Innovation contests*

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Abstract

We study innovation contests with asymmetric information and identical agents, where contestants' efforts and innate abilities generate inventions of varying qualities. The designer offers a reward to the contestant achieving the highest quality and receives the revenue generated by the innovation. We characterize the equilibrium behavior, outcomes and payoffs for both nondiscriminatory and discriminatory (where the reward is agent-dependent) contests. We derive conditions under which discrimination is optimal and describe settings where they are satisfied.

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1 Introduction

Innovative activity has traditionally been rewarded mainly through the patent system. An alternate approach to generating and rewarding innovations is to design contests that solicit proposals to solve targeted objectives (see, for instance, Suzanne Scotchmer, 2004). A sponsor interested in technological improvement can launch a contest where agents compete by submitting prototypes, the best of which will be adopted by the sponsor. Such contests have been held to obtain innovations in various fields including mathematics, food preservation and maritime navigation.[1]

Today, contests are frequently used by organizations, such as the Institute for Advanced Architecture of Catalonia and HP to promote discussion and research ... that can help us to envisage how the city and the habitat of the 21st century will turn out. Similarly, the U.S.-based Knight Foundation has set innovation contests to elicit digital news experiments that inform and engage communities. Contests have also been suggested as a future means to address a variety of issues. For example, Richard G. Newell and Nathan E. Wilson (2005) and Richard G. Newell (2008) proposed that the U.S. Department of Energy should hold contests to resolve specific technical and scientific challenges related to greenhouse gases mitigation.

In this paper, we introduce and analyze a new model of contests with asymmetric information. We describe it as an innovation contest, although it is applicable to many settings. For example, it can be used to study procurement decisions, government contracts, research budgets and promotions.

Our framework is the following. A designer wishes to obtain an innovation that can be produced by two agents. The quality of the innovation achieved by an agent depends on his ability and the effort devoted to the task. The agents' abilities are independently drawn from the same distribution function and once an agent observes his ability he decides on his effort. Both ability and effort are an agent's private information. The designer sets up a contest whereby the highest-quality innovation receives a prize. The contest is called nondiscriminatory if the prize does not depend on the winner's identity, otherwise it is called discriminatory .

[1] Curtis R. Taylor (1995) analyzed the optimal contest in an environment with symmetric information where the quality of the innovation obtained by a firm is a random variable. More recently, the design of optimal contests in an R&D environment has been studied by Yeon-Koo Che and Ian Gale (2003).

We characterize the equilibrium behavior of the two agents in a nondiscriminatory contest for arbitrary distributions of abilities and determine the regions where the equilibrium involves positive effort levels for both agents and those where the contestants exert zero effort. The boundaries of the regions depend on the distribution function of the ability parameter and the prize.

We then consider discriminatory contests where the reward depends on the identity of the winner. We provide several properties of the equilibrium behavior for arbitrary distribution functions. The agents' equilibrium behavior in discriminatory contests again involves regions with zero and positive effort levels. The main new feature is the discontinuity in the behavior of the agent with the larger reward, who moves discretely from zero to a positive effort level. We also provide a full characterization of the equilibrium when the distribution function is either convex or concave.

We conclude by determining conditions under which discrimination is optimal. Discrimination can be optimal in our set up because it leads to an increase in the aggregate effort of the agents, thereby resulting in larger expected revenues, which more than compensate for the increase in cost. In particular, we show that discrimination is optimal if the distribution of the agents' abilities is a convex function, with very low density at zero, and the designer has a high enough valuation of the quality of the innovation. We also find concave distribution functions of an agent's ability for which discrimination is optimal if the designer's valuation is low enough.

Contests in symmetric information environments have been extensively analyzed. Michael R. Baye *et al.* (1996) studied the contestants' equilibrium behavior in standard all-pay auctions with symmetric information where agents bid for an object, all bids are paid and the highest bidder receives the object. Todd R. Kaplan *et al.* (2003) investigated all-pay auctions where the size of the reward depends on the effort. In particular, they applied this framework to an analysis of R&D races. Che and Gale (2003) derived the equilibrium behavior and characterized the optimal research contest in an environment where each contestant submits a quality-price pair. The cost of producing the quality is sunk. The contestant offering the largest surplus, defined as the difference between quality and price, is paid the price. Ron Siegel (2009 and 2010) introduced a general framework encompassing a very large class of all-pay auctions and provided a general method of solving and calculating equilibrium payoffs. Kai Konrad (2009) provided an excellent survey of equilibrium and optimal design in contests.

In scenarios where agents' valuations are private information, Erwin Amann and Wolfgang Leininger (1996) characterized the equilibrium bids for all-pay auctions where valuations are independently drawn from a common distribution function. Vijay Krishna and John Morgan (1997) analyzed the case in which the bidders' information is affiliated. Benny Moldovanu and Aner Sela (2001) considered an environment where an agent's type determines his cost of bidding and agents are privately informed about their type. In their model, the designer's goal is to maximize the sum of efforts by the agents. They show that in the case of a linear or concave cost function, the designer finds it optimal to offer just one prize. However, offering two or more prizes may be optimal for a convex cost function.

Our model differs with respect to the previous literature in several ways. For one thing, the designer chooses the size of the reward and may discriminate among the contestants. Moreover, the asymmetric information is introduced in a novel way in which ability and effort are substitutes. In this sense, the model is suitable for the analysis of common day scenarios of competition where both innate ability and effort generate the final outcome. Last, our finding on the possible optimality of discrimination runs contrary to several studies that show that it is, in general, beneficial to handicap a stronger contestant so as to level the field when starting from an asymmetric contest (Michael R. Baye *et al.*, 1993, Derek J. Clark and Christian Riis, 2000, Che and Gale, 2003, Rene Kirkegaard, 2010). Discriminating among identical agents has also been shown to be optimal in models very different from ours. In particular, Eyal Winter (2004) found that discriminating is optimal in environments where a principal wants to provide several identical agents with incentives to carry out a task .

The rest of the paper proceeds as follows. In Section 2, we introduce the model. Section 3 analyzes agents' equilibrium behavior in nondiscriminatory contests and agents' behavior in discriminatory contests is analyzed in Section 4. Section 5 calculates the designer's payoff as a function of the rewards and shows conditions under which it is optimal to discriminate. Section 6 concludes and proposes directions for further research.

2 The model

We consider the problem facing an organization that wishes to procure an innovation. The benefits derived from this innovation depend on its quality q and are given by $I(q)$, with $I'(q) > 0, I''(q) < 0$.

There are two identical risk-neutral agents A and B who can realize the desired innovation. The quality of the innovation produced by an agent depends on his type and his choice of nonnegative effort. The agents' types represent their proficiency to develop the particular innovation. Both the types and choices of effort are private information. Denoting the type of agent N by θ_N and his effort by e_N, the quality of the innovation realized by agent N is given by $q_N(\theta_N, e_N) = \theta_N + e_N$.

Agents' types are independently distributed according to the same differentiable and atomless distribution function $F(.)$ on $[0, 1]$, with $F'(.) > 0$ for all $\theta \in [0, 1]$. Types are revealed to the agents prior to their choice of effort. The quality of the innovation while observed by the designer cannot be verified. However, an independent authority can verify which innovation is best.

To procure the innovation, the organization holds a *contest* among the two agents. Henceforth, we refer to the organization as the *designer* of the contest. The winner of the contest is the agent who offers the innovation of the highest quality, with ties broken by having each agent win with probability 1/2. The designer may discriminate among the agents and offer different prizes, depending on the identity of the winner. She specifies a prize R_A to agent A if he wins and a prize R_B to agent B were he to win, with $R_A \geq R_B$. A *nondiscriminatory contest* involves $R_A = R_B$.

Given a contest (R_A, R_B), the payoff of agent N when he chooses effort e is $R_N - e$ in case he wins the contest and $-e$ otherwise. The payoff to the designer is $I(q) - R_M$, where q is the quality of the innovation generated by the contest and M is the identity of the agent winning the contest.

The agents' strategies are denoted by two functions $q_A(\theta), q_B(\theta)$ for A and B respectively, with $q_N(\theta) \geq \theta$, where $q_N(\theta)$ indicates the choice of quality by agent N when his type is θ. Given the agents' strategies, the efforts exerted by agent A of type θ_A and agent B of type θ_B will be $e_A(\theta_A) = q_A(\theta_A) - \theta_A$ and $e_B(\theta_B) = q_B(\theta_B) - \theta_B$.

3 Agents' equilibrium strategies in nondiscriminatory contests

In this section, we provide the agents' equilibrium strategies when they compete in a nondiscriminatory contest, that is, $R_A = R_B = R$.

To formulate agent A's maximization problem, we let agent B's strategy be $q_B(\theta)$, and assume it is an increasing and differentiable function.[2] Then, agent A's expected profits when he is of type θ and offers quality $q \geq \theta$ are given by his probability of winning times the prize minus the effort; that is,[3]

$$\Pr_{\widetilde{\theta}}\left(q \geq q_B(\widetilde{\theta})\right) R - (q - \theta).$$

When $q \in [q_B(0), q_B(1)]$,[4] and $q \geq \theta$, agent A's expected profits can be written as

$$F\left(q_B^{-1}(q)\right) R - (q - \theta).$$

If the level of quality q that maximizes agent A's expected profits is interior, that is, $q > \theta$, the following first-order condition (FOC) must hold:

$$q_B'\left(q_B^{-1}(q)\right) = F'\left(q_B^{-1}(q)\right) R. \tag{1}$$

Similarly, the FOC for agent B in the interior case is

$$q_A'\left(q_A^{-1}(q)\right) = F'\left(q_A^{-1}(q)\right) R. \tag{2}$$

These FOCs yield the following system of differential equations that the functions $q_A(\theta)$ and $q_B(\theta)$ must satisfy if both qualities are interior solutions:

$$q_B'\left(q_B^{-1}(q_A(\theta))\right) = F'\left(q_B^{-1}(q_A(\theta))\right) R, \tag{3}$$

[2] Due to the structure of our model, a continuous and monotone equilibrium can be replicated by one where agents' strategies are not monotonic. However, these two equilibria will be payoff equivalent; hence we consider only the continuous and monotone equilibria. Note that the reasoning that Amann and Leininger (1996) use to show that the equilibrium is continuous and monotone cannot be applied here due to the fact that the strategy includes a choice of effort.

[3] Since $q_B(\theta)$ is strictly increasing, we ignore the possibility of ties as they occur with probability zero.

[4] Note that offering qualities above $q_B(1)$ cannot be optimal for agent A since $q = q_B(1)$ already ensures that he wins the contest with probability one. On the other hand, offering qualities below $q_B(0)$ might be optimal if $q_B(0) > 0$.

$$q_A'\left(q_A^{-1}\left(q_B(\theta)\right)\right)=F'\left(q_A^{-1}\left(q_B(\theta)\right)\right)R. \tag{4}$$

The solution of these two equations is given by:

$$q_A(\theta)=F(\theta)R+\eta_A, \tag{5}$$

$$q_B(\theta)=F\left(\theta\right)R+\eta_B, \tag{6}$$

for some $\eta_A,\eta_B\in\mathbb{R}$.

Equations (5) and (6) describe the equilibrium strategies over the range of θ's that lead to an interior solution, where efforts chosen by agents are strictly positive. However, there may be regions of parameters where agents choose corner solutions, where they provide zero effort, $q(\theta)=\theta$. Theorem 1 provides the explicit agents' equilibrium strategies for any differentiable and atomless distribution function $F(.)$. Prior to presenting the theorem, we outline below an intuitive explanation regarding the nature of the equilibrium strategies.

Since the equilibrium strategies are continuous, there are two types of regions for the parameter θ. We let Region I be the region where agents choose positive levels of effort, thus the equilibrium is characterized by equations (5) and (6), with $\eta_A=\eta_B=\eta$ because, as we will show, any equilibrium is symmetric.[5] We let Region C be the one where agents choose zero effort, $q_A(\theta)=q_B(\theta)=\theta$.

To derive the conditions that an equilibrium must satisfy, we consider the case where agents' strategies lie in Region I for $\theta\in[\theta_1,\theta_2)$ and in Region C for $\theta\in[\theta_2,\theta_3]$. First, the continuity of the equilibrium strategies implies that $F(\theta_2)R+\eta=\theta_2$; hence, $\eta=\theta_2-F(\theta_2)R$. Second, for any $\theta\in[\theta_2,\theta_3]$, in order for $q(\theta)=\theta$ to be the optimal choice an agent's profits for this choice cannot be lower than his profits for any $q\in(\theta,\theta_3]$, i.e., $F(\theta)R\geq F(q)R-(q-\theta)$, or

$$F(q)R-q\leq F(\theta)R-\theta \text{ for any } \theta\in\left[\theta_2,\theta_3\right],q\in\left(\theta,\theta_3\right]. \tag{7}$$

Finally, since effort is non-negative it must be the case that $F(\theta)R+\eta\geq\theta$ for any θ in $[\theta_1,\theta_2]$, or

$$F(\theta_2)R-\theta_2\leq F(\theta)R-\theta \text{ for any } \theta\in\left[\theta_1,\theta_2\right]. \tag{8}$$

Equation (7) must be satisfied for any interval of parameters where the equilibrium lies in Region C, thereby implying that the function $F(\theta)R-\theta$ is nonincreasing in such

[5] Region I can be the union of several intervals. If this is the case, the parameter η changes from one interval to another.

an interval. Equation (8) must be satisfied for any interval of parameters where the equilibrium lies in Region I. This equation requires that the value of the function $F(\theta)R-\theta$ at the upper bound of the interval cannot be higher than the value at any other θ in the interval. These conditions are shown to describe an equilibrium in Theorem 1, where the boundaries of the regions are characterized as well.[6] The theorem is stated for the case $F'(0)R-1<0$. After the theorem we indicate the necessary small changes that allow us to formulate the results for all cases.

Theorem 1 *Consider a nondiscriminatory contest with a reward R. Assume that $F'(0)R-1<0$. We define recursively two finite sequences of parameter values:*

(i) $(\alpha_n)_n : \alpha_1$ is the first parameter such that the function $F(\theta)R-\theta$ is increasing to its right; α_n, for $n \geq 2$, is the first parameter such that $F(\theta)R-\theta$ is increasing to its right for which $F(\alpha_n)R-\alpha_n < F(\alpha_{n-1})R-\alpha_{n-1}$. This sequence ends when the global minimum of $F(\theta)R-\theta$ is reached.

(ii) $(\beta_n)_n : \beta_0 = 0$; β_n, for $n \geq 1$, is the first parameter θ larger than α_n that is not a local minimum for which $F(\beta_n)R-\beta_n = F(\alpha_n)R-\alpha_n$. This sequence ends when such a β fails to exist.

We denote by $\widetilde{n}$ the length of the longest of the two sequences $(\alpha_n)_n$ and $(\beta_n)_n$. If $\beta_{\widetilde{n}}$ exists, we let $\alpha_{\widetilde{n}+1}=1$; otherwise, $\beta_{\widetilde{n}}=1$.

Then, the unique symmetric equilibrium where agents' strategies are monotonic is

$$
\begin{aligned}
q^*(\theta) &= F(\theta)R+\alpha_n-F(\alpha_n)R \text{ for } \theta \in [\alpha_n, \beta_n], n \geq 1 \text{ (Region } I\text{)} \\
q^*(\theta) &= \theta \qquad\qquad\qquad\qquad\quad \text{ for } \theta \in [\beta_{n-1}, \alpha_n], n \geq 1 \text{ (Region } C\text{)}.
\end{aligned}
$$

The proposed strategies constitute the unique equilibrium in monotonic strategies if the set $\{\theta \in [0,1] / F'(\theta) = 1/R\}$ has zero measure.

Figure 1 represents the equilibrium effort levels as a function of the parameter θ.

When $F'(0)R-1>0$, Theorem 1 still holds, and the recursive definition of the sequences $(\alpha_n)_n$ and $(\beta_n)_n$ is similar, except that we start with $\alpha_1=0$ and β_0 does not exist. Similarly, if $F'(0)R-1=0$, the sequences will be the same as in Theorem 1 unless $F(\theta)R-\theta>0$ for some interval $(0,\widehat{\theta})$, with $\widehat{\theta}>0$ in which case, the definition of the sequences starts with $\alpha_1=0$.

[6] All proofs are relegated to the appendix.

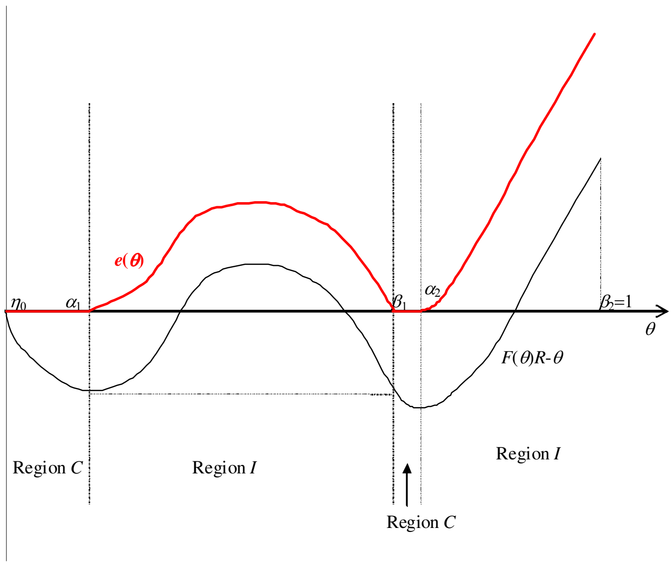

Figure 1: Equilibrium effort in a non-discriminatory contest.

4 Agents' equilibrium strategies in discriminatory contests

We now consider discriminatory contests, where $R_A > R_B$. Formulating the agents' maximization problems in the same manner as in the nondiscriminatory case, we obtain the following FOC for agent A when he exerts a positive effort:

$$q_B'\left(q_B^{-1}(q)\right) = F'\left(q_B^{-1}(q)\right) R_A. \tag{9}$$

Similarly, the FOC for agent B is

$$q_A'\left(q_A^{-1}(q)\right) = F'\left(q_A^{-1}(q)\right) R_B. \tag{10}$$

When both agents exert a positive effort, we obtain a system of differential equations whose solution is given by

$$q_A(\theta) = F(\theta)R_B + \eta_A, \tag{11}$$

$$q_B(\theta) = F(\theta) R_A + \eta_B, \tag{12}$$

for some $\eta_A, \eta_B \in \mathbb{R}$.

As we found in our analysis of the agents' equilibrium strategies in nondiscriminatory contests exerting a positive effort is not always a best response. Thus, in equilibrium there are regions where one or both agents put in zero effort. Furthermore, in discriminatory contests, discontinuities in an agent's strategy cannot be ruled out. Therefore, there may exist a quality interval that is never reached by an agent even though he offers qualities below and above that interval.

We proceed by establishing several properties of the equilibrium strategies. We note first that in equilibrium the qualities offered must satisfy $q_A(1) = q_B(1)$. Moreover, Lemma 1 rules out many possible strategy configurations in equilibrium.

Lemma 1 *Assume the sets* $\{\theta \in [0,1] \,|F'(\theta) = 1/R_A\}$ *and* $\{\theta \in [0,1] \,|F'(\theta) = 1/R_B\}$ *have zero measure. Equilibrium strategies in a contest* (R_A, R_B) *cannot give rise to a nonempty interval of qualities* $(q_1, q_2) \subseteq [\min\{q_A(0), q_B(0)\}, q_A(1)]$ *such that one of the following holds:*

(a) $q_A^{-1}(q) < q$ *and* $q_B^{-1}(q)$ *does not exist, for all* $q \in (q_1, q_2)$*,*
(b) $q_A^{-1}(q)$ *does not exist and* $q_B^{-1}(q) < q$*, for all* $q \in (q_1, q_2)$*,*
(c) both $q_A^{-1}(q)$ *and* $q_B^{-1}(q)$ *do not exist, for all* $q \in (q_1, q_2)$*,*
(d) $q_A^{-1}(q) < q$ *and* $q_B^{-1}(q) = q$*, for all* $q \in (q_1, q_2)$*,*
(e) $q_A^{-1}(q) = q$ *and* $q_B^{-1}(q) < q$*, for all* $q \in (q_1, q_2)$*,*
(f) $q_A^{-1}(q) = q$ *and* $q_B^{-1}(q)$ *does not exist, for all* $q \in (q_1, q_2)$*.*

According to parts (a) and (d), if there is an interval (q_1, q_2) such that $q_A^{-1}(q) < q$ for all $q \in (q_1, q_2)$, it is necessarily the case that also $q_B^{-1}(q) < q$ for all $q \in (q_1, q_2)$. That is, if agent A is offering qualities in the interval (q_1, q_2) by exerting a positive effort, agent B is also exerting a positive effort over that interval. A symmetric conclusion follows from parts (b) and (e). Therefore, an open interval of qualities is offered by agent A through positive effort levels if and only if it is also offered by agent B through positive effort levels. Part (c) states that the support of the qualities offered is connected. Finally, part (f) (together with parts (a) and (c)) implies that the range of qualities offered by agent B is also connected.

As in the nondiscriminatory contests, we call Region I the set of quality levels q that are reached when both agents play according to the interior solution; that is, $q \in I$ if $q_A^{-1}(q) < q$ and $q_B^{-1}(q) < q$. Also, we call Region C the set of qualities reached when both agents put in zero effort; that is, $q \in C$ if $q_A^{-1}(q) = q_B^{-1}(q) = q$. The remaining region corresponds to the quality levels reached by B through zero effort, but never offered by agent A. More precisely, Region J is given by the set of qualities q such that $q_A^{-1}(q)$ does not exist and $q_B^{-1}(q) = q$.

Following Lemma 1, we can conclude that these are the only three possible strategy configurations that can emerge in equilibrium. This is stated as part (a) of the following proposition whereas parts (b) and (c) describe the potential order in which the regions can appear.

Proposition 1 *(a) Consider an equilibrium* $(q_A(\theta), q_B(\theta))$ *of the contest* (R_A, R_B) *and assume the sets* $\{\theta \in [0,1] \,|F'(\theta) = 1/R_A\}$ *and* $\{\theta \in [0,1] \,|F'(\theta) = 1/R_B\}$ *have zero measure. Then the range of qualities offered in equilibrium* $[q_A(0), q_A(1)]$ *can be split into intervals, each of which belongs to either Region* I*,* C*, or* J*.*

(b) Consider an equilibrium where there exists a (maximal) interval (q_1, q_2) *in Region* J*. Then, it must be followed by another interval* (q_2, q_3) *in Region* I*.*

(c) Consider an equilibrium where there exists a (maximal) interval (q_1, q_2) *in Region* I*. Then, it must be preceded by another interval* (q_3, q_1) *in Region* J*.*

Parts (b) and (c) of Proposition 1 imply that Region J and Region I always appear together in equilibrium in the parameter space, with Region I following Region J. Therefore, the range of qualities in equilibrium can be split into two types of intervals, each of which belong to either Region C or Region I_γ, where Region I_γ is given by an interval of (lower) qualities that is never reached by agent A and an interval of (higher) qualities that is reached by both agents having contributed a positive effort. We note that in discriminatory contests, Region I_γ plays a similar role to Region I in nondiscriminatory contests.

So far we have demonstrated several properties of equilibrium strategies in discriminatory contests for general distribution functions. We now proceed by analyzing particular classes of distribution functions for which equilibrium strategies can be fully described in a simple manner.

First, we discuss the possibility of a corner solution. It is intuitive that such a solution would emerge if the prizes allocated in the contest are very small. Proposition 2 goes a step forward and provides a necessary and sufficient condition for Region C to constitute the (only) equilibrium in the contest. In such an equilibrium, both agents choose zero effort for every $\theta \in [0,1]$, a strategy profile that we denote by $\left(q_A^C, q_B^C\right)$, defined as follows:

$$q_A^C(\theta) = q_B^C(\theta) = \theta \text{ for all } \theta \in [0,1].$$

Proposition 2 *The strategy profile $\left(q_A^C, q_B^C\right)$ constitutes an equilibrium of the contest (R_A, R_B) if and only if the function $F(\theta)R_A - \theta$ is non-increasing in θ for all $\theta \in [0,1]$. Moreover, if the set $\{\theta \in [0,1] \,|F'(\theta) = 1/R_A\}$ has zero measure, then no other equilibrium exists.*

Second, it is also intuitive that high rewards would give the agents (at least agent A) incentives to always exert a positive effort. The next proposition states a result that complements Proposition 2, namely that in equilibrium, the qualities offered lie in Region I_γ whenever the function $F(\theta)R_A - \theta$ is nondecreasing for all $\theta \in [0,1]$, which is the case if R_A is high enough. In addition, it provides a necessary and sufficient condition for the equilibrium to always lie in Region I_γ where agents follow the strategy profile $\left(q_A^{I_\gamma}, q_B^{I_\gamma}\right)$, defined as follows:[7]

$$\begin{aligned} q_A^{I_\gamma}(\theta) &= F(\theta)R_B + \gamma && \text{for all } \theta \in [0,1] \\ q_B^{I_\gamma}(\theta) &= \theta && \text{for all } \theta \in [0,\gamma) \\ &= F(\theta)R_A + \gamma - F(\gamma)R_A && \text{for all } \theta \in [\gamma, 1] \end{aligned}$$

where γ solves $R_B = [1 - F(\gamma)]\, R_A$; hence, $\gamma = F^{-1}\left(1\text{-}\frac{R_B}{R_A}\right)$.

Proposition 3 *The strategy profile $\left(q_A^{I_\gamma}, q_B^{I_\gamma}\right)$ constitutes an equilibrium of the contest (R_A, R_B) if and only if the following two conditions hold:*

$$F(\theta)R_A - \theta \leq F(\gamma)R_A - \gamma \textit{ for all } \theta \leq \gamma \tag{13}$$

$$F(\theta)R_A - \theta \geq F(\gamma)R_A - \gamma \textit{ for all } \theta \geq \gamma. \tag{14}$$

In particular, $\left(q_A^{I_\gamma}, q_B^{I_\gamma}\right)$ constitutes an equilibrium if the function $F(\theta)R_A - \theta$ is non-decreasing in θ for all $\theta \in [0,1]$.

[7] The qualities chosen by the agents as well as the cut-off γ in the profile $\left(q_A^{I_\gamma}, q_B^{I_\gamma}\right)$ depend on the particular function $F(.)$ and the rewards R_A and R_B. We do not express this dependence in $\left(q_A^{I_\gamma}, q_B^{I_\gamma}\right)$ to keep the notation simple.

When the function $F(\theta)R_A - \theta$ is increasing in some intervals of θ's and decreasing in others, then the equilibrium will include intervals of quality that lie in Region I_γ and, often, others that lie in Region C. While a full characterization of the equilibrium strategies is cumbersome for general distribution functions, there are large families of distribution functions that allow for quite simple characterizations. Their analysis will also provide robust intuitions on the agents' equilibrium behavior. Next, we will consider two such families that are given by convex and concave distribution functions.

The following theorem characterizes the structure of equilibrium strategies when the function $F(.)$ is convex. We note that for a convex $F(.)$, it is always the case that $F'(0) < 1 < F'(1)$. To simplify the presentation of the theorem, we define the strategy profile $\left(q_A^{CI_\gamma}, q_B^{CI_\gamma}\right)$ as follows:

$$\begin{aligned} q_A^{CI_\gamma}(\theta) &= \theta && \text{for all } \theta \in [0, \alpha) \\ &= F(\theta)R_B + \gamma - F(\alpha)R_B && \text{for all } \theta \in [\alpha, 1] \\ q_B^{CI_\gamma}(\theta) &= \theta && \text{for all } \theta \in [0, \gamma) \\ &= F(\theta)R_A + \gamma - F(\gamma)R_A && \text{for all } \theta \in [\gamma, 1] \end{aligned}$$

where α and γ solve

$$[1 - F(\alpha)]\, R_B = [1 - F(\gamma)]\, R_A \tag{15}$$

$$F(\alpha)\, R_A - \alpha = F(\gamma)\, R_A - \gamma. \tag{16}$$

Under the strategy profile $\left(q_A^{CI_\gamma}, q_B^{CI_\gamma}\right)$, both agents exert zero effort up to a threshold value of θ and choose positive effort levels for higher values. Agent A's threshold is lower than agent B's, ($\alpha < \gamma$) due to the higher reward he obtains if he wins the contest. Finally, while agent B's strategy is continuous in the parameter θ, agent A's strategy entails a discrete jump from α to γ at his threshold.

Theorem 2 *Let $F(.)$ be convex.*
(a) If $R_A \leq \frac{1}{F'(1)}$, then an equilibrium is given by $\left(q_A^C, q_B^C\right)$.
(b) If either $R_A \in \left(\frac{1}{F'(1)}, 1\right)$ or both $R_A \in \left[1, \frac{1}{F'(0)}\right)$ and $F(R_A - R_B)R_A - (R_A - R_B) \leq 0$, then an equilibrium is given by $\left(q_A^{CI_\gamma}, q_B^{CI_\gamma}\right)$.
(c) If either $R_A \geq \frac{1}{F'(0)}$ or both $R_A \in \left[1, \frac{1}{F'(0)}\right)$ and $F(R_A - R_B)R_A - (R_A - R_B) > 0$, then an equilibrium is given by $\left(q_A^{I_\gamma}, q_B^{I_\gamma}\right)$.

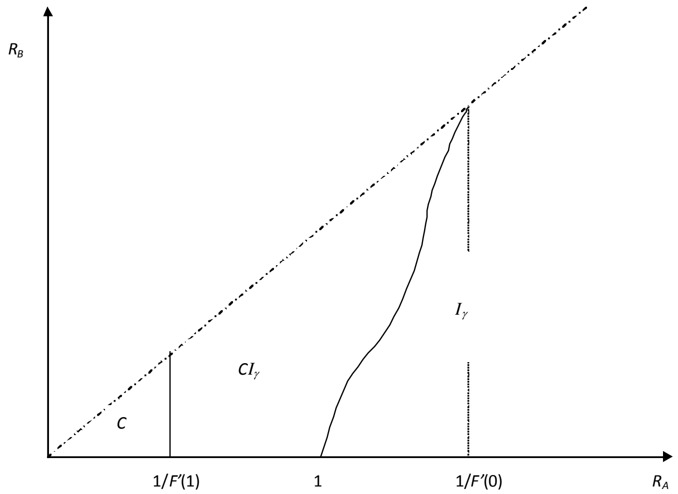

Figure 2: Equilibrium configuration when $F(.)$ is convex.

We point out that when $R_A \in \left[1, \frac{1}{F'(0)}\right)$, the condition $F\left(R_A - R_B\right) R_A - \left(R_A - R_B\right) > 0$ is equivalent to $R_A - R_B > p\left(R_A\right)$, where we denote by $p\left(R_A\right)$ the unique strictly positive p that satisfies $F(p)R_A - p = 0$. Thus, when $R_A \in \left[1, \frac{1}{F'(0)}\right)$, we are in case (c) if R_B is not "close" to R_A.

Figure 2 depicts the equilibrium configuration as a function of the prizes R_A and R_B for the class of convex distribution functions. When R_A is small enough, the whole equilibrium is in Region C, and when R_A is large enough, the equilibrium is in Region I_γ where agent A puts in a positive effort at $\theta = 0$ and agent B starts to exert a positive effort only for large enough θ's. The same occurs when R_A is intermediate but substantially larger than R_B. The intuition for these two cases is similar to that provided after Propositions 2 and 3, namely agents do not have incentives to exert any effort if the reward is low while competition in efforts arises if the reward is high.

When R_A is intermediate and, depending on R_A, not too much higher than R_B, the equilibrium entails a corner solution for low values of θ and an interior solution for high values (the Region CI_γ). The reason for a positive effort being exerted at higher values of θ can be traced back to the larger density at higher values of θ due to the convexity of the distribution function $F(\theta)$. A higher density implies a larger increase in the probability of winning the contest following a higher effort and hence a larger payoff to any increase in effort.

We now characterize the equilibrium strategies when the function $F(.)$ is concave in which case $F'(1) < 1 < F'(0)$. We also, we define the strategy profile $\left(q_A^{I_\gamma C}, q_B^{I_\gamma C}\right)$ as follows:

$$
\begin{aligned}
q_A^{I_\gamma C}(\theta) &= F(\theta)R_B + \gamma \text{ for all } \theta \in [0, \beta) \\
&= \theta \qquad\qquad\quad \text{for all } \theta \in [\beta, 1] \\
q_B^{I_\gamma C}(\theta) &= \theta \qquad\qquad\quad \text{for all } \theta \in [0, \gamma) \cup [\beta, 1] \\
&= F(\theta)R_A + \gamma - F(\gamma)R_A \text{ for all } \theta \in [\gamma, \beta)
\end{aligned}
$$

where γ and β solve

$$
\gamma = \beta - F(\beta) R_B \tag{17}
$$

$$
F(\gamma) R_A - \gamma = F(\beta) R_A - \beta. \tag{18}
$$

When agents follow the strategy profile $\left(q_A^{I_\gamma C}, q_B^{I_\gamma C}\right)$, both agents exert zero effort above the same threshold value of θ. For low values of θ, they follow an interior solution where, due to the difference in reward, agent A has an incentive to exert a strictly positive effort even when $\theta_A = 0$, whereas agent B only exerts a positive effort above a certain threshold of θ_B.

Theorem 3 *Let $F(.)$ be concave.*
(a) If $R_A \leq \frac{1}{F'(0)}$, then an equilibrium is given by $\left(q_A^C, q_B^C\right)$.
(b) If either $R_A \in \left(\frac{1}{F'(0)}, 1\right)$ or both $R_A \in \left[1, \frac{1}{F'(1)}\right)$ and $F(1-R_B)R_A-(1-R_B) > R_A-1$, then an equilibrium is given by $\left(q_A^{I_\gamma C}, q_B^{I_\gamma C}\right)$.
(c) If either $R_A \geq \frac{1}{F'(1)}$ or both $R_A \in \left[1, \frac{1}{F'(1)}\right)$ and $F(1-R_B)R_A - (1-R_B) \leq R_A - 1$, then an equilibrium is given by $\left(q_A^{I_\gamma}, q_B^{I_\gamma}\right)$.

Note that when $R_A \in \left[1, \frac{1}{F'(1)}\right)$, the condition $F(1-R_B)R_A - (1-R_B) > R_A - 1$ is equivalent to $z(R_A) < 1 - R_B$, where we denote by $z(R_A)$ the unique strictly positive z that satisfies $F(z)R_A - z = R_A - 1$. Thus, when $R_A \in \left[1, \frac{1}{F'(0)}\right)$, we are in case (b) if R_B is "small enough" or not too close to R_A.

When rewards are quite low or quite high, the agents' behavior is similar to that for the convex case. For intermediate values of reward, they play according to the interior strategy profile for low levels of θ, and they both exert zero effort for high values of θ.

Notice that the corner strategy profile emerges now for high values of θ because of the low density of θ due to the concavity of the distribution function.

We remark that the propositions and theorems derived in the current section also apply to nondiscriminatory contests. However in such contests , since $R_A = R_B$, the systems (15)-(16) and (17)-(18).do not have a unique solution. The equilibrium behavior in a nondiscriminatory contest when the distribution function is convex is given by the solution of (15)-(16) that satisfies $\gamma = \alpha$ and $F'(\alpha)R = 1$ in the strategy profile $\left(q_A^{CI_\gamma}, q_B^{CI_\gamma}\right)$; and when the distribution function is concave, it is given by the solution of (17)-(18) that satisfies $\gamma = 0$ and β given by the unique positive value for which $F(\beta)R - \beta = 0$ in the strategy profile $\left(q_A^{I_\gamma C}, q_B^{I_\gamma C}\right)$. Naturally, these two equilibria coincide with the contestants' equilibrium behavior in the nondiscriminatory contest identified in Theorem 1.

5 Designer's payoff and optimality of discrimination

In this section, we address the optimality of discrimination. First in Proposition 4 we determine the designer's payoff as a function of the rewards and the agents' strategies. We distinguish among the four possible equilibrium strategy profiles we identified in Section 4. We will then use Proposition 4 to derive conditions under which discrimination is optimal.

Proposition 4 *The designer's payoff $U(R_A, R_B)$, for $R_A \geq R_B$, as a function of the agents' strategies, is the following:*
(a) If agents follow the strategy profile $\left(q_A^C, q_B^C\right)$, then

$$U(R_A, R_B) = 2\int_0^1 I(q)F(q)F'(q)dq - \frac{1}{2}(R_A + R_B).$$

(b) If agents follow the strategy profile $\left(q_A^{I_\gamma}, q_B^{I_\gamma}\right)$, then

$$U(R_A, R_B) = \frac{1}{R_A R_B}\int_\gamma^{R_B+\gamma} I(q)\left[2(q-\gamma) + R_A - R_B\right]dq - R_A + \frac{1}{2}R_B\left(1 - \frac{R_B}{R_A}\right).$$

(c) If agents follow the strategy profile $\left(q_A^{CI_\gamma}, q_B^{CI_\gamma}\right)$, *then*

$$U(R_A, R_B) = 2\int_0^{\alpha} I(q)F(q)F'(q)dq + \int_{\alpha}^{\gamma} I(q)F(\alpha)F'(q)dq +$$
$$\frac{1}{R_A R_B}\int_{\gamma}^{[1-F(\gamma)]R_A+\gamma} I(q)\left[2\left(q-\gamma+F(\gamma)R_A\right)-(R_A-R_B)\right]dq -$$
$$\frac{1}{2}\left[R_A+R_B+(1-F(\alpha))^2(R_A-R_B)\left(1-\frac{R_B}{R_A}\right)\right].$$

(d) If agents follow the strategy profile $\left(q_A^{I_\gamma C}, q_B^{I_\gamma C}\right)$, *then*

$$U(R_A, R_B) = \frac{1}{R_A R_B}\int_{\gamma}^{\beta} I(q)\left[2(q-\gamma)+F(\gamma)R_A\right]dq +$$
$$2\int_{\beta}^{1} I(q)F(q)F'(q)dq - \frac{1}{2}\left[R_A+R_B+F(\gamma)F(\beta)(R_A-R_B)\right].$$

We now discuss the change in the designer's payoff due to discrimination separately for each strategy profile analyzed in Proposition 4. In any contest for which the agents' equilibrium strategy profile is $\left(q_A^C, q_B^C\right)$, the designer's revenue is the same, whereas the cost increases with the rewards. Therefore, such contests are dominated by the nondiscriminatory contest with $R = 0$. Proposition 5 addresses the other three cases.

Proposition 5 *Consider a nondiscriminatory contest* $R_A = R_B = R$ *for which an equilibrium is given by the strategy profile* (q_A, q_B).
(a) If $(q_A, q_B) = \left(q_A^I, q_B^I\right)$ *and marginal changes in* (R_A, R_B) *lead to* $\left(q_A^{I_\gamma}, q_B^{I_\gamma}\right)$, *then the contest is dominated by a discriminatory contest when* $R < \frac{2}{F'(0)}$.
(b) If $(q_A, q_B) = \left(q_A^{CI}, q_B^{CI}\right)$ *and marginal changes in* (R_A, R_B) *lead to* $\left(q_A^{CI_\gamma}, q_B^{CI_\gamma}\right)$, *then the contest is dominated by a discriminatory contest.*
(c) If $(q_A, q_B) = \left(q_A^{IC}, q_B^{IC}\right)$ *and marginal changes in* (R_A, R_B) *lead to* $\left(q_A^{I_\gamma C}, q_B^{I_\gamma C}\right)$, *then the contest is dominated by a discriminatory contest.*

Discrimination in all the cases addressed in Proposition 5 is optimal because it elicits higher efforts on the part of the agents, which, in turn, increases revenues, thus overcoming the increase in costs. Now, we provide an intuitive approach to examine the change in efforts and costs following a marginal move from a nondiscriminatory to a discriminatory contest.

Consider a shift from a nondiscriminatory contest R to a discriminatory contest ($R_A = R+\varepsilon, R_B = R-\varepsilon$). The increase in cost due to this shift is a second-order effect because the infinitesimal change in the rewards is multiplied by infinitesimal changes in the probability of A winning (which went up) and B winning (which went down). Therefore, the marginal shift increases profits if the change in efforts is positive.

To examine the change in efforts in case (a) of Proposition 5, note that agent A moves from the strategy $q_A^I(\theta) = F(\theta)R$ to $q_A^{I_\gamma}(\theta) = F(\theta)(R-\varepsilon)+\gamma$ for all $\theta \in [0,1]$, where $\gamma = F^{-1}\left(1\text{-}\frac{R-\varepsilon}{R+\varepsilon}\right)$. Therefore, the change in the quality offered by agent A of type θ is:$\frac{dq_A^{I_\gamma}}{d\varepsilon} = -F(\theta) + \frac{2R}{(R+\varepsilon)^2}\frac{1}{F'(\gamma)}$. Similarly, agent B moves from $q_B^I(\theta)$ to $q_B^{I_\gamma}(\theta) = F(\theta)(R+\varepsilon)+\gamma - F(\gamma)(R+\varepsilon)$ for $\theta \in [\gamma, 1]$ and $q_B^{I_\gamma}(\theta) = \theta$ for $\theta \in [0,\gamma)$, which leads to $\frac{dq_B^{I_\gamma}}{d\varepsilon} = F(\theta) + \frac{2R}{(R+\varepsilon)^2}\frac{1}{F'(\gamma)} - F(\gamma) - \frac{2R}{(R+\varepsilon)^2}(R+\varepsilon)$ for all $\theta \in [\gamma, 1]$.

Summing up both changes and evaluating it at $\varepsilon = 0$, and recalling that $\gamma = 0$ as well, we obtain

$$\frac{dq_A^{I_\gamma}}{d\varepsilon} + \frac{dq_B^{I_\gamma}}{d\varepsilon} = 2\left[\frac{2}{F'(0)R} - 1\right] \text{ for all } \theta \in [0,1].$$

Therefore, the increase in the quality offered by agent A of any type θ more than compensates for the possible decrease in the quality offered by agent B of type θ if and only if $R < \frac{2}{F'(0)}$. If this is the case, the expected quality of the innovation achieved in the contest increases, leading to larger profits by the designer.

We can proceed similarly for case (b), where the change (evaluated at $\varepsilon = 0$) in strategies from $\left(q_A^{CI}, q_B^{CI}\right)$ to $\left(q_A^{CI_\gamma}, q_B^{CI_\gamma}\right)$ following the shift to the discriminatory contest leads to

$$\frac{dq_A^{CI_\gamma}}{d\varepsilon} = -F(\theta) + 2 - F(\alpha) \text{ for all } \theta \in [\alpha, 1]$$

$$\frac{dq_B^{CI_\gamma}}{d\varepsilon} = F(\theta) - F(\alpha) \text{ for all } \theta \in [\alpha, 1].$$

Therefore, the sum of the changes is $2 - 2F(\alpha)$, which is always positive, implying that discrimination is optimal.

Finally, in case (c) a marginal shift from $\left(q_A^{IC}, q_B^{IC}\right)$ to $\left(q_A^{I_\gamma C}, q_B^{I_\gamma C}\right)$ leads at $\varepsilon = 0$ to the following derivatives:

$$\frac{dq_A^{CI_\gamma}}{d\varepsilon} = -F(\theta) + \frac{\partial\beta}{\partial\varepsilon} + F(\beta) - F'(\beta)R\frac{\partial\beta}{\partial\varepsilon} \text{ for all } \theta \in [0, \beta]$$

$$\frac{dq_B^{CI_\gamma}}{d\varepsilon} = F(\theta) + \frac{\partial\beta}{\partial\varepsilon} - F(\beta) - F'(\beta)R\frac{\partial\beta}{\partial\varepsilon} \text{ for all } \theta \in [0, \beta],$$

Summing up we obtain

$$\frac{dq_A^{CI_\gamma}}{d\varepsilon}+\frac{dq_B^{CI_\gamma}}{d\varepsilon}=2\left[1-F'(\beta)R\right]\frac{\partial\beta}{\partial\varepsilon}\text{ for all }\theta\in[0,\beta]\,.$$

This last expression has has the same sign as $\frac{\partial\beta}{\partial\varepsilon}$ because β lies in the decreasing part of the function $RF(\theta)-\theta$. Since $\frac{\partial\beta}{\partial\varepsilon}$ is positive (see the appendix for the proof) discrimination is optimal.

While Proposition 5 shows, when it is beneficial to discriminate for a given strategy profile, it fails to provide actual conditions under which a discriminatory contest is optimal. That is, it could be the case that the optimal nondiscriminatory contest would never lead to an equilibrium profile that satisfies the conditions identified in the Proposition. To show that such an equilibrium profile exists, we present the following two propositions. These propositions provide sufficient conditions for the optimality of discriminatory contests in the case of a convex distribution function (Proposition 6) or a concave distribution function (Proposition 7).

Proposition 6 *If $F(\theta)$ is a convex function with $F'(0)=0$, and $I(q)=vi(q)$, then discrimination is optimal when v is large enough.*

For the concave case, we describe a parametrized family of distribution and designer payoff functions and parameter restrictions for which discrimination is optimal

Proposition 7 *Let $F(\theta)=\theta^\lambda$ and $I(q)=vq^\mu$ with $\lambda,\mu\in(0,1)$ and $v>0$. If $3\lambda+\mu<1$ and $v<\frac{\mu+2}{2\mu}$ discrimination is optimal.*

These results show that larger rewards might lead to discrimination in the convex case whereas smaller rewards are more to likely to generate discrimination in the concave case.

6 Conclusion

We provided a new setting of contests with asymmetric information where innate abilities and effort combine to generate innovations of various qualities. Both the ability and effort of an agent are his private information. The designer, whose revenue depends on the quality of the bid, specifies a contest where the innovation of the highest quality is rewarded. We first analyzed strategic behavior in a nondiscriminatory contest, where the reward does not depend on the identity of the winner. We allowed for arbitrary distribution functions and determined the structure of equilibrium strategies and outcomes. The agents' strategies were continuous and the equilibrium consisted of two types of quality intervals: regions where both agents put in a positive effort and those where both agents put in zero effort.

We then analyzed strategic behavior in discriminatory auctions where rewards depend on the identity of the winner. Here equilibrium strategies were more complex, while the low-reward agent's strategy remained continuous, the strategy of the higher-reward agent could be discontinuous. We provided a qualitative analysis of the structure of the equilibrium strategies for general distribution functions and a full characterization of equilibrium behavior when the distribution functions are convex or concave.

We then used the equilibrium analysis to evaluate the designer's payoff. This generated several sets of conditions under which an optimal contest (which maximizes the designer's payoff) is a discriminatory contest. These conditions state that whenever an optimal nondiscriminatory contest entails a certain equilibrium behavior, it is possible to increase the designer's payoff by resorting to a discriminatory contest. To show that the results are not vacuous we provided parameterized classes of environments with convex or concave distribution functions where discrimination is optimal.

Our result that discrimination is optimal in a symmetric setting is quite surprising, and is in sharp contrast to the intuition that when agents are asymmetric, some restrictions imposed on the stronger contestant may increase the designer's payoff. In our environment, discrimination, under some circumstances, increases the efforts generated, which more than compensates for the increase in the expected sum of the rewards awarded.

Our model can handle many familiar scenarios in addition to innovation. It can be used, for example, to analyze, lobbying activity, procurement settings, promotion competitions and even the design of sporting events. It can also be used to study contest design in the presence of asymmetric contestants and shed further light on the imposition of handicaps or favoritism.

The model can be extended in several dimensions. A dynamic version would consider two-stage contests where the winners of the first round are paired against each other in the second round. A more general structure of preferences and technology would allow for risk-averse contestants and general functions, that transform effort and ability into outcomes. The designer's objective function may also be expanded to include explicit dependence on the agent's innate ability or on the sum of efforts exerted by both agents. Moreover, since the analysis was carried out only from the point of view of a single designer, a challenging task for further research would be to consider environments with several competing designers.

7 Appendix

Proof of Theorem 1. We first show through a series of claims that $q^*(\theta)$ is indeed an equilibrium.

Claim 1. The equilibrium quality is well defined, that is, $q^*(\theta) \geq \theta$ for any $\theta \in [\alpha_n, \beta_n]$ for any n.

By construction, α_n is a local minimum of the function $F(\theta)R - \theta$. β_n is the first instance for which $F(\theta)R - \theta$ also reaches this minimum for $\theta > \alpha_n$ and, if this minimum is never reached again, $\beta_n = 1$. Hence, $q^*(\theta) = F(\theta)R - (F(\alpha_n)R - \alpha_n) \geq \theta$ for any $\theta \in [\alpha_n, \beta_n]$.

Claim 2. The equilibrium quality $q^*(\theta)$ is strictly increasing in θ.

By definition, $q^*(\theta) = \theta$ over all intervals in C, and $q^*(\theta) = F(\theta)R + \eta$ over all intervals in I (where η is different for different intervals). Given that $F(\theta)$ is strictly increasing, $q^*(\theta)$ is also strictly increasing over any interval. Moreover, by construction $q^*(\theta)$ is continuous: consider an interval $[\alpha_n, \beta_n]$ in I, $q^*(\alpha_n) = F(\alpha_n)R + \alpha_n - F(\alpha_n)R = \alpha_n$ and $q^*(\beta_n) = F(\beta_n)R + \alpha_n - F(\alpha_n)R = \beta_n$. Hence, $q^*(\theta)$ is strictly increasing.

Claim 3. The profits of any agent of type θ as a function of his choice of q are constant in any interval in Region I; that is, for any $q \in [\alpha_n, \beta_n]$ with $q \geq \theta$.

For any $q \in [\alpha_n, \beta_n]$ with $q \geq \theta$, the agent's profits are $F\left(F^{-1}(\frac{1}{R}\left(q - \alpha_n + F(\alpha_n)R\right))\right) R - (q - \theta) = F(\alpha_n)R - (\alpha_n - \theta)$. These profits are independent of q.

Claim 4. The profits of any agent of type θ, as a function of his choice of q are decreasing in any interval in Region C, that is; for any $q \in \left[\beta_{n-1}, \alpha_n\right]$ with $q \geq \theta$.

For $q \in \left[\beta_{n-1}, \alpha_n\right]$ with $q \geq \theta$, the agent's profits are $F(q)R - (q - \theta)$. They are decreasing in q since $F(q)R - q$ is decreasing in q by construction of β_{n-1} and α_n.

Claim 5. The profits of any agent of type θ are non-increasing in his choice of q.

This follows from claims 3 and 4.

Claim 6. $q^*(\theta)$ is a best response for an agent of type θ.

If $q^*(\theta) = \theta$, then the only possible change in the strategy is to increase q which results, according to Claim 5, in a lower payoff,. If $q^*(\theta) > \theta$, increasing q is also non-profitable. Moreover, any decrease in q for which the effort is still non-negative implies that the agent stays within the same region (recall that effort is zero in the left boundary of the region). Hence, according to Claim 3, profits remain the same, .

Claims 1 to 6 show that $q^*(\theta)$ is an equilibrium of the nondiscriminatory contest.

We now prove that $q^*(\theta)$ is the unique symmetric equilibrium in the class of continuous and monotonic strategies. We again proceed through a series of claims. Note first that by continuity of any equilibrium strategy $q(\theta)$, the interval $[0, 1]$ can be divided into a finite sequence of intervals over which $q(\theta)$ alternates between interior and corner solutions. Consider any symmetric equilibrium $q(\theta)$.

Claim 7. If $q(\theta^o) = \theta^o$ and $q(\theta^{oo}) = \theta^{oo}$ for $\theta^{oo} \geq \theta^o$ then $F(\theta^o)R - \theta^o \geq F(\theta^{oo})R - \theta^{oo}$.

The expected payoff of an agent of type θ^o when he chooses $q(\theta^o) = \theta^o$ is equal to $F(\theta^o)R$, which must not be less than $F(\theta^{oo})R - (\theta^{oo} - \theta^o)$ which corresponds to his expected payoff if he offers quality θ^{oo}. Therefore, we obtain $F(\theta^o)R - \theta^0 \geq F(\theta^{oo})R - \theta^{oo}$.

Claim 8. Consider a maximal interval $[\theta^o, \theta^{oo}]$ where $q(\theta)$ is an interior solution. Then, $q(\theta) = F(\theta)R + \eta$ for all $\theta \in [\theta^o, \theta^{oo}]$ with $\eta = \theta^o - F(\theta^o) R$. Moreover, either $\theta^{oo} = 1$ or θ^{oo} is the first parameter which is not a local minimum of the function $F(\theta)R - \theta$ that satisfies $\theta^o - F(\theta^o) R = \theta^{oo} - F(\theta^{oo}) R$.

The property that $q(\theta) = F(\theta)R + \eta$ follows from the FOCs characterizing an interior equilibrium. To show that $\eta = \theta^o - F(\theta^o) R$ we distinguish between two cases.

(1) If $\theta^o > 0$, then there exists an interval in Region C just to the left of θ^o. By continuity of $q(\theta)$, it must be the case that $q(\theta^0) = \theta^0$ which implies that $\eta = \theta^o - F(\theta^o)R$.

(2) If $\theta^o = 0$, the probability of winning is zero because the quality offered is strictly increasing in θ. Hence, it cannot be that in equilibrium both agents choose $q(0) > 0$, since it would lead to a negative payoff. Therefore, $F(0)R + \eta = 0$, i.e., $\eta = 0$ and $\eta = \theta^o - F(\theta^o) R$ holds in this case as well.

Also, by continuity, if $\theta^{oo} < 1$, it must be the case that $F(\theta^{oo})R + \eta = \theta^{oo}$; that is, $\theta^o - F(\theta^o) R = \theta^{oo} - F(\theta^{oo}) R$. Finally, suppose by way of contradiction, that $\theta^{oo} > \widehat{\theta}$, where $\widehat{\theta}$ is the first parameter which is not a local minimum of the function $F(\theta)R - \theta$ that satisfies $\theta^o - F(\theta^o) R = \widehat{\theta} - F\left(\widehat{\theta}\right) R$. Then, $q(\theta)$ is an interior solution in an interval

$\left[\widehat{\theta}, \widehat{\widehat{\theta}}\right]$ where $F(\theta)R - \theta$ is a decreasing function. However, this is not possible because for $\theta \in \left(\widehat{\theta}, \widehat{\widehat{\theta}}\right]$, $q(\theta) = F(\theta)R + \eta = F(\theta)R + \theta^o - F(\theta^o)R = F(\theta)R + \widehat{\theta} - F\left(\widehat{\theta}\right)R < \theta$, since $F(\theta)R - \theta$ is decreasing in this interval.

Claim 9. In a maximal interval $[\theta^o, \theta^{oo}]$ where $q(\theta) = \theta$, either $\theta^{oo} = 1$ or θ^{oo} is the first parameter such that the function $F(\theta)R - \theta$ is increasing to its right of.

Suppose, by way of contradiction, that θ^{oo} is such that the function $F(\theta)R - \theta$ is not increasing to its right. Hence, $F(\theta)R - \theta$ is not increasing in an interval $\left(\theta^{oo}, \widehat{\theta}\right]$ for some $\widehat{\theta} > \theta^{oo}$. Recall that by maximality of the interval $[\theta^o, \theta^{oo}]$ it must be the case that $q(\theta) > \theta$ for $\theta \in \left(\theta^{oo}, \widetilde{\theta}\right]$ with $\widetilde{\theta} < \widehat{\theta}$. Hence, $q(\theta) = F(\theta)R + \eta = F(\theta)R + \theta^{oo} - F(\theta^{oo})R > \theta$ for $\theta \in \left(\theta^{oo}, \widetilde{\theta}\right]$. But this cannot happen if $F(\theta)R - \theta$ is not increasing. Furthermore, by claim 7, θ^{oo} must be the first parameter where this happens after θ^o.

Therefore, $q^*(\theta)$ is the unique symmetric equilibrium, given that it is the only candidate compatible with claims 7 to 9.

Finally, we prove the property that any equilibrium is necessarily symmetric if the set $\{\theta \in [0,1] / F'(\theta) = 1/R\}$ has zero measure. Claim 10 shows the main argument needed for this proof, that in equilibrium, it is not possible that an interval of qualities is reached by one agent who is offering zero effort while the other agent offers positive effort.

Claim 10. Consider the equilibrium strategies $(q_A(\theta), q_B(\theta))$. Then, there can not exist a non-empty interval (q^o, q^{oo}) such that $q_A^{-1}(q) = q$ and $q_B^{-1}(q) < q$, for all $q \in (q^o, q^{oo})$.

Suppose, by contradiction, that there exists an interval (q^o, q^{oo}) such that $q_A^{-1}(q) = q$ and $q_B^{-1}(q) < q$, for all $q \in (q^o, q^{oo})$. For any type $\theta \in \left(q_B^{-1}(q^o), q_B^{-1}(q^{oo})\right)$, $q(\theta)$ maximizes firm B's profits $F(q)R - (q - \theta)$. Therefore, the following FOC is necessarily satisfied: $F'(q_B^{-1}(\theta))R - 1 = 0$. However, this is not possible for an interval $\left(q_B^{-1}(q^o), q_B^{-1}(q^{oo})\right)$ provided the set $\{\theta \in [0,1] / F'(\theta) = 1/R\}$ has zero measure.

Consider now an equilibrium $(q_A(\theta), q_B(\theta))$. Given Claim 10, the continuity of the any equilibrium strategy $q_i(\theta)$ and the fact that $q_i(\theta)$ must be strictly increasing, it is possible to divide the interval $[0,1]$ into a series of intervals. In some of the intervals, both agents choose the corner solution $q_i(\theta) = \theta$ for both $i = A, B$; hence, the equilibrium is symmetric in those intervals. In the other intervals, both agents choose an interior solution. Therefore, their behavior is necessarily described by equations (5) and (6). The equilibrium is indeed symmetric if we prove that, over any of these intervals, the constants

η_A and η_B coincide. Denote by θ^o the lower bound of one such interval. By continuity, $F(\theta^o)R+\eta_A=\theta^o$; therefore, $\eta_A=\theta^o-F(\theta^o)R$. Similarly, $\eta_B=\theta^o-F(\theta^o)R$. This shows that $\eta_A=\eta_B$ and concludes the proof that any equilibrium in monotonic strategies is necessarily symmetric. ■

Proof of Lemma 1. We prove the six properties by way of contradiction.

(a) If such an interval (q_1,q_2) exists, then agent A of type $\theta\in(q_A^{-1}(q_1),q_A^{-1}(q_2))$ can increase his payoff by lowering the quality offered to another $q'<q(\theta)$ such $q'\geq\max\{q_1,\theta\}$. This change reduces the cost and does not affect his probability of winning the contest.

(b) The proof is similar to the proof of (a).

(c) If such an interval (q_1,q_2) exists, let $q_3=\{\inf q\mid q>q_2 \text{ and } q=q_i(\theta) \text{ for some } i=A,B \text{ and } \theta\in[0,1]\}$. If it is the case that q_3 is offered, that is, $q_A(\theta)=q_3$ for some $\theta\in[0,1]$ (we take agent A to be the one offering q_3 without loss of generality), then $\theta<q_3$ (it is certainly true if $q_2>1$, and if $q_2\leq1$ it is true since the equilibrium strategies are monotonic) and agent A of type θ can increase his payoff by lowering the quality offered to another $q'>\theta$ in the interval (q_1,q_2) because this change does not affect his probability of winning the contest. By continuity, a similar argument goes through if q_3 is not reached.

(d) and (e) The proofs are similar to the proof of Claim 10 in Theorem 1.

(f) Consider the maximal last interval (q_1,q_2) of this type. Since $q_B(\theta)\geq q_2$ for every $\theta\in(q_1,q_2)$ and there cannot be a mass point, it must be the case that $q_2<1$. Moreover, we claim that agent B must be offering quality levels arbitrarily close to q_2. Indeed, if this were not the case, the maximality of (q_1,q_2) implies that agent A is either reaching qualities just above q_2 by putting in positive effort or he is not reaching these qualities. The first possibility is ruled out by part (a) while the second is ruled out by part (c) of the lemma. Note that these qualities arbitrarily close to q_2 must be offered through positive effort levels by agent B since they are offered by types $\theta<q_2$.

Given parts (b) and (e), it is necessarily the case that if B puts in positive effort to reach a certain interval (q_2,q_3), A also puts in positive effort to reach this interval.

Consider now the largest such q_3, we show that $\theta_{A3}\equiv q_A^{-1}(q_3)>q_B^{-1}(q_3)\equiv\theta_{B3}$. Notice first that $q_2=q_A^{-1}(q_2)>q_B^{-1}(q_2)\equiv\theta_{B2}$ (we assume for convenience that both q_2 and q_3 are reached, otherwise we can make a limiting argument). Given that the qualities offered in an interior equilibrium are given by $q_A(\theta)=F(\theta)R_B+\eta_A$ and $q_B(\theta)=F(\theta)R_A+\eta_B$,

$q_3 = F(\theta_{A3})R_B + \eta_A = F(\theta_{B3})R_A + \eta_B$ and $q_2 = F(q_2)R_B + \eta_A = F(\theta_{B2})R_A + \eta_B$. Therefore, $q_3 - q_2 = [F(\theta_{A3}) - F(q_2)]\, R_B = [F(\theta_{B3}) - F(\theta_{B2})]\, R_A$, which implies that $F(\theta_{A3}) > F(\theta_{B3}) - F(\theta_{B2}) + F(q_2) > F(\theta_{B3})$, i.e., $\theta_{A3} > \theta_{B3}$ as we wanted to show. Therefore, there still exists another interval (q_3, q_4) above (q_2, q_3) where agents bid. Given that agent B is putting in positive effort to reach q_3, he cannot, since there are no atoms, switch to a region of qualities that are reached by him through zero effort. Therefore, in the new interval it is again the case that B does not offer any quality in it while A puts in zero effort. This is the type of region we started with, in contradiction to the assumption that it is the last region of this kind. Hence, such a region cannot exist in equilibrium. ■

Proof of Proposition 1. (a) It follows from Lemma 1.

(b) Consider the maximal interval (q_1, q_2) in Region J. We notice that $q_A^{-1}(q_2) \leq q_1 < 1$. Therefore, there are types of agent A (higher than q_1) that offer qualities above Region J. In this new interval just above J, agent A puts in strictly positive effort. Thus, it must be the case (according to Lemma 1) that agent B also puts in positive effort; that is, this new interval belongs to Region I.

(c) Consider the maximal interval (q_1, q_2) in Region I. We prove this part if we show that the interval can not be preceded by an interval in Region C and that it can not be the initial interval. Suppose by contradiction that either $q_1 = 0$ or that (q_1, q_2) is preceded by a interval in Region C. In both cases, $q_A(q_1) = q_B(q_1) = q_1$. Given the equilibrium strategies in an interior region, $q_A(\theta) = F(\theta)R_B + q_1 - F(q_1)R_B$ and $q_B(\theta) = F(\theta)R_A + q_1 - F(q_1)R_A$ for any $\theta \in (q_1, q_2)$. Therefore, $q_A(\theta) < q_B(\theta)$ for any $\theta \in (q_1, q_2)$, which implies that $q_B^{-1}(q_2) < q_A^{-1}(q_2)$ (or that $\lim_{q \to q_2} q_B^{-1}(q) < \lim_{q \to q_2} q_A^{-1}(q)$). In particular, there must be an interval of qualities reached above q_2 and $q_B^{-1}(q_2) < q_2$. Therefore, in the interval of qualities just above q_2, agent B exerts positive effort, which must be matched by agent A also offering positive effort, contradicting the maximality of (q_1, q_2) in Region I. ■

Proof of Proposition 2. We first prove a claim that will be used in the current proof as well as in several proofs in Section 5.

Claim 11. Suppose that agent i, for $i = A, B$, chooses $q_i(\theta) = \theta$ for all $\theta \in (\theta^o, \theta^{oo})$ and that the function $F(\theta)R_j - \theta$, for $j \neq i$, is non-increasing in θ for $\theta \in (\theta^o, \theta^{oo})$. Then, the payoff of agent j of type θ_j is non-increasing in the quality offered q, for $q \in (\theta^o, \theta^{oo})$ with $q \geq \theta_j$.

The proof of Claim 11 follows from the fact that $F(q)R_j - (q - \theta_j) \leq F(q')R_j - (q' - \theta_j)$

when $q \geq q'$ if the function $F(\theta)R_j - \theta$ is non-increasing between q and q'.

We now prove Proposition 2. If $F(\theta)R_A - \theta$ is non-increasing in θ for all $\theta \in [0,1]$, then the function $F(\theta)R_B - \theta$ is also non-increasing in θ for all $\theta \in [0,1]$ because $R_B < R_A$. Therefore, if agent i chooses $q_i(\theta) = \theta$ for all $\theta \in [0,1]$, then agent $j \neq i$ maximizes his payoff by choosing $q_j(\theta) = \theta$ as well, according to Claim 11. It follows that there is an equilibrium where the agents' strategies lie in Region C for all $\theta \in [0,1]$.

Moreover, if agents' equilibrium strategies lie in Region C for all $\theta \in [0,1]$, then it is necessarily the case that $F(\theta)R_A - \theta \geq F(q)R_A - q$ for any $\theta \in [0,1]$ and for any $q \geq \theta$. Therefore, the function $F(\theta)R_A - \theta$ is non-increasing in θ for all $\theta \in [0,1]$.

Finally, suppose by contradiction that there exists another equilibrium. It must either start with an interval in Region I_γ or with an interval in Region C followed by an interval in Region I_γ. Therefore, there is a jump, that is, there exist two values q_1 and q_2 (where q_1 is possibly 0) with $q_1 < q_2$ such that $F(q_1)R_A - q_1 \leq F(q_2)R_A - q_2$. If the inequality is strict, then this contradicts the fact that the function $F(\theta)R_A - \theta$ is non-increasing in θ. If this an equality, then $F(\theta)R_A - \theta$ is constant for all $\theta \in [q_1, q_2]$, which contradicts the property that the set $\{\theta \in [0,1] \,|F'(\theta) = 1/R_A\}$ has zero measure. ■

Proof of Proposition 3. We show first that $\left(q_A^{I_\gamma}, q_B^{I_\gamma}\right)$ is well defined. (i) $q_B^{I_\gamma}(\theta) \geq \theta$ for all $\theta \in [0,1]$ because $q_B^{I_\gamma}(\theta) = \theta$ for all $\theta \in [0,\gamma)$ and $q_B^{I_\gamma}(\theta) = F(\theta)R_A + \gamma - F(\gamma)R_A \geq \theta$ for all $\theta \geq \gamma$ according to (14). (ii) $q_A^{I_\gamma}(\theta) \geq \theta$ for all $\theta \in [0,1]$ because $q_A^{I_\gamma}(\theta) \geq q_B^{I_\gamma}(\theta)$ due to the properties that $q_A^{I_\gamma}(1) = q_B^{I_\gamma}(1)$ and $q_A^{I_\gamma\prime}(\theta) = R_B < R_A = q_B^{I_\gamma\prime}(\theta)$.

Second, we prove a claim that will be useful at several proofs:

Claim 12. Suppose that agent i, for $i = A, B$, chooses $q_i(\theta) = F(\theta)\, R_j + \eta$ for all $\theta \in (\theta^o, \theta^{oo})$, with $j \neq i$. Then, the payoff of agent j of type θ_j is constant and equal to $\theta_j - \eta$ when he offers any quality $q \in (F(\theta^o)\, R_j + \eta, F(\theta^{oo})\, R_j + \eta)$ with $q \geq \theta_j$.

Given $q_i(\theta)$, the payoff of agent j of type θ_j when he offers $q \in (F(\theta^o)\, R_j + \eta, F(\theta^{oo})\, R_j + \eta)$ with $q \geq \theta_j$ is

$$R_j \Pr_\theta(F(\theta)R_j + \eta \leq q) - (q - \theta_j) = R_j \left(\frac{q - \eta}{R_j}\right) - q + \theta_j = \theta_j - \eta.$$

Third, from Claim 12 and given agent B's strategy, the payoff of agent A of type θ when he offers any quality $q \in [\gamma, q_B(1)]$ with $q \geq \theta$ is $\theta - \gamma + F(\gamma)R_A$. Similarly, the payoff of agent B of type θ when he offers quality $q \in [\gamma, q_B(1)]$ with $q \geq \theta$ is $\theta - \gamma$, also independent of q. In particular, the strategies suggested are best responses one to the other for agents of type $\theta \in [\gamma, 1]$.

Agent B's payoff when offering quality $q(\theta) = \theta$ for $\theta \in [0, \gamma]$ is zero. His payoff would be negative if he were to offer any $q \in (\theta, \gamma]$ since he still has a probability zero of winning and it would be $\theta - \gamma < 0$ if he were to offer any $q \in (\gamma, q_B(1)]$. Therefore, agent B's strategy is a best response for all $\theta \in [0, \gamma)$ as well.

Agent A's payoff when following the strategy suggested for $\theta \in [0, \gamma)$ is $\theta - \gamma + F(\gamma)R_A$. As shown above, his payoff is the same for any $q \geq \gamma$. If he offers $q \in [\theta, \gamma)$, then his payoff is $F(q)R_A - (q - \theta)$. Hence, agent A's proposed strategy is his best response if $F(q)R_A - (q - \theta) \leq \theta - \gamma + F(\gamma)R_A$ for all $q \leq \gamma$, that is, $F(q)R_A - q \leq F(\gamma)R_A - \gamma$, which is implied by (13).

Finally, we prove that conditions (13) and (14) are necessary for $\left(q_A^{I_\gamma}, q_B^{I_\gamma}\right)$ to be an equilibrium. If $F(\theta)R_A - \theta < F(\gamma)R_A - \gamma$ for some $\theta > \gamma$, then $\left(q_A^{I_\gamma}, q_B^{I_\gamma}\right)$ can not be an equilibrium because $q_B^{I_\gamma}(\theta)$ would not be well defined ($q_B^{I_\gamma}(\theta) < \theta$). Moreover, if $F(\theta)R_A - \theta > F(\gamma)R_A - \gamma$ for some $\theta < \gamma$, then $\left(q_A^{I_\gamma}, q_B^{I_\gamma}\right)$ also can not be an equilibrium because agent A of type θ would strictly prefer θ to $q_A^{I_\gamma}(\theta)$ (because his benefits under $q_A^{I_\gamma}(\theta)$ are the same as under γ), contradicting the optimality of $q_A^{I_\gamma}$. $\blacksquare$

Proof of Theorem 2. (a) Given the convexity of $F(.)$, $F'(1)R_A \leq 1$ is a necessary and sufficient condition for $F(\theta)R_A - \theta$ to be non-increasing in θ for all $\theta \in [0, 1]$. Therefore, the proof of this part follows from Proposition 2.

(b) First, we show that α and γ are well defined in this region and that $\gamma > \alpha$. Equation (15) defines a function $\gamma^1(\alpha)$ which is increasing and such that $\gamma^1(1) = 1$ and $\gamma^1(\alpha) > \alpha$ for $\alpha \in [0, 1)$ (because $R_A > R_B$). Equation (16), together with the condition that $\gamma \geq \alpha$ defines another function $\gamma^2(\alpha)$. $\gamma^2(\alpha)$ is defined only for values of α where the function $F(\theta)R_A - \theta$ is non-increasing, but not necessarily for all of them. Note that $\gamma^2(\alpha)$ is defined for all such values when $R_A \geq 1$; furthermore, it is certainly defined for values of α close enough to the minimum of the function $F(\theta)R_A - \theta$, which we denote $\theta_{\min}$. Also note that $\gamma^2(\alpha)$ always lies in the increasing part of $F(\theta)R_A - \theta$. The function $\gamma^2(\alpha)$ is strictly decreasing and converges to $\theta_{\min}$ when α converges to $\theta_{\min}$. We distinguish between two cases.

When $R_A \in \left(\frac{1}{F'(1)}, 1\right)$, then $F(1)R_A - 1 < 0$ Therefore, there is some α^o for which $\gamma^2(\alpha^o) = 1$, from which on the function is strictly decreasing until it reaches $\theta_{\min}$, where $\gamma^2(\theta_{\min}) = \theta_{\min}$. Given that $\gamma^1(\alpha)$ is strictly increasing, $\gamma^1(1) = 1$ and $\gamma^1(\alpha) > \alpha$ for $\alpha \in [0, 1)$, then a solution to the system of equations always exists.

When $R_A \in \left[1, \frac{1}{F'(0)}\right)$, then the function $\gamma^2(\alpha)$ is defined for $\alpha \in [0, \theta_{\min})$ and it takes values always lower than 1. In this case, given that $\gamma^2(\alpha)$ is decreasing and $\gamma^1(\alpha)$ is increasing, a solution exists if and only if $\gamma^2(0) \geq \gamma^1(0)$, that is $\gamma^2(0) \geq F^{-1}\left(\frac{R_A - R_B}{R_A}\right)$, which we write as, $F(\gamma^2(0)) R_A \geq R_A - R_B$, or, $\gamma^2(0) \geq R_A - R_B$. Given that $\gamma^2(0)$ is the increasing part of $F(\theta) R_A - \theta$, the previous inequality is equivalent to $F(R_A - R_B) R_A - (R_A - R_B) \leq 0$, which we assume in Region (b).

Second, we show that agents' strategies are well defined, that is, the functions $\phi_A(\theta) \equiv q_A^{CI_\gamma}(\theta) - \theta$ and $\phi_B(\theta) \equiv q_B^{CI_\gamma}(\theta) - \theta$ are non-negative for all $\theta \in [0, 1]$. This trivially holds for all regions where players choose zero effort.

For $\theta \in [\gamma, 1]$ we have $q_B^{CI_\gamma}(\theta) = F(\theta)R_A + \gamma - F(\gamma)R_A$. Given that γ lies in the increasing part of $F(\theta)R_A - \theta$, we have $\phi_B(\theta) = F(\theta)R_A + \gamma - F(\gamma)R_A - \theta \geq 0$ for $\theta \in [\gamma, 1]$. For $\theta \in [\alpha, \gamma)$, the convexity of $\phi_A(\theta) = F(\theta)R_B + \gamma - F(\alpha)R_B - \theta$ implies that $\phi_A(\theta) \geq \phi_A(\alpha) + \phi_A'(\alpha)(\theta - \alpha) = \gamma - \alpha + (F'(\alpha)R_B - 1)(\theta - \alpha) \geq \gamma - \alpha - (\theta - \alpha) \geq 0$. For $\theta \in [\gamma, 1]$, we note that both $\phi_A(\theta)$ and $\phi_B(\theta)$ are convex functions. Furthermore, $\phi_A(\gamma) > \phi_B(\gamma) = 0$, $\phi_A(1) = \phi_B(1)$ (since $R_A - F(\gamma)R_A = R_B - F(\alpha)R_B$) and $\phi_A'(\theta) < \phi_B'(\theta)$ which implies that $\phi_A(\theta) \geq \phi_B(\theta)$ for all $\theta \in [\gamma, 1]$ and thus $\phi_A(\theta) \geq 0$ for all $\theta \in [\gamma, 1]$ as well.

Third, we prove that each agent's strategy is best response to each other.

Given agent B's strategy, the payoff of agent A of type θ when he offers quality $q \in [\gamma, q_B(1)]$ with $q \geq \theta$ is (see Claim 12) $\theta - \gamma + F(\gamma)R_A$, which is independent of q. Similarly, the payoff of agent B of type θ when offering quality $q \in [\gamma, q_B(1)]$ with $q \geq \theta$ is $\theta - \gamma + F(\alpha)R_B$, also independent of q. This implies, in particular, that the strategies suggested are best responses one to the other for agents of type $\theta \in [\gamma, 1]$.

The payoff of agent B of type $\theta \in (\alpha, \gamma)$ is decreasing in q for $q \in (\theta, \gamma)$, because no type of agent A chooses qualities in (α, γ) and the payoff is constant for $q \in [\gamma, q_B(1)]$. Therefore, $q_B^{CI_\gamma}(\theta) = \theta$ is a best response for all $\theta \in (\alpha, \gamma)$. The payoff of agent B of type $\theta \in [0, \alpha)$ is decreasing in q for $q \in [\theta, \alpha)$ because the interval $[\theta, \alpha)$ is in the decreasing part of the function $F(\theta)R_A - \theta$ (see Claim 11). Therefore, $q_B^{CI_\gamma}(\theta) = \theta$ is a best response because we also know that it is first decreasing and then constant for $q \in [\alpha, q_B(1)]$.

Agent A of type θ that chooses $q \in [0, \gamma)$, with $q \geq \theta$, obtains a payoff of $F(q)R_A - q + \theta$. The function $F(q)R_A - q$ is decreasing until α, then it further decreases, then increases until it recovers the same value $F(\alpha)R_A - \alpha$ at γ (see (16)). As we saw above, A's payoff is constant for $q \in [\gamma, q_B(1)]$. Therefore, $q_A^{CI_\gamma}(\theta) = \theta$ is a best response for all $\theta \in [0, \alpha)$ and $q_A^{CI_\gamma}(\theta) = F(\theta)R_B + \gamma - F(\alpha)R_B$ is a best response for all $\theta \in [\alpha, \gamma)$.

(c) Given the convexity of $F(\theta)$, the function $F(\theta)R_A - \theta$ is always increasing when $R_A \geq \frac{1}{F'(0)}$. Moreover, given the definition of γ, when $R_A \in \left[1, \frac{1}{F'(0)}\right)$ the condition $F(R_A - R_B) R_A - (R_A - R_B) > 0$ is equivalent to $F(R_A - R_B) R_A - F(\gamma) R_A > 0$, or $R_A - R_B > \gamma$, which is equivalent to $F(\gamma) R_A - \gamma > 0$. Given that, in this region, $F(\theta)R_A - \theta$ is first decreasing and then decreasing, $F(\gamma) R_A - \gamma > 0$ is a sufficient condition for equations (13) and (14) to hold. Therefore, part (c) holds due to Proposition 3. ■

Proof of Theorem 3. (a) The proof follows from Proposition 2 because $F'(0)R_A \leq 1$ and the concavity of F imply that $F(\theta)R_A - \theta$ is non-increasing in θ for all $\theta \in [0, 1]$.

(b) We first show that γ and β are well defined in this region and that $\beta > \gamma$. Similar to its behavior in Theorem 2, equation (18) defines a function $\beta^2(\gamma)$ for those values of γ where $F(\theta) R_A - \theta$ is non-decreasing, but not necessarily for all of them. $\beta^2(\gamma)$ is defined for all such values when $R_A \leq 1$; furthermore it is certainly defined for values of γ close enough to the maximum of the function $F(\theta) R_A - \theta$, which we denote $\theta_{\max}$. Also note that, $\beta^2(\gamma)$ always lies in the decreasing part of $F(\theta) R_A - \theta$. The function $\beta^2(\gamma)$ is strictly decreasing (in the interval of γ where it is defined) and converges to $\theta_{\max}$ when γ converges to $\theta_{\max}$.

Equation (17) defines a function $\gamma^1(\beta)$. The function is increasing at least for $\beta \geq \theta_{\max}$ because $F(\theta) R_A - \theta$ is decreasing for $\theta \geq \theta_{\max}$ and $R_A > R_B$. Moreover, $\gamma^1(1) = 1 - R_B$. We distinguish between two cases. When $\beta^2(0)$ is well defined, that is, when $R_A - 1 \leq 0$ then, since $R_B < R_A$, $\gamma^1(1) = 1 - R_B$ is positive. Therefore, the functions $\gamma^1(\beta)$ and $\beta^2(\gamma)$ intersect and a solution to equations (17) and (18) exists. When $R_A \in \left[1, \frac{1}{F'(1)}\right]$, then $F(1) R_A - 1 \leq 0$, therefore there is some γ for which $\beta^2(\gamma) = 1$. We denote this value by $z(R_A)$. The necessary and sufficient condition for (17) and (18) to intersect is that $z(R_A) < 1 - R_B$ or, equivalently, $F(1 - R_B)R_A - (1 - R_B) > R_A - 1$.

Second, we show that the functions $\delta_A(\theta) \equiv q_A^{I_\gamma C}(\theta) - \theta$ and $\delta_B(\theta) \equiv q_B^{I_\gamma C}(\theta) - \theta$ are non-negative for all $\theta \in [0, 1]$. This trivially holds if players choose zero effort.

For $\theta \in [\gamma, \beta)$, we have $F(\theta)R_A - \theta \geq F(\gamma)R_A - \gamma$ because γ lies in the increasing part of $F(\theta)R_A - \theta$ and the function takes the same value for γ and β. Hence, $\delta_B(\theta) = F(\theta)R_A + \gamma - F(\gamma)R_A - \theta \geq 0$ for $\theta \in [\gamma, \beta)$. For $\theta \in [0, \beta)$, $\delta_A(\theta) = F(\theta)R_B + \gamma - \theta > 0$ because it is a concave function of θ, $\delta(0) = \gamma > 0$ and $\delta(\beta) = 0$ by equation (17).

Third, we prove that each agent's strategy is best response to each other.

Given agent B's strategy, the payoff of agent A of type $\theta \in [\beta, 1]$ is decreasing in q for $q > \theta$ because the function $F(q)R_A - q$ is decreasing (see Claim 11); thus $q_A^{I_\gamma C}(\theta) = \theta$ is agent A's best response. For $\theta \in [0, \beta)$, the payoff of agent A is equal to $\theta + F(\gamma)R_A - \gamma$ for any $q \in [\gamma, \beta]$ with $q \geq \theta$ (by Claim 12) and it is decreasing for $q \in [\beta, 1]$ (by Claim 11). If agent A offers quality $q \in [0, \gamma]$ with $q \geq \theta$ his payoff is $F(q)R_A - (q - \theta)$ which is smaller than $\theta + F(\gamma)R_A - \gamma$ because the function $R_A F(\theta) - \theta$ is increasing in that interval. Therefore, $q_A^{I_\gamma C}(\theta)$ is an agent A's best response.

Given agent A's strategy, the payoff of agent B of type $\theta \in [\beta, 1]$ when offering $q_B^{I_\gamma C}(\theta) = \theta$ is $F(\theta)R_B$, which is higher than his payoff for any $q > \theta$ because we are in the decreasing part of $F(\theta)R_A - \theta$. The payoff of agent B of type $\theta \in [\gamma, \beta]$ when offering $q_B^{I_\gamma C}(\theta)$ is $\theta - \gamma$, which is higher than his payoff if he offers quality $q \in [\beta, 1]$ because $F(\theta)R_B - \theta$ is decreasing in θ for $\theta \geq \beta$. Finally, the payoff of agent B of type $\theta \in [0, \gamma]$ when offering quality θ is zero. It would be negative for any $q \in [0, \gamma]$ with $q > \theta$ and, as shown above, the payoff would be first constant and then decreasing as q is higher. Therefore, $q_B^{I_\gamma C}(\theta)$ constitutes a best response to agent A's strategy.

(c) We prove (c) using Proposition 3, which we can apply directly if $R_A \geq \frac{1}{F'(1)}$ because, due to the concavity of $F(\theta)R_A - \theta$, the function $F(\theta)R_A - \theta$ is always increasing. If $R_A \in \left[1, \frac{1}{F'(1)}\right)$, then conditions (13) and (14) hold (given the concavity of $F(\theta)R_A - \theta$) if and only if $F(\gamma)R_A - \gamma \leq R_A - 1$ which, given that $F(\gamma)R_A = R_A - R_B$ by definition of γ, is equivalent to $\gamma \geq 1 - R_B$. We rewrite the inequality as $F\left(1 - \frac{R_B}{R_A}\right) \geq 1 - R_B$, or, $R_A - R_B \leq F(1 - R_B) R_A$, which is the condition we require in (c). ■

Proof of Proposition 4. (a) If agents follow $\left(q_A^C, q_B^C\right)$, the quality q that wins the contest is the max $\{\theta_A, \theta_B\}$. Therefore, the distribution of q is $F^*(q) = F(q)^2$ and

$$dF^*(q) = 2F(q)F'(q)dq.$$

The expression for $U(R_A, R_B)$ provided in the proposition follows immediately from the fact that either agent wins the contest with probability $\frac{1}{2}$.

(b) If agents follow $\left(q_A^{I_\gamma}, q_B^{I_\gamma}\right)$, the interval of qualities q that may be offered is $[\gamma, R_B + \gamma]$, following the distribution function

$$F^*(q) = \frac{1}{R_B}(q - \gamma)\frac{1}{R_A}\left[(q - \gamma) + R_A - R_B\right],$$

$$dF^*(q) = \frac{1}{R_A R_B}\left[2(q - \gamma) + R_A - R_B\right]dq.$$

Therefore, the designer's expected income is the first part of the expression $U(R_A, R_B)$. The expected cost depends on the probability that either agent wins the contest. An agent A of type θ wins the contest with probability

$$\Pr_{\theta_B}\left(F(\theta_B)R_A + \gamma - (R_A - R_B) \leq F(\theta)R_B + \gamma\right) = \frac{1}{R_A}\left[F(\theta)R_B + R_A - R_B\right].$$

It follows that the probability that agent A wins the contest is

$$\int_0^1 \frac{1}{R_A}\left[F(\theta)R_B + R_A - R_B\right]F'(\theta)d\theta = \frac{R_B}{R_A}\frac{1}{2}\left[F(\theta)^2\right]_0^1 + \frac{(R_A - R_B)}{R_A}\left[F(\theta)\right]_0^1 = 1 - \frac{1}{2}\frac{R_B}{R_A}$$

while the probability that B wins the contest is $\frac{1}{2}\frac{R_B}{R_A}$. Therefore, the designer's expected cost is $R_A\left(1 - \frac{1}{2}\frac{R_B}{R_A}\right) + R_B\frac{1}{2}\frac{R_B}{R_A}$, from which the second part of the expression $U(R_A, R_B)$ is obtained.

(c) If agents follow $\left(q_A^{CI_\gamma}, q_B^{CI_\gamma}\right)$, the set of qualities that is reached is $[0, (1 - F(\gamma))R_A + \gamma]$. For $q \in [0, \alpha)$, $dF^*(q) = 2F(q)F'(q)dq$. For $q \in [\alpha, \gamma)$, $F^*(q) = F(\alpha)F(q)$ and

$$dF^*(q) = F(\alpha)F'(q)dq.$$

Finally, for $q \in [\gamma, (1 - F(\gamma))R_A + \gamma]$,

$$F^*(q) = \frac{1}{R_B}\left(q - \gamma + F(\alpha)R_B\right)\frac{1}{R_A}\left[q - \gamma + F(\gamma)R_A\right],$$

$$dF^*(q) = \frac{1}{R_A R_B}\left[2(q - \gamma) + F(\alpha)R_B + F(\gamma)R_A\right]dq = \frac{1}{R_A R_B}\left[2\left(q - \gamma + F(\gamma)R_A\right) - (R_A - R_B)\right]dq$$

and the expression for the designer's income follows. Concerning the probability that either agent wins the contest, agent A of type $\theta \in [0, \alpha)$ wins with probability $F(\theta)$ whereas, if his type is $\theta \in [\alpha, 1]$, he wins with probability

$$\Pr_{\theta_B}\left(F(\theta_B)R_A + \gamma - F(\gamma)R_A \leq F(\theta)R_B + \gamma - F(\alpha)R_B\right) = \frac{1}{R_A}\left[F(\theta)R_B - F(\alpha)R_B + F(\gamma)R_A\right] = \frac{1}{R_A}\left[F(\theta)R_B + R_A - R_B\right].$$

Therefore, the probability that A wins the contest is

$$\int_0^\alpha F(\theta)F'(\theta)d\theta + \int_\alpha^1 \frac{1}{R_A}\left[F(\theta)R_B + R_A - R_B\right]F'(\theta)d\theta = \frac{1}{2}\left[F(\theta)^2\right]_0^\alpha + \frac{R_B}{R_A}\frac{1}{2}\left[F(\theta)^2\right]_\alpha^1 + \frac{(R_A - R_B)}{R_A}\left[F(\theta)\right]_\alpha^1 =$$

$$1 - \frac{1}{2}\frac{R_B}{R_A} + \frac{1}{2}\left(1 - \frac{R_B}{R_A}\right)F(\alpha)^2 - \left(1 - \frac{R_B}{R_A}\right)F(\alpha) = \frac{1}{2} + \frac{1}{2}\left[1 - F(\alpha)\right]^2\left(1 - \frac{R_B}{R_A}\right)$$

and the designer's expected cost is

$$R_A\left[\frac{1}{2}+\frac{1}{2}\left(1-\frac{R_B}{R_A}\right)(1-F(\alpha))^2\right]+R_B\left[\frac{1}{2}-\frac{1}{2}\left(1-\frac{R_B}{R_A}\right)(1-F(\alpha))^2\right]=$$
$$\frac{1}{2}\left[R_A+R_B+(1-F(\alpha))^2(R_A-R_B)\left(1-\frac{R_B}{R_A}\right)\right],$$

which corresponds to the last term of $U(R_A, R_B)$ in part (c) of the proposition.

(d) If agents follow the strategy profile $\left(q_A^{I_\gamma C}, q_B^{I_\gamma C}\right)$, the space of qualities that is reached is $[\gamma, 1]$. For $q \in [\gamma, \beta)$,

$$F^*(q)=\frac{1}{R_B}(q-\gamma)\frac{1}{R_A}[q-\gamma+F(\gamma)R_A],$$

$$dF^*(q)=\frac{1}{R_A R_B}[2(q-\gamma)+F(\gamma)R_A]\,dq.$$

For $q\in[\beta,1]$, $dF^*(q)=2F(q)F'(q)dq$. Therefore, the expression for the designer's income follows. We compute now the probability that agent A wins the contest. If his type is $\theta\in[0,\beta)$, he wins with probability

$$\Pr_{\theta_B}(F(\theta_B)R_A+\gamma-F(\gamma)R_A\leq F(\theta)R_B+\gamma)=\frac{1}{R_A}[F(\theta)R_B+F(\gamma)R_A].$$

Moreover, agent A with type $\theta\in[\beta,1]$ wins with probability $F(\theta)$. Therefore, the probability that A wins the contest is

$$\int_0^\beta\frac{1}{R_A}[F(\theta)R_B+F(\gamma)R_A]F'(\theta)d\theta+\int_\beta^1 F(\theta)F'(\theta)d\theta=$$
$$\frac{R_B}{R_A}\frac{1}{2}\left[F(\theta)^2\right]_0^\beta+F(\gamma)\left[F(\theta)\right]_0^\beta+\frac{1}{2}\left[F(\theta)^2\right]_\beta^1=\frac{R_B}{R_A}\frac{1}{2}F(\beta)^2+F(\gamma)F(\beta)+\frac{1}{2}-\frac{1}{2}F(\beta)^2=$$
$$\frac{1}{2}+\frac{1}{2R_A}[F(\beta)R_B+2F(\gamma)R_A-F(\beta)R_A]F(\beta)=\frac{1}{2}+\frac{1}{2}F(\gamma)F(\beta)$$

(where the last equality is derived from the two equations that define γ and β) and the designer's expected costs are

$$\frac{1}{2}[R_A+R_B+F(\gamma)F(\beta)(R_A-R_B)],$$

which corresponds to the expression for the cost in part (d) of the proposition. ■

Proof of Proposition 5. (a) Consider a marginal change from a nondiscriminatory contest where agents play $\left(q_A^I, q_B^I\right)$ in equilibrium to a discriminatory contest where the new equilibrium is $\left(q_A^{I_\gamma}, q_B^{I_\gamma}\right)$. To evaluate the optimality of such a change, we take the partial derivatives of the designer's payoff function $U(R_A, R_B)$ when agents play $\left(q_A^{I_\gamma}, q_B^{I_\gamma}\right)$ with respect to R_A and R_B.

$$\frac{\partial U}{\partial R_A}(R_A, R_B) = -\frac{1}{R_A^2 R_B}\int_\gamma^{R_B+\gamma} I(q)\left[2(q-\gamma)+R_A-R_B\right]dq+$$
$$\frac{1}{R_A R_B}\int_\gamma^{R_B+\gamma} I(q)\left[-2\frac{\partial\gamma}{\partial R_A}+1\right]dq + \frac{1}{R_A R_B} I(R_B+\gamma)\left[R_A+R_B\right]\frac{\partial\gamma}{\partial R_A} -$$
$$\frac{1}{R_A R_B} I(\gamma)\left[R_A-R_B\right]\frac{\partial\gamma}{\partial R_A} - 1 + \frac{1}{2}\frac{R_B^2}{R_A^2}.$$

In particular, when $R_A = R_B = R$, then $\gamma = 0$ and

$$\frac{\partial U}{\partial R_A}(R_A = R, R_B = R) = -\frac{2}{R^3}\int_0^R I(q)q dq + \frac{1}{R^2}\left[1 - 2\frac{\partial\gamma}{\partial R_A}\right]\int_0^R I(q)dq + \frac{2}{R^2}I(R)R\frac{\partial\gamma}{\partial R_A} - \frac{1}{2}.$$

Similarly,

$$\frac{\partial U}{\partial R_B}(R_A, R_B) = -\frac{1}{R_A R_B^2}\int_\gamma^{R_B+\gamma} I(q)\left[2(q-\gamma)+R_A-R_B\right]dq+$$
$$\frac{1}{R_A R_B}\int_\gamma^{R_B+\gamma} I(q)\left[-2\frac{\partial\gamma}{\partial R_B}-1\right]dq + \frac{1}{R_A R_B} I(R_B+\gamma)\left[R_A+R_B\right]\left[1+\frac{\partial\gamma}{\partial R_B}\right] -$$
$$\frac{1}{R_A R_B} I(\gamma)\left[R_A-R_B\right]\frac{\partial\gamma}{\partial R_B} + \frac{1}{2} - \frac{R_B}{R_A}.$$

Therefore,

$$\frac{\partial U}{\partial R_B}(R_A = R, R_B = R) = -\frac{2}{R^3}\int_0^R I(q)q dq + \frac{1}{R^2}\left[-2\frac{\partial\gamma}{\partial R_B} - 1\right]\int_0^R I(q)dq+$$
$$\frac{2}{R^2}I(R)R\left[1+\frac{\partial\gamma}{\partial R_B}\right] - \frac{1}{2}.$$

Consider now a nondiscriminatory contest R. If we marginally increase R_A and simultaneously marginally decrease R_B, then the total effect is

$$\left[\frac{\partial U}{\partial R_A} - \frac{\partial U}{\partial R_B}\right](R_A = R, R_B = R) = \frac{2}{R^2}\left[1 - \frac{\partial\gamma}{\partial R_A} + \frac{\partial\gamma}{\partial R_B}\right]\int_0^R I(q)dq -$$
$$\frac{2}{R^2}I(R)R\left[1 - \frac{\partial\gamma}{\partial R_A} + \frac{\partial\gamma}{\partial R_B}\right] = \frac{2}{R^2}\left[1 - \frac{\partial\gamma}{\partial R_A} + \frac{\partial\gamma}{\partial R_B}\right]\left[\int_0^R I(q)dq - I(R)R\right].$$

The integral $\int_0^R I(q)dq - I(R)R < 0$ because $I(q)$ is an increasing function. Therefore, $\left[\frac{\partial U}{\partial R_A} - \frac{\partial U}{\partial R_B}\right](R_A = R, R_B = R) > 0$, that is, discriminating marginally increases the designer's payoff, if and only if $1 - \frac{\partial\gamma}{\partial R_A} + \frac{\partial\gamma}{\partial R_B} < 0$. We know that $\gamma = F^{-1}\left(1 - \frac{R_B}{R_A}\right)$; then,

$$\frac{\partial\gamma}{\partial R_A}(R_A, R_B) = \frac{R_B}{R_A^2}\frac{1}{F'(\gamma)} \text{ and } \frac{\partial\gamma}{\partial R_B}(R_A, R_B) = -\frac{1}{R_A}\frac{1}{F'(\gamma)}.$$

When we evaluate these derivatives at $R_A = R_B = R$, we obtain

$$1 - \frac{\partial \gamma}{\partial R_A} + \frac{\partial \gamma}{\partial R_B} = 1 - \frac{1}{R}\frac{1}{F'(0)} - \frac{1}{R}\frac{1}{F'(0)} = 1 - \frac{2}{RF'(0)}$$

and the result follows.

(b) We proceed as in part (a).

$$\begin{aligned}\frac{\partial U}{\partial R_A}(R_A, R_B) = {} & 2I(\alpha)F(\alpha)F'(\alpha)\frac{\partial \alpha}{\partial R_A} + \int_\alpha^\gamma I(q)F'(\alpha)F'(q)\frac{\partial \alpha}{\partial R_A}dq + \\ & I(\gamma)F(\alpha)F'(\gamma)\frac{\partial \gamma}{\partial R_A} - I(\alpha)F(\alpha)F'(\alpha)\frac{\partial \alpha}{\partial R_A} - \\ & \frac{1}{R_A^2 R_B}\int_\gamma^{[1-F(\gamma)]R_A+\gamma} I(q)\left[2\left(q - \gamma + F\left(\gamma\right)R_A\right) - \left(R_A - R_B\right)\right]dq + \\ & \frac{1}{R_A R_B}\int_\gamma^{[1-F(\gamma)]R_A+\gamma} I(q)\left[2\left(-1 + F'\left(\gamma\right)R_A\right)\frac{\partial \gamma}{\partial R_A} + \left[2F\left(\gamma\right) - 1\right]\right]dq + \end{aligned}$$

$$\begin{aligned}& \frac{1}{R_A R_B} I\left(\left[1 - F(\gamma)\right]R_A + \gamma\right)\left(R_A + R_B\right)\left[1 - F(\gamma) + \left(1 - F'(\gamma)R_A\right)\frac{\partial \gamma}{\partial R_A}\right] - \\ & \frac{1}{R_A R_B} I\left(\gamma\right)\left(2F\left(\gamma\right)R_A - \left(R_A - R_B\right)\right)\frac{\partial \gamma}{\partial R_A} - \\ & \frac{1}{2}\left[1 + \left(1 - \frac{R_B^2}{R_A^2}\right)\left(1 - F\left(\alpha\right)\right)^2 - 2\left(R_A - 2R_B + \frac{R_B^2}{R_A}\right)\left(1 - F\left(\alpha\right)\right)F'\left(\alpha\right)\frac{\partial \alpha}{\partial R_A}\right].\end{aligned}$$

When $R_A = R_B = R$, then $\gamma = \alpha$ and α satisfies $F'(\alpha)R = 1$. Therefore,

$$\begin{aligned}\frac{\partial U}{\partial R_A}\left(R_A = R, R_B = R\right) = {} & \frac{1}{R}I(\alpha)F(\alpha)\frac{\partial \alpha}{\partial R_A} - \frac{1}{R}I(\alpha)F(\alpha)\frac{\partial \gamma}{\partial R_A} - \\ & \frac{2}{R^3}\int_\alpha^{[1-F(\alpha)]R+\alpha} I(q)\left[q - \alpha + F\left(\alpha\right)R\right]dq + \frac{1}{R^2}\int_\alpha^{[1-F(\alpha)]R+\alpha} I(q)\left[2F\left(\alpha\right) - 1\right]dq + \\ & \frac{2}{R}I\left(\left[1 - F(\widehat{\theta})\right]R + \alpha\right)\left[1 - F(\alpha)\right] - \frac{1}{2}.\end{aligned}$$

The derivative of the designer's payoff with respect to R_B is

$$\begin{aligned}\frac{\partial U}{\partial R_B}(R_A, R_B) = {} & 2I(\alpha)F(\alpha)F'(\alpha)\frac{\partial \alpha}{\partial R_B} + \int_\alpha^\gamma I(q)F'(\alpha)F'(q)\frac{\partial \alpha}{\partial R_B}dq + \\ & I(\gamma)F(\alpha)F'(\gamma)\frac{\partial \gamma}{\partial R_B} - I(\alpha)F(\alpha)F'(\alpha)\frac{\partial \alpha}{\partial R_B} - \\ & \frac{1}{R_A R_B^2}\int_\gamma^{[1-F(\gamma)]R_A+\gamma} I(q)\left[2\left(q - \gamma + F\left(\gamma\right)R_A\right) - \left(R_A - R_B\right)\right]dq + \\ & \frac{1}{R_A R_B}\int_\gamma^{[1-F(\gamma)]R_A+\gamma} I(q)\left[2\left(-1 + F'\left(\gamma\right)R_A\right)\frac{\partial \gamma}{\partial R_B} + 1\right]dq + \end{aligned}$$

$$\frac{1}{R_A R_B}\left(1-F'(\gamma)R_A\right)\frac{\partial\gamma}{\partial R_B}I\left(\left[1-F(\gamma)\right]R_A+\gamma\right)\left(R_A+R_B\right)-$$
$$\frac{1}{R_A R_B}\frac{\partial\gamma}{\partial R_B}I\left(\gamma\right)\left(2F\left(\gamma\right)R_A-\left(R_A-R_B\right)\right)-$$
$$\frac{1}{2}\left[1+\left(-2+2\frac{R_B}{R_A}\right)\left(1-F\left(\alpha\right)\right)^2-2\left(R_A-2R_B+\frac{R_B^2}{R_A}\right)\left(1-F\left(\alpha\right)\right)F'\left(\alpha\right)\frac{\partial\alpha}{\partial R_B}\right],$$

which implies

$$\frac{\partial U}{\partial R_B}\left(R_A=R,R_B=R\right)=\frac{1}{R}I(\alpha)F(\alpha)\frac{\partial\alpha}{\partial R_B}-\frac{1}{R}I(\alpha)F(\alpha)\frac{\partial\gamma}{\partial R_B}-$$
$$\frac{2}{R^3}\int_\alpha^{[1-F(\alpha)]R+\alpha}I(q)\left(q-\alpha+F\left(\alpha\right)R\right)dq+\frac{1}{R^2}\int_\alpha^{[1-F(\alpha)]R+\alpha}I(q)dq-\frac{1}{2}.$$

A marginal increase in R_A and a simultaneous marginal decrease in R_B from a nondiscriminatory contest R leads to

$$\left[\frac{\partial U}{\partial R_A}-\frac{\partial U}{\partial R_B}\right]\left(R_A=R,R_B=R\right)=\frac{1}{R}I(\alpha)F(\alpha)\left[\frac{\partial\alpha}{\partial R_A}-\frac{\partial\gamma}{\partial R_A}-\frac{\partial\alpha}{\partial R_B}+\frac{\partial\gamma}{\partial R_B}\right]-$$
$$\frac{2}{R^2}\int_\alpha^{[1-F(\alpha)]R+\alpha}I(q)\left[1-F\left(\alpha\right)\right]dq+\frac{2}{R}I\left(\left[1-F(\alpha)\right]R+\alpha\right)\left[1-F(\alpha)\right].$$

To compute the partial derivatives of α and γ, we use equations (15) and (16) that implicitly define these variables. Then,

$$\begin{pmatrix} -F'\left(\alpha\right)R_B & F'\left(\gamma\right)R_A \\ F'\left(\alpha\right)R_A-1 & -F'\left(\gamma\right)R_A+1 \end{pmatrix}\begin{pmatrix} d\alpha \\ d\gamma \end{pmatrix}=\begin{pmatrix} 1-F\left(\gamma\right) & -1+F\left(\alpha\right) \\ F\left(\gamma\right)-F\left(\alpha\right) & 0 \end{pmatrix}\begin{pmatrix} dR_A \\ dR_B \end{pmatrix}$$

from which,

$$\begin{pmatrix} d\alpha \\ d\gamma \end{pmatrix}=\frac{1}{\Delta}\begin{pmatrix} -F'\left(\gamma\right)R_A+1 & -F'\left(\gamma\right)R_A \\ -F'\left(\alpha\right)R_A+1 & -F'\left(\alpha\right)R_B \end{pmatrix}\begin{pmatrix} 1-F\left(\gamma\right) & -1+F\left(\alpha\right) \\ F\left(\gamma\right)-F\left(\alpha\right) & 0 \end{pmatrix}\begin{pmatrix} dR_A \\ dR_B \end{pmatrix}$$

where

$$\Delta=F'\left(\alpha\right)R_B\left[F'\left(\gamma\right)R_A-1\right]+F'\left(\gamma\right)R_A\left[1-F'\left(\alpha\right)R_A\right].$$

We notice that $\Delta>0$ because α is in the decreasing part, while γ is in the increasing part, of $F\left(\theta\right)R_A-\theta$, that is, $F'\left(\alpha\right)R_A-1<0$ and $F'\left(\gamma\right)R_A-1>0$. Therefore, $\frac{\partial\alpha}{\partial R_A}-\frac{\partial\gamma}{\partial R_A}-\frac{\partial\alpha}{\partial R_B}+\frac{\partial\gamma}{\partial R_B}=\frac{\Omega}{\Delta}$, where

$$\Omega=\left[-F'\left(\gamma\right)R_A+1\right]\left[1-F\left(\gamma\right)\right]-F'\left(\gamma\right)R_A\left[F\left(\gamma\right)-F\left(\widehat{\theta}_A\right)\right]-$$
$$\left[-F'\left(\alpha\right)R_A+1\right]\left[1-F\left(\gamma\right)\right]+F'\left(\alpha\right)R_B\left[F\left(\gamma\right)-F\left(\alpha\right)\right]-$$
$$\left[-F'\left(\gamma\right)R_A+1\right]\left[-1+F\left(\alpha\right)\right]+\left[-F'\left(\alpha\right)R_A+1\right]\left[-1+F\left(\alpha\right)\right]=$$

$$
\begin{aligned}
&-F'(\gamma)\,R_A + F'(\gamma)\,F(\alpha)\,R_A + F'(\alpha)\,[1-F(\gamma)]\,R_A + \\
&\quad F'(\alpha)\,[F(\gamma)-F(\alpha)]\,R_B + F'(\gamma)\,[-1+F(\alpha)]\,R_A - F'(\alpha)\,[-1+F(\alpha)]\,R_A = \\
&-2F'(\gamma)\,[1-F(\alpha)]\,R_A + F'(\alpha)\,[2R_A - F(\gamma)\,R_A + F(\gamma)\,R_B - F(\alpha)\,R_B - F(\alpha)\,R_A]\,.
\end{aligned}
$$

Both Δ and Ω depend on (R_A, R_B) and we need to compute $\frac{\Omega}{\Delta}$ at $(R_A = R, R_B = R)$. We note that $\Omega\,(R_A = R, R_B = R) = 0$ and $\Delta\,(R_A = R, R_B = R) = 0$. We use that $\lim_{R_B \longrightarrow R_A} \frac{\Omega}{\Delta}\,(R_A, R_B) = \frac{\lim_{R_B \longrightarrow R_A} \frac{\partial \Omega}{\partial R_B}}{\lim_{R_B \longrightarrow R_A} \frac{\partial \Delta}{\partial R_B}}\,(R_A, R_B)\,.$

$$
\begin{aligned}
\frac{\partial \Omega}{\partial R_B} &= -2F''(\gamma)\,[1-F(\alpha)]\,R_A \frac{\partial \gamma}{\partial R_B} + 2F'(\gamma)\,F'(\alpha)\,R_A \frac{\partial \alpha}{\partial R_B} + \\
&\quad F''(\alpha)\,[2R_A - F(\gamma)\,R_A + F(\gamma)\,R_B - F(\alpha)\,R_B - F(\alpha)\,R_A]\,\frac{\partial \alpha}{\partial R_B} + \\
&\quad F'(\alpha)\left[F(\gamma) - F'(\gamma)\,(R_A - R_B)\,\frac{\partial \gamma}{\partial R_B} - F(\alpha) - F'(\alpha)\,(R_B + R_A)\,\frac{\partial \alpha}{\partial R_B}\right],
\end{aligned}
$$

which, taking into account that $\gamma = \alpha$ and $F'(\alpha) = \frac{1}{R}$ when $R_A = R_B = R$, implies

$$
\frac{\partial \Omega}{\partial R_B}\,(R_A = R, R_B = R) = 2F''(\alpha)\,[1-F(\alpha)]\,R\left[\frac{\partial \alpha}{\partial R_B} - \frac{\partial \gamma}{\partial R_B}\right].
$$

Similarly,

$$
\begin{aligned}
\frac{\partial \Delta}{\partial R_B} &= F'(\alpha)\,[F'(\gamma)\,R_A - 1] + F''(\alpha)\,R_B\,[F'(\gamma)\,R_A - 1]\,\frac{\partial \alpha}{\partial R_B} + \\
&F'(\alpha)\,R_B F''(\gamma)\,R_A \frac{\partial \gamma}{\partial R_B} + F''(\gamma)\,R_A\,[1 - F'(\alpha)\,R_A]\,\frac{\partial \gamma}{\partial R_B} - F'(\gamma)\,R_A F''(\alpha)\,R_A \frac{\partial \alpha}{\partial R_B},
\end{aligned}
$$

hence,

$$
\frac{\partial \Delta}{\partial R_B}\,(R_A = R, R_B = R) = F''(\alpha)\,R\left[\frac{\partial \gamma}{\partial R_B} - \frac{\partial \alpha}{\partial R_B}\right].
$$

We notice that $\gamma > \alpha$ as soon as $R_A > R_B$, which implies that $\frac{\partial \gamma}{\partial R_B} - \frac{\partial \alpha}{\partial R_B} > 0$ at $R_A = R_B = R$. Therefore,

$$
\frac{\Omega}{\Delta} = \frac{2F''(\alpha)\,R\,[1-F(\alpha)]\left[\frac{\partial \alpha}{\partial R_B} - \frac{\partial \gamma}{\partial R_B}\right]}{F''(\alpha)\,R\left[\frac{\partial \gamma}{\partial R_B} - \frac{\partial \alpha}{\partial R_B}\right]} = -2\,[1-F(\alpha)]\,.
$$

We substitute $\frac{\partial \alpha}{\partial R_A} - \frac{\partial \gamma}{\partial R_A} - \frac{\partial \alpha}{\partial R_B} + \frac{\partial \gamma}{\partial R_B}$ in the derivative $\left[\frac{\partial U}{\partial R_A} - \frac{\partial U}{\partial R_B}\right](R_A = R, R_B = R)$ to obtain

$$
\begin{aligned}
&\left[\frac{\partial U}{\partial R_A} - \frac{\partial U}{\partial R_B}\right](R_A = R, R_B = R) = -2\frac{1}{R} I(\alpha) F(\alpha)\,[1-F(\alpha)] - \\
&\quad \frac{2}{R^2}\int_{\alpha}^{[1-F(\alpha)]R+\alpha} I(q)\,[1-F(\alpha)]\,dq + \frac{2}{R}\,[1-F(\alpha)]\,I\left([1-F(\alpha)]\,R + \alpha\right) =
\end{aligned}
$$

$$\frac{2}{R^2}\left[1-F(\alpha)\right]\left[I\left(\left[1-F(\alpha)\right]R+\alpha\right)R-I(\alpha)F(\alpha)R-\int_{\alpha}^{[1-F(\alpha)]R+\alpha}I(q)dq\right]>$$

$$\frac{2}{R^2}\left[1-F(\alpha)\right]\left[\left(I\left(\left[1-F(\alpha)\right]R+\alpha\right)-I(\alpha)\right)R-\int_{\alpha}^{[1-F(\alpha)]R+\alpha}I(q)dq\right]>$$

$$\frac{2}{R^2}\left[1-F(\alpha)\right]\left[\left(I\left(\left[1-F(\alpha)\right]R+\alpha\right)-I(\alpha)\right)\left[1-F(\alpha)\right]R-\int_{\alpha}^{[1-F(\alpha)]R+\alpha}I(q)dq\right]>0$$

given that $I(q)$ is increasing. Therefore, discriminating marginally always increases the designer's payoff.

(c) Proceeding as in the previous cases, the marginal change in costs due to marginal discrimination is zero. Therefore, it is enough to examine the change in revenues. Rather than proceeding directly through the designer's revenue function, we continue the line of argument that we presented after Proposition 5. We show that $\frac{\partial\beta}{\partial\varepsilon}>0$. The equations determining γ are β are (17) and (18). Differentiating the equations with respect to ε, at $\varepsilon=0$, yields

$$\begin{pmatrix} 1 & -1+F'(\beta)R \\ 1-RF'(\beta) & -1+F'(\beta)R \end{pmatrix}\begin{pmatrix} d\gamma \\ d\beta \end{pmatrix}=\begin{pmatrix} F(\beta) \\ 0 \end{pmatrix}d\varepsilon$$

hence,

$$\frac{d\beta}{d\varepsilon}=\frac{F(\beta)}{F'(\beta)R}>0.$$

■

Proof of Proposition 6. By Proposition 5 (b), discriminating is optimal if the equilibrium strategy profile in the nondiscriminatory contest is $\left(q_A^{CI},q_B^{CI}\right)$ and if marginal changes in (R_A,R_B) lead to $\left(q_A^{CI_\gamma},q_B^{CI_\gamma}\right)$. Given that $F(\theta)$ is convex and $F'(0)=0$, the equilibrium profile is $\left(q_A^{CI},q_B^{CI}\right)$ if the optimal R satisfies $R>\frac{1}{F'(1)}$. To show that this is the case if v is large enough, we compare the profits that the designer obtains by choosing an $R\geq\frac{1}{F'(1)}$ with those obtained for $R=0$ ($R=0$ is the optimal choice among all the rewards that lead to the equilibrium profile of $\left(q_A^{C},q_B^{C}\right)$). By Proposition 4 (c), the designer's payoff if the equilibrium profile is $\left(q_A^{CI},q_B^{CI}\right)$ is

$$U(R)=2v\int_0^{\alpha}i(q)F(q)F'(q)dq+\frac{2v}{R^2}\int_{\alpha}^{[1-F(\alpha)]R+\alpha}i(q)\left[q-\alpha+F(\alpha)R\right]dq-R$$

where $F'(\alpha)R=1$, whereas

$$U(0)=2v\int_0^1 i(q)F(q)F'(q)dq.$$

The designer's payoff is higher for some $R > \frac{1}{F'(1)}$ than for $R = 0$ if

$$2vh(R) > R,$$

where we denote

$$h(R) \equiv \frac{1}{R^2}\int_{\alpha}^{[1-F(\alpha)]R+\alpha} i(q)\left[q-\alpha+F(\alpha)R\right]dq - \int_{\alpha}^{1} i(q)F(q)F'(q)dq.$$

Since v can be arbitrarily large, it suffices to show that $h(R) > 0$. We rewrite $h(R)$ as

$$h(R) = \frac{1}{2}\int_{\alpha}^{[1-F(\alpha)]R+\alpha} i(q)\frac{d}{dq}\left(\frac{q-\alpha+F(\alpha)R}{R}\right)^2 - \frac{1}{2}\int_{\alpha}^{1} i(q)\frac{d}{dq}\left(F(q)\right)^2.$$

Note that $\frac{q-\alpha+F(\alpha)R}{R} > F(q)$ because $q > \alpha$ and as such, it is in the increasing part of $RF(\theta) - \theta$. Therefore, the distribution function $\left(\frac{q-\alpha+F(\alpha)R}{R}\right)^2$ first-order stochastically dominates $(F(q))^2$. Given that $i(q)$ is strictly increasing in q, $h(R) > 0$ for all $R > \frac{1}{F'(1)}$.

Finally, the proposition is proved if we show that marginal changes from a nondiscriminatory contest $R_A = R_B = R > \frac{1}{F'(1)}$ lead to $\left(q_A^{CI_\gamma}, q_B^{CI_\gamma}\right)$. According to Theorem 2, this property certainly holds if $R_A F(R_A - R_B) - (R_A - R_B) < 0$. Taking $R_A = R + \varepsilon$ and $R_B = R - \varepsilon$, the inequality is equivalent to $(R+\varepsilon)F(2\varepsilon) - 2\varepsilon < 0$. The inequality holds for ε small enough because $F'(0) = 0$. ■

Proof of Proposition 7. By Proposition 5 (c), discrimination is optimal if the equilibrium strategy profile in the optimal nondiscriminatory contest is $\left(q_A^{IC}, q_B^{IC}\right)$ and if marginal changes in (R_A, R_B) lead to $\left(q_A^{I_\gamma C}, q_B^{I_\gamma C}\right)$.

To determine the optimal R in the nondiscriminatory contest, we note that F is concave with $F'(0) = \infty$. By Theorem 3, the discussion following it, and $F'(0) = \infty$, we have to consider three possible scenarios: (i) $R = 0$, (ii) $0 < R < 1$, and (iii) $R > 1$.

When $R = 0$, the effort levels offered are zero and the payoff to the designer is $\frac{2v\lambda}{(\mu+2\lambda)}$.

When $0 < R < 1$, the equilibrium strategy profile is $\left(q_A^{IC}, q_B^{IC}\right)$ and by Proposition 4 part (d), the designer's payoff is given by

$$U(R) = \frac{2v}{R^2}\int_0^{\beta} q^{\mu+1}dq + 2v\lambda\int_{\beta}^{1} q^{\mu+2\lambda-1}dq - R =$$
$$\frac{2v}{(\mu+2)}\frac{1}{R^2}\beta^{\mu+2} + \frac{2v}{(\mu+2\lambda)}\lambda - \frac{2v}{(\mu+2\lambda)}\lambda\beta^{\mu+2\lambda} - R$$

where $\beta > 0$ satisfies $F(\beta)R - \beta = 0$, that is, $\beta = R^{\frac{1}{1-\lambda}}$. Therefore,

$$U(R) = 2v\left(\frac{1}{(\mu+2)} - \frac{\lambda}{(\mu+2\lambda)}\right)R^{\frac{\mu+2\lambda}{1-\lambda}} + \frac{2v\lambda}{(\mu+2\lambda)} - R.$$

Differentiating the designer's payoff we obtain $U'(R) = 2v\left(\frac{(1-\lambda)\mu}{(\mu+2)(\mu+2\lambda)}\right)\frac{\mu+2\lambda}{1-\lambda}R^{\frac{\mu+2\lambda}{1-\lambda}-1} - 1 = \frac{2\mu v}{(\mu+2)}R^{\frac{\mu+3\lambda-1}{1-\lambda}} - 1$. We see that $U'(0) = \infty$ because $\mu + 3\lambda - 1 < 0$ and $U'(1) = \frac{2\mu v}{(\mu+2)} - 1 < 0$ because $v < \frac{\mu+2}{2\mu}$.

We now examine the case $R \geq 1$. The equilibrium strategy profile is (q_A^I, q_B^I) and by Proposition 4 part (b) the designer's payoff is given by

$$U(R) = \frac{2v}{R^2}\int_0^R q^{\mu+1}dq - R = \frac{2v}{(\mu+2)}R^{\mu} - R.$$

The function $U(R)$ is continuously differentiable at $R = 1$ ($U'(R)$ is also $\frac{2\mu v}{(\mu+2)} - 1 < 0$ using the expression above). Moreover, $U''(R) = \frac{2\mu(\mu-1)v}{(\mu+2)}R^{\mu-2} < 0$ for all $R \geq 1$. The function is also concave for $0 < R < 1$. Therefore, the function $U(R)$ obtains a unique maximum at some $0 < R < 1$.

Finally, the proposition is proved if we show that marginal changes from a nondiscriminatory contest $R_A = R_B = R$ with $0 < R < 1$ lead to $\left(q_A^{I,C}, q_B^{I,C}\right)$. According to Theorem 3, this property holds if $R_A \in \left(\frac{1}{F'(0)}, 1\right)$. Taking $R_A = R + \varepsilon$ and $R_B = R - \varepsilon$, the inequality is satisfied for ε small enough since $F'(0) = \infty$ and $R < 1$. ■

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