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Endogenous strength in conflicts☆

Carmen Beviá [a,b,*], Luis C. Corchón [c]

[a] Universitat Autònoma de Barcelona, Spain
[b] Barcelona GSE, Spain
[c] Departamento de Economía, Universidad Carlos III, Spain

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ABSTRACT

In this paper we study a two period contest where the strength of players in the second period depends on the result of the contest in the first stage. We show that in contrast to one-shot contests in the same setting, heterogeneous players exert different efforts in the first stage and rent dissipation in the first period may be large. We study the conditions under which the discouragement effect holds. In addition, new issues emerge like the evolution of the strengths and the shares of the prize during the game.

1. Introduction

The theory of contests analyzes situations in which several contenders expend effort to win a prize. The theory developed from the initial papers by Tullock (1967), Krueger (1974) and Becker (1983), see also Hirshleifer (1991), assumed in the main that the effort of different players had an identical impact in the contest. We will refer to this impact as the strength of a player. Static models in which players have different strengths were considered by Hillman and Riley (1989), Gradstein (1995), Corchón (2000) and Cornes and Hartley (2005).

Dynamic contests have been studied in a number of papers focusing on infinite horizon models (Cairns, 1989; Leininger and Chun-Lei, 1994; McBride and Skaperdas, 2007; Wirl, 1994), two period models of war and settlement (Garfinkel and Skaperdas, 2000; Skaperdas and Syropoulos, 1996) and models in which players have to win a number of contests in order to win a grand contest (Konrad and Kovenock, 2009; see also the surveys of Konrad, 2009, Chpt. 8, and Konrad, 2010). All these papers assume that the strength of players does not vary during the contest.

In this paper we present a two period, two players contest in which the strength of players is endogenous. The contest in each period is modeled by an asymmetric Tullock contest success function (CSF) where effective effort in the contest is determined by the strength of the player and her effort. At the end of each period, players receive their share in the contested resource. This departs from the usual interpretation of a CSF in which the outcome of the contest is probabilistic.

We assume that the strength of a player in the second period depends on the share obtained in the first period. This assumption captures situations such as wars in which the strength of a country depends on the fraction of the territory owned by this country. Another example might be the cold war between the USSR and the US in which the relative strength of each side could be measured by the territories (or the population) under its control. Also a firm with greater market share today could build its "brand" for the future; and a team that wins today can receive more money that will, in turn, make it more competitive in the future. Finally, in a political campaign, the first period contest is a poll which determines the strength of the two candidates in the election.

We prove the existence of a Subgame Perfect Nash Equilibrium which, under some additional assumptions, is unique. In equilibrium, the player with the largest relative strength exerts the largest effort. The latter does not hold in one shot games with two players and Tullock CSF where players with different strength exercise the same effort. Relative strengths count here because the second period

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* Corresponding author at: Universitat Autònoma de Barcelona, Spain.
E-mail address: carmen.bevia@gmail.com (C. Beviá).

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creates different incentives for players with different relative strengths. We show that the ratio of the effort of player 1 with respect to player 2 in period 1 is increasing in the strength of player 1. Thus, when the effort in the first period is also an investment for the second period, the stronger player exerts more effort in both absolute and relative terms than the weaker player.

The previous properties prompt us to compare the effort made in the first period of our game with the effort made if the game were one shot. This issue has been studied in several papers and discussed in Konrad (2010). In many cases, multi-stage contests involve a "discouragement effect" in which weak players exert less effort in early stages than they would if the contest were one shot. We find that the discouragement effect also holds in our framework when the weak player is sufficiently weak. But it does not always hold. Even if a player is three times stronger than the other the latter exerts more effort than in a one shot game. This is because in our framework players receive a prize in each period and not only at the end of the grand contest.

The Matthew effect is the phenomenon where "the rich get richer and the poor get poorer". To study this effect in our model, we distinguish between the trajectory of strength and the dynamics of the share of the prize. When the link between periods is strong (no discount and the strength in the second period equals the share in the prize in the first period) an initially strong player will be even stronger in the second period. We call this the "avalanche effect" because the initial advantage of a player is amplified later on. However when the link between periods is not strong the avalanche effect only occurs when initial strengths are similar. When initial strengths are unequal the relative strength of the strong player decreases in the second period. We call this the "level-off" effect. It is caused by an increase in the relative effort of the weak player. When the link between periods is weak the avalanche effect disappears, so in the second period relative strengths are leveled off with respect to what they were in the first period.

The trajectory of the share of the prize, does not follow the behavior of strengths: the player having initially more than half of the prize ends having a smaller share in the second period than in the first one. This is because the trajectory of the prize is determined by two forces. First, in the second period both players exert the same effort and therefore their shares coincide with their strength in this period. And two, the transition function is a contraction which means that it translates the impact of shares on strength in a moderate way.

Finally we study rent dissipation. We show that only when players have identical initial strengths and the link between periods is the strongest, rents are completely dissipated. When players are very similar and the link between periods is strong, there is more rent dissipation in the two period game than in the one shot game. But rent dissipation is not monotonic with the link between periods. Weak links can be associated with more rent dissipation than strong links due to the discouragement effect.

There are papers which also endogenize the strength of the players, see Nti (2004) and Franke et al. (2009) for a model where the strength is chosen by a planner. In other papers the CSF is not determined by a planner. Fearon (1996) (see also Leventoglu and Slantchev, 2007) presented a model in which the bargaining power is endogenous and determined by the size of the territory and the threat of a war in which one of the countries would disappear. In our model there is no final battle but a protracted conflict like in the multi-battle models.

The closest paper to ours is by Klumpp and Polborn (2006). In their model, candidates to office have to win a certain number of elections in order to win the grand contest. They show that the outcome of the first election creates an asymmetry in later rounds which might be decisive for the grand contest. They provide an explanation based on rational players for the "momentum effect" which is the tendency of early winners in preliminary contests to win the grand contest. The main difference with our paper is that the prize is obtained at the end of the grand conflict and that the strengths of players are exogenous. In their case the expected value of the prize at each moment is the variable which changes as the game is unfolding.

The rest of the paper goes as follows. Section 2 presents the model. Section 3 gathers our results on the existence and the uniqueness of equilibrium. The properties of equilibrium are shown in Section 4. Section 5 concludes.

2. The model

2.1. Players and payoffs

Two players, $i \in \{1,2\}$, fight for a divisible prize in two periods, $t \in \{1,2\}$. Each player ends each period with a fraction p_i^t of the prize. The value of the prize for each player in each period is V. The interpretation is that the resource under conflict produces a certain surplus each period that can be expropriated by the owner (harvest, money, slave population, human capital, etc.) and that this surplus does not depend on the intensity of conflict.

Player i exerts an effort e_i^t in period t. We assume that the marginal cost of effort is constant and equal to 1. Payoffs in period t are denoted by π_i^t and equal $p_i^t V - e_i^t$, $i \in \{1,2\}$. Payoffs for the whole game are $\sum_{t=1}^{2} \delta^{t-1} \pi_i^t \equiv \Pi_i$ where $\delta \in [0,1]$ is the discount rate of the players.

Players have relative strengths which determine the impact of their effort. We denote by $\alpha^t \in [0,1]$ the relative strength of player 1 at t, and by $1-\alpha^t$ the relative strength of player 2 at t. The contest success function (CSF) maps efforts and strengths in a period into the fraction of the prize owned by the players in this period. This departs from the usual interpretation of the CSF in which the outcome of the conflict is a probability of winning it. Let p (resp. $1-p$) be the fraction obtained by player 1 (resp. 2). We assume the CSF takes the asymmetric general Tullock form:

$$p^t = \frac{\alpha^t \left(e_1^t\right)^\gamma}{\alpha^t (e_1^t)^\gamma + (1-\alpha^t)(e_2^t)^\gamma} \text{ if } e_1^t > 0; \quad p^t = \alpha^t \text{ otherwise.} \tag{2.1}$$

$$1-p^t = \frac{\left(1-\alpha^t\right)\left(e_2^t\right)^\gamma}{\alpha^t (e_1^t)^\gamma + (1-\alpha^t)(e_2^t)^\gamma} \text{ if } e_1^t + e_2^t > 0; 1-p^t = 1-\alpha^t \text{ otherwise.} \tag{2.2}$$

The parameter γ measures the sensitivity of the probability of winning to the efforts. When $\gamma = 0$, the outcome of the contest is independent of efforts. When $\gamma = 1$, the CSF is *proportional*. It seems reasonable to require that the CSF is homogeneous of degree zero, so winning probabilities do not depend on how resources are measured (euros or dollars, thousands or millions of soldiers, etc.). Clark and Riis (1998), following Skaperdas (1996), have shown that under certain assumptions the only functional form that is homogeneous of degree zero is precisely the one above.

Efforts and relative strength enter multiplicatively in the CSF. Think of the relative strength as capital (social or physical) or territory and of $\alpha^t (e_i^t)^\gamma$ as the (Cobb–Douglas) production function of the influence of player i in the contest. Thus influence in the contest is produced by capital and labor. This interpretation of the influence of a player in the contest as a production function that depends of multiple inputs was already pointed out by Nti (2004), Kolmar and Wagener (2005), Cornes and Hartley (2005) and Ray and Sarin (2009).

Finally, note that the only source of asymmetry among players in payoffs and strategies comes from relative strength in period one which is exogenously given.

2.2. The transition function

The relative strength of player 1 changes from period one to period two according to the following transition function:

$$\alpha^t = f\left(p^{t-1}\right). \tag{2.3}$$

According to the production function interpretation in which α was thought as an input, the share of the prize received today allows for the accumulation of the alpha input that can be used next period.

We assume that f: $[0,1]\to[0,1]$ fulfills the following properties:

i) $f(1/2)=1/2$,
ii) $0<f'(p)\le 1, f''(p)\le 0$, for all $p\in[0,1]$.

Property (i) says that when both players have identical resources they have identical strength. Property (ii) says that $f(\cdot)$ is an increasing, and concave contraction. In order to motivate these properties, consider the following linear transition function:

$$f\left(p^{t-1}\right) = ap^{t-1}+b,\ \text{with}\ 0<a\le 1,\ \alpha = 1-2b,\ b\ge 0, \tag{2.4}$$

where a measures the importance of the share of the resource in the previous period and b the strength of country 1 which does not depend on the share. Since $a>0$ the ownership of the resource contributes positively to the relative strength, i.e. more people to draft or more/better sources of food, money, etc. The condition $a=1-2b$ makes p and $1-p$ symmetric because the strength for player 2 evolves according to

$$1-\alpha^t = 1-ap^{t-1}-b = a\left(1-p^{t-1}\right)+1-a-b. \tag{2.5}$$

Even if a country has a zero share in the resource it has a non-negative relative strength. Thus,

$$b\ge 0 \quad \text{and} \quad 1\ge a+b. \tag{2.6}$$

It is also natural to assume that the relative strength of a country is not maximal when it owns zero of the resource. Thus

$$b\le 0 \quad \text{and} \quad b+a\ge 0. \tag{2.7}$$

Conditions (2.6) and (2.7) imply $a\in[0,1]$ which corresponds to the assumption that $f(\cdot)$ is an increasing contraction.

3. Equilibrium

We look for a Subgame Perfect Nash Equilibrium of the game described in the previous section. Since there are only two periods, the game is solved backwards.

In what follows and in order to simplify the notation we will denote with a hat the variables in the second period and without a hat the variables in the first period.

In the second period, since the game ends, players play the one shot Nash equilibrium. Thus,

$$\hat{e}_1 = \hat{e}_2 = \gamma(1-\hat{a})\hat{\alpha}V, \tag{3.1}$$

and the fraction of the prize that player 1 gets in the second period is given by:

$$\hat{p} = \hat{\alpha} = f(p). \tag{3.2}$$

Payoffs in the second period, given Eq. (3.2), are:

$$\hat{\pi}_1 = f(p)V-\gamma f(p)(1-f(p))V = \tag{3.3}$$

$$= f(p)V(1-\gamma(1-f(p))). \tag{3.4}$$

$$\hat{\pi}_2 = (1-f(p))V-\gamma f(p)(1-f(p))V = \tag{3.5}$$

$$= (1-f(p))V(1-\gamma f(p)). \tag{3.6}$$

In the first period, each player solves:

$$\max_{e_1} pV-e_1+\delta f(p)V(1-\gamma(1-f(p))) \tag{3.7}$$

$$\max_{e_2} (1-p)V-e_2+\delta(1-f(p))V(1-\gamma f(p)). \tag{3.8}$$

First order conditions of payoff maximization for both players are:

$$\frac{\partial p}{\partial e_1}V\left[1+\delta f'(p)(1-\gamma+2\gamma f(p))\right] = 1, \tag{3.9}$$

$$-\frac{\partial p}{\partial e_2}V\left[1+\delta f'(p)(1+\gamma-2\gamma f(p))\right] = 1. \tag{3.10}$$

In Appendix A we show the concavity of the payoff function in the player's own strategy.

Note first that p is as a function of relative efforts and relative strengths. Let $x=e_1/e_2$. And let $h_1(\cdot,\cdot)$ and $h_2(\cdot,\cdot)$ be

$$h_1(x,\alpha) = 1+\delta f'(p)(1-\gamma+2\gamma f(p)), \tag{3.11}$$

$$h_2(x,\alpha) = 1+\delta f'(p)(1+\gamma-2\gamma f(p)). \tag{3.12}$$

Thus, the first order conditions can be rewritten as:

$$\frac{\gamma\alpha(1-\alpha)e_1^{\gamma-1}e_2^{\gamma}}{(\alpha e_1^{\gamma}+(1-\alpha)e_2^{\gamma})^2}Vh_1(x,\alpha) = 1; \tag{3.13}$$

$$\frac{\gamma\alpha(1-\alpha)e_1^{\gamma}e_2^{\gamma-1}}{(\alpha e_1^{\gamma}+(1-\alpha)e_2^{\gamma})^2}Vh_2(x,\alpha) = 1. \tag{3.14}$$

Thus,

$$e_2h_1(x,a) = e_1h_2(x,\alpha). \tag{3.15}$$

Dividing the above equation by e_2 we get:

$$h_1(x,\alpha)-xh_2(x,\alpha) = 0. \tag{3.16}$$

We show in Appendix A that the above equation has a solution. Let $x=x(\alpha)$ be one of the solutions of this equation. Thus, from Eq. (3.13) we get that

$$e_1(\alpha) = \frac{\gamma\alpha(1-\alpha)(x(\alpha))^{\gamma}}{(\alpha(x(\alpha))^{\gamma}+(1-\alpha))^2}Vh_1(x(\alpha),\alpha); \tag{3.17}$$

$$e_2(\alpha) = \frac{\gamma\alpha(1-\alpha)(x(\alpha))^{\gamma-1}}{(\alpha(x(\alpha))^{\gamma}+(1-\alpha))^2}Vh_1(x(\alpha),\alpha). \tag{3.18}$$

which are the equilibrium efforts. Thus we have shown,

Proposition 1. *A Subgame Perfect Nash Equilibrium exists.*

Note that periods are linked by the discount rate δ and the transition function. When δ is zero, or the transition function is constant, this link is severed and our equilibrium is just the one shot equilibrium. Indeed from Eqs. (3.11) to (3.12) we obtain $h_1(x,\alpha)=h_2(x,\alpha)=$

1 so $x(\alpha)=1$ and efforts in Eqs. (3.17) and (3.18) collapse in the one shot equilibrium values which are

$$e_1^{os}(\alpha) = e_2^{os}(\alpha) = \gamma\alpha(1-\alpha)V. \tag{3.19}$$

Even if the strength of players is different, the effort made in equilibrium in the one shot game is the same for both players. This property holds as long as there are two players with identical valuations and the CSF is homogeneous of degree zero (Corchón, 2000). In our two period game this property does not hold in the first period, reflecting the different strategic opportunities for both players in the continuation game.

In general, we cannot guarantee uniqueness of equilibrium. Uniqueness is obtained if the transition function is linear and the contest success function is proportional to weighted efforts. We formally state this in the following proposition. The proof is in Appendix A.

Proposition 2. *If $\gamma=1$ and $f(p)$ is linear, there exist a unique Subgame Perfect Nash Equilibrium.*

To close this section, note that in the case described in Proposition 2, plugging Eqs. (3.17) and (3.18) in Eq. (2.1) we obtain that the fraction of the resource owned by player 1 in period 1 is

$$p(\alpha) = \frac{\alpha x(\alpha)}{\alpha x(\alpha) + 1 - \alpha}. \tag{3.20}$$

It is easy to show that since $x(\cdot)$ is increasing (see Proposition 4 below) $p(\cdot)$ is increasing. So, as in the one shot game –where $p(\alpha)=\alpha$– the fraction of resources owned by player 1 in period 1 depends positively on the initial strength (as intuition suggests), though in a more complicated way.

Finally, notice that V does not affect the equilibrium distribution of the prize between players in both periods. This also happens in the one shot game.

In what follows we restrict the analysis of the properties of equilibrium to the special case described in Proposition 2. This assures uniqueness of equilibrium which seems a sensible requirement when exploring the properties of equilibrium.

4. Properties of equilibrium

4.1. Preliminary properties

We first state and prove some properties of equilibrium efforts that will be useful later on. We will see that some of these properties differ from the corresponding properties in a one shot game. All the proofs are gathered in Appendix A.

Proposition 3. *The equilibrium efforts in the first period satisfy the following:*

(i) $e_2(\alpha)=e_1(1-\alpha)$;
(ii) $e_1(\alpha)=e_2(\alpha)$ for $\alpha=1/2$, $\alpha=0$, $\alpha=1$;
(iii) $e_1(\alpha)>e_2(\alpha)$ if and only if $\alpha>1/2$.

Proposition 3 says that individual efforts display symmetry properties inherited from the symmetry of the basic data of the problem. Part (i) says that the effort of player 1 is the mirror image of the effort of player 2 when her relative strength α is substituted by $1-\alpha$. Part (ii) says that the effort of both players are identical either when they have the same relative strength ($\alpha=1/2$) or when one of them has zero strength. Part (iii) says that the player with larger strength exerts larger effort. Notice that this is not true in the one shot game, so this fact is explained by the existence of a second period.

The next result studies the ratio of efforts.

Proposition 4. *The ratio of the equilibrium efforts in the first period, $x(\alpha)$, is increasing in α.*

Proposition 4 says that relative efforts are increasing with relative strength. Thus, the strong player exerts more effort in the first period than the weaker player, which leaves her in better shape for the conflict in the second period. This contrasts with the one shot game where $x(\alpha)=1$ for all $\alpha\in[0,1]$.

4.2. The discouragement effect

We now address the question of when players exert more effort in our two period game than in the one shot game. We start by considering the following example.

Example 1. Suppose that $a=1$, $b=0$ and $V=10$. In this case we obtain a closed form solution for efforts and x, namely

$$x(\alpha) = \frac{2\alpha-2\delta+4\alpha\delta-1+\sqrt{4\delta+4\delta^2+16\alpha^2\delta^2-16\alpha\delta-16\alpha\delta^2+16\alpha^2\delta+1}}{2\alpha}. \tag{4.1}$$

In Fig. 1 below, we show the effort in the first period for both players as a function of α. We draw the case of $\delta=1$. The solid line corresponds to player 1 and the dashed line to player 2. Note the symmetry of the two lines, as proved in 3 part i. The dotted line corresponds to the effort of each player in the one shot game.

When the strength of a player is very large or very small, this player exerts little effort. This is because the outcome of the contest is very biased for her. When the contest is "fair" in the sense that similar efforts have similar impacts on the contests, efforts are larger.

We can see the effect of introducing a second period. If a player has little strength (approximately less than .3 in the figure for player 1) she is discouraged by the existence of a second period in the sense of exerting less effort in the two-period game than in the one period game. However, for larger values of strength, the existence of a second period encourages players to exercise more effort than in the one period game.

The example above exhibits a "discouragement effect" which is when weak players "reduce their incentives to expend effort in early rounds," (Konrad and Kovenock, 2010, p. 95 see the references there for earlier analysis of this effect and Konrad (2009, pp. 189–191) for a survey). This effect runs counter to the intuition that in a multiperiod game, players exert more effort than in a single period game because each period adds more return to the effort and thus incentives to expend more effort are enhanced by the existence of additional periods. This intuition is correct when first order

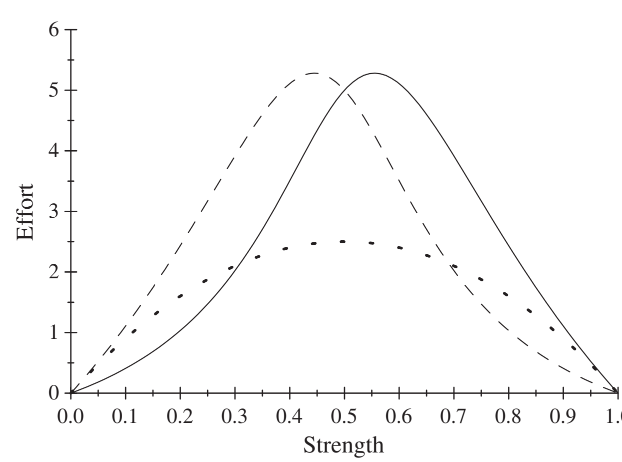

Fig. 1. The discouragement effect with $a=1$ and $\delta=1$.

conditions of payoff maximization are unaffected by the effort of other players. But when this is not the case the situation might be reversed. The next proposition analyzes this effect for player 1. The analysis for player 2 would be totally symmetric.

Proposition 5. *If a, $\delta > 0$, there exist $\alpha^* \in (0,1/2)$ such that for all $\alpha \in (0,\alpha^*)$, the equilibrium effort of player 1 in period 1 is smaller than the equilibrium effort in the one shot game.*

This result says that the discouragement effect happens when one of the players is sufficiently weak. But as Fig. 1 makes clear, even for reasonably low values of the strength of the weak player, say $\alpha = 1/3$, the discouragement effect does not hold.

The discouragement effect is less and less severe as a or δ become smaller. In the limit case ($a=0$ or $\delta=0$) the effect disappears because the equilibrium values of efforts collapse in the value corresponding to the one shot equilibrium. In Fig. 2, we represent the effort of player 1 for different values of a and $\delta=1$. The solid line corresponds to $a=1$, the dashed line corresponds to $a=0.8$, and the dotted line corresponds to the one shot game which is equal to $a=0$. Similar effects are obtained when δ decreases.

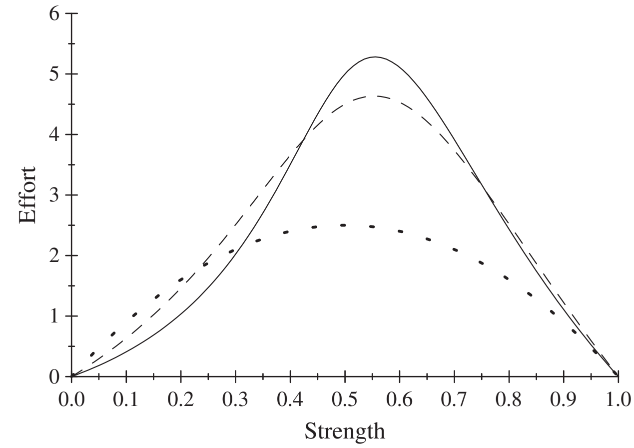

Fig. 2. The discouragement effect is less severe as a becomes smaller.

4.3. Avalanches or level off?

The second question that we address is the trajectory of strengths. Since $\hat{a}$ is increasing in p which in turn is increasing in α, it follows that $\hat{a}$ is increasing in α. But this does not imply anything about whether $\alpha \gtrless \hat{\alpha}$.[1]

A possibility is that when player 1 is initially strong ($\alpha > 1/2$) she will be even stronger in the second period ($\hat{\alpha} > \alpha$). We call this situation the *avalanche effect* of the second period because the strength of strong (resp. weak) players is amplified.[2] We see that this is the case when $a=\delta=1$ and $b=0$. This follows from the fact that $\hat{\alpha} = \alpha x/(\alpha x + 1 - \alpha)$ is increasing in x and for $\alpha > 1/2$, $x > 1$. This is represented in Fig. 3 below by a sinusoid solid line. The straight solid line is the 45° line.

But when $a=0.8$, $\delta=1$ and $b=0.1$ –represented in Fig. 3 by the dotted line– this line intersects the 45° line in three points. From 1/2 to the intersection to the right of 1/2 (or from the intersection to the left of 1/2 to zero) the avalanche effect still holds. However for α close to one, $\hat{\alpha} < \alpha$ and for α close to zero $\hat{\alpha} > \alpha$. Thus the existence of a second period *levels off* relative strengths.

Finally, the dashed line in Fig. 3 represents the case $a=0.5$, $\delta=1$ and $b=0.25$. In this case the avalanche effect disappears completely and starting from any position the relative strength of players is leveled off in the second period.

In fact these three cases exhaust all the possibilities that might arise in our framework. This is shown in the next proposition where the case a) corresponds to the solid line, the case b) corresponds to the dotted line and the case c) corresponds to the dashed line in Fig. 3.

Proposition 6.

a) *If $b=0$ there is an avalanche effect for all $\alpha \in [0,1] \setminus \{1/2\}$.*
b) *If $0<b<1/4$ and $\delta > 2b/(1-2b)(1-4b)$, there exist $\bar{\alpha} < 1/2$ (resp. $\tilde{\alpha} > 1/2$) such that for all $\alpha \in (0,\bar{\alpha})$ (resp. $\alpha \in (\tilde{\alpha},1)$) there is a level-off effect. For all $\alpha \in (\bar{\alpha},1/2)$ (resp. $\alpha \in (1/2,\tilde{\alpha})$) there is an avalanche effect.*
c) *If $0<b<1/4$ and $\delta < 2b/(1-2b)(1-4b)$, or if $b \geq 1/4$ and $\delta \in [0,1]$ there is a level-off effect for all $\alpha \in [0,1]$ except, possibly, for two isolated values of α.*

The condition $\delta > 2b/(1-2b)(1-4b)$ and $b<1/4$ is equivalent to $d\hat{\alpha}/d\alpha > 1$ at $\alpha = 1/2$. In this case the curve relating α and $\hat{a}$ crosses the 45° from below like the solid line (where $d\hat{\alpha}/d\alpha = 2$ at $\alpha=1/2$) and the dotted line (where $d\hat{\alpha}/d\alpha = 1.2414$ at $\alpha = 1/2$) in Fig. 3. Finally, the conditions $b<1/4$, and $\delta < 2b/(1-2b)(1-4b)$, or $b \geq 1/4$ and $\delta \in [0,1]$ imply that $d\hat{\alpha}/d\alpha < 1$ at $\alpha=1/2$ like the dashed line in Fig. 3 (where $d\hat{\alpha}/d\alpha = 0.6$ at $\alpha = 1/2$).

4.4. The domino effect

The third question that we address is the trajectory of the share of the prize in the hands of player 1. This share summarizes the equilibrium outcome of our game. One would expect that this share follows the behavior of α. We see that this is not the case.

Following the ideas introduced in the previous subsection consider the possibility that when player 1 is having initially more than half of the prize ($p > 1/2$) she will have even a larger share in the second period ($\hat{p} > p$). We call this situation the *domino effect* of the second period because the initial share of a strong (resp. weak) player is amplified later on in the game. Notice that

$$\hat{p} = ap + b = (1-2b)p + b. \tag{4.2}$$

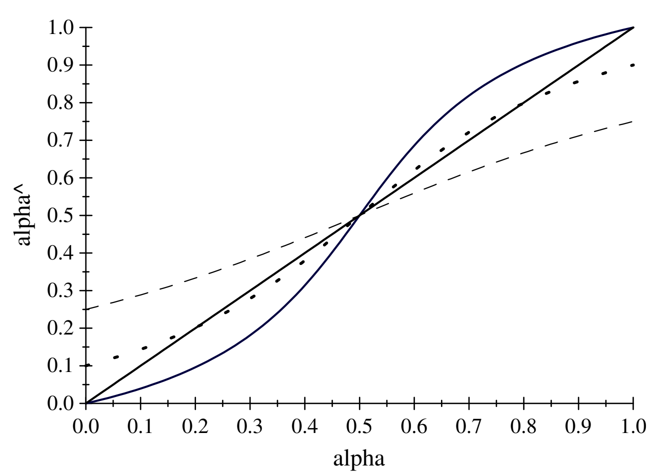

Fig. 3. The avalanche effect.

[1] Clearly, if $\alpha = 1/2$, $\hat{\alpha} = 1/2$ too.

[2] This effect has consequences similar to the momentum effect in Klumpp and Polborn (2006). But the momentum effect operates through the value of the prize and the avalanche effect through the strength of players.

Rearranging Eq. (4.2) we obtain

$$\hat{p}-p = b(1-2p). \tag{4.3}$$

Thus we have two cases. In the extreme case in which only the outcome in the first period is relevant to determine the strength next period (i.e. $b=0$), $\hat{p}=p$ so shares are invariant in time. In any other case, $b>0$ and $p>1/2$ imply $\hat{p}<p$, irrespective of whether there is an avalanche or a level off effect. The explanation of this result is that trajectory of the prize reflects, on the one hand that in the second period both players exert the same effort and therefore their shares coincide with their strength in this period. On the other hand we assumed that the transition function is a contraction.

Our result suggests that protracted conflicts tend to end up in an impasse in which players have to spend resources period after period in order to maintain their position. Examples like the Roman empire vs. Germanic tribes or vs. the Persian Empire, the first World War (until the entry of US in the conflict) or the cold war come to our mind. However, a full proof of this conjecture would take a model with several periods which is not attempted here. We do not enter in the discussion of what kind of modeling is preferable, a two period model or an infinite horizon model. For an enthusiastic defense of the former see Shapiro (1989).

4.5. Rent dissipation

Our final question is the impact of the second period on the rent dissipation in the first period. In the second period since efforts equal those in a one shot game rent dissipation is like in a one shot game.

Total effort in the first period amounts to

$$\frac{\alpha(1-\alpha)x(\alpha)}{(\alpha x(\alpha)-(1-\alpha))^2}Vh_1(x(\alpha),\alpha)+\frac{\alpha(1-\alpha)}{(\alpha x(\alpha)+(1-\alpha))^2}Vh_1(x(\alpha),\alpha). \tag{4.4}$$

Since in equilibrium

$$h_1(x(\alpha),\alpha)=x(\alpha)h_2(x(\alpha),\alpha), \text{ and } h_1(x(\alpha),\alpha)+h_2(x(\alpha),\alpha) = 2(1+\delta a), \tag{4.4}$$

can be written as:

$$\frac{\alpha(1-\alpha)x(\alpha)2(1+\delta a)V}{(\alpha x(\alpha)+(1-\alpha))^2}. \tag{4.5}$$

Call this function $B(\alpha,d,x)$ where $d\equiv\delta a$. We now study the maxima of $B()$ with respect to α,d and x. Given that $B()$ does not take into account the dependence of x with respect to the other variables, the maxima of $B()$ is always larger or equal than the maximum amount of effort. We see immediately that $B()$ is increasing in d so in the maximum $d=1$ (which implies that $a=\delta=1$). We also see that the maximum with respect to α cannot be at the boundaries of [0,1] because there, the function takes the value 0. Also, the maximum cannot be at either $x=0$ (where the function takes the value 0) or at an arbitrarily large value of x where the function takes a value arbitrarily close to 0. Thus the maximum with respect to α and x must be interior. Computing

$$\frac{\partial B(\alpha,d,x)}{\partial\alpha}=0 \text{ yields } \alpha=\frac{1}{x+1} \tag{4.6}$$

$$\frac{\partial B(\alpha,d,x)}{\partial x}=0 \text{ yields } x=\frac{1-\alpha}{\alpha}. \tag{4.7}$$

Eqs. (4.6) and (4.7) are identical so there is a continuum of solutions. We now introduce the fact that x is increasing in α and it is always positive. Thus $1/(x(\alpha)+1)$ is decreasing in α and strictly positive. So Eq. (4.6) has a unique solution. Note that for $\alpha=1/2$, $x(\alpha)=1$, and this is always a solution of Eq. (4.6). So, this must be the unique solution. We have proved the following.

Proposition 7. *Rents are completely dissipated iff $\alpha=1/2$, $\delta=a=1$.*

The previous result calls for a comparison of the rent dissipation in our game and in the one shot game. In the latter total efforts are

$$2\alpha(1-\alpha)V. \tag{4.8}$$

In this case, rents are never completely dissipated. Thus we have the following

Proposition 8. *For α close to 1/2 and δ and a close to 1, there is more rent dissipation in the two period game than in the one shot game.*

The result follows from the fact that the correspondence mapping α, δ and a into efforts has a closed graph in $(0,1)\times[0,1]\times[0,1]$. Since this correspondence is a function (because equilibrium is unique) this function is continuous and the result follows.[3]

Thus, when the link between periods is stronger (no discount and strengths are derived directly from the share in the first period) competition among players dissipates the prize entirely. In this case competition is tougher because to the effect of fighting for the prize in the first period, we have to add the effect of maintaining relative strengths in the second period. Clearly, as strength in the second period depends less on effort in the first period, this second effect vanishes. In Fig. 4, we show how total effort in the first period changes with a for the case of $\delta=1$. The solid line corresponds to $a=1$, the dashed line corresponds to $a=0.5$, and the dotted line corresponds to $a=0$. We note that, due to the discouragement effect, in some cases, conflict is less severe than in the one shot game.

5. Final comments

In this paper we have developed a theory of endogenous strength. We assumed that the strength in a period is a function of the fraction of the resources enjoyed by a player. We have found that equilibrium displays some features different from the one shot game. In particular rents might be completely dissipated in the first period and players with different strengths exert different efforts in the first period. Our model also differs from other multi-contest models in which the discouragement effect is pervasive. Finally new issues appear like the avalanche/level-off effect and the domino effect.

In order to get a tractable model, we assume two players, two periods and a linear transition function. The assumption that $a\le1$ plays also an important role in our proofs. Therefore, it would be interesting to investigate a model in which $\alpha>1$ or in which the transition function is not always increasing reflecting that too much territory might be disadvantageous for strength. But this is outside the scope of this paper. Here we try to make a first cut in the issue of the evolution of strength when it depends on past outcomes. Our conclusions are, of course, tentative.

Our model does not pay attention to issues which play an important role in dynamic conflicts. Among them we note the following two.

1. There are no resource constraints in the model. Consequently there are no bankruptcies. But the history of Europe has plenty of examples where conflict was ended by bankruptcy: the bankruptcy of 1607 which sealed the fate of the Spanish Habsburgs in their fight against France, the bankruptcy of France in 1788 –caused by the war with Great Britain– which paved the way for the French

[3] Notice that this result cannot be obtained directly from the function $B()$ since this function does not incorporate the restriction that $x=x(\alpha)$.

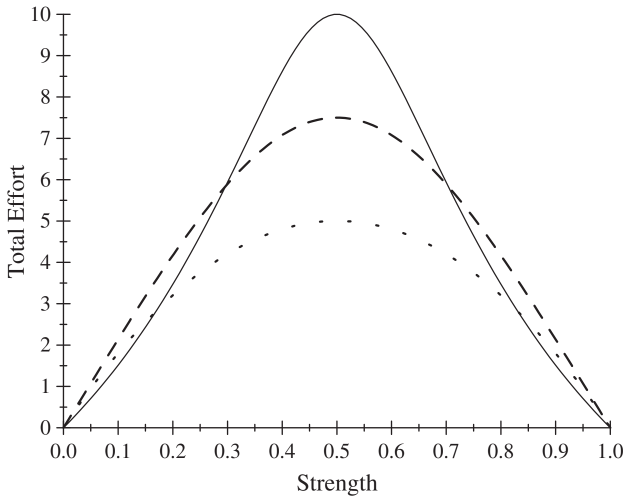

Fig. 4. Rent dissipation.

revolution and the British dominance in the next hundred years, and the bankruptcy of the USSR in the late eighties of the past century –caused by the military expenses– that led to the collapse of the socialist block.

2. The role of chance. The importance of random events in conflicts cannot be underestimated. von Clausewitz (1832) devoted the seventh chapter of his book to highlighting the influence of "frictions" on the outcome of war. Also there is a sizeable literature of contest in which the CSF arises as a reduced form of the effort of players and a random variable (Baye and Hoppe, 2003; Dixit, 1987; Fullerton and McAfee, 1999; Hillman and Riley, 1989; Jia, 2008; Lazear and Rosen, 1981). In our case, a possible way to introduce random events would be by making the parameter b a random variable.

We plan to study these aspects in the near future.

Appendix A

Second order conditions

In what follows, we show the concavity of the payoff function in a player's own strategy. Indeed,

$$\frac{\partial^2 \Pi_i}{\partial e_1^2} = \frac{\partial^2 p}{\partial e_1^2} V\left[1+\delta f^{'}(p)(1-\gamma+2\gamma f(p))\right] + \left(\frac{\partial p}{\partial e_1}\right)^2 V\delta 2\gamma\left(f^{'}(p)\right)^2 + \tag{6.1}$$

$$+\left(\frac{\partial p}{\partial e_1}\right)^2 V\delta f^{''}(p)(1-\gamma+2\gamma f(p)). \tag{6.2}$$

Note that,

$$\frac{\partial p}{\partial e_1} = -\frac{\gamma\alpha(1-\alpha)e_1^{\gamma-1}e_2^{\gamma}}{(\alpha e_1^{\gamma}+(1-\alpha)e_2^{\gamma})^2}, \tag{6.3}$$

which is positive, and

$$\frac{\partial^2 p}{\partial e_1^2} = -\frac{\gamma\alpha(1-\alpha)e_1^{\gamma-2}e_2^{\gamma}}{(\alpha e_1^{\gamma}+(1-\alpha)e_2^{\gamma})^3}\left(e_2^{\gamma}(1-\alpha)(1-\gamma)+e_1^{\gamma}\alpha(1+\gamma)\right), \tag{6.4}$$

which is negative.

We show first that $\partial^2 p/\partial e_1^2 + 2(\partial p/\partial e_1)^2 \leq 0$. Note that $\partial^2 p/\partial e_1^2 + 2(\partial p/\partial e_1)^2$ can be written as:

$$\frac{\gamma\alpha(1-\alpha)e_1^{\gamma-2}e_2^{\gamma}}{(\alpha e_1^{\gamma}+(1-\alpha)e_2^{\gamma})}\left[2e_1^{\gamma}e_2^{\gamma}\gamma\alpha(1-\alpha)-\left(e_2^{\gamma}(1-\alpha)(1-\gamma)+e_1^{\gamma}\alpha(1+\gamma)\right)\left(\alpha e_1^{\gamma}+(1-\alpha)e_2^{\gamma}\right)\right]. \tag{6.5}$$

Note that the term in brackets can be rewritten as:

$$2e_1^{\gamma}e_2^{\gamma}\alpha(1-\alpha)(\gamma-1)-e_2^{2\gamma}(1-\alpha)^2(1-\gamma)-e_1^{2\gamma}\alpha^2(1+\gamma), \tag{6.6}$$

which is negative because $\gamma \leq 1$. Thus, $\partial^2 p/\partial e_1^2 + 2(\partial p/\partial e_1)^2 \leq 0$.

Note that since $f''(p) \leq 0$, and $\gamma \leq 1$, the last term in Eq. (6.1) is less or equal than zero. Since $f'(p) \geq 0$, and $\partial^2 p/\partial e_1^2 \leq 0$, the first term is less or equal to $V(\partial^2 p/\partial e_1^2)$, and since $f'(p) \leq 1$, $\delta \leq 1$, and $\gamma \leq 1$, $\partial p/\partial e_1 > 0$, the second term is less or equal to $2V(\partial p/\partial e_1)^2$. Finally, since $\partial^2 p/\partial e_1^2 + 2(\partial p/\partial e_1)^2 \leq 0$ we obtain that $\partial^2 \Pi_i/\partial e_1^2 \leq 0$, as we wanted to show.

Existence of $x(\alpha)$

Existence: Recall that $x(\alpha)$ is defined as the solution of

$$1+\delta f^{'}(p)(1-\gamma+2\gamma f(p)) = x\left[1+\delta f^{'}(p)(1+\gamma-2\gamma f(p))\right]. \tag{6.7}$$

Suppose $x \to 0$. Then, the left hand side is larger than the right hand side (which tends to zero). But if $x \to \infty$ the right hand side tends to infinite (note that, because of the assumptions on the transition function, the term in brackets is bounded) and is larger than the left hand side which tends to a positive real number. By the intermediate value theorem there is an x such that both sides are identical, so Eq. (6.7) has indeed a solution.

Proof of Proposition 2

Existence of equilibrium is guarantee as we proved in the last section. We show that in the case of $\gamma = 1$, and a linear transition function the equilibrium is unique. For that it is enough to show that the solution to $h_1(x,\alpha) - xh_2(x,\alpha) = 0$ is unique. For $\gamma = 1$, and a linear transition function, $h_1(x,\gamma) - xh_2(x,\alpha) = 0$ can be written as:

$$1+\delta 2a\left(a\frac{\alpha x}{\alpha x+(1-\alpha)}+b\right) = x\left(1+\delta 2a\left(1-a\frac{\alpha x}{\alpha x+(1-\alpha)}-b\right)\right). \tag{6.8}$$

Write Eq. (6.8) as follows

$$1+\delta 2a^2\frac{\alpha x}{\alpha x+(1-\alpha)}(x+1)+\delta 2ab = x(1+2\delta a-2\delta ab). \tag{6.9}$$

The right hand side of Eq. (6.9) is linear and increasing, being zero when $x = 0$. The left hand side of Eq. (6.9) takes a positive value when $x = 0$. Furthermore, when $\alpha > 1/2$ it is strictly concave. A linear function and a strictly concave function can intersect, at most twice. But given the behavior of both functions at $x = 0$ the intersection is unique. If $\alpha \leq 1/2$ the left hand side of Eq. (6.9) is convex (linear if $\alpha = 1/2$), thus the slope of the curve is increasing with x. When x tends to infinity the slope tends to $\delta 2a^2$. But notice that since $a+b \leq 1$, $1-2\delta ab+2\delta\alpha \geq 1+\delta 2\alpha^2$, which implies that $\delta 2\alpha^2 < 1-2\delta ab+2\delta a$ thus the slope of the convex curve is always smaller than the slope of the linear function. Thus, given the behavior of both functions at $x = 0$, the linear function and the convex function intersect just once.

Proof of Proposition 3

(i) If the strength of player 1 is $1-\alpha$, the first order conditions of the maximization problem for each player can be written as:

$$\frac{\alpha(1-\alpha)e_2}{((1-\alpha)e_1+\alpha e_2)^2}Vg_1(y,1-\alpha)=1; \tag{6.10}$$

$$\frac{\alpha(1-\alpha)e_1}{((1-\alpha)e_1+\alpha e_2)^2}Vg_2(y,1-\alpha)=1, \tag{6.11}$$

where $y=e_2/e_1$, and

$$g_1(y,1-\alpha)=1+\delta 2a(ap+b), \tag{6.12}$$

$$g_2(y,1-\alpha)=1+\delta 2a(a(1-p)+b), \tag{6.13}$$

$$p=\frac{(1-\alpha)}{(1-\alpha)+\alpha y};\quad 1-p=\frac{\alpha y}{(1-\alpha)+\alpha y}. \tag{6.14}$$

Thus, from Eqs. (6.10), to (6.11) we get that

$$yg_1(y,1-\alpha)-g_2(y,1-\alpha)=0. \tag{6.15}$$

Notice that $g_1(y,1-\alpha)=h_2(y,\alpha)$, and $g_2(y,1-\alpha)=h_1(y,\alpha)$, thus Eq. (6.15) is identical to Eq. (3.16), which implies that

$$y(1-\alpha)=x(\alpha). \tag{6.16}$$

Thus, from Eq. (6.10) and the definition of y we obtain that

$$e_1(1-\alpha)=\frac{\alpha(1-\alpha)y(1-\alpha)}{((1-\alpha)+\alpha y(1-\alpha))^2}Vg_1(y,1-\alpha)= \tag{6.17}$$

$$=\frac{\alpha(1-\alpha)x(\alpha)}{((1-\alpha)+\alpha x(\alpha))^2}Vh_2(x,\alpha)= \tag{6.18}$$

$$=\frac{\alpha(1-\alpha)}{((1-\alpha)+\alpha x(\alpha))^2}Vh_1(x,\alpha)=e_2(\alpha), \tag{6.19}$$

where we have made use of the fact that $g_1(y,1-\alpha)=h_2(y,\alpha)$ and Eq. (3.16).

(ii) Trivially, if $\alpha=1$, or $\alpha=0$, $e_1=e_2=0$. And since $e_2(\alpha)=e_1(1-\alpha)$, $e_1(1/2)=e_2(1/2)$. Thus $x(1/2)=1$.

(iii) We finally show that when $\alpha>1/2$, $x(\alpha)>1$ which implies that $e_1(\alpha)>e_2(\alpha)$. Recall that $x(\alpha)$ is the solution of $0=h_1(x,a)-xh_2(x,\alpha)$. Since $h_1(x,\alpha)$ is increasing in α, $h_2(x,\alpha)$ is decreasing in α, and $\alpha>1/2$, we have that $h_1(x,\alpha)-xh_2(x,\alpha)>h_1(x,1/2)-xh_2(x,1/2)$. Since $x(1/2)=1$, $h_1(1,1/2)-h_2(1,1/2)=0$. Thus, $h_1(x,1/2)-xh_2(x,1/2)>h_1(1,1/2)-h_2(1,1/2)$. But note that $h_1(x,1/2)-xh_2(x,1/2)$ can be written as

$$1+2\delta a^2\frac{x}{x+1}+2\delta ab-x\left(1+2\delta a-2\delta a^2\frac{x}{x+1}-2\delta ab\right). \tag{6.20}$$

Rearranging terms,

$$1+2\delta ab+x(2\delta a(a+b-1)-1), \tag{6.21}$$

which is decreasing in x because $a+b\le 1$. Therefore, $h_1(x,1/2)-xh_2(x,1/2)$ is decreasing in x. Thus, $x(\alpha)>1$ for $\alpha>1/2$. ■

Proof of Proposition 4

Since $x(\alpha)$ is given by $h_1(x,\alpha)-xh_2(x,\alpha)=0$,

$$x'(\alpha)=\frac{-\dfrac{\partial h_1}{\partial\alpha}+x\dfrac{\partial h_2}{\partial\alpha}}{\dfrac{\partial h_1}{\partial x}-h_2-x\dfrac{\partial h_2}{\partial x}}. \tag{6.22}$$

The sign of $\partial h_1/\partial\alpha$ depends on the sign of $\partial p/\partial\alpha$ which is positive. The sign of $\partial h_2/\partial\alpha$ depends on the sign of $-\partial p/\partial\alpha$ which is negative. Thus, the numerator in Eq. (6.22) is negative. We show next that the denominator is also negative. Note first that the denominator can be written as:

$$2\delta a^2\frac{\partial p}{\partial x}-1-2\delta a(a(1-p)+b)+x2\delta a^2\frac{\partial p}{\partial x}. \tag{6.23}$$

Eq. (6.23) can be rewritten as:

$$2\delta a^2\frac{\partial p}{\partial x}(1+x)-(1-p)-1-2\delta ab. \tag{6.24}$$

Since $\partial p/\partial x=(\alpha(1-\alpha))/(\alpha x+(1-\alpha))^2$ Eq. (6.24) can be rewritten as

$$2\delta a^2\frac{1-\alpha}{\alpha x+(1-\alpha)}\left(\frac{\alpha(1+x)}{\alpha x+(1-\alpha)}-1\right)-1-2\delta ab. \tag{6.25}$$

Simplifying Eq. (6.25) we obtain

$$2\delta a^2\frac{1-\alpha}{\alpha x+(1-\alpha)}\left(\frac{2\alpha-1}{\alpha x+(1-\alpha)}\right)-1-2\delta ab. \tag{6.26}$$

Since the expression in brackets is negative for $\alpha\le 1/2$, Eq. (6.25) is negative as we wanted to prove.

We show next that this is also the case for $\alpha>1/2$. If $\alpha>1/2$, $x(\alpha)>1$ and since Eq. (6.26) is decreasing in x it is smaller than

$$2\delta a^2(1-\alpha)(2\alpha-1)-1-2\delta ab \tag{6.27}$$

which has a maximum at $\alpha=3/4$ then Eq. (6.27) is smaller than

$$\frac{\delta a^2}{4}-1-2\delta ab \tag{6.28}$$

which is always negative. ■

Proof of Proposition 5

Recall that e_i^{os} denotes the equilibrium effort of player i in the one shot game. Note that in the one shot game both players spend the same effort and $e_1^{os}=e_2^{os}=\alpha(1-\alpha)V$. We show that there exists $\alpha^*<1/2$ such that for all $\alpha\in(0,\alpha^*)$, $e_1(\alpha)<e_1^{os}(\alpha)$. Note first that by Eq. (3.17) the equilibrium effort of player 1 can be written as:

$$e_1(\alpha)=e_1^{os}(\alpha)\frac{x(\alpha)}{(\alpha x(\alpha)+(1-\alpha))^2}h_1(x(\alpha),\alpha). \tag{6.29}$$

Let us see that there exists a unique $\alpha^*<1/2$ such that

$$\frac{x(\alpha)}{(\alpha x(\alpha)+(1-\alpha))^2}h_1(x(\alpha),\alpha)=1. \tag{6.30}$$

For $\alpha=1/2$, $x(\alpha)=1$ and therefore Eq. (6.30) is equal to $h_1(x(\alpha),\alpha)$. Recall that $h_1(x(\alpha),\alpha)=1+\delta 2a(ap+b)>1$. Thus, for $\alpha=1/2$ the left hand side of Eq. (6.30) is bigger than 1. When α is close to zero, the left hand side of Eq. (6.30) is close to zero. Thus, by the intermediate value theorem there exists $\alpha^{*}<1/2$ such that Eq. (6.30) is satisfied. We show that the left hand side of Eq. (6.30) is strictly increasing for all $\alpha\leq 1/2$ which guarantees that α^{*} is unique. Since $h_1(x(\alpha),\alpha)$ is strictly increasing, it only remains to be proved that $x(\alpha)/(\alpha x(\alpha)+(1-\alpha))^2$ is increasing for all $\alpha\leq 1/2$. The first derivative of $x(\alpha)/(\alpha x(\alpha)+(1-\alpha))^2$ can be written as:

$$\frac{x^{'}(\alpha)(-\alpha x(\alpha)+(1-\alpha))+2x(\alpha)(1-x(\alpha))}{(\alpha x(\alpha)+(1-\alpha))^3}. \tag{6.31}$$

Since $\alpha<1/2$, $x(\alpha)<1$, and $(1-\alpha)>\alpha>\alpha x(\alpha)$. Thus, Eq. (6.31) is positive as we wanted to show. Thus, there exists a unique $\alpha^{*}<1/2$ such that for all $\alpha\in(0,\alpha^{*})$, $e_1(\alpha)<e_1^{os}(\alpha)$. ■

Proof of Proposition 6

We first recall the equations that we will use here, namely:

$$\hat{\alpha}=ap+b, \tag{6.32}$$

$$p=\frac{\alpha x}{\alpha x+1-\alpha}, \tag{6.33}$$

$$x=\frac{1+2\delta a(ap+b)}{1+2\delta a(1-ap-b)}, \tag{6.34}$$

$$a=1-2b. \tag{6.35}$$

Using Eq. (6.32), Eqs. (6.33) and (6.34) can be written as

$$\alpha x(\hat{\alpha}-b-a)=(b-\hat{\alpha})(1-\alpha). \tag{6.36}$$

$$x=\frac{1+2\delta a\hat{\alpha}}{1+2\delta a(1-\hat{\alpha})}. \tag{6.37}$$

Substituting the value of x in Eq. (6.37) in Eq. (6.36) we obtain that

$$\alpha(\hat{\alpha}-b-a)(1+2\delta a\hat{\alpha})-(b-\hat{\alpha})(1-\alpha)(1+2\delta a(1-\hat{\alpha}))=0 \tag{6.38}$$

which will be our main equation in this proof.

Our first step is to study the roots of Eq. (6.38) when $\alpha=\hat{\alpha}$. Notice that in this case Eq. (6.38) is a cubic function of α

$$\alpha(\alpha-b-a)(1+2\delta a\alpha)-(b-\alpha)(1-\alpha)(1+2\delta a(1-\alpha))=0. \tag{6.39}$$

Note that $\alpha=1/2$ is always a solution of Eq. (6.39), and if $\bar{\alpha}$ is a solution of Eq. (6.39), then $1-\bar{\alpha}$ is also a solution of Eq. (6.39). Also note that Eq. (6.39) can be written as

$$4\delta(1-2b)\alpha^3+6\delta(2b-1)\alpha^2+2\left(\delta-4\delta b^2+b\right)\alpha+b(4b\delta-1-2\delta)=0. \tag{6.40}$$

By using numerical methods it can be shown that Eq. (6.40) has, at most, three solutions in α.

Our second step is to compute $d\hat{\alpha}/d\alpha$. Let us call the left hand side of Eq. (6.38) $F(\alpha,\hat{\alpha})$. Totally differentiating Eq. (6.38) we obtain that

$$\frac{d\hat{\alpha}}{d\alpha}=\frac{-\frac{\partial F(\alpha,\hat{\alpha})}{\partial\alpha}}{\frac{\partial F(\alpha,\hat{\alpha})}{\partial\alpha'}}. \tag{6.41}$$

We now compute

$$-\frac{\partial F(\alpha,\hat{\alpha})}{\partial\alpha}=(1+2\delta a\hat{\alpha})(a+b-\hat{\alpha})+(\hat{\alpha}-b)(1+2\delta a(1-\hat{\alpha})). \tag{6.42}$$

Since $\hat{\alpha}>b$ (from Eq. (6.32)) and $\alpha=\hat{\alpha}=1/2$ (from Eq. (6.36)), both terms in the right hand side of Eq. (6.42) are positive.

Let us now study the denominator of Eq. (6.41). We compute

$$\frac{\partial F(\alpha,\hat{\alpha})}{\partial\hat{\alpha}}=2\delta a(4\alpha\hat{\alpha}-2\alpha-2\hat{\alpha}+1+b)+1. \tag{6.43}$$

Thus, using Eqs. (6.42) and (6.43), Eq. (6.41) can be written as

$$\frac{d\hat{\alpha}}{d\alpha}=\frac{(1+2\delta a\hat{\alpha})(1-b-\hat{\alpha})+(\hat{\alpha}-b)(1+2\delta a(1-\hat{\alpha}))}{2\delta a(4\alpha\hat{\alpha}-2\alpha-2\hat{\alpha}+1+b)+1}. \tag{6.44}$$

Next we compute $d\hat{\alpha}/d\alpha$ evaluated at $\alpha=\hat{\alpha}=1/2$ which amounts to

$$\frac{d\hat{\alpha}}{d\alpha}=\frac{2(1+\delta a)(1/2-b)}{2\delta ab+1}. \tag{6.45}$$

Rearranging the previous expression we obtain that

$$\frac{d\hat{\alpha}}{d\alpha}>1 \text{ if and only if } \delta>\frac{2b}{(1-2b)(1-4b)} \text{ and } b<1/4, \tag{6.46}$$

$$\frac{d\hat{\alpha}}{d\alpha}<1 \text{ if and only if } \delta \mathrm{b}\frac{2b}{(1-2b)(1-4b)} \text{ and } b<1/4, \textit{ or } b\geq 1/4 \text{ and } \delta\in[0,1]. \tag{6.47}$$

Finally when $\alpha=0$, Eq. (6.38) which defines $\hat{\alpha}$ as a function of α is $-(b-\hat{\alpha})(1+2\delta a(1-\hat{\alpha}))=0$. This equation has only one root $\hat{\alpha}=b$.

Now we have all the necessary ingredients to prove the proposition. We will do it for the case $\alpha<1/2$. The case $\alpha<1/2$ is totally symmetric.

Part a) If $b=0, d\hat{\alpha}/d\alpha>1$ so the curve relating α with $\hat{\alpha}$ cuts the 45° degree line from below. In this case Eq. (6.39) has three solutions in alpha, namely 0,1/2 and 1. Given the geometry of the problem, the avalanche effect occurs for all $\alpha\in[0,1]\backslash\{1/2\}$.

Part b) If $\delta>2b/(1-2b)(1-4b)$ and $b<1/4, d\hat{\alpha}/d\alpha>1$. Since when $\alpha<1/2$, but sufficiently close to 1/2, $\hat{\alpha}<\alpha$ and when $\alpha=0$, $\hat{\alpha}=b$, by continuity the function relating α to $\hat{\alpha}$ must cut the 45° line so the existence of $\bar{\alpha}$ is guaranteed. The symmetry of the function around 1/2 and the existence of at most three solutions to Eq. (6.40) imply that this intersection is unique in (0,1/2). Thus for all $\alpha\in(0,\bar{\alpha})$ there is a level-off effect and for $\alpha\in(\bar{\alpha},1/2)$ there is an avalanche effect.

Part c) If $b<1/4$ and $\delta>2b/(1-2b)(1-4b)$, or if $b\geq 1/4$ and $\delta\in[0,1]$, $d\hat{\alpha}/d\alpha<1$. Thus when α is less than 1/2, but sufficiently close to 1/2, $\hat{\alpha}>\alpha$ and when $\alpha=0$, $\hat{\alpha}=b$. The function relating α and $\hat{\alpha}$ does not fall below the 45°. There might be a point at which $\alpha=\hat{\alpha}$ but just one because if this function cuts twice the 45°, by symmetry, there would be 5 solutions to Eq. (6.40) which is impossible. Thus the level-off effect holds for all $\alpha\in[0,1]$ except, possibly, for two values of α. ■

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