}_j$.

**Proof of (c).** Define $U$ as in Claim 2. Then, for all $\mathbf{X} \in \mathcal{X}$, equation (A10) yields

$$U(\mathbf{X}) \quad = \quad \sum_{i \in N} \sum_{j \in M} c_j^i \cdot x_j^i \quad = \quad \sum_{i \in N} \sum_{j \in M} q_j\, p^i\, x_j^i \quad = \quad W(\mathbf{X}).$$

Thus, Claim 2 implies that $W$ represents $\succeq$ on $\mathcal{X}$.

**Proofs of (b) and (d).** Similar to the proofs of Theorem 3(b,d). $\Box$

*Proof of Proposition 6.* The "if" direction is obvious; we will prove the "only if" direction. If $n = 2$, then $|L| = 2$. Then an ordering $\succeq$ on $\mathbb{R}^L$ satisfies Coordinate Monotonicity if and only if it is totally separable; the additive representation then follows from Debreu (1960).

So, suppose $n \geq 3$. Then the proof is very similar to the proof of Proposition 2, except that Lemma A4 is replaced with the following claim.

**Claim 1:** *Let $n \geq 3$, and $\succeq$ be a continuous preference order on an open box $\mathcal{B} \subseteq \mathbb{R}^L$. For all $i \in N$ and $j \in M$, suppose the sets $L^i$ and $L_j$ are $\succeq$-separable, and the set $\{(i,j)\}$ is $\succeq$-strictly essential. Then $\succeq$ is totally separable.*

The proof of Claim 1 is very similar to the proof of Lemma A4 (suitably adapted to the $n \times n$ square minus the diagonal). The proof of part (a) now follows the proof of Proposition 2 verbatim, only using Claim 1 in place of Lemma A4. (To see this, observe that the proof of Proposition 2 makes no reference to the structure of the set $N \times M$. Thus, the same argument works if we replace $N \times M$ with $L$. The separability of the subsets $L^i$ and $L_j$ again follows from Row Monotonicity and Column Monotonicity. Lemmas A1 and A2 apply to any abstract Cartesian product.)

**Proof of (b).** Let $\pi \in \Pi$. Since $\pi(\mathcal{X}) = \mathcal{X}$, we have $\mathcal{X}^{\pi(i)}_{\pi(j)} = \mathcal{X}^i_j$ for all $i, j \in N$. Repeating this for all $\pi \in \Pi$, we conclude that $\mathcal{X}^i_j = \mathcal{X}^h_k$ for all pairs $(h, i) \in L$ and $(j, k) \in L$ which are in the same $\Pi$-orbit. But it is easy to see that $\Pi$ acts transitively on $L$. Thus, we obtain $\mathcal{X}^*_* \subseteq \mathbb{R}$ such that $\mathcal{X}^i_j = \mathcal{X}^*_*$ for all $(i, j) \in L$.

Now, for any $\pi \in \Pi$, define $V_\pi := V \circ \pi^{-1} : \mathcal{X} \longrightarrow \mathbb{R}$. Thus, for all $\mathbf{X} \in \mathcal{X}$, we have

$$V_\pi(\mathbf{X}) \quad = \quad \sum_{(i',j') \in L} v^{i'}_{j'}\left(x^{\pi^{-1}(i')}_{\pi^{-1}(j')}\right) \quad = \quad \sum_{(i,j) \in L} v^{\pi(i)}_{\pi(j)}(x^i_j).$$

(Here, the last step is by the change of variables $i := \pi^{-1}(i')$ and $j := \pi^{-1}(j')$, because $\pi$ is a bijection of $N$.) But IMPARTIALITY implies that $V_\pi$ also represents the order $\succeq$. Thus, by uniqueness up to affine transformations, we obtain some constant $a > 0$ and constants $b^i_j$ for all $i, j \in N \times N$ such that $v^{\pi(i)}_{\pi(j)} = a\, v^i_j + b^i_j$ for all $(i, j) \in N$. It follows that $v^{\pi^2(i)}_{\pi^2(j)} = a^2\, v^i_j$+[a constant], and $v^{\pi^3(i)}_{\pi^3(j)} = a^3\, v^i_j$+[a constant], and so on. But $\pi^n$ is the identity map on $L$. Thus, we get $v^i_j = a^n\, v^i_j$+[a constant], which means that $a^n = 1$, which means $a = 1$.

Thus, $v^{\pi(i)}_{\pi(j)} = v^i_j + b^i_j$ for all $(i, j) \in N$. Repeating this argument for all $\pi \in \Pi$, we conclude that there is a single continuous increasing function $v : \mathcal{X}^*_* \longrightarrow \mathbb{R}$ and constants $c^i_j$ for all $(i, j) \in L$ such that $v^i_j = v + c^i_j$ for all $(i, j) \in L$.

Since adding a constant does not change the representation, we can remove the constants $c^i_j$, and assume without loss of generality that $v^i_j = v$ for all $(i, j) \in L$. □

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