Liquidity Scarcity, Project Selection and Volatility

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Abstract

The severe contraction that followed the recent financial crisis highlighted the exposure of the real sector to financial markets and the volatility in credit conditions. Unreliability of future funding influences the way in which firms balance risks when choosing investment projects and designing financial arrangements. The present paper studies the behavior of project choice in an environment with financial frictions and its consequences for the aggregate behavior of the economy. I focus on responses to fluctuations in the external supply of liquidity and in the liquidity created by the entrepreneurial projects themselves. When shocks occur to external liquidity sources, such as changes in the cash-flows that support mortgage-backed securities or other non-corporate assets, these are transmitted through financial arrangements towards the real sector. The anticipation of these shocks and its reflection in asset prices influence project selection and change the pattern of fluctuations, creating additional comovement. Likewise, the anticipation of variations in the internal liquidity of firms, resulting from shocks to their productivity, changes their choice of projects. For moderate liquidity scarcity, the effect through project choice is shown to lead to the dampening of these underlying productivity shocks; while for more severe shortages, amplification emerges. Despite the possibility of excess exposure to risk being generated endogenously, allocations are constrained efficient. Policy implications are then discussed in light of this result.

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1 Introduction

The recent financial crisis was followed by one of the sharpest credit contractions since the Great Depression. Major drops were experienced in syndicated lending, down by 79% of its peak volume (Ivashina and Scharfstein (2010)), and in industrial and commercial loans by U.S. commercial banks, which dropped by approximately a quarter from the Oct/2008 peak to the Oct/2009 bottom[1]. Concerns about a market freeze in commercial paper also led to an unconventional intervention, with the creation of the Commercial Paper Funding Facility by the Federal Reserve system at the height of the crisis. This special purpose vehicle, by acting as buyer of last resort in the commercial paper market, eventually held up to approximately U$350 billion in commercial paper (Adrian, Kimbrough and Marchioni (2011)). Shock-waves of the crisis were felt across multiple sectors of the economy and the severe recession that followed highlighted the exposure of the real sector to financial factors and to the volatility in credit conditions. A few important questions emerge. First, how can the financial system be made more resilient, to prevent other such crises from emerging? Second, how does the anticipation of unreliability in future funding affect decisions of non-financial firms regarding their exposure to both real and financial risks? Last, is this exposure excessive, creating a case for future intervention?

The elusive answer to the first of these questions has attracted a number of important contributions[2]. The present paper attempts to address the remaining set of questions. To do so, it is necessary to study an environment in which unreliable financial conditions and fluctuations in asset markets matter for real economy activity. Also, one in which agents in the real sector face trade-offs in their exposure to the different risks involved in production and its financing.

I build on the framework of Holmström and Tirole (1998, 2001), which provide a model environment in which liquidity conditions affect investment, asset prices and output. There, however, investment prospects are fixed. I extend their baseline model to incorporate the choice over different investment projects and to endogenize the economy's response to a set of shocks which includes asset return fluctuations, productivity volatility and financial distress possibilities.

There are three time periods. Investments on projects are made over the first two and they only mature, generating revenues, on the third one. Projects differ in how their productivity, costs and capacity to attract external funding respond to shocks. Entrepreneurs and lenders design financial

[1]Board of Governors of the Federal Reserve System, Release H.8

[2]Some examples include Adrian and Shin (2009); Acharya, Mehran and Thakor (2010); Brunnermeier (2009); Curdia and Woodford (2010); Diamond and Rajan (Forthcoming); Farhi and Tirole (2011); Geanakoplos (2009); Gertler, Kiyotaki and Queralto (2011); Hanson, Kashyap and Stein (2010); Kurlat (2010); Lorenzoni (2008); Shleifer and Vishny (2010); Stein (2011)

contracts to cover the random costs of these projects and need to take into account constraints that arise from both sides of the agreement. As a consequence, decisions regarding project selection and financial arrangements are intertwined.

From the entrepreneurial side, only a limited share of the cash flows generated can be credibly committed to the repayment of other agents, i.e., there is limited pledgeability of output. Consequently, a project's potential to guarantee the funding necessary for its own completion is compromised and there is limited *internal liquidity*. Since financial needs of projects might exceed available internal liquidity, there is a demand for pre-arranged transfers of resources from lenders.

From the lenders' side, limited commitment constrains their promises to transfer resources in the future to help fund the project. As a result, other assets available in the economy play a role in this arrangement, as they can serve as collateral and help back reinvestment promises. These assets serve as *external liquidity* and are demanded as part of the optimal financial contract. Important practical examples of non-corporate assets which are either held directly by firms for contingent liquidation or back funding delivery promises include cash, sovereign bonds and mortgage-backed securities.

Jointly, the availability of internal and external liquidity in the economy determine its aggregate liquidity conditions and asset prices. In turn, these asset prices influence optimal financial contracts and project choices. Through these interactions, endogenous project selection and general equilibrium effects are key determinants of the behavior of the aggregate economy and its responses to shocks.

My first main result originates from an application in which I study the choice over projects which differ in the volatility of their capacity to generate output and revenues. As only a fraction of this output is pledgeable, internal liquidity drops in case of a negative productivity shock and financial needs, which need to be backed by external assets, increase. The opposite occurs when positive productivity shocks hit projects. The optimal financial contract specifies which project is chosen, under which conditions it is completed, downsized or terminated, as a well as all relevant transfers and the asset acquisitions that are necessary for their backing. The possibility of controlling the exposure to productivity risk, by choosing among different projects, is shown to work in this environment as an imperfect substitute for external asset purchases.

When the economy features a single risk-less asset that can be used for backing transfer promises, its price signals its scarcity and determines how liquidity-constrained entrepreneurs are in equilibrium. Project selection and financial contract design work together in ensuring that pledgeable resources are available in the states where they are the most valuable. When asset prices are low, entrepreneurs purchase enough of these assets to be constrained only in states with low productivity. Therefore, choosing projects with lower volatility helps move resources to those states. However, when assets are sufficiently scarce and prices are high, entrepreneurs find themselves constrained even in states with higher productivity. The relative value of resources across those different states determines in which direction they want to bias project choice. As prices increase and entrepreneurs become more liquidity constrained, they choose projects with higher volatility, to make sure they have resources to finance ongoing investments at least in the situations in which the project is the most productive. Therefore, the deterioration of aggregate liquidity conditions leads to the choice of riskier projects, showing that endogenous project selection can be a powerful determinant of aggregate volatility.

I then turn to the consequences of fluctuations which are driven by changes in the values of non-corporate assets, i.e., by shocks to external liquidity. Some examples of central relevance given recent events include the possibility of a drop in house prices leading to a collapse in mortgage-backed securities or sudden changes in the value of sovereign bonds. In the model studied, such fluctuations are introduced as variations in payouts from a set of trees which are in fixed supply. Contingent claims are traded, serve as external liquidity and are backed by these trees. Asset trades are sufficiently sophisticated and allow for positions that include but are not restricted to the holding of risk-less claims. I study how shocks to the payouts of these trees are transmitted towards corporate investment policies and also how endogenous project selection, by generating additional comovement of entrepreneurial output and asset values, can work as an amplification mechanism for these shocks.

In this setting, liquidity premia[3] are always higher for assets that pay out in states where tree output scarcer. Additionally, completion rates for entrepreneurial projects and their final output are always non-decreasing in the trees' output. Since, claims on trees play the role of a financial input in an entrepreneurial sector which is liquidity constrained, a lower payout from them is transmitted towards entrepreneurial output whenever there is a shortage of internal liquidity. This is a natural transmission mechanism and generates some output comovement on its own.

When project choice is introduced in this environment, an additional degree of comovement arises endogenously. Whenever internal liquidity falls short of the necessary costs of investment, investment opportunities and external assets payouts are complementary. Therefore, a project that offers these opportunities in future states in which external liquidity is more plentiful and, consequently, cheaper to acquire in advance is preferred by entrepreneurs. As a result, the entrepreneurial sector biases its investment towards projects that comove positively with the trees' output and ends up being endogenously more exposed to the factors which determine that level.

A third set of results relates to constrained efficiency in the environments studied. Despite the

[3]Liquidity premia are present as assets might sell above their value for consumption purposes. They are defined as a ratio of asset prices to their expected payouts.

possibility of additional exposure to risk and amplification of fluctuations emerging endogenously through project selection, all outcomes are constrained Pareto efficient. Therefore, a planner that does not have advantages in the creation of liquidity nor in its contingent reallocation across firms cannot improve the overall efficiency of production nor increase welfare. This generates a characterization of which classes of policies cannot lead to improvements. Examples of such policies are the ones which ban projects deemed excessively risky, mandate minimum liquid asset holding levels or preclude the use of risky assets as part of financial arrangements. On the other hand, that does not imply the inexistence of policies that could lead to improvements; but if they do exist, they need to rely on an governmental advantage in the creation of liquid assets[4], on its greater flexibility in reallocating resources after realizations of aggregate states of the economy or on its capacity of improving the underlying contractual environment.

The paper also includes a series of additional results. First, a general model is introduced. A few closed-form criteria for project selection in this environment are analyzed. For instance, as a consequence of these frictions, there is an important departure from standard net-present-value criteria largely used in corporate practice. Output generated in a given period is treated differently and needs to be decomposed according to its shares which can be credibly pledged to outsiders and the one that needs to be claimed by entrepreneurs. Also, given credit constraints, optimal leverage determination plays a key role. After this analysis, specialized environments are proposed, to illustrate the different aggregate consequences of the interactions between liquidity scarcity and project selection. The central conclusions from most of these have been reported in the previous paragraphs. The last environment studies the consequences of enriching the set of assets trades in the economy with endogenous choice of output volatility. It shows that although allocations change in interesting ways, the main qualitative conclusions regarding incentives for the amplification or dampening of productivity fluctuations are robust to these more sophisticated trades and are, thus, not a consequence of the single risk-less asset assumption initially made.

Related Literature- The present paper is related to different strands of economic literature. As anticipated, it is the most closely related to the literature on liquidity asset pricing which follows from Holmström and Tirole (1998, 2001). My focus, however, is on the joint determination of the exposure to real and financial risks which occurs when real investments have to be selected and financed by arrangements which need to take into account the frictions that arise from both sides of a relationship. This focus brings into light interplays between technological and financial decisions,

[4] As in Holmström and Tirole (1998), which discusses how exclusive ("regalian") enforcement powers give the public sector a unique opportunity to create liquidity backed by its ability to tax citizens in the future.

as well as their aggregate consequences.

Difficulties in securing future funding and the need to manage liquidity buffers also seem to be a growing concern in corporate practice. A dramatic increase in corporate liquid holdings has been observed in the last few decades, through a a steep growth in the cash-to-asset ratio of U.S. industrial firms, which more than doubled in the 1980-2006 period (Bates, Kahle and Stulz (2009)). Indeed, that study also reports that the average corporation has enough cash to withdraw all of its debt and that a common measure of leverage which nets out cash holdings, the net debt ratio[5], has suffered a substantial secular decline. The picture becomes even more impressive when we take into account additional instruments for liquidity hoarding beyond cash. For instance, Campello et al. (2011) report results of a survey which shows that the average firm has credit line access amounting to 24% of the value of their total assets, about twice the volume of cash they additionally hold[6].

Some recent empirical papers have also studied the behavior of the mix between cash and credit lines and highlighted the importance of covenants in determining the availability of these pre-committed funds[7]. In particular, Acharya, Almeida and Campello (2010) discusses the importance of aggregate risk in triggering covenant violations and reducing the amount of resources available for covering corporate expenses. The recent financial crisis has also provided rich data on the interaction between liquidity dry-ups and the responses in corporate investment policy, employment and financial management[8]. The current paper focuses on how production decisions and financial arrangements anticipate these possibilities and, especially, on the consequences on the aggregate behavior of economy in face of real and financial shocks.

It is thus also related to another set of papers which have addressed the broad issue of project selection, or investment composition, in environments with financial frictions. For example, Matsuyama (2004, 2007a,b), which study deterministic aggregate implications of imperfect credit markets, such as credit cycles, leapfrogging, aggregate demand spill-overs, reverse international capital flows and traps. Or Aghion et al. (2010), which studies the choice between a low volatility, but financially exposed investment, versus a more volatile short-term investment, across economies with borrowing constraints of different severity. It shows that the determination of investment composition can help account for empirical patterns in levels of cross-country growth rates and their volatility. The present work differs from these papers in studying how the joint selection of projects and financial

[5]Defined as debt minus cash, divided by book assets.

[6]It is worth noting a significant discrepancy in cash ratios in Bates, Kahle and Stulz (2009) and Campello et al. (2011), due to sampling among firms with different characteristics.

[7]Data on credit line availability and drawdowns have typically been hard to obtain. A few recent papers such as Sufi (2009) and Acharya, Almeida and Campello (2010) made progress in its obtainment and analysis.

[8]See, for instance, Almeida et al. (2009); Campello et al. (2011); Ivashina and Scharfstein (2010).

arrangements respond to the scarcity of aggregate liquidity and the importance of this mechanism in determining the pattern of fluctuations in the economy. Its conclusions add a new perspective to this broad set of macroeconomic consequences of imperfect financial markets.

One of the central results of the paper regards the emergence of the choice of riskier projects in economies which face severe liquidity scarcity. I identify a form of risk-seeking behavior on entrepreneurial decisions. To the best of my knowledge, it significantly differs from previously known channels, such as an agency problem leading to asset substitution and risk-shifting (Jensen and Meckling (1976)) and non-convexities in the entrepreneur's value function derived from a combination of credit constraints and occupational choice (Vereshchagina and Hopenhayn (2009)). The key driver of this risk-seeking mechanism lies in the partial pledgeability of output and on the way through which output fluctuations move pledgeable resources across states of the world. These resources are useful for backing the financing of investment, substitute for costly asset hoarding and are especially valuable when investment is more productive. Unlike in the risk-shifting literature, the contracts between lender and borrower that I study offer sufficient state contingency and the choice for riskier projects is an ex ante decision on which both lender and borrower agree as the best response to the constraints and environment they face.

The paper is also related to the literature on optimal risk management[9], but takes a less common approach by studying how investment decisions interact with financial arrangements; and also by doing that in a general equilibrium environment. The first of these elements is present in a recent paper by Almeida, Campello and Weisbach (2011)(ACW), which studies investment and risk management when future financing involves frictions. Its main insight is that the possibility of future financing shortfalls leads to investment in projects with earlier payouts and lower risk exposure. Some key distinctions are responsible for generating different analysis and complementary results between our papers. In ACW, investment opportunities are independent across periods. The potential for funding shortages on an upcoming decreasing returns to scale investment opportunity creates a form of risk-aversion[10] and, without temporal dependence in productivity across projects, biasing is always towards safer projects. In the environment I study in Section 4.1, the same project is financed sequentially, which naturally introduces an inter-temporal dependence in investment productivity. A more volatile project, while more severely affected by shocks on the downside, generates more pledgeable output, which backs its own financing, exactly in the situations in which reinvestment is more productive. This mechanism is at the heart of the emergence of the form of risk-seeking

[9]For instance, Froot, Scharfstein and Stein (1993); Leland (1998); Holmström and Tirole (2000); Rampini and Viswanathan (2010).

[10]As first illustrated by Froot, Scharfstein and Stein (1993).

behavior which is identified in that section. Other sources of complementarity lie in the study of the aggregation of multiple firms in general equilibrium on the current paper, which is essential for its focus on aggregate consequences, and also on the presence a richer set of macroeconomic shocks.

Last, it can also be related to the literature on financial development and volatility, when different comparative statics on the magnitude of the underlying frictions and the availability of non-corporate assets are conducted. For instance, the result linking liquidity scarcity, if interpreted as low financial development, to the choice of riskier projects is consistent with the finding that less developed countries specialize in riskier sectors (Koren and Tenreyro (2007)), which is difficult to reconcile with models based on optimal portfolio choice approaches in face of a mean and volatility trade-off.

Structure- The rest of the paper is organized as follows. Section 2 proposes the general model. Section 3 studies incentives driving project choice in this environment. Section 4 discusses the interactions and aggregate consequences in specialized environments. Section 5 proves the constrained Pareto optimality result and policy consequences, while section 6 concludes. All proofs omitted from the main text are in the appendix.

2 The Model

The central features of the model are the presence of a set of agents with a menu of investment opportunities, entrepreneurs, and a set of agents without these opportunities, who act as lenders. They design a financial contract subject to constraints from both sides of the arrangement: limited pledgeability from the entrepreneur side and limited commitment from lenders. The presence of random costs before the completion of projects occurs creates a need to ensure the availability of resources for those situations, generating a rationale for liquidity insurance and management. Entrepreneurs try to make sure they have resources in situations in which they can be used productively, for salvaging a project under distress or for taking advantage of investment opportunities. Limited commitment constrains liquidity insurance by lenders and creates a role for asset purchases from third parties in enabling some limited insurance. The markets for these assets are potentially incomplete, to allow for potential difficulties in fully state-contingent liquidity trades.

Time and uncertainty

Time is described by $t = 0, 1, 2$. There is a single good in each period, which can be used for both consumption and investment.

The state of the world is fully described by $\omega \in \Omega$, where Ω is a finite set with $\#\Omega$ elements. All uncertainty is realized at time $t = 1$ and $\pi : \Omega \to [0, 1]$ is a probability mass function.

Agents

Entrepreneurs - The economy is populated by a continuum measure one set of identical entrepreneurs, indexed by $j \in [0, 1]$. They are the only agents in the economy with access to a menu of investment technologies, soon to be described. Each one has initial net worth A at $t = 0$ and no endowment in future periods. They are risk neutral, with utility given by $U(c_0, c_1, c_2) = E[c_0 + c_1 + c_2]$.

Lenders/consumers - There is a continuum of agents without direct access to investment opportunities, but with large endowments in the first two periods, A_0^L and A_1^L and no endowments in the last period. We assume that the measure of this set is strictly greater than one, so there are more lenders than entrepreneurs available. They are also risk neutral and also evaluate consumption streams according to $U(c_0, c_1, c_2) = E[c_0 + c_1 + c_2]$. The large endowment assumption ensures that scarcity of resources does not limit investment, leaving the determination of scale to be a consequence contractual frictions and not resource scarcity. Lenders are not able to commit to payments at $t = 1, 2$.

Assets

There are K assets in fixed supply $L \in \mathbb{R}_+^K$, which are initially held by lenders. The payout vector at state ω is given by $z(\omega) \in \mathbb{R}_+^K$. To emphasize the role as stores of value and not as physical inputs, let us assume that these resources are only available for consumption at $t = 2$. Additionally, let the payoff matrix have full rank K and $K \leq \#\Omega$. Therefore, there are no redundant assets and, for each asset k, $z_k(\omega) > 0$ for at least some $\omega \in \Omega$.

These assets are traded at prices $q \in \mathbb{R}_{++}^K$ at time zero, with q_k representing the price of asset k. For simplicity, there is no market for such assets at $t = 1$. Given the ability to pledge payoffs from assets in financial contracts and common preferences, this assumption is innocuous.

This general formulation nests the case in which there are only real assets that can be purchased with the purpose of backing promises of transfers across agents, as well as an economy in which a complete set of Arrow-Debreu state contingent financial securities can be traded.

Definition 1. A **liquidity premium** on asset k is defined as the excess payment made for this asset at $t = 0$ relative to its expected output, that is, $\frac{q_k}{E[z_k]} - 1$.

Projects

Entrepreneurs choose from a menu of projects. These are described by the choice of $\gamma \in \Gamma$, where Γ is a compact subset of $\mathbb{R}^n$. Each of these projects involves a constant-returns-to-scale technology that generates $\rho_1(\omega, \gamma)$ units of output per-unit of investment if brought to completion. Investment is made at time $t = 0$ and output becomes available at $t = 2$. However, due to a contractual friction, only $\rho_0(\omega, \gamma) < \rho_1(\omega, \gamma)$ can be pledged to lenders. This friction can be motivated using a moral hazard problem, limited commitment or other distortions. The set-up cost of these projects is $\phi(\gamma)$ per-unit at $t = 0$[11].

Project choice can be narrowly interpreted as a technological decision, describing different ways to produce a final good or as alternative investment possibilities in different sectors of an economy. More broadly, it can also involve choices over different costly actions that can be taken by management during the implementation of a single enterprise which lead to changes in its returns, costs and responses to risks.

The projects involve a time-to-build component and might suffer additional cost shocks at $t = 1$, which make projects require essential reinvestment before completion occurs. These reinvestment shocks are denoted by $\rho(\omega, \gamma)$. Each unit of project γ will only be brought to completion and deliver output at $t = 2$ if an additional amount $\rho(\omega, \gamma)$ of resources is invested in the intermediate period, $t = 1$. An incomplete unit does not generate any output.

Partial continuation at any state contingent scale $x(\omega) \in [0, 1]$ is possible. That means that if entrepreneurs face liquidity shortages that render them unable to fully continue the project, a downsizing possibility exists. By downsizing the projects to a fraction $x(\omega)$ of their initial scale, total and pledgeable returns can still be collected for the relevant share of completed units.

Financial Contract

At the beginning of period $t = 0$, each entrepreneur competitively offers a financial contract to be accepted by a single lender. The contract specifies $\left\{I, \{x(\omega)\}_{\omega\in\Omega}, \gamma, a\right\}$, where I is an investment scale, $\{x(\omega)\}_{\omega\in\Omega}$ is the fully-state-contingent continuation policy, γ is the project chosen and $a \in \mathbb{R}^K$ is the portfolio of external assets held by the entrepreneur-lender pair as part of the financial arrangement. This contract also determines time and state contingent transfers $\tau = \{\tau^0, \tau^1(\omega), \tau^2(\omega)\}$ from the lender to the entrepreneur. Given limited commitment, the lender can walk away at $t = 1$, losing rights to any payoffs from the project or external assets that are held as part of the financial arrangement. Since lenders lose access to the payoffs from assets in case they do not deliver the specified

[11] Given constant returns to scale, assuming that projects have different set-up costs is equivalent to normalizing this cost to be one and scaling all relevant returns and liquidity shocks by a multiplicative factor of $\phi(\gamma)^{-1}$. The additional function is left for convenience in the applications that follow.

transfers to entrepreneurs, external assets play the role of collateral in the financial arrangement.

Taking as given an outside option of $\underline{\tau}$, lender participation at $t = 0$ requires

$$\underline{\tau} \geq E\left[\tau^0 + \tau^1(\omega) + \tau^2(\omega)\right]. \tag{1}$$

The lender commitment problem, imposes an interim participation constraint for each $\omega \in \Omega$ at $t = 1$ of the form

$$0 \geq \tau^1(\omega) + \tau^2(\omega). \tag{2}$$

That means that, in order for the lender not to walk away from the contract at $t = 1$ when state $\omega \in \Omega$ is realized, the sum of continuation transfers from the lender to the entrepreneur has to be non-positive.

Feasibility of the plan and entrepreneurial consumption $\left(c^{0,E}, c^{1,E}, c^{2,E}\right)$ requires

$$\tau^0 + A = \phi(\gamma) I + q \cdot a + c^{0,E}, \tag{3}$$

which means that transfers from lenders plus initial entrepreneurial wealth need to cover the costs of investment, portfolio purchases and any entrepreneurial consumption,

$$\tau^1(\omega) = \rho(\omega, \gamma) x(\omega) I + c^{1,E}(\omega), \; for\, each\, \omega \in \Omega, \tag{4}$$

that is, transfers from lenders need to cover any additional project costs at $t = 1$ plus any entrepreneurial consumption at that stage and, last,

$$\rho_1(\omega, \gamma) x(\omega) I + \tau^2(\omega) = c^{2,E}(\omega), \; for\, each\, \omega \in \Omega, \tag{5}$$

total output generated by the project plus any additional transfers equal entrepreneurial consumption at $t = 2$.

The cases of interest are the ones in which $\tau^2(\omega) < 0$, indicating that there is repayment from entrepreneurs to lenders. These repayments are bounded by limited pledgeability, which imposes that $-\tau^2(\omega)$ needs to be covered by pledgeable income from the project and the assets held,

$$\rho_0(\omega, \gamma) x(\omega) I + z \cdot a \geq -\tau^2(\omega). \tag{6}$$

Entrepreneurs therefore solve

$$\max_{c^E,\tau,I,\{x(\omega)\}_{\omega\in\Omega},\gamma,a} E\left[c^{0,E}+c^{1,E}+c^{2,E}\right] \tag{7}$$

subject to constraints (1)-(6).

The timing of consumption and the transfers between lenders and entrepreneurs are not particularly interesting in this environment, given perfect substitution in consumption. As a consequence, the study of the environment and allocations can be much simplified once we work in terms of surpluses from investment, which are defined below. Additionally, there are two simplified formulations of the entrepreneur's problem, which do not depend on these elements, and are justified through the use of Lemma 1, which follows shortly.

Definition 2. We define the **total unit surplus** of an investment and portfolio plan as

$$B_1(\omega;q;\gamma,x,\hat{a}) \equiv \rho_1(\omega,\gamma)\,x(\omega)-\rho(\omega,\gamma)\,x(\omega)-\phi(\gamma)-[q-z(\omega)]\cdot\hat{a}.$$

The **pledgeable unit surplus** is

$$B_0(\omega;q;\gamma,x,\hat{a}) \equiv \rho_0(\omega,\gamma)\,x(\omega)-\rho(\omega,\gamma)\,x(\omega)-\phi(\gamma)-[q-z(\omega)]\cdot\hat{a}.$$

The **non-pledgeable component** of investment is

$$B_{1-0}(\omega,\gamma) \equiv [\rho_1(\omega,\gamma)-\rho_0(\omega,\gamma)]\,x(\omega)\,.$$

The total surplus, $B_1(\omega;q;\gamma,x,\hat{a})$, is simply the final output per unit generated by the investment at $t=2$, taking into account the completion rate $x(\omega)$, plus the payout from the portfolio of assets $z(\omega)\cdot\hat{a}$ net of all opportunity costs of generating this value. By investing and buying a portfolio at $t=0$, entrepreneurs and lenders forgo $\phi(\omega)+q\cdot\hat{a}$ units of consumption. At $t=1$, an additional $\rho(\omega)\,x(\omega)$ are spent to ensure completion. Given identical and linear preferences, consumption in any period is evaluated at a one-to-one rate by the lenders or entrepreneurs. The pledgeable unit surplus, $B_0(\omega;q;\gamma,x,\hat{a})$, is analogously defined, with pledgeable output replacing total output. Finally the non-pledgeable component of investment, a wedge $B_{1-0}(\omega,\gamma)$, is simply the difference between total output and total pledgeable output per unit. These surpluses are useful in simplifying the entrepreneurs' problem, dropping all the determination of transfers, and simplifying the set of constraints, as done below.

Lemma 1. *Whenever the entrepreneur's problem admits a solution, the optimal entrepreneurial choice relative to investment, continuation decision and asset purchases,* $\left\{I, \{x(\omega)\}_{\omega\in\Omega}, \gamma, a\right\}$*, solves*

$$\max_{\left\{I,\{x(\omega)\}_{\omega\in\Omega},\gamma,\hat{a}\right\}} E_{\omega}\left[B_1(\omega; q; \gamma, x, \hat{a})\right] I \tag{8}$$

s.t.

$$E\left[B_0(\omega; q; \gamma, x, \hat{a})\right] I + A \geq -\underline{t} \tag{9}$$

$$z(\omega)\cdot \hat{a}I + \rho_0(\omega,\gamma)\, x(\omega)\, I \geq \rho(\omega,\gamma)\, x(\omega)\, I,\ for\, each\, \omega\in\Omega \tag{10}$$

and

$$\max_{\left\{I,\{x(\omega)\}_{\omega\in\Omega},\gamma,\hat{a}\right\}} E_{\omega}\left[B_{1-0}(\omega; \gamma, x)\right] I \tag{11}$$

s.t. (9) and (10). Where we define $\hat{a}$ *from* $a \equiv \hat{a}I$*, so that it acts as a normalization of the portfolio by the investment scale.*

According to Lemma 1, entrepreneurs can be thought of as solving either one of two problems. The first one is the maximization of total surplus subject to the constraints to be explained in detail momentarily. The second equivalent formulation leads to the maximization of non-pledgeable benefits, that have to be consumed by entrepreneurs, subject the the same two constraints.

The first constraint (9) is derived from a combination of feasibility constraints and the participation constraint for the lender. It determines that investment is limited by the entrepreneur's capacity to generate pledgeable surplus and initial net worth A. The cases of interest are the ones in which despite its efficiency, investment is limited by difficulties in generating sufficient pledgeable surplus. Whenever projects are sufficiently productive, entrepreneurs would like to lever up by pledging all that is possible from the project to outsiders. As such, (9) is a leverage constraint, which pins down the maximum scale of investment given initial entrepreneurial net worth and the choices made regarding project selection, continuation policies and asset purchases.

The second set of constraints (10) follow from the lack of commitment from lenders. They can be interpreted as liquidity constraints in the following way. On the left-hand side there are the two sources of pledgeable income that entrepreneurs can rely on to ensure that they get funding for continuation. The $z(\omega)\cdot a$ term is the payout from assets acquired that are external to the project, therefore external liquidity. The second term, $\rho_0(\omega,\gamma)\, x(\omega)$, is the pledgeable output that can be still generated by the project if a continuation share $x(\omega)$ is guaranteed. On the right-hand side, there are the resource requirements from the project in state $\omega\in\Omega$ at time $t=1$.

Whenever $\rho_0(\omega,\gamma) \geq \rho(\omega,\gamma)$, the project alone offers enough pledgeable income (internal liquidity) to guarantee full continuation of all units, even without reliance on the payoffs from the assets. Pledgeable income is sufficient to ensure sufficient financing from the lender and the project is said to be *self-refinancing*. Those are the states against which entrepreneurs would typically like to borrow to finance investment at $t = 0$.

On the other hand, whenever $\rho_0(\omega,\gamma) < \rho(\omega,\gamma)$, pledgeable income from the project itself does not fully cover the additional cost at $t = 1$. Project γ is not self-refinancing in state ω and is said to be under *financial distress*. In the absence of any external assets, lenders would be unwilling to transfer any additional amounts to fund continuation of the project. This possibility is responsible for generating a demand for external liquidity. The purchases of external assets can be either interpreted as entrepreneurial savings towards those states or as the acquisition of collateral to enable transfers from lenders in state ω at $t = 1$ and, consequently, insurance of continuation possibilities.

Allocations and Equilibrium

Let Σ be the space in which entrepreneurial decisions towards investment, portfolio and continuation decisions lie with $\sigma = \left\{I, \{x(\omega)\}_{\omega\in\Omega}, \gamma, \hat{a}\right\} \in \Sigma$ being the typical element[12]. In all examples analyzed, Σ can be made compact to ensure the existence of solutions to the entrepreneurs' problem[13].

We then define an allocation as a mapping from the set of entrepreneurs to their decision space Σ. This definition is purposely leaving out specifics of the borrower-lender relationship, which determine the timing of consumption and all potential transfers. Whenever an equilibrium under the definition to follow exists, these elements can be easily obtained.

Definition 3. An allocation is mapping $\sigma : [0,1] \to \Sigma$ such that each coordinate is Lebesgue measurable over $[0,1]$. An allocation naturally defines a probability measure F_σ over Σ, that is, a distribution of entrepreneurs over their decision space.

Definition 4. A competitive equilibrium consists of an allocation σ, an outside option for lenders $\underline{\tau}$ and asset prices $q \in \mathbb{R}^k_{++}$, so that:

1. For every entrepreneur $j \in [0,1]$, $\sigma(j)$ is a solution to the entrepreneur's problem given asset prices and the outside option of lenders $\underline{\tau}$.

[12]Note that $\Sigma \equiv \mathcal{I} \times [0,1]^{\#\Omega} \times \Gamma \times \mathcal{A}$, where $\mathcal{I} \subset \mathbb{R}$is the space of allowed scales and$\mathcal{A} \subset \mathbb{R}^K$ is the space of allowed asset holdings

[13]$x(\omega)$ and γ belong to compact sets and a and I can be restricted to lie in sufficiently large closed intervals without loss of generality.

2. For each asset $k \in \{1, ..., K\}$,

$$
\begin{aligned}
q_k &= E\left[z_k(\omega)\right] \text{ and } \int a_k dj \leq L_k, && (12)\\
&\text{or} \\
q_k &> E\left[z_k(\omega)\right] \text{ and } \int a_k dj = L_k. && (13)
\end{aligned}
$$

3. The presence of excess lenders drives their outside option $\underline{\tau}$ to zero.

The definition of a competitive equilibrium requires entrepreneurial maximization, allowing for indifference between several equilibrium strategies. It is important that it allows for ex post heterogeneity, which emerges in some of the applications studied. Even with indeterminacy at the individual level, aggregates are uniquely defined in these cases. Market clearing conditions take into account that consumers are willing to hold any amount of assets as long as their prices equal their expected payouts (as in condition 12). Otherwise, when a liquidity premium emerges for any given asset, this has to be held exclusively by entrepreneurs as part of financial arrangements (as in condition 13).

3 Project Choice

The two frictions introduced have the potential to drive up asset prices and change the costs of ensuring reinvestment in the different states of the world. Additionally, pledgeable and non-pledgeable income offer different benefits to entrepreneurs. Pledgeable income, can be promised to lenders, helping raise more funds to finance the project's costs, increasing leverage possibilities. However, entrepreneurs can only consume any non-pledgeable resources generated by the projects, as these cannot be credibly transferred to other agents. As projects differ in their liquidity requirements, pledgeable income and non-pledgeable income, all these factors are taken into account in the optimal choice of projects.

In this section, we analyze general criteria for the choice of projects in this environment. Given the constant-returns-to-scale property of the production function and the linearity of the entrepreneurs problem, an optimal project is one that offers the highest shadow value on entrepreneurial wealth or , equivalently, one which has the highest Lagrange multiplier associated to the leverage constraint. Under complete markets, that shadow value is a ratio of expected non-pledgeable benefits to the net liquidity costs of the project, properly weighted by the prices for liquidity delivery in all states of the world. That multiplier can also be interpreted as the product of a leverage ratio and the non-pledgeable returns on investment.

3.1 Project choice under complete markets

The central assumption for this section is that the set of assets is composed of a full set of Arrow-Debreu securities and that entrepreneurs are allowed to short those, as long as there is sufficient pledgeable income to back this sale. When entrepreneurs are borrowing constrained, they invest all their net worth in the project and pledge all that is possible to lenders. As such, they consume only the non-pledgeable component $B_{1-0}(\omega,\gamma) = [\rho_1(\omega,\gamma) - \rho_0(\omega,\gamma)]\,x(\omega)$. To simplify the expressions derived, I introduce the wedge between total and pledgeable income per unit $\rho_{1-0}(\omega,\gamma) \equiv \rho_1(\omega,\gamma) - \rho_0(\omega,\gamma)$.

Under this situation, the entrepreneur's problem for a **fixed** project γ can be written as

$$\max_{\left\{I,\{x(\omega)\}_{\omega\in\Omega},\gamma,\hat{a}\right\}} \sum_{\omega} \pi(\omega)\,\rho_{1-0}(\omega,\gamma)\,x(\omega)\,I \tag{14}$$

s.t.

$$a_\omega \geq (\rho(\omega,\gamma) - \rho_0(\omega,\gamma))\,x(\omega)\,I,\ for\,each\,\omega \in \Omega, \tag{15}$$

$$A - \sum \pi(\omega)\,(\rho(\omega,\gamma) - \rho_0(\omega,\gamma))\,x(\omega)\,I - \sum_{\omega}(q(\omega) - \pi(\omega))\,a_\omega - \phi(\gamma)\,I = 0, \tag{16}$$

$$I \geq x(\omega)\,I \geq 0. \tag{17}$$

Taking the necessary first-order conditions for an optimum while treating $x(\omega)\,I$ as a single choice variable, we obtain

$$x(\omega)\,I:\ \pi(\omega)\,\rho_{1-0}(\omega,\gamma) - (\lambda\pi(\omega) + \mu(\omega))\,(\rho(\omega,\gamma) - \rho_0(\omega,\gamma)) \begin{cases} = \eta_{x(\omega)I} > 0, & if\,x(\omega) = 1, \\ = \eta_{x(\omega)I} = 0, & if\,x(\omega) \in (0,1), \\ < 0, \eta_{x(\omega)I} = 0, & if\,x(\omega) = 0. \end{cases} \tag{18}$$

$$a_\omega:\ \mu(\omega) = \lambda\,(q(\omega) - \pi(\omega)), \tag{19}$$

and

$$I: \phi(\gamma)\lambda = \sum_{\omega} \eta_{x(\omega)I}, \tag{20}$$

where $\mu(\omega)$, λ and $\eta_{x(\omega)I}$ are respectively the multipliers on constraints 15, 16 and $I \geq x(\omega) I$.

A few insights emerge from these conditions. First, after optimization, only a subset of states enter the expression for the marginal value of wealth to the entrepreneur (λ). These are the states in which entrepreneurs strictly prefer to fully continue the project and which are associated to a multiplier $\eta_{x(\omega)I} > 0$, represents the shadow benefit of a scale expansion in a given state. Let this subset be denoted by $\Omega_+(\gamma, q)$. States in this subset are all the states in which,

$$\pi(\omega)\rho_{1-0}(\omega,\gamma) - \lambda q(\omega)(\rho(\omega,\gamma) - \rho_0(\omega,\gamma)) > 0,$$

that is, states in which the private benefit from completion outweighs the opportunity cost in terms of liquidity consumption necessary for continuation, $(\rho(\omega,\gamma) - \rho_0(\omega,\gamma))$, properly weighted by the price $q(\omega)$. This condition is naturally satisfied for all states in which financial distress does not occur, as both $\rho_{1-0}(\omega,\gamma) > 0$ and $\rho(\omega,\gamma) - \rho_0(\omega,\gamma) < 0$.

The shadow value of entrepreneurial wealth can be rewritten as

$$\lambda^*(\gamma, q) = \frac{\sum_{\omega\in\Omega_+(\gamma,q)} \pi(\omega)\rho_{1-0}(\omega,\gamma)}{\phi(\gamma) + \sum_{\omega\in\Omega_+(\gamma,q)} q(\omega)(\rho(\omega,\gamma) - \rho_0(\omega,\gamma))}. \tag{21}$$

Given linearity of the entrepreneur's problem, the value obtained from investing in project γ and choosing optimal continuation policies and portfolios is given by $\lambda^*(\gamma, q) A$. Project choice is then a matter of choosing the project $\gamma^* \in \Gamma$ which is associated to the highest multiplier $\lambda^*(\gamma, q)$. From equation 21, we can study which characteristics make a project more desirable. For instance, fixing all other elements, an increase in the non-pledgeable benefits has that effect. Alternatively, a project with a lower requirement of expensive liquidity ($\rho(\omega,\gamma) - \rho_0(\omega,\gamma) > 0$ in states associated to high $q(\omega)$) is reduced. Both $t = 1$ costs $\rho(\omega,\gamma)$ and pledgeable income $\rho_0(\omega,\gamma)$ enter the denominator and are weighted by the price of the relevant Arrow-Debreu security. The set-up cost at $t = 0$ also consumes net worth and enters additively the denominator of the shadow value of wealth. This multiplier increases in a state price whenever the project is a net liquidity supplier in that state and decreases in prices whenever the project is in financial distress in that event but still taken to completion.

Notice that pledgeable and non-pledgeable income are treated significantly differently according to this project selection criterion. This procedure, based on the shadow value of pledgeable income, also reflects a departure from a net-present-value criterion, which indicates how projects should be optimally chosen in a frictionless environment. The essential distinction are different roles played by pledgeable and non-pledgeable income of the project. While pledgeable income is evaluated at the same prices as costs at $t = 1$, since they enter the same liquidity constraints, non-pledgeable income enters in the numerator, as in a rate of return calculation.

Notice also that $\lambda^* (\gamma^*, q) = \frac{E_\omega [B_1(\omega;q;\gamma^*,x^*,\hat{a}^*)]}{E[-B_0(\omega;q;\gamma^*,x^*,\hat{a}^*)]}$, which gives rise to a leverage interpretation. Given the leverage constraint of the form $E [B_0 (\omega; q; \gamma, x, \hat{a})] I + A \geq 0$, under the optimal policy, entrepreneurs lever up their net worth by a factor of $\frac{I}{A} = \frac{1}{E[-B_0(\omega;q;\gamma,x^*,\hat{a}^*)]}$ and can reap all the social benefits from completion of the project. All elements in the denominator, which include set-up costs, additional costs at $t = 1$ and pledgeable income can be viewed in light of the effects they have on leverage of the entrepreneurial net worth and, therefore, on the determination of the scale of the project.

3.2 Project choice under incomplete markets

Under incomplete markets, in the entrepreneur's problem, constraint

$$A - \sum \pi (\omega) (\rho (\omega, \gamma) - \rho_0 (\omega, \gamma)) x (\omega) I - \sum_{k \leq K} [q_k (\omega) - E (z_k)] a_k - \phi (\gamma) I = 0 \tag{22}$$

replaces constraint (16).

The first-order conditions (18) and (20) are unchanged. The conditions relative to asset purchases become

$$\sum_\omega \mu (\omega) z_k (\omega) = \lambda (q_k - E (z_k)), \tag{23}$$

for each asset k. On the left-hand side, we see the benefits of relaxing liquidity constraints which is a product of the relevant Lagrange multipliers and the asset returns on the different states. That benefit term is equalized to the term on the right-hand side, the cost of tightening the leverage constraint, that arises from purchasing an asset which features prices that are above its expect payouts. That naturally implies that $\frac{\mu^*(\omega;\gamma^*,q)}{\pi(\omega)\lambda^*(\omega;\gamma^*,q)} + 1$ works as a stochastic discount factor, for each entrepreneur j and for every project $\gamma^* (j)$ that is selected in equilibrium. Despite all agents having linear preferences, this stochastic discount factor can be above unity, given the presence of a stochastic liquidity premium $\frac{\mu^*(\omega;\gamma^*,q)}{\pi(\omega)\lambda^*(\omega;\gamma^*,q)}$ which is reflected on asset prices.

Additionally, it follows from (20) and (23) that

$$\lambda = \frac{\sum \eta_{x(\omega)I}}{\phi(\gamma)} = \frac{\sum_\omega \mu(\omega) z_k(\omega)}{q_k - E(z_k)}$$

indicating a trade-off between the two possible uses of pledgeable income. Entrepreneurs might use pledgeable income to expand scale, which leads to a shadow benefit of $\frac{\sum \eta_{x(\omega)I}}{\phi(\gamma)}$, or alternatively, to purchase more assets for liquidity insurance purposes, for a shadow benefit of $\frac{\sum_\omega \mu(\omega) z_k(\omega)}{q_k - E(z_k)}$ which is obtained from relaxing the liquidity constraints.

Again, given linearity of the entrepreneur's problem, the criterion for project selection is one of choosing the investment prospect that leads to the highest shadow value for entrepreneurial wealth or, equivalently, on pledgeable income generated by the project.

A stronger characterization can be obtained when the economy features a single risk-less asset. In that case, the optimality condition for the asset purchase can be reduced to

$$\sum_\omega \mu(\omega) = \lambda(q-1),$$

indicating that the purchase of the only asset available helps relax all the liquidity constraints. Given asset prices and a project chosen, we can partition the set of states in three disjoint sets: $\Omega_+(q,\gamma)$, the set of states in which entrepreneurs strictly prefer to fully continue and exhibit a multiplier $\eta_{x(\omega)I} > 0$ indicating a gain from a scale increase; $\Omega_p(q,\gamma)$, the set of states with partial continuation and $\eta_{x(\omega)I} = 0$; and last, $\Omega_0(q,\gamma)$, representing states in which the entrepreneur would prefer to fully terminate the project.

For all states in which partial continuation occurs and the liquidity constraint binds, it is easy to find and interpret the multiplier on that constraint. Simple algebraic manipulation allows us to write

$$\pi(\omega)\left\{\frac{\rho_{1-0}(\omega,\gamma)}{\rho(\omega,\gamma) - \rho_0(\omega,\gamma)} - \lambda\right\} = \mu(\omega) > 0.$$

In those states, the binding liquidity constraint imposes that $x(\omega) I = \frac{a}{\rho(\omega,\gamma)-\rho_0(\omega,\gamma)}$. Notice a leverage effect in place, as a units of asset payouts used to ensure completion at state ω can generate $\frac{a}{\rho(\omega,\gamma)-\rho_0(\omega,\gamma)}$ completed project units. In any of those states, payoffs from assets have an opportunity cost: if they were simply pledged to outsiders and the project were fully terminated, they would generate an expected $\pi(\omega) a$ units of fully pledgeable income. That has a shadow value of $\lambda\pi(\omega) a$ to entrepreneurs. Completion of the project to the maximum extent allowed by the liquidity constraint consumes some net worth, since in a distress state $\rho(\omega,\gamma) - \rho_0(\omega,\gamma) > 0$. On the other hand, it enables a total non-pledgeable benefit of $\frac{\rho_{1-0}(\omega,\gamma)}{\rho(\omega,\gamma)-\rho_0(\omega,\gamma)}$ to be collected by the entrepreneur, if that state is reached. Therefore, the shadow value of the liquidity constraint in states where entrepreneurs choose to partially continue up to the point in which the liquidity constraint binds is the levered non-pledgeable component of income ($\frac{\rho_{1-0}(\omega,\gamma)}{\rho(\omega,\gamma)-\rho_0(\omega,\gamma)}$), net of the opportunity cost of the pledgeable income dissipated (λ), all of which multiplied by the probability of state ω.

Notice that for all states with full continuation

$$a \geq \left[\rho\left(\omega,\gamma\right) - \rho_0\left(\omega,\gamma\right)\right] I.$$

This constraint can only bind for a single state: the one with the largest financial shortfall $\rho\left(\omega,\gamma\right) - \rho_0\left(\omega,\gamma\right)$. Let that state be called $\tilde{\omega}$. For all other states with full continuation, liquidity constraints are slack and $\mu\left(\omega\right) = 0$. As a consequence of the existence of a single non-entrepreneurial asset used for liquidity management purposes, for a fixed project γ, there is a single state $\tilde{\omega}$ that can have both a positive shadow value on scale increases and a binding liquidity constraint. We can write further that

$$\phi\left(\gamma\right)\lambda = \sum_{\Omega_+} \pi\left(\omega\right)\left\{\rho_{1-0}\left(\omega,\gamma\right) - \lambda\left(\rho\left(\omega,\gamma\right) - \rho_0\left(\omega,\gamma\right)\right)\right\} - \left(\rho\left(\tilde{\omega},\gamma\right) - \rho_0\left(\tilde{\omega},\gamma\right)\right)\mu\left(\omega\right),$$

indicating the shadow value of wealth used in a increase in scale as being the sum over all states with full continuation of the private benefit net of liquidity opportunity costs minus the shadow value of the tightening of the liquidity constraint on $\tilde{\omega}$. Also,

$$\lambda\left(q-1\right) = \sum_{\Omega_p} \pi\left(\omega\right)\left\{\frac{\rho_{1-0}\left(\omega,\gamma\right)}{\rho\left(\omega,\gamma\right) - \rho_0\left(\omega,\gamma\right)} - \lambda\right\} + \mu\left(\tilde{\omega}\right).$$

A purchase of external liquidity, in the form of the single risk-less asset, helps relax all the binding relevant liquidity constraints, both in the states with partial continuation, as in the single full continuation state with a binding liquidity constraint. Indeed, using $\hat{a} = a/I$, the normalized asset holdings, it is possible to rewrite the shadow value of entrepreneurial wealth as

$$\lambda^*\left(q,\gamma\right) = \frac{\sum_{\Omega_+} \pi\left(\omega\right)\rho_{1-0}\left(\omega,\gamma\right) + \sum_{\Omega_p} \pi\left(\omega\right)\frac{\rho_{1-0}(\omega,\gamma)}{\rho(\omega,\gamma)-\rho_0(\omega,\gamma)}\hat{a}}{\phi\left(\gamma\right) + \left(q-1\right)\hat{a} + \sum_{\Omega_p} \pi\left(\omega\right)\hat{a} + \sum_{\Omega_+} \pi\left(\omega\right)\left(\rho\left(\omega,\gamma\right) - \rho_0\left(\omega,\gamma\right)\right)}.$$

The interpretation of $\lambda^*\left(q,\gamma\right)$ is similar to the case with complete markets. The shadow value of the entrepreneurial wealth is a ratio of private benefits collected, both in states with full continuation and in states with partial continuation (in which a leverage on liquidity, $\frac{\rho_{1-0}(\omega,\gamma)}{\rho(\omega,\gamma)-\rho_0(\omega,\gamma)}\hat{a}$, term emerges), relative to all costs of the project implementation and risk-less asset purchases in terms of pledgeable income. As stated before, project selection boils down to choosing $argmax_\gamma \lambda^* (q, \gamma)$.

Once a significant departure from a standard Arrow-Debreu benchmark is acknowledged in the selection of projects, a remaining question concerns whether it leads to significant macroeconomic consequences. That question is addressed in the next section, which illustrates the macroeconomic effects of the interactions between liquidity scarcity and endogenous exposure to risks through project selection and optimal financial arrangements.

4 Macroeconomic Consequences

In this section, I specialize the general model into particular cases to analyze the aggregate consequences of the interactions between liquidity scarcity and project selection. In the first environment, entrepreneurs face ex ante and ex post credit rationing and choose projects which differ in the volatility of their output. As a consequence of partial pledgeability of output, these projects also differ in their ability to guarantee their own financing in future events. In this environment, the main source of fluctuations is in the corporate sector itself and works through variations in the internal liquidity of projects. Therefore, it is useful for understanding how aggregate liquidity scarcity interact with corporate liquidity fluctuations in determining the endogenous degree of volatility that firms face in the economy.

In the second environment, the fluctuations studied arise from the supply of external liquidity, which is stochastic. There, shocks which are external to the entrepreneurial sector, such as a housing market collapse, are transmitted towards it through their impacts on financial arrangements. I then introduce project choice, in which some projects are allowed to co-vary more strongly or weakly with the factors behind fluctuations in external liquidity, and show that project selection responses lead to additional endogenous comovement. Behind this comovement results lies a a complementarity between projects, which might need additional investments before completion, and assets payouts that back reinvestment promises. This environment illustrates how this complementarity is responsible for biasing project choice in a direction which makes corporate investment and output covary strongly with external factors that determine aggregate liquidity conditions of the economy. Due to this positive comovement, fluctuations are also intensified in this set up.

Finally, I go back to a variant of the first environment and introduce a more complex set of instruments for liquidity distribution. As a consequence, financial arrangements and project choices change, highlighting a complementarity between different forms of contingent liquidity and different projects. Strong forms of specialization might emerge, despite the initial homogeneity of entrepreneurs. Although this leads to changes in the allocations and the possible specialization of firms with the introduction of richer asset trades, the qualitative results regarding dampening or amplification of productivity fluctuations remain similar. For example, for sufficient liquidity scarcity there is still amplification at an aggregate level. Therefore, this environment illustrates that amplification of fluctuations in economies with severe liquidity scarcity is robust to the introduction of more sophisticated financial arrangements.

4.1 Environment 1: Liquidity Scarcity and Volatility, with a single riskless asset

When output is partially pledgeable to outsiders, variations in how much output can be generated, such as the ones caused by productivity shocks, change the volume of resources that can be used to finance both the project set-up and continuation in case of distress. The impact on the latter is of particular importance. As such, productivity shocks have the potential to change how much internal liquidity is available across states of the world and the depth of the financial shortfalls that need to be covered by external liquidity. When there are negative shocks to total output that can be produced, pledgeable output which is useful for ensuring external financing is reduced and creates more difficulties for funding the continuation of the project. The opposite is true for a shock that leads to an increase in total and pledgeable output.

By choosing among projects with different levels of volatility in productivity, entrepreneurs alter their liquidity needs across states of the world and, indirectly, the value which external assets have in their financial arrangements. Therefore, project selection interacts with liquidity management. If projects differ in their output volatility, the choice over this variable is of particular importance.

In this section, we study an environment in which there is a single risk-less asset that can be held as a buffer of external liquidity[14]. The equilibrium price of this asset is shown to be of central importance for the joint determination of which projects are selected, the quantity of the asset that is purchased and which continuation policies are implemented. In particular, while for low prices, entrepreneurs choose full continuation and low volatility of productivity, once prices are higher,

[14]The case in which trades on external liquidity can be made state-contingent is studied later. This example is particularly useful for its simplicity and for contrasting its results with what is achieved when richer contracts for external liquidity trades are feasible. Formally, if one wants a justification for the absence of contracts for deliveries of risk-less assets at $t = 1$ across firms, we could resort to spatial separation or commitment problems.

partial continuation emerges and high volatility might be chosen.

The intuition for this mechanism is that by controlling volatility, entrepreneurs move pledgeable resources across states of the world and this works as an imperfect substitute for asset purchases. When asset prices are low, entrepreneurs are only constrained in a low productivity state. Therefore, at the margin, it is worth to choose projects with less volatility and relax that constraint. On the other hand, when acquiring assets is too expensive, entrepreneurs purchase less of these and end up liquidity constrained in multiple states. Pledgeable resources can then be the most valuable at the margin in states with higher productivity, but binding funding needs. Choosing higher volatility, in that case, helps move resources to those states.

Uncertainty- There are four states of nature. Financial needs at $t = 1$ are given by an aggregate shock which belongs to $\{0, \rho\}$, with $\rho > 0$. Their realization is ρ with probability π_ρ. Additionally, the aggregate determinants of the productivity of projects belong to $\{g, b\}$ and occur with respective probabilities of π_g and $\pi_b \equiv 1-\pi_g$. In the g (good, higher productivity) states each unit of each project is capable of delivering its highest possible output, while in the b (bad, lower productivity) states it is capable of delivering a lower output. Productivity and reinvestment need shocks are assumed to be independent. Therefore, the state of the world is fully described by $\omega \in \Omega \equiv \{g, b\} \times \{0, \rho\}$.

Projects- There is a continuum of projects which differ in their initial set-up costs and in the magnitude of their productivity fluctuations. Formally, there is a compact set of projects indexed by $\gamma \in \Gamma \equiv \left[\underline{\gamma}, \overline{\gamma}\right] \in \mathbb{R}_{++}$. The project specific parameter γ measures the dispersion of output of a given project across the aggregate productivity states. Let

$$\rho_1(\omega, \gamma) = \begin{cases} \rho_1 + \frac{\gamma}{\pi_g} & , \text{ for } \omega \in \{g\} \times \{0, \rho\} \\ \rho_1 - \frac{\gamma}{\pi_b} & , \text{ for } \omega \in \{b\} \times \{0, \rho\} \end{cases}.$$

Notice that, conditional on full completion, all units of all projects have the same expected output of ρ_1, but differ in their variance, which is increasing in γ. Therefore, the higher γ, the more volatile the output of a project is. In particular, the extreme project $\overline{\gamma}$ is the one that is the most adversely affected by the realization of an event with low productivity, while also the one that is the most positively affected by a high productivity event.

Assume that there exists a baseline, lowest cost project, γ_0 with $\phi(\gamma_0) = 1$. The output process of this project provides the benchmark level of fluctuations, around which amplification and dampening are defined. A project involving $\gamma > \gamma_0$ features more output fluctuation than the baseline project and is said to lead to *amplification*. Analogously, a project with $\gamma < \gamma_0$ fluctuates less than the benchmark project and is said to lead to *dampening*. Let $\phi(\gamma)$ be C^2 and strictly convex.

Let us assume that limited pledgeability is caused by an agency problem, with a severity which does not vary across states of nature. That means that a constant, state independent, private benefit $\rho_{1-0} > 0$ per continued unit of the project has to be offered to entrepreneurs in order to ensure diligent behavior. As a consequence, pledgeable output will move one-to-one with total output and $\rho_0(\omega,\gamma) = \rho_1(\omega,\gamma) - \rho_{1-0}$, for each state ω and project γ. This assumption is chosen for two reasons. The first is that private benefits are not made pro-cyclical or counter-cyclical in themselves, so that all incentives guiding project choice are entirely financial and related to liquidity costs. The second is that it generates greater tractability by allowing the entrepreneurs' problem to take an average-cost formulation, in which dependence on total output produced, ρ_1, disappears.

Assets- There is a single risk-less asset that pays out a certain unit of consumption in all states $\omega \in \Omega$ at $t = 2$.

Additional Assumptions- The following set of assumptions about parameters in the production functions is made:

A1 $0 < \rho_0 < \rho < \rho_1$,

A2 $\rho + \frac{\overline{\gamma}}{\pi_b} < 1 + \pi_\rho \rho$,

A3 $\rho_1 > \frac{1}{1-\pi_\rho}$,

A4 $\rho_0 + \frac{\overline{\gamma}}{\pi_g} < \rho$.

Assumption $A1$ ensures that financial distress occurs when the refinancing shock is ρ. Assumption $A2$, that it is optimal to fully continue even the project that is most adversely affected by a negative productivity shock if there is no premium on the risk-less asset. Jointly, $A1$ and $A2$ imply finite leverage. Assumption $A3$ implies that entrepreneurs are willing to undertake the project even when unable to continue in the distress states. Assumption $A4$ ensures that not even the project that is the most positively affect by a high productivity realization becomes self-financing in the (g,ρ) state.

Analysis

The purchase of the existing asset will enable entrepreneurs to simultaneously relax the two relevant liquidity constraints, which are

$$a + \left(\rho_0 - \frac{\gamma}{\pi_b}\right) x(b,\rho) I \geq \rho x(b,\rho) I \tag{24}$$

and

$$a+\left(\rho_0+\frac{\gamma}{\pi_g}\right)x(g,\rho)I \geq \rho x\left(g,\rho\right)I, \tag{25}$$

since $\{(g,\rho),(b,\rho)\}$ are the two states in which financial distress occurs[15]. The first term on the left-hand side of both constraints is the level of purchases of risk-less assets (a) or, alternatively, how much externality was acquired at $t=0$. The second term is the amount of internal liquidity available after the realization of the productivity shock is learned, for a continuation at scale $x\left(\omega\right)I$. To back financing, given the absence of commitment from lenders, the sum of those two terms needs to cover the required disbursement of ρ for the $x\left(\omega\right)I$ units of the project that should be taken to completion.

Output shocks create and destroy internal liquidity across states of nature, as seen in second term on the left-hand side of the liquidity constraints (24) and (25). Therefore, the negative productivity shock state (b) always involves a more stringent liquidity constraint than the positive state (g). For any level of asset purchases, continuation in the b state needs to be weakly lower than in the positive state, g. Therefore, productivity shocks induce supply shocks on aggregate liquidity and represent a force towards higher liquidity premia on the asset. By choosing less productive projects that involve dampening (lower γ), entrepreneurs can shift internal liquidity to the b state, where it is scarcer. This provides a rationale for why output shocks can lead to incentives for the dampening of fluctuations.

However, there is another force in place. The return on liquidity hoarding towards a given state is also influenced by the productivity shocks. A unit of liquidity in a state ω where a liquidity constraint binds enables the completion of project units that have a social surplus of $\rho_1\left(\omega,\gamma\right)-\rho$. From the liquidity constraints, the completion of each of these requires $\rho-\rho_0(\omega,\gamma)$ units of external liquidity in that state. Notice that a multiplier or leverage effect is in place: a unit of liquidity brought into state ω can enable the production of $\frac{1}{\rho-\rho_0(\omega,\gamma)}$ units of output, which create a non-pledgeable benefit of $\frac{\rho_{1-0}}{\rho-\rho_0(\omega,\gamma)}$.

When the aggregate productivity shock is more favorable, both total, $\rho_1\left(\omega,\gamma\right)$, and pledgeable outputs, $\rho_0\left(\omega,\gamma\right)$, are higher for all projects. Completion becomes more valuable in the g state, given that a unit of the project completed delivers more social surplus. The multiplier effect on external liquidity is larger, meaning that each completed unit offers the entrepreneur more pledgeable income on which she will be able to lever up at $t=1$. These effects are in place as long as constraint (25) binds, which occurs when external liquidity is still necessary at the margin in the higher output state.

[15]Assumption A4 and $x\left(\omega\right)I>0$ imply that $a\geq 0$ from 25. As a consequence, the liquidity constraints relative to all states in which the reinvestment shock is 0 are always slack.

These two combined generate a higher potential return on liquidity in the high productivity state and provide a rationale for the choice of projects that offer more liquidity in the g state. Those are the projects with higher γ, potentially involving amplification.

The proposition below helps understand the results that follow, by characterizing optimal decisions regarding asset holding and continuation policies, when entrepreneurs are restricted to an arbitrary fixed project.

Proposition 1. *For any fixed project $\gamma \in \Gamma$, there exist two cutoffs, $\underline{q}(\gamma)$ and $\bar{q}(\gamma)$, such that*

1. For $1 \leq q \leq \underline{q}(\gamma)$, full continuation in all states is optimal. Asset holdings are exactly sufficient to ensure full continuation in the $\omega{=}(b,\rho)$ state, with $\frac{a}{I} = \left(\rho - \rho_0 + \frac{\gamma}{\pi_b}\right)$. Constraint (24) holds with equality and constraint (25) is slack.

2. For $\underline{q}(\gamma) \leq q \leq \bar{q}(\gamma)$, an optimal policy features full continuation in the (g,ρ) state and partial continuation in the (b,ρ) state. Asset holdings are pinned down by $\frac{a}{I} = \left(\rho - \rho_0 - \frac{\gamma}{\pi_g}\right)$. Constraints (24) and (25) hold with equality.

3. For $q \geq \bar{q}(\gamma)$, it is optimal to set $a = 0$ and fully terminate the project in the distress states.

Whenever the costs of liquidity hoarding are relatively low, it is optimal to purchase enough assets to guarantee full continuation even in the worst possible state of nature. In that situation, entrepreneurs are only effectively liquidity constrained in the (b,ρ)-state[16].

In this full continuation regime, there are incentives towards dampening, as the choice of a project with lower volatility reduces the need for asset hoarding. This can be seen in constraint (24), which is relaxed with the choice of lower γ. The forces pushing towards amplification are absent, given that the liquidity constraint for the (g,ρ) state does not bind.

However, once the liquidity premium is sufficiently high, it is optimal to switch to a policy of limited liquidity hoarding. In that situation, resorting to partial liquidation in case reinvestment needs coincide with low productivity helps reduce asset purchases, economizing on the use of expensive external liquidity. Since both liquidity constraints bind, there are forces in place both in the direction of dampening (relaxing constraint (24) by choosing a project with lower volatility) and of amplification (relaxing constraint (25) by choosing a project with higher volatility) if project choice is permitted. Which one dominates depends on the costs of asset hoarding.

[16] As $x(b,\rho) = x(g,\rho) = 1$, it follows that $\left(\rho_0 + \frac{\gamma}{\pi_g}\right) I + a > \left(\rho_0 - \frac{\gamma}{\pi_b}\right) I + a = \rho I$, implying that constraint (25) is slack.

Finally, for every project, there are sufficiently high prices leading to optimality of full liquidation in case of distress. Enabling insurance through the accumulation of stores of value becomes too expensive from the entrepreneurs' perspective. Later, when equilibrium conditions are taken into account, prices cannot rise beyond the point in which entrepreneurs stop demanding liquidity.

This general behavior remains similar once project choice is incorporated: there will be three relevant regimes for the solution of the individual problem as a function of asset prices. A first one with full continuation in all states, a second with full continuation only in case of high productivity shock and a third with full termination in case of distress. Within each one of these, which project is optimal can be determined by a first-order condition. Determining actual cut-offs for the switch between these regimes requires comparisons across solutions and either closed-form examples or a computational approach. Nonetheless, the essential qualitative properties can be proved without resorting to those. They are summarized by the proposition below.

Proposition 2. *The solution to the individual problem features three regions, delimited by the prices* $\underline{q}$ *and* $\overline{q}$*. The following properties hold:*

1. For $1 < q < \underline{q}$, there exists a unique optimal project, which features dampening and full continuation. In this region, optimal project choice is a decreasing function of q.

2. For $\underline{q} < q < \overline{q}$, there exists a unique optimal project with full continuation in the (g, ρ) state and partial continuation in the (b, ρ) state. Project choice is an increasing function of q and, for sufficiently high q within this range, there is amplification.

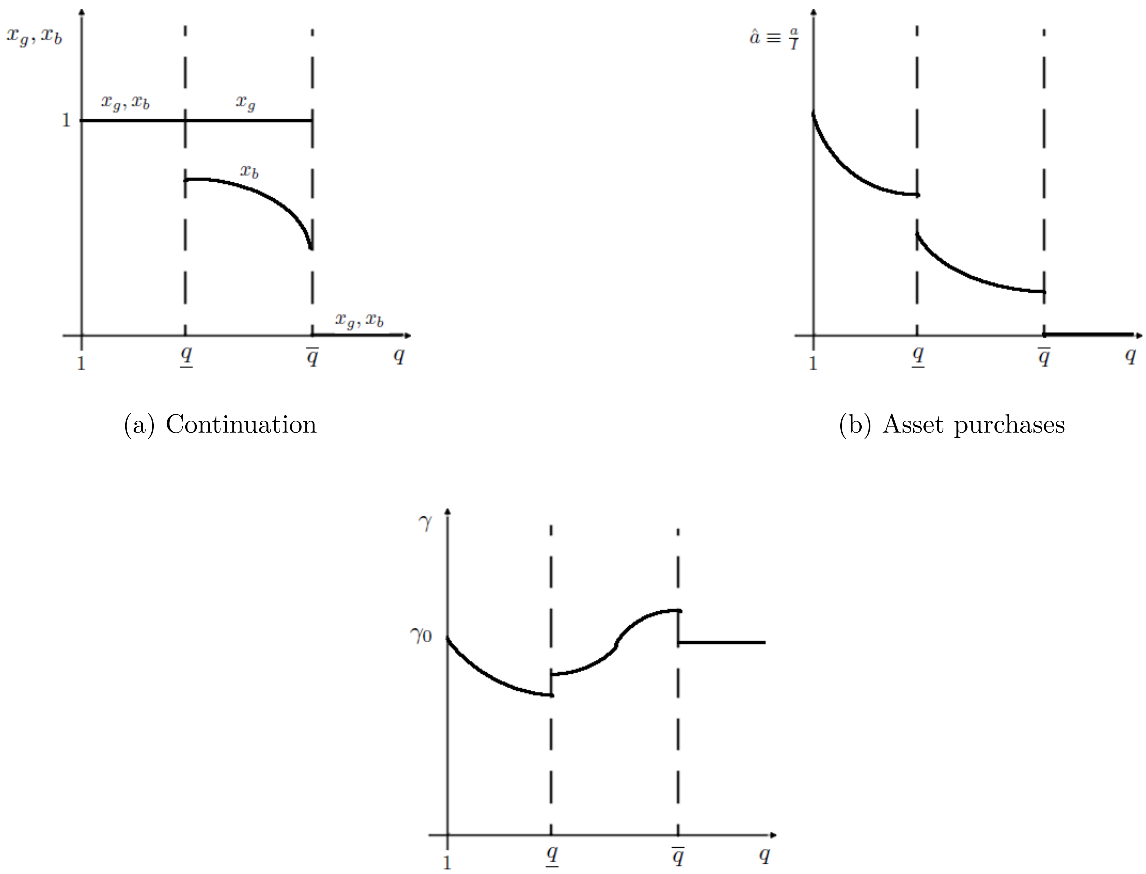
3. For $q > \overline{q}$, there exists a unique optimal project, involving full termination. It is the lowest cost project, γ_0.

4. For $q = 1$, γ_0 and full continuation are optimal.

5. At the thresholds $\underline{q}$ and $\overline{q}$, two projects with respective different continuation policies are optimal.

Figure 1 below illustrates the main results from Proposition 2, regarding continuation shares, asset purchases and project choice as a function of the price of the risk-less asset's.

When prices are interpreted as a partial equilibrium measure of the intensity of the external liquidity scarcity, a characterization of the behavior of the equilibria emerges. If liquidity is plentiful enough, no premium is present $(q = 1)$ and, as a consequence, continuation decisions and project



(a) Continuation

(b) Asset purchases

(c) Project choice

Figure 1: Optimal entrepreneurial decisions, as functions of asset price (liq. premium)

choice are efficient: full continuation always occurs and the most productive project, γ_0, is chosen by all entrepreneurs.

As liquidity becomes moderately scarce, the economy moves to the behavior described in Proposition 2, part 1. The relevant binding liquidity constraint is in the low productivity state and, as a consequence, entrepreneurial choice over projects takes the relaxation of that constraint into account. That leads to projects that, although less productive due to the higher cost, fluctuate less and dampen the underlying shock. As such, they require less hoarding of liquidity, even for full continuation. In these economies, no financial crises involving termination of projects are ever observed.

If liquidity is more severely scarce, entrepreneurs opt to sacrifice continuation in the worst state of nature as in part 2 of Proposition 2. Asset purchases are just enough to ensure full continuation in the best financial distress state. By choosing projects that fluctuate more, entrepreneurs can save on the costly asset hoarding necessary to ensure this level of continuation, but end up sacrificing efficient continuation in the worst financial distress state. When liquidity is scarce enough, as signaled by a high liquidity price, the gains from moving internal liquidity to the high productivity state to economize on external liquidity become large. The losses in terms of continuation in the bad productivity state are more than offset and projects leading to amplification end up being preferred.

Eventually, with sufficiently high prices, it becomes optimal for entrepreneurs not to buy any assets and to choose the most efficient project γ_0 again. Conditional on the policy of full termination in case of distress, there is no role for the relaxation of liquidity constraints through the choice of any other projects.

Moving back towards general equilibrium, we can construct an aggregate demand for assets from the behavior of the individual demand. Aggregation helps smooth out the discontinuities around the two thresholds, $\underline{q}$ and $\overline{q}$, by exploiting the indifference of entrepreneurs between different projects and associated continuation policies. Together, the aggregate demand for assets and the inelastic supply of external liquidity pin-down the asset price and liquidity premium, q. A sketch of the equilibrium determination is provided in Figure 2.

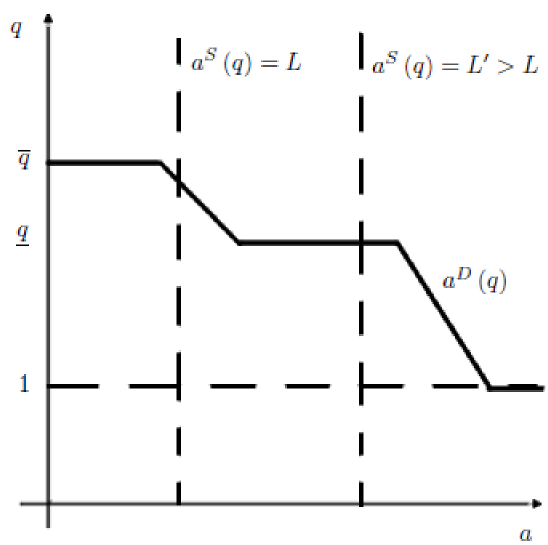


Figure 2: Sketch of equilibrium determination for economies with different levels for the supply of stores of value, L and L'. In the flat parts of the aggregate asset demand, two projects are optimal and the supply of risk-less assets determines the fraction of entrepreneurs that chooses each one the possible optimal policies, with their different underlying project choices and continuation decisions.

Changes in the liquidity supply, L, or in the pledgeable component of income, ρ_0, have similar qualitative effects. Both can be thought of as measures of financial development and a reduction in either moves equilibrium asset prices towards higher levels. By doing so, the economy moves within and across the regions described in the partial equilibrium analysis.

Economies with severe financial underdevelopment experience higher fluctuations of output and severe financial crises, with the discontinuation of many projects. Economies with moderate financial development experience dampened fluctuations relative to first-best, but at cost in terms of lower productivity in entrepreneurial projects.

4.2 Environment 2: Fluctuations in non-corporate assets; Transmission and Synchronization

When financial conditions matter for economic activity, an important source of fluctuations lies outside the corporate sector itself. Liquidity conditions in the economy fluctuate as the value of assets that back promises of reinvestment in the economy change. The sudden drop in the prices of mortgage-backed securities in mid-2007 and the ensuing contraction of credit and investment highlight the practical importance of this particular channel.

In the framework proposed here, these fluctuations occur through the realization of different payouts for the assets available. A variation in external liquidity can be captured by the introduction of a mass L of trees, which produce a stochastic payout, or fruit. Suppose the output from these trees can be high (z_u) or low(z_d), with $z_u > z_d$. I allow for the contingent trading of this output through assets, to be described shortly. As these assets back financial arrangements and enable some insurance for the continuation of projects, competition for them might drive their prices above their expected output, creating liquidity premia.

In the first section to follow, I study the transmission of the fluctuation in the value of these assets into output from the entrepreneurial projects. In the presence of aggregate distress, the value of tree output determines how much reinvestment the economy as a whole can afford. Asset prices, investment scale and continuation policies are jointly determined in equilibrium. In this environment, premia on assets are always decreasing in the realization of tree output, while continuation shares of projects under distress and, consequently, entrepreneurial output are increasing. In this sense, shocks to the trees' capacity to generate fruit are transmitted towards the entrepreneurial sector and influence its capacity to take investment under distress to completion. In itself, this creates comovements across sectors of the economy, in which one sector is the provider of liquidity (trees) and another sector is the one that needs that liquidity in its financial arrangements (entrepreneurial projects).

I then show that if entrepreneurs face a choice of exposure to the same risks that drive tree output, they endogenously choose to increase their exposure to these risks. This is done in section 4.2.2, the synchronization result. The intuition underlying the mechanism is that when reinvestment shocks are proportional to the output that can be generated, projects that comove positively with the trees will have more plentiful, and therefore, cheaper external liquidity as complements. As a consequence, this economy creates an even stronger comovement across sectors than the transmission mechanism alone.

4.2.1 Transmission

We specialize the structure of the general model to the following particular case.

Uncertainty- Assume that financial distress and the realization of the trees' output are independent. The economy features four relevant states of nature: $\Omega = \{u, d\} \times \{\rho, 0\}$. Projects might suffer an aggregate refinancing shock $\rho > 0$ with a probability $\pi_\rho \in [0, 1]$. Otherwise, no additional costs have to be paid to ensure continuation of investments towards completion. Additionally, total external liquidity, derived from the fruits of the trees can take two realizations in $\{z_u, z_d\}$ with $z_u > z_d$. Let π_u be the probability of the high realization and $\pi_d \equiv 1 - \pi_u$. Independence of external liquidity shocks and refinancing needs is imposed and the probability of each realization of the state of the world $\pi(\omega)$ is naturally defined as the product of the relevant marginal probabilities.

Project- Suppose there is a single project available in the economy, by setting $\Gamma = \{1\}$, and normalize its cost to unit,$\phi(1) = 1$. Both total output per unit completed and the pledgeable component do not vary across states of nature. That means $\rho_1(\omega, 1) = \rho_1$ and $\rho_0(\omega, 1) = \rho_0$. When the aggregate reinvestment need shock happens, firms find themselves under financial distress. In that case, they are required to pay an additional $\rho > \rho_0$ to bring each unit of the project to completion. Otherwise, no additional cost has to be paid and firms are not under financial distress.

Assets- Let two assets be traded[17] which are contingent on the output of the tree: u (d) is a claim on all the output of a tree contingent on it being revealed to be z_u (z_d) and is traded at price q_u (q_d) at $t = 0$. Therefore, asset u pays out z_u in states $\omega \in \{(u, \rho), (u, 0)\}$ and 0 otherwise. Similarly, asset d pays out z_d in $\omega \in \{(d, \rho), (d, 0)\}$ and 0 in other states. Let a_u and a_d be the quantities of these claims purchased as part of a financial arrangement between an entrepreneur and a lender. Both of these assets are available in fixed supply L.

Parameter Assumptions- We make the following assumptions about the parameters of the production function:

[17] Trade in these two assets is sufficient to fully span both all contingencies of the trees output and to allow independent continuation decisions in all distress states.

A1 $0 < \rho_0 < \rho < \rho_1$

A2 $\rho < \frac{1}{1-\pi_\rho}$

A3 $\rho_1 > \frac{1}{1-\pi_\rho}$

Assumption $A1$ ensures that financial distress occurs when the refinancing shock is ρ. Assumption $A2$ guarantees that it is optimal to fully continue the project if liquidity premia are sufficiently close to zero. Jointly, A1 and A2 imply finite leverage. Assumption $A3$ implies that entrepreneurs are willing to undertake the project even if unable to continue in the distress states.

Analysis

We proceed in the following way. First, we analyze the solution to the entrepreneurs problem and describe it graphically. Then, equilibrium conditions are imposed over that graphic description. Different regimes are possible for the behavior of the equilibrium and the total availability of liquidity in the economy determines which one holds. Finally, the key proposition of the section, concerning the transmission mechanism, is analyzed.

In both states that do not involve additional investment needs, projects are self-financing at $t = 1$ and offer excess liquidity. It is easily shown that liquidity constraints cannot bind in those states and that full continuation under those contingencies is optimal. Therefore, without loss of generality one can restrict attention to policies that set $x(\omega) = 1$ for $\omega \in \{(u, 0), (d, 0)\}$. The two possibly binding liquidity constraints that entrepreneurs face are given by

$$z_i a_i + \rho_0 x(i, \rho) I \geq \rho x(i, \rho) I, \text{ for } i = u, d. \tag{26}$$

The minimum purchase of these contingent assets that needs to be made to enable continuation is obtained by solving for an equality in the conditions (26). By proceeding this way, one obtains the minimal amount of asset purchases necessary to enable continuation of a share $x(i, \rho)$ of investment, denoted as $\hat{a}_i(x) = \frac{(\rho-\rho_0)x(i,\rho)}{z_i}$, which can be plugged into the entrepreneur's problem to write it as

$$\max_{I, x(\omega)} E[B_1(\omega; q; \gamma, x, \hat{a}(x))] I \tag{27}$$

s.t.

$$A + E[B_0(\omega; q; \gamma, x, \hat{a}(x))] I \geq 0, \tag{28}$$

where again B_1 and B_0 represent, respectively, the total and the pledgeable surpluses from investment as defined in section 2[18].

A more complete description of the equilibrium behavior is offered in the appendix. Figure 3 describes the key elements of the characterization. In a graphical representation of the entrepreneurs' problem, there are four main regions, with the liquidity premia on both assets being the key elements for determining optimal continuation policies.

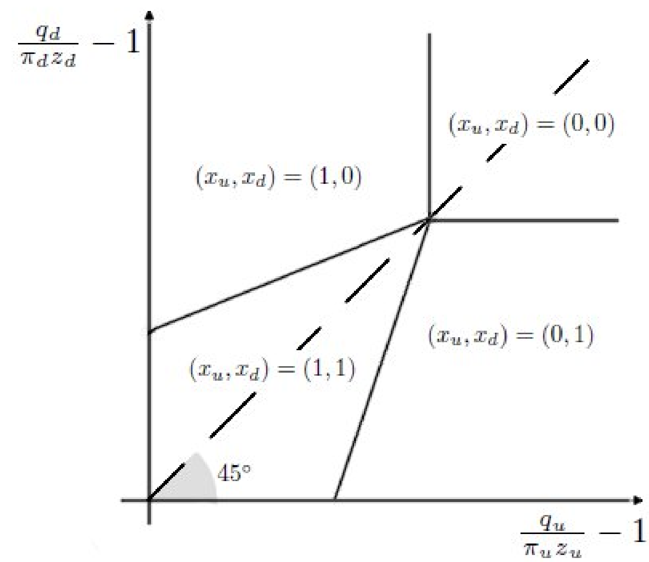


Figure 3: Optimal Continuation Policy Regions, as function of liquidity premia.

Around the lower-left corner of Figure 3 is found the region with low liquidity premia on both assets, in which full continuation in all states is optimal. When premia are sufficiently close to zero, liquidity hoarding is relatively inexpensive and entrepreneurs choose to fully insure against distress shocks. Thus, a policy of full continuation described by $(x_u, x_d) = (1, 1)$ is optimal.

The region above it, marked with $(x_u, x_d) = (1, 0)$, displays the area in the liquidity premium space in which full termination in the (d, ρ) state and full continuation in the (u, ρ) state is optimal. There, the premium in the d asset is sufficiently high while the premium on asset u is relatively low. The frontier between these two areas, described by a segment, is the locus of points where

[18]In the entrepreneurs' problem: $E\left[B_1\left(\omega, \gamma, q, x, \hat{a}\left(x\right)\right)\right] = \left(1 - \pi_r\right)\rho_1 + \pi_r\left(\rho_1 - \rho\right)\left[\pi_u x\left(u, \rho\right) + \pi_d x\left(u, \rho\right)\right] + \sum_{i=u,d}\left(q_i - \pi_i z_i\right)\frac{(\rho - \rho_0)x(i,\rho)}{z_i} - 1$. $E\left[B_0\left(\omega, \gamma, q, \hat{a}\left(x\right)\right)\right]$ can be analogously obtained by substituting ρ_0 for $\rho_1$in that expression.

entrepreneurs would be willing to partially liquidate in case of a combination of low output and high liquidity needs. This will be an important object in the characterization of the equilibrium.

Analogously, the region in the lower-right corner is the one in which the premium on asset u would be sufficiently high as to induce termination, while the premium on asset d is not. As liquidity is scarcer in the (d, ρ) state than on (u, ρ), it will not be a relevant region once equilibrium conditions are taken into account. Finally, the rectangular region in the upper-right corner describes the area in which both liquidity premia are so high that it is optimal for entrepreneurs to fully liquidate in both financial distress states, choosing continuation shares $(x_u, x_d) = (0, 0)$.

Aggregate liquidity scarcity in a distress state has two possible equilibrium consequences. It might constrain the scale of investment, by creating an equilibrium liquidity premium which ensures that liquidity insurance becomes sufficiently costly and consumes a fraction of the initial entrepreneurial net worth, as seen in equation (28). This way it would reduce the average entrepreneurial leverage and the scale of the average project. Alternatively, it might drive a liquidity premium to such a high level that entrepreneurs become indifferent regarding liquidity insurance for a given state of the world and some termination happens in equilibrium. In the brief description to follow, liquidity scarcity in one state of the world might be responsible for limiting the aggregate scale of investment. If that is not the worst possible state in terms of aggregate liquidity supply, all states with worse conditions will involve some termination of investment.

Equilibrium with a positive asset payout in both states imposes that full termination in a given state can never be uniquely optimal. Therefore, the border segments between the four regions represented in the previous figure are the loci where equilibrium prices have to lie. As formalized in the appendix, possible equilibrium price and allocation behavior can be described by the numbered (i)-(v) points and segments displayed in Figure 4.

At point (i), liquidity even in the state with the most severe degree of scarcity, (d, ρ), is sufficient to allow for full insurance of a scale of investment which is the highest possible. Liquidity premia on both assets are zero and the scale of investment is only constrained by entrepreneurial net worth, not by costs of asset hoarding. In all alternative cases to follow, liquidity premia are a feature of the equilibrium.

At point (ii), a liquidity premium in asset d alone is sufficient to increase the cost of liquidity insurance, crowding out investment in scale while still guaranteeing full continuation in all states. However, if the required premium is too high, as at point (iii) and above, entrepreneurs become indifferent regarding liquidation in that state. For all points above (iii), such as loci (iv) and (v),scarcity in the (d, ρ) state no longer limits aggregate investment scale and liquidity crises with termination

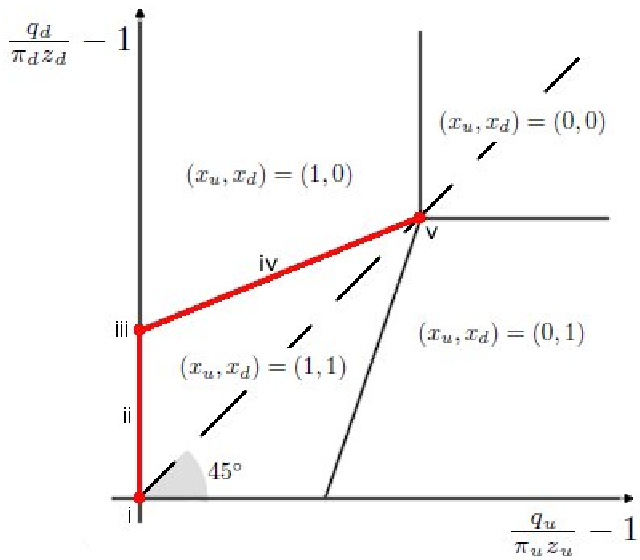


Figure 4: Loci of possible equilibria: premia and policies.

of projects happen on that state.

For sufficiently low Lz_d, changes in the supply of the scarcer liquidity induced by either a change in the number of trees, L, or in the payout in low output states, z_d, change aggregate availability of insurance for that state and, as a consequence, the share of projects that face termination. This equilibrium reduction frees up entrepreneurial net worth, which is spent on an increase in the scale of the average project, as opposed to being spent on costly liquidity insurance. For very low tree output, it is then possible the aggregate liquidity constraint also binds in the higher liquidity availability distress state and that a liquidity premium emerges on asset u as well as on d, as in regions (iv) and (v).

Region (iv) involves full continuation in the (u, ρ) state under all policies. There, in equilibrium, aggregate investment scale is such that there is just enough liquidity in that state as to enable continuation of all projects. Last, at point (v), liquidity premia on both assets are equalized and set to the maximum that entrepreneurs are willing to pay to enable insurance of continuation possibilities. Entrepreneurs are just indifferent between hoarding any asset or not being capable of withstanding financial shocks at $t = 1$.

Notice that liquidity premia are always higher in the d-contingent asset, as illustrated by the fact that all possible equilibria have prices that lie on or above the 45-degree line. Higher liquidity availability in states with higher tree output is always translated into lower premia on the u-contingent asset than on the d-contingent one.

Additionally, in regions (iii)-(v), some aggregate liquidation of projects occurs when financial distress happens. In these areas, aggregate continuation of investment under financial distress at $t = 1$ is constrained by the limited availability of external liquidity in the economy. Therefore, higher returns from the trees are always translated into strictly higher continuation shares in the state involving $z_u > z_d$ once the economy is in any of these regimes, in which liquidity is scarce enough. These elements are the essence of Proposition 3.

Proposition 3. *In the competitive equilibrium of this economy:*

1. *Liquidity premia are higher for the asset contingent on low tree output:* $\frac{q_d}{\pi_d z_d} \geq \frac{q_u}{\pi_u z_u}$.
2. *Continuation shares are higher in the high output state than in the low output state:* $x(u, \rho) \geq x(d, \rho)$*, with a strict inequality if some termination ever occurs.*

Output from the trees serves both as a consumption good itself and as backer of the liquidity reserves held by the entrepreneurs. Resources from the trees are not consumed directly in the production process (as fruits are only available at $t = 2$), but the holding of claims on trees helps guarantee reinvestment for projects under distress. At $t = 1$, news about more availability of fruits in the future enable more continuation of projects if distress happens.

Whenever the aggregate economy is ex post liquidity-constrained and partial continuation of projects occurs in equilibrium, negative shocks to the trees' output imply strictly lower continuation shares and lower output from the investment projects in the economy. In this sense, shocks to the sector of the economy which is a net supplier of liquidity (trees) are naturally transmitted towards the sector of the economy which holds it. In this environment, unlike in the one to follow in the next subsection where which project choice is endogenous, that transmission and comovement of output only works through the aggregate distress states.

Notice that the economy has always enough resources at $t = 1$ to enable full continuation of all projects, given the large consumer endowment assumption. The output from trees is not necessary as a physical input in the entrepreneurial projects (it is not even available yet when distress happens), but the claims it backs are an essential input into their financing plans.

The economy analyzed was assumed to be closed, but the use of international liquidity can be easily incorporated. As pointed out by Caballero and Krishnamurthy (2001, 2003) and Holmström and Tirole (2011), even with relatively plentiful international liquidity, specific economies might be constrained in their capacity to access foreign financial markets, especially due to limited capacity to generate internationally pledgeable income or tradable goods.

As such, the supply of aggregate liquidity can also incorporate some foreign component and shocks to the trees can also include fluctuations in international liquidity itself or in an economy's ability to access those markets. Therefore, the mechanism for the transmission of liquidity shocks into output fluctuations highlighted can also work across different countries, when there are international markets for liquid assets. This is specially clear when one central country provides a less financially developed economy with liquid stores of value.

In addition to these transmission effects, there might be incentives in place for entrepreneurs to select more pro-cyclical investment projects, synchronizing entrepreneurial output with external liquidity cycles, as the next example highlights.

4.2.2 Synchronization

The structure of the economy is mostly the same as in the previous section, with one important distinction. Now entrepreneurs can choose how intensely the output of the projects covaries with the output of the trees. In the case under study, pledgeable output and potential reinvestment shocks are proportional to total output that a project can generate in a given state. That production structure is interpreted as a sequential investment problem in which in the first stage ($t = 0$) entrepreneurs choose projects that give them investment opportunities at $t = 1$ which comove in different ways with the output from trees.

Uncertainty, Assets and Parameter assumptions- Same as in Section 4.2.1.

Projects- There is a continuum of projects $\Gamma = [-\pi_u, \pi_d]$ and each entrepreneur can choose any $\gamma \in \Gamma$. Total output of project γ in state ω, conditional on full completion, is given by

$$\rho_1(\omega, \gamma) = \rho_1 n(\omega, \gamma). \tag{29}$$

Analogously, pledgeable output is $\rho_0(\omega, \gamma) = \rho_0 n(\omega, \gamma)$ and the costs of completion under the financing shock are $\rho(\omega, \gamma) = \rho n(\omega, \gamma)$, for $\omega \in \{u, d\} \times \{\rho\}$.[19] The mapping from $\gamma \in [-\pi_u, \pi_d]$ into $n(\omega, \gamma)$ is given by

$$n(\omega, \gamma) = \begin{cases} 1 + \frac{\gamma}{\pi_u}, & \text{for } u \text{ states} \\ 1 - \frac{\gamma}{\pi_d}, & \text{for } d \text{ states} \end{cases}.$$

Notice that $E_\omega[n(\omega, \gamma)] = 1$. We assume there exists a baseline benchmark, a "neutral investment" $\gamma = 0$, which has the lowest possible cost per unit, $\phi(0) = 1$, and that $\phi : \Gamma \to \mathbb{R}_+$ is twice

[19]Notice that in all states in which the financing shock does not happen, i.e. in $\omega \in \{u, d\} \times \{0\}$, $\rho(\omega, \gamma) = 0$.

continuously differentiable and weakly convex, so that

$$\phi'(\gamma)\begin{cases}\geq 0, \text{ if } \gamma \geq 0,\\ \leq 0, \text{ if } \gamma \leq 0,\end{cases}$$

and $\phi''(\gamma) \geq 0$.

This production structure can be motivated in the following way. An initial investment at $t = 0$ generates a fixed expected number of profitable investment opportunities ("ideas") at $t = 1$. The cost of taking advantage each one of these opportunities is determined by an aggregate shock: it can be ρ with probability π_ρ or zero with probability $1 - \pi_\rho$.

The output from each implemented project is not fully pledgeable, as incentives have to be provided for diligent entrepreneurial behavior. Each investment opportunity which is implemented generates a total output of ρ_1, of which only a component ρ_0 is pledgeable to outsiders.

Whenever the cost of additional investment is zero, these investment opportunities are self-financing at $t = 1$. Otherwise, entrepreneurs need to have assets in place to back investment, since $\rho > \rho_0$, and pledgeable income from projects themselves is not sufficient to finance all costs of investing at $t = 1$.

Project $\gamma = 0$, the neutral investment, provides entrepreneurs with one investment opportunity in each state of nature. Importantly, for this neutral project, the emergence of these opportunities is independent from the realization of the trees' output. At a higher cost, entrepreneurs might choose projects that give them investment opportunities which comove more positively or negatively with the trees' output. That is, entrepreneurs might choose within a menu of projects that differ by offering these opportunities in a more strongly pro-cyclical or more strongly anti-cyclical way, where the cycle is defined relative to the aggregate shock causes high or low tree output. Thus, a higher γ biases the distribution of ideas towards being more strongly related to the cycle from the trees, but keeps the expected number of ideas constant. For instance, setting $\gamma = \pi_d$ makes all ideas appear in the states where tree output is learned to be at its highest possible value, Lz_u.

Besides the constant number of expected investment opportunities, all projects also share the same structure per-unit in terms of refinancing costs at $t = 1$, total and pledgeable returns. They differ however on the set-up cost and, as a consequence, also on their returns per-unit of the consumption good invested. In an environment without liquidity premia, $\gamma = 0$ would dominate all other investment possibilities, due to its lower $\phi(\gamma)$.

Analysis

The central synchronization result is summarized by the proposition below:

Proposition 4. *Suppose the set-up cost function* $\phi(\gamma)$ *is strictly convex. Then, in any equilibrium with a premium differential, i.e. where* $\frac{q_d}{\pi_d z_d} > \frac{q_u}{\pi_u z_u}$, *there is a unique optimal project* $\gamma^* > 0$, *which features biasing of investment opportunities towards the high tree output states.*

In this environment, there exists a complementarity between investment that leads to profitable ideas at $t = 1$ and liquidity that allows the implementation of these ideas when they turn out not to be self-financing. Once a liquidity premium differential emerges, reflecting the relative scarcity of liquidity delivered in the event of low tree output, projects that offer investment opportunities in a more strongly pro-cyclical way will have as complements a cheaper form of liquidity.

As such, there will be incentives for this synchronization of liquidity needs and liquidity supply, which is made possible by choosing a project with $\gamma > 0$. Despite having a lower return per-unit of the good invested in the technology itself, pro-cyclical projects offer an advantage, as the portfolios that enable their continuation involve the use of cheaper, pro-cyclical liquidity.

In the previous section, output from the entrepreneurial sector covaried with the trees output only once partial continuation occurred. Under those circumstances, completion rates exhibited a cyclical behavior and this was inherited by the entrepreneurial output. This transmission mechanism only worked through the financial distress states.

The present example highlights that once more strongly pro-cyclical projects are chosen at $t = 0$, their output covaries positively with the trees' return. This happens not only on the financial distress states, but even in the states where projects turn out to be self-financing. This comovement does not depend on a lower completion rate as in the previous example, but on a ex ante preference towards projects that require more plentiful liquidity as complements.

4.3 Environment 3: Liquidity Scarcity and Volatility with multiple assets

In this section, I study the result of the introduction of a more complete set of assets in the economy described in section 4.1, in which exposure to output fluctuations is endogenous. The assets introduced allow entrepreneurs to purchase external liquidity in a way which is contingent on the realization of the productivity states.

A complementarity between specific assets and projects is shown to exist. As a consequence, in economies with severe liquidity scarcity and high premia, entrepreneurs specialize in two projects and holdings of a single asset. One of these projects has strong output volatility, while the other has mild volatility. In case of aggregate financial distress, one of these projects is fully taken to completion, while the other is fully terminated. In that sense, these economies feature extreme levels of partial insurance and crises are always associated to failures in a large set of firms.

Additionally, these economies feature aggregate amplification of productivity fluctuations due to two forces. First, firms which hold liquidity which is contingent on positive productivity shocks are willing to pay high costs to economize on those, which they can do by choosing highly volatile projects. Second, there are more of these highly volatile firms than firms which choose to have projects with low levels of productivity fluctuations.

The environment is a special case of the general model proposed and is summarized by the following:

Uncertainty, Projects and Parameter assumptions- Same as in 4.1.

Assets- Now, unlike in Section 4.1, there are two assets that can be traded. Assets g and b, which pay out respectively in the high and low productivity contingencies. Asset g pays out 1 unit of consumption in states $\omega \in \{(g,\rho),(g,0)\}$ and 0 otherwise. Asset b pays out 1 unit of consumption in states $\omega \in \{(b,\rho),(b,0)\}$ and 0 otherwise. Both assets are in fixed supply L.

As in the previous section, these two assets are sufficient to fully span the subspace of states of the world in which external liquidity is essential for continuation. That means that there entrepreneurs can independently move liquidity into each state where projects are under financial distress and external assets are essential for continuation. As a consequence, equilibrium outcomes are the same that would be achieved if asset markets were complete, with the presence of four state contingent securities, while we avoid indeterminacy problems on asset holdings.

Analysis

We first analyze the asset pricing consequences of this enriched set of assets when project choice is shut down, i.e., when all entrepreneurs are constrained to managing the baseline γ_0 project. In section 4.2.1, the consequences of the shocks to external liquidity, through the output of the tree, generated all pricing and equilibrium characterization results. It is useful to contrast the behavior of the optimal continuation policies when productivity shocks induce shocks to internal liquidity, as they now do, with that case.

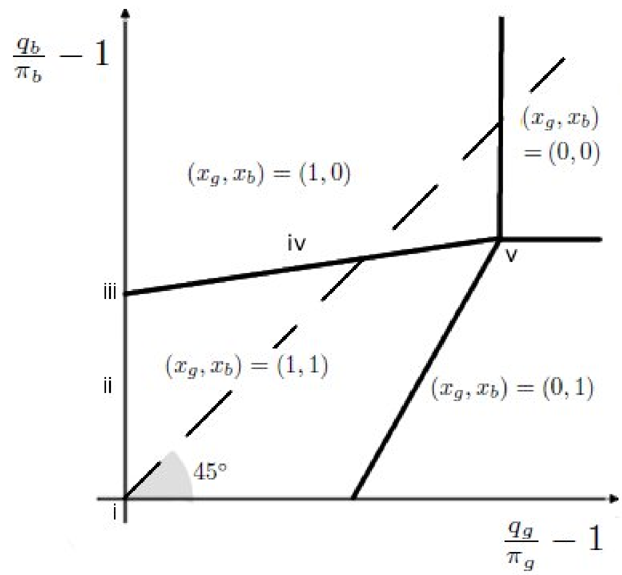


Figure 5: Optimal continuation policy regions, as functions of liquidity premia. For a fixed project γ.

As illustrated in Figure 5, the two effects that the productivity shock have on the possible prices of external liquidity are present. Following the same reasoning from the previous examples, the competitive equilibrium of a given economy has to lie in regions (i)-(v). Liquidity in the b state is always less plentiful than in the g, so if asset g carries a liquidity premium, the scarcer asset b will also need to carry one.

However, locus (iv), the set of points in which entrepreneurs are indifferent regarding hoarding liquidity to withstand the refinancing shock in the presence of low productivity, crosses the 45°line. Therefore, liquidity in the g asset might be valued at a premium above that embedded in asset b. This reflects the higher return on liquidity hoarding in the high productivity state, which had already emerged in the single-asset environment of Section 4.1.

Once project choice is available under the same assumptions as in the previous section, a few new features emerge. First, it is no longer the case that entrepreneurs can be indifferent between choosing a full continuation or a full termination policy, as in point (v) in Figure 5.

If this were the case, a joint deviation in choosing a project with a different degree of intrinsic volatility and holding only one of the contingent assets would dominate these options. That is due to the fact that project choice exhibits a complementarity with liquidity purchase decisions: a project that which involves amplified fluctuations leads to a higher willingness to pay for the pro-cyclical

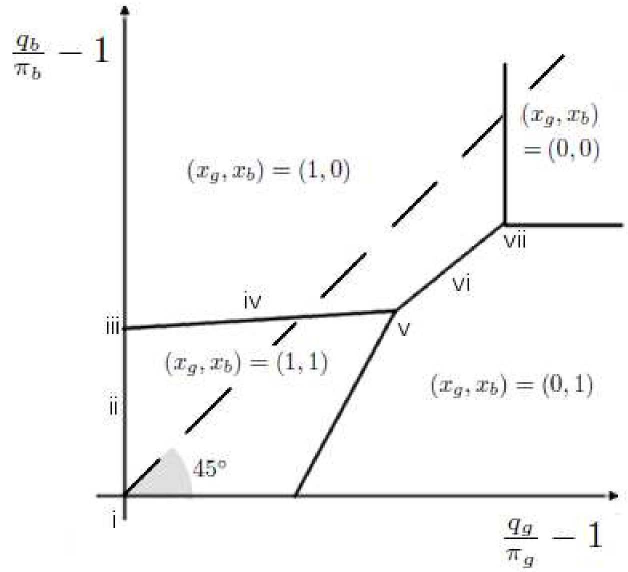


Figure 6: Optimal continuation policies, conditional on optimal policy choice, as functions of liquidity premia.

form of liquidity (asset g). The reverse is true for projects involving dampening, which display a higher reservation value for the b contingent asset.

The graph representing the optimal continuation policy as a function of liquidity premia on the two assets displays a typical behavior as the one in Figure 6. Behind each of the possibly optimal policies, there lies an associated optimal project choice.

Due to the complementarity between project choice and liquidity portfolio decisions, a region in which policies leading to continuation in only one of the two financial distress states dominate both full continuation and partial continuation emerges. It illustrated by locus (vi) in Figure 6.

Given that the economy has a set of assets which is rich enough to allow for the contingent allocation of external liquidity, equilibria in this region lead to the emergence of fully specialized firms: one with a project which fluctuates more severely, featuring amplification, and which is insured against financial distress only in the event of high aggregate productivity and another that fluctuates less, featuring dampening, and is insured against financial distress only in low aggregate productivity states.

The fact that locus (vi) lies below the 45-degree line has important consequences for project choice: in this region, firms choosing the pro-cyclical continuation policies favor projects leading to amplification and, given that they face a higher liquidity premium on the relevant asset, have stronger incentives for choosing projects away from the baseline γ_0 than do firms which choose less-cyclical projects and anti-cyclical continuation policies.

Indeed, that can be clearly seen from the entrepreneur's problem for fixed continuation policies. An entrepreneur that fully continues on state (g, ρ) and all non-distress states, but fully terminates on state (b, ρ) needs only to purchase asset u, not b. The relevant binding liquidity constraint is then

$$a_u = \left(\rho - \rho_0 - \frac{\gamma}{\pi_g}\right) I,$$

implying that asset purchases have to be just enough to cover the gap between refinancing needs, ρI, and pledgeable income in that state, $\left(\rho_0 + \frac{\gamma}{\pi_g}\right) I$. It can then be easily shown that the entrepreneur's problem, subject to this fixed continuation policy, can be rewritten as

$$min_{\gamma_{10}} c_{10} \left(q_g, \gamma\right),$$

where

$$c_{10} \left(q_g, \gamma\right) \equiv \frac{\phi\left(\gamma\right) + \pi_\rho \pi_g \left(\rho - \frac{\gamma}{\pi_g}\right) + \left(q_g - \pi_g\right)\left(\rho - \rho_0 - \frac{\gamma}{\pi_g}\right)}{\left(1 - \pi_\rho\right) + \pi_\rho \pi_g}$$

is an average cost per project unit completed project. An interior optimum for project choice immediately implies that

$$\phi'\left(\gamma_{10}\right) = \pi_\rho + \left(\frac{q_g}{\pi_g} - 1\right).$$

Fixing the continuation policy, entrepreneurial incentives regarding project choice take into account the possibility of saving on the relevant asset, by choosing projects with a different level of productivity fluctuations. The liquidity premium on asset g, $\left(\frac{q_g}{\pi_g} - 1\right)$, naturally appears in the expression.

An analogous procedure shows that the optimal project choice for a policy of full termination in distress state (g, ρ) and full continuation in all other states is given by the first-order condition

$$\phi'\left(\gamma_{01}\right) = -\pi_\rho - \left(\frac{q_b}{\pi_b} - 1\right),$$

which takes into account the liquidity premium on asset b, the relevant one for this policy. The fact that the locus of indifference between these policies lies below the 45-degreee line immediately implies that incentives for entrepreneurs holding the pro-cyclical form of liquidity (the g asset, with a higher premium) to deviate from the baseline γ_0 are stronger.

Despite the possible specialization of firms in holding a single form of liquidity and choosing a complementary project, which could not happen under the single-asset environment, the main results regarding amplification and dampening are maintained. Therefore, the emergence of amplification or dampening of fluctuations at the aggregate level is not a direct consequence of the absence of sophisticated contingent arrangements, but a response to the value to liquidity scarcity in accordance to its value across different states of the world.

For a sufficiently high supply of liquidity, the economy features dampening of the underlying shock. That is, a single project with $\gamma < \gamma_0$ is chosen by all entrepreneurs, as in locus (ii).

Once liquidity shortages are more severe, entrepreneurs specialize into two different projects. In regions (vi) and (vii), where policies $(x_g, x_b) = (1, 0)$ and $(x_g, x_b) = (0, 1)$ coexist, there is aggregate amplification. Entrepreneurs running the pro-cyclical projects, involving amplification, face higher incentives for deviating from γ_0, given that by doing so they save on a form of liquidity with higher equilibrium premium. Additionally, in equilibrium, there is a higher mass of these projects than of the project involving dampening.

These results can be summarized by the proposition below.

Proposition 5.

1. An equilibrium regime in which a pro-cyclical ($\gamma_{10} > \gamma_0$) project with associated continuation decisions $(x_g, x_b) = (1, 0)$ and an anti-cyclical project ($\gamma_{01} < \gamma_0$) with associated continuation decisions $(x_g, x_b) = (0, 1)$ coexist occurs for sufficiently scarce external liquidity, L.

2. Assume that $\phi(\gamma)$ is strictly convex and symmetric around γ_0 and that project choice is interior. Then, in regimes (vi) and (vii), there is aggregate amplification: $\|\gamma_{10} - \gamma_0\| > \|\gamma_{01} - \gamma_0\|$ and a higher share of entrepreneurs choose γ_{10} and $(x_g, x_b) = (1, 0)$.

In both environments, with a single asset or with two contingent assets, there are in place two mechanisms for the equilibrium determination of prices, liquidity premia and project choice.

The first mechanism is mostly a supply-contraction effect: a shock to productivity reduces internal liquidity and makes liquidity scarcity more severe in the low productivity states. Aggregate liquidity constraints are always tighter in these states, in the sense that if any continuation share is feasible in that state, it is also feasible in the higher productivity distress state.

On the opposite direction, there is an effect on the entrepreneurs' willingness to pay for liquidity: a demand-side effect. When the economy is constrained in all distress states, liquidity in states with higher productivity is more valuable than liquidity in states with lower productivity. Higher productivity makes a unit of liquidity have a higher multiplier effect into units of the project salvaged and also increases the total output of each one of these units. Therefore, when scarcity is sufficient to make prices be driven by this demand-side effect, the liquidity premium is higher on the asset that offers payments conditional on the high productivity events.

For moderate liquidity shortfalls, the scarcity effect dominates and influence project choice towards costly dampening. Once these shortfalls are more severe, as they are likely to be in economies with less financial development, termination of projects becomes more common and projects with more intrinsic volatility become more desirable.

The main consequence from a richer asset structure is the emergence of specialized firms. With sufficiently severe liquidity scarcity, all projects face the threat of termination in equilibrium, but external liquidity is efficiently used for salvaging projects with the best prospects in a given state of the world. In a distress state with lower productivity, projects involving dampening offer an advantage in their productivity, as they suffer less from a negative shock. On the other hand, the opposite is true in a distress state with higher productivity: projects with higher volatility have an advantage when relatively better shocks occur. Projects are also chosen in a complementary way to the form of liquidity that entrepreneurs choose to hold. This insight goes beyond the model with homogenous entrepreneurs, which endogenously specialize, and would also apply to examples that involve some original heterogeneity in exposure to productivity fluctuations. Specialization in the use of liquidity is a response to financial frictions and, once present, introduces a feedback into technological decisions that affect the economy's behavior under aggregate risk. Partially insured firms have higher incentives to manipulate their business-cycle exposure to economize only on the few assets they need to buy.

One might be surprised by the stark level of specialization of firm holdings of liquidity, which have a corner solution behavior. In equilibrium, projects are either fully insured or fully uninsured against financial distress shocks in a given state. This is a direct consequence of the constant-returns-to-scale assumption: if a set of firms holds a marginal unit of liquidity and is responsible for driving its price above the willingness to pay of firms with different projects, then these firms will also be the holders of all infra-marginal units. The insights about the complementarity between project choice and liquidity holdings and some degree of specialization in liquidity holdings should extend to economies with decreasing returns to scale, where less extreme forms of partial insurance would emerge.

It is also worth noting that the same allocation could be implemented without state contingent trades, as long as multi-project firms or financial intermediation (a lender with multiple entrepreneurs) are allowed. This way, external liquidity can be ex post allocated in the most efficient way.

5 Constrained Pareto Optimality and Policy

I first ask the question of whether a planner which is subject to the same constraints as private agents regarding bilateral contracts, limited pledgeability, limited commitment and asset market incompleteness, but which can choose contracts, projects and reallocate assets across the whole economy, can create a Pareto improvement over the original allocation. As previously anticipated, the answer is negative. Therefore, conditional on other frictions, a planner which has no advantages in the creation or distribution of liquidity over the private sector[20] cannot improve project choices or asset allocations.

Definition 5. An allocation is constrained Pareto optimal if there is no other set of pairwise financial arrangements (transfers, project choice, continuation decisions and asset holdings) and transfers of pledgeable resources at $t = 0$ that respects feasibility and constraints on pairwise financial arrangements (constraints 2 to 6), which Pareto dominates it[21].

Proposition 6. *Every competitive equilibrium of the economies described in Section 2 is constrained Pareto optimal.*

The constrained efficiency result highlights that incentives regarding project choice and asset holdings are properly aligned, not only at the lender and entrepreneur level, but also relative to the rest of the agents in the economy. When agents decide on their project choice, continuation decisions and asset holdings, they have internalized all impacts on other agents. For example, in section 4.1, when agents decide to dampen or amplify productivity fluctuations to relax their relevant liquidity constraints, they purchase less risk-less assets and free up this valued collateral to be used by other agents.

The use of external assets for backing transfers between a pair of agents excludes other agents from using these, but that effect is properly reflected in equilibrium asset prices and, as a consequence, asset reallocation at $t = 0$ can never lead to a Pareto improvement. Most importantly, there are no fire-sale externalities in the interim stage ($t = 1$) which a planner could address. Those are present in similar models in which constrained efficiency fails, such as Lorenzoni (2008), Shleifer and Vishny (1992) or Stein (2011).

[20] These advantages would be such as being able to issue assets agents cannot or use instruments for the contingent delivery of liquidity across firms which are more complete than the ones allowed by the original asset markets.

[21] Given the continuum of agents, for a Pareto improvement, a strict utility increase is required for a positive mass of agents.

The constrained optimality result in this environment generates a clear policy message related to which policies have the potential of creating improvements in this environment and which ones do not. For instance, no policy distorting or mandating project choice alone can generate a Pareto improvement over the original allocation. That includes banning the choice of the riskiest projects in an economy as the one described in section 4.1. The unintended consequences of such policy, in economies in which it effectively constrains project selection, would be a drop in the value of external assets, thus hurting their initial holders, and a potential increase in the number of firms that have to discontinue projects in case of financial distress, as other firms are not allowed to economize on their use of external assets by choosing more volatile projects.

Analogously, another policy that cannot lead to any Pareto improvement is one that forces the exclusive use of risk-less assets in backing financial arrangements. Although it could reduce the transmission of external financial shocks into the corporate sector and eliminate any endogenous increase in exposure the sources of such shocks generated by project choice, it leads to the wasting of valuable resources that fail to pay out in all contingencies. It therefore leaves the entrepreneurial sector excessively exposed to financial needs that could at least be satisfied in some instances.

On the other hand, the constrained optimality result does not imply that there is no room for policy at all in this environment. It does however determine clearly conditions which any improving policy needs to satisfy. Any such policy needs to feature an advantage from some central authority in creating or distributing liquidity which is not shared by the private sector. One example is the use of exclusive taxation powers that could make a bigger share of resources in the economy pledgeable, as suggested by Holmström and Tirole (1998). That relies on the assumption that the government has some enforcement power, such as non-pecuniary penalties for tax evasion, which is not shared by other agents in the economy. Another set of policies that could be welfare improving have to do with changes the legal and corporate governance environments, which determine the shares of resources which are pledgeable or not in the economy. An increase in enforcement abilities of private agents induced by policy can lead to a Pareto improvement[22].

6 Conclusion

The present paper analyzed project choice in an environment with financial frictions, with particular emphasis on its aggregate consequences. Given unreliability of future funding, which stems

[22]It is essential, however, to compensate the initial holders of external liquidity for a possible decrease in the rents they derived from the presence of premia on those assets.

from a limited commitment problem from lenders, real investment decisions and financial policies are intertwined. Arrangements for future investment have to be backed by transferable cash flows from projects (internal liquidity) or external assets that play a role as collateral (external liquidity). Therefore, shocks to these two influence the economy's fundamentals and create business cycles. As project selection is endogenous, in a general equilibrium environment, it interacts with asset prices and is a key determinant of the pattern of fluctuations of the aggregate economy.

In the environments studied, different forms of endogenous increase in exposure to risk can be driven by project selection. When there is severe scarcity of non-corporate assets which work as essential inputs for financial arrangements, endogenous project selection is biased towards riskier projects. Essentially, the economy is sufficiently constrained so that resources in the downside, or the worst states of nature, become less valuable than resources in situations in which constraints are less stringent but still present. The use of higher volatility emerges as a natural instrument for dealing with these constraints and transferring resources to where they are the most productive.

Alternatively, fluctuations might also originate from crunches in the value of these non-corporate assets. Under these conditions, there is not only a natural financial transmission mechanism that translates reductions in payouts of non-corporate assets into lower investment and output from corporations, but also an additional comovement effect that emerges in anticipation of these fluctuations, through project selection. The fundamental cause of this additional comovement is a complementarity between asset payouts that enable future investment and specific projects that deliver these investment opportunities.

All outcomes in the economies studied are constrained efficient. Project choice and financial policies respond adequately to the constraints faced by agents in the economy. There are no gains for a planner in distorting asset allocations, as prices properly reflect the assets' scarcity and their shadow value. Nor are there gains in targeting project choices or financial policies. Any gains from policy can only come from natural advantages in the creation of liquidity, in its contingent reallocation in ways that are not allowed by the original assets or in generating improvements in the fundamentals of the contracting environment.

An interesting extension would include the study of situations in which project selection cannot be perfectly contracted upon, so that financial policies and asset purchases play an additional role in providing incentives for project selection. A question to be addressed in that environment is whether asset prices would still properly reflect their value for society or whether an intervention through their reallocation or taxation could then lead to an improvement. Another interesting extension would include the study of overlapping projects, so that their liquidity demand could create spill-overs across the different vintages of investment. Whether constrained efficiency would survive in that environment is also an interesting open question.

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Appendix

A. General Lemmas

Proof. (1) - If the problem admits a solution $\tilde{c}, \tilde{t}, \tilde{\sigma}$, it admits a solution $\hat{c}, \hat{t}, \tilde{\sigma}$ in which entrepreneurial consumption at $t = 1$ and $t = 2$ is as low as possible (as much anticipation as possible is done), in which

$$\begin{aligned} \hat{c}_2(\omega) &= [\rho_1(\omega, \tilde{\gamma}) - \rho_0(\omega, \tilde{\gamma})] \tilde{x}(\omega) \tilde{I}, \\ \hat{c}_1(\omega) &= 0, \end{aligned}$$

It follows that

$$\rho_0(\omega, \tilde{\gamma}) \tilde{x}(\omega) \tilde{I} + z \cdot \tilde{a} = -\hat{t}_2(\omega)$$

and

$$\hat{t}_1(\omega) = \rho(\omega, \tilde{\gamma}) \tilde{x}(\omega) \tilde{I}.$$

It suffices to set $\hat{t}_0 = \tilde{t}_0 + E\left[\tilde{t}_2(\omega) + \tilde{t}_1(\omega) - \hat{t}_2(\omega) - \hat{t}_1(\omega)\right]$ and $\hat{c}_0 = \tilde{c}_0 + E\left[\tilde{c}_2(\omega) + \tilde{c}_1(\omega) - \hat{c}_2(\omega) - \hat{c}_1(\omega)\right].$ No constraint is violated and the same value for the objective function is achieved.
When we restrict attention to solutions with these properties, the problem can be simplified into the one of finding an optimal c_0 and σ, in which all consumption in other periods and transfers can be substituted away:

$$\max_{c_0 \geq 0, \sigma} c_0 + E\left[[\rho_1(\omega, \gamma) - \rho_0(\omega, \gamma)] x(\omega) I\right]$$

s.t.

$$E\left[\rho_0(\omega, \gamma) x(\omega) I - \rho(\omega, \gamma) x(\omega) I - [q - E(z)] \cdot a\right] - A - c_0 \geq -\underline{t},$$

$$z(\omega) \cdot \hat{a} I + \rho_0(\omega, \gamma) x(\omega) I \geq \rho(\omega, \gamma) x(\omega) I, \; for\, each\, \omega \in \Omega$$

This problem can only admit a solution when the first constraint binds and has a shadow-value $\lambda^* \geq 1$. In both the $\lambda^* = 1$ and $\lambda^* > 1$ possible cases $c_0 = 0$ is part of an admissible solution. Therefore, σ needs to solve (11). Also, the constraint can be added to the objective function and simplifications carried out to write the problem as (8).

□

Lemma 2. *Whenever $\rho_1(\omega, \gamma) > \rho_0(\omega, \gamma) > 0$, for all (ω, γ), one can restrict attention to policies with full continuation, $x(\omega) = 1$, in the states in which financial distress does not occur, i.e., for all $\omega \in \Omega$ with $\rho(\omega, \gamma) < \rho_0(\omega, \gamma)$.*

Proof. Suppose σ^* is an optimal plan. For a contradiction, assume that in some state ω in which financial distress does not happen, $x_{\sigma^*}(\omega) < 1$. Then a plan σ' which coincides with σ^*, except for setting $x_{\sigma'}(\omega) = 1$, leads to a strictly greater value for the objective function, without violating any of the constraints.

□

B. Proofs of results in Section 4.1

To simplify notation, let $x_g \equiv x(g, \rho)$ and $x_b \equiv (b, \rho)$. Using Lemma 2, we restrict attention to strategies setting $x(i, 0) = 1$, for $i = g, b$.

Now, $E\left[B_1\left(\omega;q;\gamma,x,\hat{a}\right)\right]$ can be written as

$$E\left[B_1\left(\omega;q;\gamma,x,\hat{a}\right)\right]=\left\{\rho_1\left(1-\pi_\rho\right)+\pi_\rho\left[\pi_g\left(\rho_1+\frac{\gamma}{\pi_g}\right)x_g+\pi_b\left(\rho_1-\frac{\gamma}{\pi_b}\right)x_b\right]\right\}$$
$$-\left\{\pi_\rho\rho\left(\pi_g x_g+\pi_b x_b\right)+\phi\left(\gamma\right)\right\}-\left(q-1\right)\hat{a}$$

and $E\left[B_0\left(\omega;q;\gamma,x,a\right)\right]$ is defined analogously.

Substitution of the leverage constraint into the objective function and the definition of $\hat{a}\equiv\frac{a}{I}$ can transform the entrepreneurs problem in

$$\max_{\gamma,x,\hat{a}}\frac{\rho_1-c\left(q;\gamma,x,\hat{a}\right)}{c\left(q;\gamma,x,\hat{a}\right)-\rho_0}A \tag{30}$$

s.t.

$$\hat{a}+\left(\rho_0+\frac{\gamma}{\pi_g}\right)x_g \geq \rho x_g \tag{31}$$

$$\hat{a}+\left(\rho_0-\frac{\gamma}{\pi_b}\right)x_b \geq \rho x_b, \tag{32}$$

in which

$$c\left(q;\gamma,x,\hat{a}\right)\equiv\frac{\left\{\phi\left(\gamma\right)+\pi_\rho\left[\pi_g x_g\left(\rho-\frac{\gamma}{\pi_g}\right)+\pi_b x_b\left(\rho+\frac{\gamma}{\pi_b}\right)\right]\right\}+\left(q-1\right)\hat{a}}{\left(1-\pi_\rho\right)+\pi_\rho\left[\pi_g x_g+\pi_b x_b\right]} \tag{33}$$

is similar to an average cost function, where the equivalence of productivity shocks learned at $t=1$ and refinancing shocks becomes clear.

Lemma 3. *The value function defined in*

$$\begin{aligned} c\left(q;\gamma\right) &\equiv \min_{x,a'} c\left(q;\gamma,x,a'\right) \\ s.t. &\quad (31);(32) \end{aligned}$$

for $q\geq 1$ is bounded from below by $c\left(q=1;\gamma\right)=\phi\left(\gamma\right)+\pi_\rho\rho\geq\phi\left(\gamma_0\right)+\pi_\rho\rho$ and from above by $\frac{\phi(\gamma)}{1-\pi_\rho}$.

Proof. Since $\hat{a}\geq 0$, $c\left(q;\gamma\right)$ is increasing and will have a minimum at $c\left(q=1;\gamma\right)$. At that point $x_g=x_b=1$ and $\hat{a}$ set to satisfy (32) with equality are optimal, leading to $c\left(q=1;\gamma\right)=\phi\left(\gamma\right)+\pi_\rho\rho\geq\phi\left(\gamma_0\right)+\pi_\rho\rho$. Finally, since $x_g=x_b=\hat{a}=0$ is always feasible and leads to $c\left(q;\gamma,x,\hat{a}\right)=\frac{\phi(\gamma)}{1-\pi_\rho}$, it follows that $c\left(q;\gamma\right)\leq\frac{\phi(\gamma)}{1-\pi_\rho}$. □

Proof. (Proposition 1) We can solve the entrepreneur's problem by minimizing (33) subject to (31) and (32). The FOCs are

$$x_g:\ \frac{\pi_\rho\pi_g\left(\rho-\frac{\gamma}{\pi_g}\right)}{D^*}-c^*\left(q;\gamma\right)\frac{\pi_\rho\pi_g}{D^*}+\left(\rho-\rho_0-\frac{\gamma}{\pi_g}\right)\mu_g\pi_g\begin{cases}\leq 0 & \text{, if } x_g=1\\ =0 & \text{,if } x_g\in(0,1)\\ \geq 0 & \text{, if } x_g=0\end{cases} \tag{34}$$

$$x_b: \frac{\pi_\rho \pi_b \left(\rho + \frac{\gamma}{\pi_b}\right)}{D^*} - c^*(q;\gamma)\frac{\pi_\rho \pi_b}{D^*} + \left(\rho - \rho_0 + \frac{\gamma}{\pi_b}\right)\mu_b \pi_b \begin{cases} \leq 0 & \text{, if } x_b = 1 \\ = 0 & \text{,if } x_b \in (0,1) \\ \geq 0 & \text{, if } x_b = 0 \end{cases} \tag{35}$$

$$\hat{a}: \frac{(q-1)}{D^*} - \mu_g \pi_g - \mu_b \pi_g \begin{cases} = 0 & \text{, if } \hat{a} > 0 \\ \geq 0 & \text{, if } \hat{a} = 0 \end{cases} \tag{36}$$

where $\mu_g \pi_g$ and $\mu_b \pi_b$ are the multipliers on constraints (31) and (32) and D^* is the denominator of the average cost evaluated at the point studied. Additionally,

$$\left[\hat{a} - \left(\rho - \rho_0 - \frac{\gamma}{\pi_g}\right) x_g\right] \mu_g = 0 \tag{37}$$

$$\left[\hat{a} - \left(\rho - \rho_0 + \frac{\gamma}{\pi_b}\right) x_b\right] \mu_b = 0 \tag{38}$$

We study possible solutions through three mutually exclusive cases: x_b being 1, 0 or interior.

i) $x_b = 1$. If that is the case, Constraint (32) implies that constraint (31) is slack. Thus, $\mu_g = 0$. Therefore, given A2 and the previous lemma, the FOC for x_g holds with the "<" inequality, meaning that $x_g = 1$. Additionally,

$$q - 1 = D^* \mu_b \pi_g, \tag{39}$$

which implies that

$$\pi_\rho \pi_b \left(\rho + \frac{\gamma}{\pi_b}\right) - c^*(q;\gamma)\pi_\rho \pi_b + \left(\rho - \rho_0 + \frac{\gamma}{\pi_b}\right)(q-1) \leq 0. \tag{40}$$

Assuming $x_g = x_b = 1$ is a solution also leads to $c^*(q;\gamma) = \phi(\gamma) + \pi_\rho \rho + (q-1)\left(\rho - \rho_0 + \frac{\gamma}{\pi_b}\right)$. Therefore, expression 40 is violated for sufficiently large q.

ii) $x_b = 0$.

$\hat{a} > 0$ would imply a slack liquidity constraint (32) and $\mu_b = 0$. The combination of A2' and Lemma (3) implies that this cannot be the case or 35 would lead to a contradiction. Therefore, $a = 0$, which also forces x_g= 0 from the liquidity constraint (31). Note that $x_g = x_b = 0$ as a solution leads to $c^*(q;\gamma) = \frac{\phi(\gamma)}{1-\pi_\rho}$.

iii) x_b is interior.

$\hat{a} > 0$ from (32). $\mu_g > 0$ is necessary otherwise for the same reasons as before, 34 would lead to a contradiction.

From (35) and the usual combination of A2' and the lemma, $\mu_b > 0$. Therefore, (32) and (31) imply that $x_b = x_g \frac{\left(\rho - \rho_0 - \frac{\gamma}{\pi_g}\right)}{\left(\rho - \rho_0 + \frac{\gamma}{\pi_b}\right)}$.

Therefore there are 3 regimes to consider: full continuation ($x_g = x_b = 1$), full termination upon distress ($x_g = x_b = 0$) and some continuation with $\hat{a} = \left(\rho - \rho_0 - \frac{\gamma}{\pi_g}\right) x_g = \left(\rho - \rho_0 + \frac{\gamma}{\pi_b}\right) x_b$ and an interior x_b.

To better characterize thresholds of transitions between the three behaviors, it is useful to solve two auxiliary, restricted, optimization problems. Problem 1 assumes that both liquidity constraints bind as in

$$c^*_{x_g, \kappa x_g}(q;\gamma) \equiv \min_{x_g \in [0,1]} \frac{\left\{\phi(\gamma) + \pi_\rho \left[\pi_g x_g \left(\rho - \frac{\gamma}{\pi_g}\right) + \pi_b x_g \kappa(\gamma)\left(\rho + \frac{\gamma}{\pi_b}\right)\right]\right\} + (q-1)\left(\rho - \rho_0 - \frac{\gamma}{\pi_g}\right) x_g}{(1-\pi_\rho) + \pi_\rho\left[\pi_g x_g + \pi_b x_g \kappa(\gamma)\right]}, \tag{41}$$

with $\kappa(\gamma) \equiv \frac{\left(\rho-\rho_0-\frac{\gamma}{\pi_g}\right)}{\left(\rho-\rho_0+\frac{\gamma}{\pi_b}\right)}$ being such that $x_b = x_g \kappa(\gamma)$. The FOC for x_g is

$$\pi_\rho \left[\pi_g \left(\rho - \frac{\gamma}{\pi_g}\right) + \pi_b \kappa(\gamma) \left(\rho + \frac{\gamma}{\pi_b}\right)\right] - \pi_\rho c^*_{x_g,\kappa x_g}(q;\gamma)\left[\pi_g + \pi_b \kappa(\gamma)\right] + (q-1)\left(\rho - \rho_0 - \frac{\gamma}{\pi_g}\right) \begin{cases} \leq 0 & , if\, x_g = 1 \\ = 0 & , if\, x_g \in [0,1] \\ \geq 0 & , if\, x_g = 0 \end{cases} \tag{42}$$

Notice that whenever $x_g = 0$ is a solution, $c^*_{x_g,\kappa x_g}(q;\gamma) = \frac{\phi(\gamma)}{1-\pi_\rho}$. Therefore, by substituting in that value, a threshold where indifference between following $(x_g, x_b) = (1, \kappa(\gamma))$ and full termination occurs is defined implicitly, as in

$$\pi_\rho \left[\pi_g \left(\rho - \frac{\gamma}{\pi_g}\right) + \pi_b \kappa(\gamma) \left(\rho + \frac{\gamma}{\pi_b}\right)\right] + (\overline{q}(\gamma) - 1)\left(\rho - \rho_0 - \frac{\gamma}{\pi_g}\right) = \pi_\rho\left[\pi_g + \pi_b \kappa(\gamma)\right] \frac{\phi(\gamma)}{1-\pi_\rho}. \tag{43}$$

For any $q < \overline{q}(\gamma)$, $c^*_{1,\kappa}(q;\gamma) < \frac{\phi(\gamma)}{1-\pi_\rho}$ and $(x_g, x_b) = (1, \kappa(\gamma))$ dominates both full termination and any other policy involving an interior x_b.

Problem 2 assumes that $x_g = 1$ and optimizes on x_b, assuming that the liquidity constraint only binds in the bad state. It is written as

$$c^*_{1,x_b}(q;\gamma) \equiv \min_{x_b \in [0,1]} \frac{\left\{\phi(\gamma) + \pi_\rho \left[\pi_g \left(\rho - \frac{\gamma}{\pi_g}\right) + \pi_b x_b \left(\rho + \frac{\gamma}{\pi_b}\right)\right]\right\} + (q-1)\left(\rho - \rho_0 + \frac{\gamma}{\pi_b}\right) x_b}{(1-\pi_\rho) + \pi_\rho\left[\pi_g + \pi_b x_b\right]}$$

and has a FOC given by

$$\pi_\rho \pi_b \left(\rho + \frac{\gamma}{\pi_b}\right) - \pi_\rho \pi_b c^*_{1,x_b}(q;\gamma) + (q-1)\left(\rho - \rho_0 + \frac{\gamma}{\pi_b}\right) \begin{cases} \leq 0 & , if\, x_b = 1 \\ = 0 & , if\, x_b \in [0,1] \\ \geq 0 & , if\, x_b = 0 \end{cases}$$

Let $c_{11}(q;\gamma) = \phi(\gamma) + \pi_\rho \rho + (q-1)\left(\rho - \rho_0 + \frac{\gamma}{\pi_b}\right)$, be the cost associated to full continuation. An interior solution for x_b will be admissible on Problem 2 iff

$$c^*_{1,x_b}(q;\gamma) = c_{11}(q;\gamma) = \left(\rho + \frac{\gamma}{\pi_b}\right) + (\pi_\rho \pi_b)^{-1}(q-1)\left(\rho - \rho_0 + \frac{\gamma}{\pi_b}\right).$$

This defines implicitly another threshold as $\underline{q}(\gamma)$ to the left of which full continuation is preferred to any strategy with $x_g = 1$ and interior x_b. At this threshold, any policy with partial continuation in x_b is equivalent. In particular,$(x_g, x_b) = (1, 1)$ and $(x_g, x_b) = (1, \kappa(\gamma))$ lead to the same value for the objective function. To the right of this threshold, $x_b = 1$ cannot happen in the solution and regime (i) is not possible.

Finally, notice that $c_{11}(q;\gamma)$ crosses the cost of termination upon distress $\frac{\phi(\gamma)}{1-\pi_\rho}$ at the point where

$$c_{11}(\tilde{q}(\gamma);\gamma) = \frac{\phi(\gamma)}{1-\pi_\rho} = \rho + (\pi_\rho)^{-1}(\tilde{q}(\gamma) - 1)\left(\rho - \rho_0 + \frac{\gamma}{\pi_b}\right).$$

$c_{11}(q;\gamma) < \frac{\phi(\gamma)}{1-\pi_\rho}$, if q is less than the value $\tilde{q}(\gamma)$ implicitly defined above. It is then easily verified by comparing their two implicit definitions that $\underline{q}(\gamma) < \tilde{q}(\gamma)$ and it follows that at $c_{11}\left(\underline{q}(\gamma);\gamma\right) = c^*_{1,x_b}\left(\underline{q}(\gamma);\gamma\right) < \frac{\phi(\gamma)}{1-\pi_\rho}$, which proves that $\underline{q}(\gamma) < \overline{q}(\gamma)$.

$\square$

Proof. Proposition 2:

For a fixed project, the last proposition has shown that at least one of $(x_g, x_b) = (1,1)$, $(x_g, x_b) = (1, \kappa(\gamma))$ or $(x_g, x_b) = (0,0)$ solves the optimization problem, with the first one being strictly preferred for a low price range, the second for an intermediate price range and the third for high prices.

One can therefore solve the entrepreneur's problem in two stages. One optimizes on the project for each of these three continuation policies of interest. Then, one optimizes over the three policies.

Let us define cost functions for fixed policies and projects as

$$\begin{aligned}
c_{11}(q;\gamma) &\equiv \phi(\gamma) + \pi_\rho \rho + (q-1)\left(\rho - \rho_0 + \frac{\gamma}{\pi_b}\right) \\
c_{1\kappa}(q;\gamma) &\equiv \frac{\left\{\phi(\gamma) + \pi_\rho\left[\pi_g\left(\rho - \frac{\gamma}{\pi_g}\right) + \pi_b\kappa(\gamma)\left(\rho + \frac{\gamma}{\pi_b}\right)\right]\right\} + (q-1)\left(\rho - \rho_0 - \frac{\gamma}{\pi_g}\right)}{(1-\pi_\rho) + \pi_\rho\left[\pi_g + \pi_b\kappa(\gamma)\right]} \\
c_{00}(q;\gamma) &\equiv \frac{\phi(\gamma)}{1-\pi_\rho}
\end{aligned}$$

Analogously, we define the value functions of the first stage of this optimization, which minimizes cost for a fixed continuation policy, as in

$$\begin{aligned}
c_j(q) &\equiv \min_\gamma c_j(q;\gamma), \\
&\quad \text{for } j = 11, 1\kappa, 00
\end{aligned}$$

First, notice that $c_{00}(q) = \frac{\phi(\gamma_0)}{1-\pi_\rho}$, as the problem is trivially solved at the lowest cost project γ_0.

Additionally, $c_{11}(q) \geq \phi(\gamma_0) + \pi_\rho + (q-1)(\rho - \rho_0)$, which is the cost of full continuation if shocks were not present in the most efficient project. This crosses $c_{00}(q) = \frac{\phi(\gamma_0)}{1-\pi_\rho}$ at $\hat{q}$ implicitly defined in

$$\underbrace{\rho} + \frac{(\hat{q}-1)}{\pi_\rho}\underbrace{(\rho - \rho_0)} = \frac{\phi(\gamma_0)}{1-\pi_\rho}. \tag{44}$$

As a consequence $c_{11}(q) > c_{00}(q)$ for all $q > \hat{q}$.

Now, $c_{1\kappa}(q) \leq c_{1\kappa}(q;\gamma_0)$. $c_{1\kappa}(q;\gamma_0)$ crosses $c_{00}(q)$ at $\overline{q}(\gamma_0)$ implicitly defined in

$$\underbrace{\frac{\left[\pi_g\left(\rho - \frac{\gamma_0}{\pi_g}\right) + \pi_b\kappa(\gamma_0)\left(\rho + \frac{\gamma_0}{\pi_b}\right)\right]}{\left[\pi_g + \pi_b\kappa(\gamma_0)\right]}} + \frac{1}{\pi_\rho}\left(\overline{q}(\gamma_0) - 1\right)\underbrace{\frac{\left(\rho - \rho_0 - \frac{\gamma_0}{\pi_g}\right)}{\left[\pi_g + \pi_b\kappa(\gamma_0)\right]}} = \frac{\phi(\gamma_0)}{1-\pi_\rho}. \tag{45}$$

As a consequence, $c_{1\kappa}(q) < c_{00}(q)$ for all $q < \overline{q}(\gamma_0)$.

We compare the terms with braces in equation (45) versus equation (44). The first term in braces in (45) is a weighted average of terms with mean ρ, with a higher weight put in the lower term. Therefore, it is less than ρ. The second term in braces can be rearranged to make clear that it is the probability weighted harmonic mean of $\left(\rho - \rho_0 - \frac{\gamma}{\pi_g}\right)$ and $\left(\rho - \rho_0 + \frac{\gamma}{\pi_b}\right)$ which is less than the expectation $(\rho - \rho_0)$. As a consequence, $\hat{q} < \overline{q}(\gamma_0)$. This proves that for some intermediate range of prices, $c_{1\kappa}(q) < c_{11}(q)$.

From the Proposition 1, at $q=1$, for any project, full continuation dominates $(1,\kappa)$ and $(0,0)$. Therefore, $c_{11}(q) < c_{1\kappa}(q)$ and $c_{11}(q) < c_{00}(q)$ for q sufficiently close to 1.

It is also possible to show that crossings occur only once, as using an Envelope Theorem and the inequality involving the arithmetic and harmonic means,

$$
\begin{aligned}
c'_{11}(q) &= \left(\rho-\rho_0+\frac{\gamma^*_{11}}{\pi_b}\right) > \rho-\rho_0 > 0 \\
c'_{1\kappa}(q) &= \frac{\left(\rho-\rho_0-\frac{\gamma^*_{1\kappa}}{\pi_g}\right)}{(1-\pi_\rho)+\pi_\rho\left[\pi_g+\pi_b\kappa\left(\gamma^*_{1\kappa}\right)\right]} \\
&\in (0,\rho-\rho_0) \\
c'_{00}(q) &= 0.
\end{aligned}
$$

As a consequence, there exist two thresholds $\overline{q}$ and $\underline{q}$, delimiting areas of optimality for the three continuation policies.

Next, we prove the behavior of the optimal policy regarding γ in each of the three regions.

i) For the region with $x_g = x_b = 1$, $\gamma^*_{11}(q)$ satisfies the first-order condition $\phi'\left(\gamma^*_{11}\right) = -\frac{(q-1)}{\pi_b}$. Therefore, $\gamma^*_{11}(q) \le \gamma_0$ and, given convexity of ϕ, it is an decreasing function of q.

ii) For the region with $x_g = 1$ and $x_b = \frac{\left(\rho-\rho_0-\frac{\gamma}{\pi_g}\right)}{\left(\rho-\rho_0+\frac{\gamma}{\pi_b}\right)}$, it can be shown that $\frac{\partial^2 c_{1\kappa}}{\partial\gamma\partial q} < 0$, indicating that the solution $\gamma^*_{1\kappa}(q)$ it a increasing function of q.

iii) For the region with full termination, it has already been argued that the optimal choice for is $\gamma^*_{00}(q) = \gamma_0$.

Last, it is necessary to show that for sufficiently high q between the $\underline{q}$ and $\overline{q}$, amplification is optimal. Notice that at $\overline{q}(\gamma_0)$ this will be the case, as

$$
c_{1\kappa}\left(\overline{q}(\gamma_0);\gamma_0\right) = c_{00}\left(\overline{q}(\gamma_0);\gamma_0\right).
$$

However, [23]

$$
\frac{\partial c_{1\kappa}\left(\overline{q}(\gamma_0);\gamma\right)}{\partial\gamma}\Big|_{\gamma_0} < 0,
$$

which implies that once project choice is incorporated, $c_{1\kappa}\left(\overline{q}(\gamma_0)\right) < c_{00}\left(\overline{q}(\gamma_0)\right)$.

Finally, I show that at $\overline{q}(\gamma_0)$, all projects $\gamma < \gamma_0$ are dominated by the choice of γ_0 and termination upon distress. For that purpose, note that

$$
c_{1\kappa}\left(\overline{q}(\gamma_0);\gamma\right) > \frac{\left\{\phi(\gamma_0)+\pi_\rho\left[\pi_g\left(\rho-\frac{\gamma}{\pi_g}\right)+\pi_b\kappa(\gamma)\left(\rho+\frac{\gamma}{\pi_b}\right)\right]\right\}+\left(\overline{q}(\gamma_0)-1\right)\left(\rho-\rho_0-\frac{\gamma}{\pi_g}\right)}{(1-\pi_\rho)+\pi_\rho\left[\pi_g+\pi_b\kappa(\gamma)\right]},
$$

since $\phi(\gamma) > \phi(\gamma_0)$. The term on the right-hand side is at least as great as $c_{00}\left(\overline{q}(\gamma_0);\gamma_0\right)$ iff

$$
\frac{\left[\pi_g\left(\rho-\frac{\gamma}{\pi_g}\right)+\pi_b\kappa(\gamma)\left(\rho+\frac{\gamma}{\pi_b}\right)\right]}{\left[\pi_g+\pi_b\kappa(\gamma)\right]} + \frac{1}{\pi_\rho}\left(\overline{q}(\gamma_0)-1\right)\frac{\left(\rho-\rho_0-\frac{\gamma}{\pi_g}\right)}{\left[\pi_g+\pi_b\kappa(\gamma)\right]} \ge \frac{\phi(\gamma_0)}{1-\pi_\rho}.
$$

The LHS is decreasing in γ and equality is reached at $\gamma=\gamma_0$, from the definition of $\left(\overline{q}(\gamma_0)-1\right)$. Therefore,

---

[23] Writing $c_{1\kappa}\left(\overline{q}(\gamma_0);\gamma_0\right) = \dfrac{\left\{\phi(\gamma)\left(\rho-\rho_0-\frac{\gamma}{\pi_g}\right)^{-1}+\pi_\rho\left[\pi_g\left(\rho-\frac{\gamma}{\pi_g}\right)\left(\rho-\rho_0-\frac{\gamma}{\pi_g}\right)^{-1}+\pi_b\left(\rho+\frac{\gamma}{\pi_b}\right)\left(\rho-\rho_0+\frac{\gamma}{\pi_b}\right)^{-1}\right]\right\}+(q-1)}{(1-\pi_\rho)\left(\rho-\rho_0-\frac{\gamma}{\pi_g}\right)^{-1}+\pi_\rho\left[\pi_g\left(\rho-\rho_0-\frac{\gamma}{\pi_g}\right)^{-1}+\pi_b\left(\rho-\rho_0+\frac{\gamma}{\pi_b}\right)^{-1}\right]}$ and using $c_{1\kappa}\left(\overline{q}(\gamma_0);\gamma_0\right) = c_{00}\left(\overline{q}(\gamma_0);\gamma_0\right)$ is helpful for showing this inequality.

$c_{1\kappa}\left(\overline{q}\left(\gamma_0\right);\gamma\right) > c_{00}\left(\overline{q}\left(\gamma_0\right);\gamma_0\right)$ and no policy involving dampening of fluctuations can be optimal. As a consequence, the optimal project choice is given by some $\gamma_{1p}^{*}\left(\overline{q}\left(\gamma_0\right)\right) > \gamma_0$. From monotonicity of $\gamma_{1p}^{*}\left(q\right)$, this will also hold for higher levels of q.

$\square$

C. Proofs of Results from Section 4.2

Proofs and Equilibrium Characterization for Section 4.2.1

A note on notation: I will make use of the Lemma 2 and restrict attention to $x(\omega)$ only for the financial distress states. To simplify notation, x_u and x_d will denote $x\left(u,\rho\right)$and $x\left(d,\rho\right)$.

Given assumptions A1-A2, constraint 9will always bind. Otherwise, no solution would exist, as the surplus of a policy of continuation if and only if financial distress does not occur goes to infinity as leverage goes to infinity. Therefore, one can substitute the constraint into the objective function to write it as $-\frac{E[B_1(\omega;q;\gamma,x,a)]}{E[B_0(\omega;q;\gamma,x,a)]}A$.

One should also note that with liquidity premia in place, over-hoarding of liquidity is dominated, i.e., entrepreneurs need only to hoard enough state-contingent liquidity to set the liquidity constraints (10) to hold with equality. Liquidity is only valuable as long as it is useful for enabling a decision $x\left(\omega\right)$. Therefore, one can restrict attention to

$$a_i\left(x,I\right) = \frac{\left(\rho-\rho_0\right)x\left(i,r\right)I}{z_i}.$$

Thus, problem 27 can be re-written, as

$$\max_{x}\frac{\rho_1 - c\left(q;x\right)}{c\left(q;x\right)-\rho_0}A,$$

where

$$c\left(q;x_u,x_d\right) \equiv \frac{1+\pi_\rho\rho\left[\pi_u x_u + \pi_d x_d\right] + \sum_{i=u,d}\left(q_i - \pi_i z_i\right)\frac{\left(\rho-\rho_0\right)x_i}{z_i}}{\left(1-\pi_\rho\right)+\pi_\rho\left[\pi_u x_u + \pi_d x_d\right]} \tag{46}$$

represents an average cost per project unit completed. Given that each unit completed generates the same social surplus and there are constant returns to scale, the entrepreneurs' problem can be written in terms of minimizing this average cost function.

Let the value function of the cost minimization problem be written as

$$c^{*}\left(q\right) \equiv \min_{0\leq x_u,x_d\leq 1} c\left(q;x_u,x_d\right), \tag{47}$$

and notice that for its partial derivatives

$$\frac{\partial c}{\partial x_i} \propto \pi_\rho\rho + \left(\frac{q_i}{\pi_i z_i}-1\right)\left(\rho-\rho_0\right) - \pi_\rho c^{*}\left(q\right). \tag{48}$$

The solution to the individual problem can be represented in terms of 4 regions, as seen in Figure 3. We first demonstrate the propositions below.

Lemma 4. *Let q_0 be the price vector free of liquidity premia, in which $q_i = \pi_i z_i$, for $i = u, d$. At this level, a policy of full continuation leads to the lowest possible equilibrium level of $c\left(q;x\right)$, $c^{*}\left(q_0\right) = 1+\pi_\rho\rho$.*

Proof. Notice that A1 implies that,

$$c\left(q_0,0,0\right)=\frac{1}{1-\pi_\rho}\geq c\left(q_0;x\right)\geq c\left(q_0;1,1\right)=1+\pi_\rho\rho,$$

where inequalities are strict for any x with an interior x_i component. Since $c\left(q,x\right)$ is increasing in q, and equilibrium requires $q\geq q_0$, this is the lowest value that $c\left(q,x\right)$ can achieve.

$\square$

Lemma 5. *The cost function is bounded by $\frac{1}{1-\pi_\rho}$, i.e., $c^*\left(q\right)\leq\frac{1}{1-\pi_\rho}$. And for sufficiently high q, in a vector sense, a policy of full termination in case of financial distress is optimal.*

Proof. For a policy that leads to full termination in case of distress, $c\left(q;0,0\right)=\frac{1}{1-\pi_\rho}$. Since that policy is always feasible, the first part of the lemma follows. For the second part, notice that the first-order relation48 has a single negative component, the third one, which is bounded at $c^*\left(q\right)=\frac{1}{1-\pi_\rho}$. Since the second term grows unbounded in q_i, for sufficiently high liquidity premia, the sign of the expression becomes positive, meaning that it is optimal to set the continuation shares $x\left(\omega,\rho\right)$ to the corner level of 0.

$\square$

By continuity, for sufficiently low liquidity premia, a policy of full continuation is optimal. We proceed to determine thresholds where partial or total continuation becomes optimal.

Lemma 6. *Indifference between a policies $\left(x_u,x_d\right)=\left(1,1\right)$ and $\left(x_u,x_d\right)=\left(1,0\right)$ is given by a straight line in the liquidity premium, $\left(\frac{q_u}{\pi_u z_u},\frac{q_d}{\pi_d z_d}\right)$-space, with a slope lower than one. Above that line termination in $\left(x_u,x_d\right)=\left(1,0\right)$ is preferred to $\left(x_u,x_d\right)=\left(1,1\right)$ and the opposite is true below it.*

Indifference between full continuation and full termination in $\left(u,\rho\right)$ is given by another line, with a slope above 1. To the right of it, $\left(x_u,x_d\right)=\left(1,0\right)$ is preferred to $\left(x_u,x_d\right)=\left(1,1\right)$ and the opposite is true for $\left(\frac{q_u}{\pi_u z_u},\frac{q_d}{\pi_d z_d}\right)$ combinations that lie to the left of it.

These two indifference loci cross over the $45°$ line, at the point where $\pi_\rho\rho+\left(\frac{q_i}{\pi_i z_i}-1\right)\left(\rho-\rho_0\right)-\pi_\rho\frac{1}{1-\pi_\rho}=0$, for $i=u,d$.

Proof. The first indifference is reached in the locus in which

$$c\left(q,1,1\right)=c\left(q,1,0\right).$$

That implies

$$\begin{aligned}1+\pi_\rho\rho+\sum_{i=u,d}\pi_i\left(\frac{q_i}{\pi_i z_i}-1\right)\left(\rho-\rho_0\right) &= \rho+\left(\frac{q_d}{\pi_d z_d}-1\right)\frac{\left(\rho-\rho_0\right)}{\pi_\rho}\\ 1-\left(1-\pi_\rho\right)\rho+\pi_u\left(\frac{q_u}{\pi_u z_u}-1\right)\left(\rho-\rho_0\right) &= \left(\frac{1-\pi_\rho\pi_d}{\pi_\rho}\right)\left(\frac{q_d}{\pi_d z_d}-1\right)\left(\rho-\rho_0\right).\end{aligned}$$

Which means that indifference is a locus of the form

$$\left(\frac{q_d}{\pi_d z_d}-1\right)=a+b\left(\frac{q_u}{\pi_u z_u}-1\right),$$

with $a \equiv \frac{1-(1-\pi_\rho)\rho}{(\rho-\rho_0)}\left(\frac{1-\pi_\rho\pi_d}{\pi_\rho}\right)^{-1} > 0$ and $b \equiv \frac{\pi_\rho\pi_u}{1-\pi_\rho\pi_d} = \frac{\pi_\rho\pi_u}{(1-\pi_\rho)+\pi_\rho\pi_u} < 1$. It crosses the 45-degree line at

$$\begin{aligned}\left(\frac{q_d}{\pi_d z_d}-1\right) &= \left(\frac{q_u}{\pi_u z_u}-1\right) = \frac{a}{1-b} \\ &= \frac{1-(1-\pi_\rho)\,\rho}{(\rho-\rho_0)}\frac{\pi_\rho}{1-\pi_\rho}.\end{aligned}$$

Since $c\,(q,1,1)$ is strictly increasing in $\left(\frac{q_d}{\pi_d z_d}\right)$ while $c\,(q,0,0)$ does not depend on it, this locus divides the space in regions where $c\,(q,1,1) < c\,(q,0,0)$, which lies below it, and where the opposite is true, which lies above it.

An analogous procedure leads to the locus of indifference between termination in (u,ρ), given full continuation in (d,ρ) being described by

$$\left(\frac{q_u}{\pi_u z_u}-1\right) = a' + b'\left(\frac{q_d}{\pi_d z_d}-1\right),$$

where $b' \equiv \frac{\pi_\rho\pi_d}{(1-\pi_\rho)+\pi_\rho\pi_d} < 1$ and $a' \equiv \frac{1-(1-\pi_\rho)\rho}{(\rho-\rho_0)}\left(\frac{1-\pi_\rho\pi_u}{\pi_\rho}\right)^{-1}$. It crosses the 45° line at the same point, which is their single intersection. In an analogous way to the first part, this locus also divides the space in a region of dominance of $(0,1)$ over $(0,0)$, to the right of it, and the opposite to its left.

□

Lemma 7. $\frac{q_u}{\pi_u z_u} \leq \bar{q}$, *for* $\bar{q}$ *implicitly defined in* $\pi_\rho\rho + (\bar{q}-1)(\rho-\rho_0) - \pi_\rho\frac{1}{1-\pi_\rho} = 0$, is necessary and sufficient for $c\,(q;1,0) \leq c\,(q;0,0)$. Analogously, $\frac{q_d}{\pi_d z_d} \leq \bar{q}$ *is necessary and sufficient for* $c\,(q;0,1) \leq c\,(q;0,0)$.

Proof. Indeed, $c\,(q;1,0) \leq c\,(q;0,0) \iff \rho + \left(\frac{q_u}{\pi_u z_u}-1\right)\frac{(\rho-\rho_0)}{\pi_\rho} \leq \frac{1}{1-\pi_\rho} \iff \frac{q_u}{\pi_u z_u} \leq \bar{q}$.

□

Characterization of optimal continuation decisions

Notice also that given linearity of the entrepreneurs' problem, at the thresholds of indifference between the four policies in the corners of the continuation possibilities square[24], any convex combination of these extreme optimal policies is also optimal. The combination of the lemmas above, justifies the characterization of the entrepreneurs' optimal continuation choice as functions of liquidity premia which is represented in Figure 3 in the main text.

Characterization of Equilibria

Proposition 7. *The following four statements hold:*

1. Equilibrium prices cannot lie at any point $\left(\frac{q_u}{\pi_u z_u}, \frac{q_d}{\pi_d z_d}\right)$ where $(x_u, x_d) = (1,1)$ is not in the set of optimal policies.

2. If in region where (1,1) is the unique optimal policy, there cannot be a liquidity premium on a_u, i.e., equilibrium $\frac{q_u}{\pi_u z_u} = 1$.

[24] $(x_u, x_d)] \in \{(0,0),(0,1),(1,0),(1,1)\}$

3. Equilibrium prices cannot lie in the segment where (1,1) and (0,1) are the only optimal policies.

4. Therefore, equilibria can only lie in loci (i)-(v) as depicted in Figure 4.

Proof. (1) Equilibria cannot lie in the interior of dominance regions of $(0,0)$, $(0,1)$ and $(1,0)$ as aggregate entrepreneurial demand for an asset carrying a liquidity premium would be zero, which contradicts market clearing. The same reasoning holds for the segments of indifference between (0,0) and (1,0) and between (0,0) and (0,1).

(2) If in this region, aggregate entrepreneurial demand for a_u is $\frac{(\rho-\rho_0)I^*}{z_u}$ and for a_d is $\frac{(\rho-\rho_0)I^*}{z_d}$. Market clearing in a_d requires

$$\frac{(\rho-\rho_0)\,I^*}{z_d}\leq L,$$

which given $z_u > z_d$ forces market clearing in a_u to be satisfied with a strict inequality, implying that $\frac{q_u}{\pi_u z_u}=1$.

(3) If this were the case aggregate demand for a_u would be strictly less than aggregate demand for asset a_d. Since supply of both assets is given by L, this would again imply that $\frac{q_u}{\pi_u z_u}=1$, which is a contradiction of the necessary indifference condition.

□

Proofs of results from 4.2.2

Again we make use of Lemma 2 to restrict attention to policies that always lead to full continuation when the projects are self-financing. To make notation less cumbersome, we will define $n_u(\gamma)$ and $n_d(\gamma)$ for the number of investment opportunities offered by project γ in the states of nature involving u and d payout from the tree.

In a similar manner as in the previous section, it is possible to rewrite the entrepreneur's problem in an average-cost-minimization form. First, one writes the minimum level of asset purchases as function of $\{I,\gamma,x_u,x_d\}$. Therefore,

$$a_i=(\rho-\rho_0)\,x_i n(\omega,\gamma)\,I \tag{49}$$

can be used to write the entrepreneur's problem as

$$\min_{\gamma;0\leq x_u,x_d\leq 1} c(q;\gamma,x_u,x_d) \tag{50}$$

where

$$c(q;\gamma,x_u,x_d)\equiv\frac{\phi(\gamma)+\pi_\rho\rho\left[\pi_u x_u n_u(\gamma)+\pi_d x_d n_d(\gamma)\right]+\sum_{i=u,d}(q_i-\pi_i z_i)\frac{(\rho-\rho_0)x_i n_i(\gamma)}{z_i}}{1-\pi_\rho+\pi_\rho\left[\pi_u x_u n_u(\gamma)+\pi_d x_d n_d(\gamma)\right]}. \tag{51}$$

We can define a change of variables to make the cost minimization problem 50 even more in line with Problem 47, which was extensively analyzed in the previous section. We set

$$\tilde{x}_u\equiv x_u n_u(\gamma)$$

and

$$\tilde{x}_d=x_d n_d(\gamma)$$

as the relevant choice variables, so that problem 50 for a fixed project becomes

$$c\left(q;\gamma\right)\equiv\min_{\tilde{x}_u,\tilde{x}_d}\frac{\phi\left(\gamma\right)+\pi_\rho\rho\left[\pi_u\tilde{x}_u+\pi_d\tilde{x}_d\right]+\sum_{i=u,d}\left(q_i-\pi_i z_i\right)\frac{(\rho-\rho_0)\tilde{x}_i}{z_i}}{1-\pi_\rho+\pi_\rho\left[\pi_u\tilde{x}_u+\pi_d\tilde{x}_d\right]}$$

s.t.

$$\begin{aligned}0 &\leq \tilde{x}_u\leq n_u\left(\gamma\right),\\ 0 &\leq \tilde{x}_d\leq n_d\left(\gamma\right).\end{aligned}$$

Proof. Proposition 4:

Using the change of variables suggested above, one can write the entrepreneurs' problem as

$$\max_{\tilde{x},\gamma}E\left[B_1\left(q;\tilde{x},\gamma\right)\right]I \tag{52}$$

s.t.

$$A+E\left[B_0\left(q;\tilde{x},\gamma\right)\right]I\geq 0$$

$$\begin{aligned}n_u\left(\gamma\right) &\geq \tilde{x}_u\geq 0\\ n_d\left(\gamma\right) &\geq \tilde{x}_d\geq 0.\end{aligned}$$

Here

$$\begin{aligned}E\left[B_1\left(q;\tilde{x},\gamma\right)\right] &= \rho_1\left[1-\pi_\rho+\pi_\rho\left(\pi_u\tilde{x}_u+\pi_b\tilde{x}_d\right)\right]\\ &- \phi\left(\gamma\right)-\pi_\rho\rho\left(\pi_u\tilde{x}_u+\pi_b\tilde{x}_d\right)-\sum_{i=u,d}\left(q_i-\pi_i z_i\right)\frac{\left(\rho-\rho_0\right)\tilde{x}_i}{z_i}\end{aligned}$$

and analogously for $E\left[B_0\left(q;\tilde{x},\gamma\right)\right]$ with ρ_0 replacing ρ_1.

Let μ be the multiplier associated to the leverage constraint and μ_i, for $i=u,d$, be the multipliers associated to the constraints of the form $n_i\left(\gamma\right)\geq\tilde{x}_i$. Then, the first-order condition on $\tilde{x}_i$ is given by

$$\frac{\partial E\left[B_1\right]}{\partial x_i}+\mu\frac{\partial E\left[B_0\right]}{\partial x_i}-\mu_i\leq 0,\, with\, =\, for\, x_i>0,$$

where $\frac{\partial E[B_1]}{\partial x_i}+\mu\frac{\partial E[B_0]}{\partial x_i}=\pi_\rho\pi_i\left(\rho_1+\mu\rho_0\right)-\left(1+\mu\right)\left(\rho\pi_\rho\pi_i+\pi_i\left(\frac{q_i}{\pi_i z_i}-1\right)\left(\rho-\rho_0\right)\right)$.

Optimization on γ is associated with

$$\frac{\partial E\left[B_1\right]}{\partial\gamma}+\mu\frac{\partial E\left[B_0\right]}{\partial\gamma}-\mu_u n_u'-\mu_d n_d'\begin{cases}\geq 0 & ,\, if\,\gamma=\overline{\gamma}\\ =0 & ,\, if\,\gamma\in\left(\underline{\gamma},\overline{\gamma}\right)\\ \leq 0 & ,\, if\,\gamma=\underline{\gamma}\end{cases},$$

where the left-hand side becomes

$$-\left(1+\mu\right)\phi'\left(\gamma\right)-\frac{\mu_u}{\pi_u}+\frac{\mu_d}{\pi_d}.$$

Note that

$$\frac{q_i}{\pi_i z_i}<\frac{q_{-i}}{\pi_{-i}z_{-i}}\implies\frac{\mu_i}{\pi_i}>\frac{\mu_{-i}}{\pi_{-i}}.$$

Therefore,

$$\begin{aligned}\frac{q_u}{\pi_u z_u} &< \frac{q_d}{\pi_d z_d} \implies \phi'(\gamma) > 0 \\ &\implies \gamma > 0.\end{aligned}$$

Uniqueness in project choice follows from the observation that, even with multiple solutions for the entrepreneurs' problem, constant returns to scale implies that each one of the solutions achieves the same optimal value $-\frac{E[B_1(q;\tilde{x},\gamma)]}{E[B_0(q;\tilde{x},\gamma)]}A = \mu A$. As such, they also have to share all μ_i and the same unique solution to the project choice problem.

Last, we rule out $\frac{q_u}{\pi_u z_u} > \frac{q_d}{\pi_d z_d}$ as this would generate $n_d > n_u$ for every individual entrepreneur and $\tilde{x}_d > \tilde{x}_u$. Since $Lz_u > Lz_d$, this is not compatible with asset market equilibrium.

□

Proofs of Section 4.3

As before, we simplify notation by using Lemma 2 and restriction attention to the characterization of $x_g \equiv x(g,\rho)$ and $x_b \equiv x_b(b,\rho)$. Given the presence of the two assets described in the text, we can restrict attention to strategies setting

$$a_g = \left(\rho - \rho_0 - \frac{\gamma}{\pi_g}\right) x_g I$$

and

$$a_b = \left(\rho - \rho_0 + \frac{\gamma}{\pi_b}\right) x_b I,$$

since asset purchases are only valuable as long as they help relax a liquidity constraint in one the states. Again, we might work with the minimization of modified average cost functions of the form

$$c(q;x,\gamma) \equiv \frac{\phi(\gamma) + \pi_\rho\left[\pi_g x_g\left(\rho - \frac{\gamma}{\pi_g}\right) + \pi_b x_b\left(\rho + \frac{\gamma}{\pi_b}\right)\right] + (q_g - \pi_g)\left(\rho - \rho_0 - \frac{\gamma}{\pi_g}\right)x_g + (q_b - \pi_b)\left(\rho - \rho_0 + \frac{\gamma}{\pi_b}\right)x_b}{(1-\pi_\rho) + \pi_\rho\left[\pi_g x_g + \pi_b x_b\right]} \tag{53}$$

or with a formulation as

$$\max_{I,x(\omega),\gamma} E\left[B_1(\omega;q;\gamma,x,\hat{a}(x))\right] I \tag{54}$$

s.t.

$$A + E\left[B_0(\omega;q;\gamma,\hat{a}(x))\right] I \quad \geq \quad 0.$$

Lemma 8. *With non-degenerate project choice, no optimal investment plan will ever feature an interior solution for continuation shares x_g or x_b.*

Proof. Suppose some optimal σ^* features an interior x_i^* for i equal to g or b. Then, $\sigma'_{x_i=1}$ and $\sigma'_{x_i=0}$ which coincide with σ^* except for, respectively, setting x_i to 1 or 0 will lead to the same value for the objective function as the original optimal strategy σ^*. Common optimality and constant returns to scale would mean that the Lagrangian associated to Program 54, $\mathcal{L}(q;\sigma)$ would feature the same multiplier μ^* for the leverage constraint for $\sigma = \sigma^*, \sigma'_{x_i=1}, \sigma'_{x_i=0}$. However, either $\mathcal{L}_\gamma\left(q;,\sigma'_{x_i=1}\right) > \mathcal{L}_\gamma(q;,\sigma^*) > \mathcal{L}_\gamma\left(q;,\sigma'_{x_i=0}\right)$

or $\mathcal{L}_\gamma\left(q;,\sigma^{'}_{x_i=1}\right) < \mathcal{L}_\gamma\left(q;,\sigma^*\right) < \mathcal{L}_\gamma\left(q;,\sigma^{'}_{x_i=0}\right)$ hold showing that the use of one of the corner x_i and an associated re-optimization over the project γ can lead to an improvement in Program 54 contradicting the optimality of σ^*.

$\square$

Proof. (Proposition 5) First, we show that for some sufficiently high prices on the state contingent assets, entrepreneurs would prefer to specialize in two projects, one leading to amplification and another one to dampening, and to be fully exposed to the risks of financial distress in one of the productivity states, while fully insured in the other one.

A equilibrium with fully specialized firms requires that for

$$\begin{aligned}
c_{10}\left(q_g,\gamma\right) &\equiv \frac{\phi\left(\gamma\right)+\pi_\rho\pi_g\left(\rho-\frac{\gamma}{\pi_g}\right)+\left(q_g-\pi_g\right)\left(\rho-\rho_0-\frac{\gamma}{\pi_g}\right)}{\left(1-\pi_\rho\right)+\pi_\rho\pi_g} \\
c_{01}\left(q_b,\gamma\right) &\equiv \frac{\phi\left(\gamma\right)+\pi_\rho\pi_b\left(\rho+\frac{\gamma}{\pi_b}\right)+\left(q_b-\pi_b\right)\left(\rho-\rho_0+\frac{\gamma}{\pi_b}\right)}{\left(1-\pi_\rho\right)+\pi_\rho\pi_b} \\
c_{11}\left(q,\gamma\right) &\equiv \left\{\phi\left(\gamma\right)+\pi_\rho\rho\right\}+\left(q_g-\pi_g\right)\left(\rho-\rho_0-\frac{\gamma}{\pi_g}\right)+\left(q_b-\pi_b\right)\left(\rho-\rho_0+\frac{\gamma}{\pi_b}\right) \\
c_{00}\left(q;\gamma\right) &\equiv \frac{\phi\left(\gamma\right)}{1-\pi_\rho}
\end{aligned}$$

$$\min_\gamma c_{10}\left(q_g,\gamma\right)=\min_\gamma c_{01}\left(q_b,\gamma\right)\leq \min_\gamma c_{11}\left(q,\gamma\right),\min_\gamma c_{00}\left(q;\gamma\right). \tag{55}$$

Additionally, let

$$\gamma_i\left(q\right)\equiv argmin_\gamma c_i\left(q;\gamma\right),\ for\ i\in\left\{11,01,10,00\right\}.$$

Notice that $\gamma_{00}\left(q\right)=\gamma_0$.

I show that there exist prices that satisfy condition 55.

Let $c_i\left(q\right)\equiv \min_\gamma c_i\left(q,\gamma\right)$. Notice that

$$\frac{\partial c_{10}}{\partial q_g},\frac{\partial c_{01}}{\partial q_b}>0, \tag{56}$$

since no project is ever self-financing in a financial distress state. Then,

$$c_{10}\left(q_g\right)=c_{01}\left(q_b\right) \tag{57}$$

defines a path in (q_g,q_b)-space which is strictly increasing. That represents the locus of asset prices that would lead to indifference between policies that involve to full continuation and full termination in opposing states of the world. Given 56, for a sufficiently high $\tilde{q}=\left(\tilde{q}_g,\tilde{q}_b\right)$ pair, $c_{10}\left(\tilde{q}_g\right)=c_{01}\left(\tilde{q}_b\right)=c_{00}\left(\tilde{q}\right)=\frac{\phi(\gamma_0)}{1-\gamma}$.

We show that at $\tilde{q}$, $c_{11}\left(\tilde{q}\right)>c_{00}\left(\tilde{q}\right)$. Suppose towards a contradiction that $c_{11}\left(\tilde{q}\right)\leq\frac{\phi(\gamma_0)}{1-\gamma}$, which implies

$$\rho+\frac{\left(\tilde{q}_g-\pi_g\right)\left(\rho-\rho_0-\frac{\gamma_{11}(\tilde{q})}{\pi_g}\right)+\left(\tilde{q}_b-\pi_b\right)\left(\rho-\rho_0+\frac{\gamma_{11}(\tilde{q})}{\pi_b}\right)+\phi\left(\gamma_{11}\left(\tilde{q}\right)\right)-\phi\left(\gamma_0\right)}{\pi_\rho}\leq\frac{\phi\left(\gamma_0\right)}{1-\pi_\rho}$$

and therefore either

$$\left(\rho-\frac{\gamma_{11}\left(\tilde{q}\right)}{\pi_g}\right)+\frac{\left(\tilde{q}_g-\pi_g\right)\left(\rho-\rho_0-\frac{\gamma_{11}\left(\tilde{q}\right)}{\pi_g}\right)+\phi\left(\gamma_{11}\left(\tilde{q}\right)\right)-\phi\left(\gamma_0\right)}{\pi_\rho\pi_g}\leq\frac{\phi\left(\gamma_0\right)}{1-\pi_\rho}$$

or

$$\left(\rho+\frac{\gamma_{11}\left(\tilde{q}\right)}{\pi_b}\right)+\frac{\left(\tilde{q}_b-\pi_b\right)\left(\rho-\rho_0+\frac{\gamma_{11}\left(\tilde{q}\right)}{\pi_b}\right)+\phi\left(\gamma_{11}\left(\tilde{q}\right)\right)-\phi\left(\gamma_0\right)}{\pi_\rho\pi_b}\leq\frac{\phi\left(\gamma_0\right)}{1-\pi_\rho}.$$

This, in turn, implies that $c_{10}\left(\tilde{q}_g,\gamma_{11}\left(\tilde{q}\right)\right)\leq c_{00}\left(\tilde{q}\right)$ or $c_{01}\left(\tilde{q}_b,\gamma_{11}\left(\tilde{q}\right)\right)\leq c_{00}\left(\tilde{q}\right)$. Since neither $c_{10}\left(\tilde{q}_g,\gamma\right)$ nor $c_{01}\left(\tilde{q}_b,\gamma\right)$ feature $\gamma_{11}\left(\tilde{q}\right)$ as a critical point, re-optimization around γ ensures that $c_{10}\left(\tilde{q}_g\right)<c_{00}\left(\tilde{q}\right)$ or $c_{01}\left(\tilde{q}_b\right)<c_{00}\left(\tilde{q}\right)$, achieving the desired contradiction.

By lowering q_g and q_b away from $\tilde{q}$ along the path defined by (57), one achieves points where policies with $x_i=1$ and $x_{-i}=0$ are strictly preferred to full termination and partial termination. Eventually, a lower bound $\underset{\sim}{q}$where equality (and indifference) with respect to $c_{11}\left(q\right)$ is reached. Above this lower bound, all demand for assets comes from these extreme policies reaching the value functions c_{01} and c_{11}.

From the definition of $B_0\left(\omega;q;\gamma,x\right)$ and $c\left(q;\gamma,x\right)$, the leverage constraint can also be written as

$$I=\left\{\left(1-\pi_\rho\right)+\pi_\rho\left[\pi_g x_g+\pi_b x_b\right]\right\}^{-1}\left(c\left(q;\gamma,x\right)-\rho_0\right)^{-1}A.$$

Demand for assets in this price path (57) can be described by

$$\begin{aligned}a_g^D\quad\left(q\right)&=\quad\left(\rho-\rho_0-\frac{\gamma_{10}\left(q\right)}{\pi_g}\right)I_{10}\left(\tilde{q}\right)\\ a_b^D\quad\left(q\right)&=\quad\left(\rho-\rho_0+\frac{\gamma_{01}\left(q\right)}{\pi_b}\right)I_{01}\left(\tilde{q}\right),\end{aligned}$$

where the aggregate scales $I_{10}\left(\tilde{q}\right)$ and $I_{01}\left(\tilde{q}\right)$ solve

$$\begin{aligned}I_{10}\left(\tilde{q}\right)&=\left\{\left(1-\pi_\rho\right)+\pi_\rho\pi_g\right\}^{-1}\left(c_{10}\left(q\right)-\rho_0\right)^{-1}A_{10}\\ I_{01}\left(\tilde{q}\right)&=\left\{\left(1-\pi_\rho\right)+\pi_\rho\pi_b\right\}^{-1}\left(c_{01}\left(q\right)-\rho_0\right)^{-1}A_{01}\end{aligned}$$

and A_{01} and A_{10} represent the distribution of entrepreneurs (and their net worth) across these two policies and, besides positiveness, have to satisfy $A_{01}+A_{10}=A$ if $q<\tilde{q}$ and $A_{01}+A_{10}\leq A$ if $q=\tilde{q}$ (as a full termination policy is also optimal at $q=\tilde{q}$). Notice that A_{10} and A_{01} will be able to adjust freely to make sure that $\frac{a_g^D}{a_b^D}=1$, since the supply of external liquidity does not vary across states of the world.

Equilibrium with $L=0$ is trivially constructed by setting $q=\tilde{q}$ and $A_{10}=A_{01}=0$. For higher L, prices might adjust downward and asset demands will increase continuously, as long as $q\geq\underset{\sim}{q}$, the price point at which the policy involving $x_g=x_b=1$ becomes relevant for the equilibrium.

(PART 2)

With interior project choice, the optimal decisions associated to the $\left(x_g,x_b\right)=\left(1,0\right)$ and $\left(x_g,x_b\right)=\left(0,1\right)$ policies are given by first-order conditions

$$\begin{aligned}\phi'\left(\gamma_{10}\left(q\right)\right)&=\pi_\rho+\left(\frac{q_g}{\pi_g}-1\right)\\ \phi'\left(\gamma_{01}\left(q\right)\right)&=-\pi_\rho-\left(\frac{q_b}{\pi_b}-1\right).\end{aligned}$$

Using symmetry of ϕ, I will show that whenever indifference (57) holds liquidity premia are higher in the

a_g asset, so that $\frac{q_g}{\pi_g} > \frac{q_b}{\pi_b}$ and the rest of the proposition follows. First, notice that

$$c_{10}\left(q_g\right) \equiv \frac{\phi\left(\gamma_{10}\left(q_g\right)\right) + \pi_\rho \pi_g \left(\rho - \frac{\gamma_{10}(q)}{\pi_g}\right) + \left(q_g - \pi_g\right)\left(\rho - \rho_0 - \frac{\gamma_{10}(q)}{\pi_g}\right)}{\left(1 - \pi_\rho\right) + \pi_\rho \pi_g}$$

and

$$c_{01}\left(q_b\right) \equiv \frac{\phi\left(\gamma_{01}\left(q_b\right)\right) + \pi_\rho \pi_b \left(\rho + \frac{\gamma_{01}(q_b)}{\pi_b}\right) + \left(q_b - \pi_b\right)\left(\rho - \rho_0 + \frac{\gamma_{01}(q_b)}{\pi_b}\right)}{\left(1 - \pi_\rho\right) + \pi_\rho \pi_b}.$$

If $\frac{q_g}{\pi_g} \leq \frac{q_b}{\pi_b}$ was true in the locus defined by (57), then

$$\left\|\phi'\left(\gamma_{10}\left(q_g\right)\right)\right\| \leq \left\|\phi'\left(\gamma_{01}\left(q_b\right)\right)\right\|,$$

which would imply, given symmetry, that

$$\phi\left(\gamma_{10}\left(q_g\right)\right) \leq \phi\left(\gamma_{01}\left(q_b\right)\right).$$

That cannot be true, otherwise both

$$\frac{\phi\left(\gamma_{10}\left(q\right)\right)}{\left(1 - \pi_\rho\right)} \leq \frac{\phi\left(\gamma_{01}\left(q_b\right)\right)}{\left(1 - \pi_\rho\right)}$$

and

$$\left(\rho - \frac{\gamma_{10}\left(q\right)}{\pi_g}\right) + \frac{\left(\frac{q_g}{\pi_g} - 1\right)\left(\rho - \rho_0 - \frac{\gamma_{10}(q)}{\pi_g}\right)}{\pi_\rho} < \left(\rho + \frac{\gamma_{01}\left(q_b\right)}{\pi_b}\right) + \frac{\left(\frac{q_b}{\pi_b} - 1\right)\left(\rho - \rho_0 + \frac{\gamma_{01}(q_b)}{\pi_b}\right)}{\pi_\rho}.$$

As $c_{10}\left(q_g\right)$ and $c_{01}\left(q_b\right)$ are weighted averages involving, respectively, the terms on the left-hand side and right-hand side of the inequalities above, this would lead to a contradiction. Therefore, $\frac{q_g}{\pi_g} > \frac{q_b}{\pi_b}$ when (57) holds, implying that individual amplification is higher for the entrepreneurs choosing to be pro-cyclical, $\left(x_g, x_b\right) = (1, 0)$, as in $\left\|\phi'\left(\gamma_{10}\left(q_g\right)\right)\right\| > \left\|\phi'\left(\gamma_{01}\left(q_b\right)\right)\right\|$.

Another force for aggregate amplification is in place since, in equilibrium in this regime,

$$\begin{aligned} L = a_g^D \quad (q) = & \quad \left(\rho - \rho_0 - \frac{\gamma_{10}\left(q\right)}{\pi_g}\right) I_{10}\left(\tilde{q}\right) \\ L = a_b^D \quad (q) = & \quad \left(\rho - \rho_0 + \frac{\gamma_{01}\left(q\right)}{\pi_b}\right) I_{01}\left(\tilde{q}\right), \end{aligned}$$

which implies

$$I_{10}\left(\tilde{q}\right) > I_{01}\left(\tilde{q}\right),$$

more investment is made in the pro-cyclical projects.

□

D. Constrained Efficiency (Section 5)

The planner has asset reallocation, net worth redistribution at $t = 0$ and project choice as possible instruments, but is subject to the same constraints on the initial bilateral arrangements and resource feasibility. Before proceeding to the proof, it is necessary to add some notation that was not required for the rest of the text. Let A_t^L represent the aggregate endowment of lenders/consumers at period t. For simplicity, let it not depend on the aggregate state. Also, let $C_t^L(\omega)$ be the aggregate consumer/lender consumption at time t and state $\omega \in \Omega$.

The proof use is similar to the usual proof of the first welfare theorem and its version available in Holmström and Tirole (2011). However, it exploits the fact that complete markets for external assets are not necessary in the environment studied, given equivalence between consumption across different periods.

Proof. (Constrained Pareto Optimality - Proposition 6) - Suppose there is another set of pairwise financial arrangements, that leads to a Pareto improvement over allocation σ. Let $\tilde{\sigma}_j = \left\{\tilde{I}_j, \{\tilde{x_j}(\omega)\}_{\omega\in\Omega}, \tilde{\gamma_j}, \tilde{a}_j\right\}$ and $\tilde{\tau}_j = \left\{\tilde{\tau}_j^0, \tilde{\tau_j}^1(\omega), \tilde{\tau_j}^2(\omega)\right\}$ be the financial contract decisions (scale, continuation, project choice, asset holdings and transfers) involved, as indexed by entrepreneur j and the associated lender. From the feasibility of investment and consumption for entrepreneur j we have

$$\tilde{\tau}_j^0 + A_j = \phi(\tilde{\gamma}_j)\tilde{I}_j + \tilde{c}_j^{0,E} \tag{58}$$

$$\tilde{\tau}_j^1(\omega) = \rho(\omega, \tilde{\gamma}_j)\tilde{x}_j(\omega) I + c_j^{1,E}(\omega) \tag{59}$$

$$z(\omega)\cdot\tilde{a}_j + \rho_1(\omega,\hat{\gamma}_j)\hat{x_j}(\omega)\hat{I} \quad + \tilde{\tau}_j^2(\omega) = \tilde{c}_j^{2,E}(\omega) \tag{60}$$

That leads to the following level of utility being achieved by entrepreneur j:

$$E\left[\rho_1(\omega,\tilde{\gamma_j})\tilde{x_j}(\omega) - \rho(\omega,\tilde{\gamma})\tilde{x_j}(\omega) - \phi(\tilde{\gamma_j})\right]\tilde{I} + \tilde{A}_j + E\left[\sum_t \tilde{\tau}_j(\omega)\right] + E\left[z(\omega)\cdot\tilde{a_j}\right].$$

In order to satisfy interim incentive compatibility of a lender associated to entrepreneur j, this new allocation needs to satisfy, for each $\omega \in \Omega$,

$$\tau_j^1(\omega) + \tau_j^2(\omega) \leq 0$$

and, given limited pledgeability, it also needs to satisfy

$$-\tilde{\tau}_j^2(\omega) \leq z(\omega)\cdot\tilde{a_j} + \rho_0(\omega,\tilde{\gamma_j})\tilde{x_j}(\omega)\tilde{I}_j. \tag{61}$$

The last two constraints, combined with non-negativity of entrepreneurial consumption, imply that

$$\rho(\omega,\tilde{\gamma})\tilde{x}_j(\omega)\tilde{I}_j \leq z(\omega)\cdot\tilde{a} + \rho_0(\omega,\tilde{\gamma})\tilde{x}_j(\omega)\tilde{I}_j.$$

As a consequence, any allocation implemented by the planner also needs to satisfy the same liquidity constraints that entrepreneurs face. From (58), (59) and (61) and non-negativity of entrepreneurial consumption, we have that $A_j + E\left[\sum_t \tilde{\tau}_j(\omega)\right] + E\left[\rho_0(\omega,\tilde{\gamma})\tilde{x}_j(\omega) - \rho(\omega,\tilde{\gamma})\tilde{x}_j(\omega) - \phi(\tilde{\gamma_j})\right]\tilde{I}_j + E\left[z(\omega)\cdot\tilde{a_j}\right] \geq 0$.

Implementation of the decision $\tilde{\sigma_j}$, if feasible, under the original competitive equilibrium would lead to a value $E\left[\rho_1(\omega,\tilde{\gamma})\tilde{x}_j(\omega) - \rho(\omega,\tilde{\gamma})\tilde{x}_j(\omega) - \phi(\tilde{\gamma_j})\right]\tilde{I}_j + A - (q - z(\omega))\cdot\tilde{a_j}$ for the entrepreneur (from Lemma 1). There are three possibilities to consider. If $\tilde{A}_j + E\left[\sum_t \tilde{\tau}_j(\omega)\right] < A - q\cdot\tilde{a}$, the plan $\tilde{\sigma}$ would have been feasible under the competitive equilibrium for entrepreneur j, but would have failed to make the leverage constraint bind, being dominated by the equilibrium plan. That leads to a contradiction of a possible improvement. In the case in which $\tilde{A}_j + E\left[\sum_t \tilde{\tau}_j(\omega)\right] = A$, the $\tilde{\sigma}(j)$ plan would also have been feasible, while no strict gains can be made for entrepreneur j and the leverage constraint would hold with equality $A + E\left[B_0\left(\omega; q; \tilde{\gamma_j}, \tilde{x_j}, \tilde{\hat{a}_j}\right)\right] I = 0$. Therefore, for strict gains to be possible for entrepreneur j, we need $\tilde{A}_j + E_\omega\left[\sum_t \tilde{\tau}_j(\omega)\right] > A - q\cdot\tilde{a_j}$, which implies that the leverage constraint under the competitive equilibrium is violated by plan $\tilde{\sigma}$, or $A + E\left[B_0\left(\omega; q; \tilde{\gamma_j}, \tilde{x_j}, \tilde{\hat{a}_j}\right)\right] I < 0$. As a consequence, improvements for entrepreneurs require $A+\int E_{\omega}\left[B_0\left(\omega;q;\tilde{\gamma}_j,\tilde{x}_j,\tilde{\tilde{a}}_j\right)\right]\tilde{I}_j dj\leq 0$, with strict inequality if a positive mass of entrepreneurs is made better-off.

Under the previous equilibrium, average consumer/lender utility was given by $C_0^L+E\left[C_1^L\left(\omega\right)+C_2^L\left(\omega\right)\right]=\sum_t A_t^L+q\cdot L$, as they did not participate in the surplus of investment, but could consume their endowments and the value of of assets sold. Another necessary condition for a Pareto improvement over the original allocation follows, with $C_0^L+E\left[C_1^L\left(\omega\right)+C_2^L\left(\omega\right)\right]\geq\sum_t A_t^L+q\cdot L$, and a strict inequality being necessary if a positive mass of consumer/lenders is made better-off. Combining the two, we obtain the necessary condition for a Pareto improvement over the initial allocation

$$C_0^L+E\left[C_1^L\left(\omega\right)+C_2^L\left(\omega\right)\right]-\int E\left[B_0\left(\omega;q;\tilde{\gamma}_j,\tilde{x}_j,\tilde{\tilde{a}}_j\right)\right]\tilde{I}_j dj>A+\sum_t A_t^L+q\cdot L. \tag{62}$$

Feasibility at aggregate levels at $t=0$ and $t=1$ requires

$$\begin{aligned} C_0^L+C_0^E+\int\phi\left(\tilde{\gamma}_j\right)I_j dj &\leq A_0^L+A \\ C_1^L\left(\omega\right)+C_1^E\left(\omega\right)+\int\rho\left(\omega,\tilde{\gamma}_j\right)\tilde{I}_j dj &\leq A_1^L \end{aligned}$$

Since only the pledgeable component of output can be transferred to lenders, lender consumption at $t=2$ is bounded by

$$C_2^L\left(\omega\right)\leq\int\rho_0\left(\omega,\tilde{\gamma}_j\right)\tilde{x}_j\left(\omega\right)I_j dj+A_2^L+z\left(\omega\right)\cdot L.$$

Weighting the three previous constraints by their event probabilities and adding them up, we get

$$C_0^L+E\left[C_1^L\left(\omega\right)+C_2^L\left(\omega\right)\right]-E\left[\int\left[\left(\rho_0\left(\omega,\tilde{\gamma}_j\right)-\rho\left(\omega,\tilde{\gamma}_j\right)\right)\tilde{x}_j\left(\omega\right)-\phi\left(\tilde{\gamma}_j\right)\right]dj\right]\leq A+\sum_t A_t^L+E\left[z\left(\omega\right)\cdot L\right].$$

Finally, the constrained planner cannot create any of the assets, given the underlying lack of commitment of consumers, which forces $\int\tilde{a}_j dj\leq L_k$. Multiplying each of these constraints by its positive price q_k from the competitive equilibrium and adding them to the previous inequality, we obtain a reversal of 62 and the desired contradiction.

□