A residual-based ADF test for stationary cointegration in $I(2)$ settings*

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Abstract

We present a residual-based ADF test that allows for detection of stationary cointegration within a system that may contain both $I(2)$ and $I(1)$ observables. The test can also detect situations of multicointegration, where first differences of the $I(2)$ observables enter the cointegrating relationships. We find the null limiting distribution of the test statistic and show that our procedure is more generally applicable than previous proposals. Critical values are computed for a variety of situations. Finally, a small Monte Carlo experiment is carried out and an empirical application is provided as an illustrative example.

JEL Classification: C12, C22, C32.

Keywords: $I(2)$ systems; stationary cointegration; multicointegration; residual-based tests.

1 Introduction

The concept of cointegration has received much attention in the last two decades. Its importance stems from the fact that cointegration provides the link between the economic concept of (long-run) equilibrium relationships and the statistical notions

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of nonstationarity and trending behavior: nonstationary variables may display relationships that are representative of long-run equilibria, in that deviations from the equilibrium are short-lasting. These ideas can be formalized using the concepts of integrated and cointegrated processes. Formally, we say that a scalar or vector process ζ_t, $t = 1, 2, ...$, is integrated of order zero ($I(0)$) if $\zeta_t - E(\zeta_t)$ is covariance stationary with nonzero and bounded spectral density at all frequencies. Alternatively, a scalar or vector ζ_t, $t = 1, 2, ...$, is integrated of order d ($I(d)$), $d = 1, 2, ...$, if the d-difference of $\zeta_t - E(\zeta_t)$, $t = d+1, d+2, ...$, is a zero-mean $I(0)$ process. These are processes which arise naturally from model (1) proposed below. For completeness, although overdifferenced processes will not be the focus of our analysis, we say that a scalar or vector ζ_t, $t = 1, 2, ...$, is $I(-d)$, $d = 1, 2, ...$, if $\zeta_t - E(\zeta_t)$ is d-differences of a zero-mean $I(0)$ process for $t = d+1, d+2, ...$ Note that if a vector ζ_t is $I(d)$, $d = 1, 2, ...$, our definition implies that at least one of the individual components of ζ_t must be $I(d)$. The rest of the components might also be $I(d)$ or, alternatively, they might have a smaller order of integration. In this sense, our definition is similar to that of Johansen (1995a). Note also that this definition does not preclude the existence of components of an $I(d)$ vector which are fractional processes ($I(c)$, c being a real number smaller than d), but the model proposed below will exclude this possibility. Next, we define cointegration for an $I(d)$, $d = 1, 2, ...$, process. Given a $p \times 1$ process $z_t \sim I(d)$, z_t is cointegrated if there exists a $p \times 1$ vector $\gamma \neq 0$ such that $\gamma' z_t \sim I(c)$, with $c < d$, prime denoting transposition. Again, this definition permits the existence of fractional linear combinations of the observables ($I(c)$, c being a real number smaller than d), but our model below excludes this possibility. Thus, the focus on the present paper will be on observables and cointegrating errors with integer (not fractional) orders of integration. Our definition of cointegration is similar to that of Johansen (1995a) and it is significantly more general than the standard notion of Engle and Granger (1987), where all observables are required to have identical integration orders. Note that according to our definition some of the cointegrating vectors might be unit vectors, just indicating that a particular observable has an integration order smaller than the order of the vector. As usual, the cointegrating rank among the elements of z_t is the number of linearly independent cointegrating vectors, and the space generated by these vectors will be denoted as cointegrating space.

Since the seminal contributions of Engle and Granger (1987) and Johansen (1995a), cointegration has been quite well studied both in uni-equation and system settings where the observables may behave like $I(1)$ or stationary variables. However, many observables (especially nominal variables such as price indexes) are smoother than what $I(1)$ behavior would suggest. For example, inflation rates have a behavior close to that of an $I(1)$ variable, so that (log)price indexes might be characterized as $I(2)$. Thus, structural models that involve aggregate prices could be combining variables with different integration orders (see Juselius, 1995, or Banerjee et al., 2001, for two different illustrations of such settings). A similar rationale applies to nominal GDP or, maybe, to nominal wealth, which is the result of the time-accumulation of nominal income. In general, models that involve both stock and flow variables may present a mixture of $I(2)$ and $I(1)$ variables (Granger and Lee, 1989, Lee 1992, Engsted et al., 1997). When dealing with $I(2)$ systems, where some (or all) of the observables are $I(2)$, the cointegrating structure of the data might be very rich even omitting the possibility of fractional processes. Indeed, $I(2)$ variables might cointegrate to $I(1)$ or to stationary relations ($I(c)$, with $c \leq 0$), and, in addition, these $I(1)$ relations might combine with $I(1)$ observables and/or first differences of the $I(2)$ observables to stationary relations. The situation where first differences of $I(2)$ observables combine with levels of the observables to achieve stationarity is popularly known as (polynomial) multicointegration.

As in the $I(1)$ case, two different approaches have been developed in order to examine cointegration in $I(2)$ systems. Johansen (1995b), Paruolo (1996), Nielsen and Rahbek (2007), among others, proposed cointegration tests within a vector autoregressive framework, which includes also the possibility of detecting multicointegration (see also, Gregoir and Laroque, 1994, Engsted and Johansen, 1999, Juselius, 1995, or Banerjee et al., 2001 for empirical applications). An alternative procedure is to rely on a regression-based analysis. This methodology extends the Phillips and Ouliaris (1990) residual based tests for cointegration to the $I(2)$ setting, and has been pursued in uni-equation settings by Haldrup (1994), and in the particular case of multicointegration, by Granger and Lee (1989, 1990), Lee (1992) and Engsted et al. (1997). The aim in these papers is to detect stationary cointegration within an $I(2)$ cointegrated vector of observables. In general, cointegration in $I(2)$ systems can be assessed by residual-based methods (see Section 3 below), but the standard approach is not informative about the departures from the null of no cointegration, that is, whether the cointegrating errors are $I(1)$ or stationary. Hence the interest of a test which might discern between these two possibilities, especially because stationary linear combinations are usually those with empirical relevance. Note also that if $I(1)$ observables are part of an $I(2)$ system, there is necessarily cointegration, so in many interesting applications (see Section 3 below), an $I(2)$ vector of observables z_t can be assumed to be cointegrated and the relevant question is whether there are stationary relations.

In the present paper we focus on regression-based methods. Obviously, as in the $I(1)$ setting, a residual-based test for cointegration offers a more limited description of the cointegrating structure of the system than a likelihood-based system approach. However, residual-based methods can be useful in at least two relevant contexts. First, there are situations where the solution of an economic model has one main equation of interest, so for empirical purposes uniequation methods might suffice. Alternatively, Gomez-Biscarri and Hualde (2010) (GBH hereafter) showed that the residual-based Augmented Dickey-Fuller (ADF) test of Phillips and Ouliaris (1990) can be used as the main tool to infer the whole cointegrating structure in $I(1)$ systems. In the same vein, a residual-based test might serve as the main tool to unveil a corresponding structure in $I(2)$ settings. In Section 3 below, we briefly elaborate on the precise contexts in which our proposal will be useful.

We propose a test which relates directly to that of Haldrup (1994). This author developed a residual-based ADF test for the null of $I(1)$ versus the alternative of stationary cointegration among a set of $I(1)$ and $I(2)$ observables. In particular, in Haldrup's model, the $I(2)$ observables cointegrate (with rank one) to an $I(1)$ cointegrating error, which under the null does not further cointegrate with the $I(1)$ observables. The test is carried out by regressing an $I(2)$ observable on the $I(1)$ observables and the rest of $I(2)$ series (which are assumed to be non-cointegrated). In view of the results of Haldrup's (1994) Theorem 4, the null limiting distribution of the test just depends on the number of $I(1)$ and $I(2)$ regressors. We find that there are two empirically relevant limitations of this test. First, and more importantly, the result appears to be valid only in the case where the coherence at frequency zero between the $I(0)$ error input processes generating the $I(1)$ and $I(2)$ components of the system, respectively, is zero. This is a very stringent requirement, which is not in general satisfied if, e.g., this $I(0)$ is a vector autoregressive and moving average process. Therefore, in general, the null limiting distribution of Haldrup's ADF test statistic is not free of nuisance parameters. Second, the test assumes that the $I(2)$ variables cointegrate with rank exactly equal to one, which in systems with several $I(2)$ observables might not be the case.

Our aim is to develop a generally applicable residual-based test for stationary cointegration in $I(2)$ settings which does not suffer from the drawbacks of Haldrup's proposal. The main novelty of our approach is that allowing for nonzero coherence (at frequency zero) requires implementing a correction in the cointegrating regression. Nicely, this correction is intimately related to the issue of multicointegration. In

Haldrup's setting multicointegration is not allowed, but Engsted et al. (1997) applied Haldrup's results to the multicointegration case in a simple bivariate setting, and suggested that Haldrup's (1994) critical values might be used. We, however, believe that this is not the case, given that one $I(2)$ observable appears as regressor both in levels and first differences, a circumstance which must affect the limiting distribution of the test statistic and it is not captured by Haldrup's (1994) setting.

Our proposed correction can be viewed as a way of obtaining a nuisance parameter free null limiting distribution of the ADF test statistic. However, given the nature of such correction, our test is also consistent to the alternative of multicointegration, covering therefore the case of Engsted et al. (1997) and providing a unified treatment of stationary cointegration in $I(2)$ settings. In addition, we show that the distribution of the test depends on the number of $I(2)$ common trends present in the system (or, alternatively, on the cointegrating rank of the $I(2)$ vector), and tabulate corresponding critical values in various scenarios. Finally, we also justify that both the required correction and a correct specification of the cointegrating regression may follow from data-based information.

The outline of the rest of the paper is as follows. Section 2 presents the model, the residual-based ADF test statistic and develops its null asymptotic distribution. Section 3 comments on some issues regarding the empirical implementation of the test, placing special emphasis on describing the contexts in which the test might be a useful tool. Section 4 presents the results of a small Monte Carlo experiment analyzing the power of the test, and an illustrative empirical example is discussed in Section 5. Section 6 concludes. Proofs are provided in the Appendix.

2 The ADF test: the model, assumptions and properties

Our purpose is to present an ADF statistic to test the null hypothesis of no stationary cointegration in a p-dimensional cointegrated $I(2)$ vector of observables z_t, which is composed of $I(2)$ and possibly also of $I(1)$ individual series. We assume that the cointegrating rank of z_t is r, where $0 < r < p$. Under the null, z_t is assumed to be generated by the model

$$\begin{pmatrix} I_r & B \\ 0 & I_{p-r} \end{pmatrix} (z_t - E(z_t)) = \begin{pmatrix} \Delta^{-1} I_r & 0 \\ 0 & \Delta^{-2} I_{p-r} \end{pmatrix} \{\zeta_t 1(t > 0)\}, \quad t = 1, 2, 3, ..., \tag{1}$$

where $\Delta = 1 - L$, L is the lag operator, I_p is the p-rowed identity matrix, ζ_t is a zero-mean $I(0)$ vector process whose spectral density at all frequencies is finite and nonsingular, B is an $r \times (p - r)$ matrix and $1(\cdot)$ is the indicator function. The truncation on the right side of (1) ensures that z_t is well defined in mean square sense. The presence of deterministic components might be allowed by nonzero $E(z_t)$, although for simplicity we will consider that $E(z_t) = 0$. Partition $z_t = \left(z'_{(1)t}, z'_{(2)t}\right)'$, where $z_{(1)t}$ collects the first r components of z_t (and $z_{(2)t}$ the rest). Model (1) captures a variety of situations where the cointegrating rank of z_t is r. If none of the rows of B is identically zero, all individual observables in z_t are $I(2)$. Alternatively, if $B = 0$, the r individual components in $z_{(1)t}$ are $I(1)$, and the $I(2)$ components $(z_{(2)t})$ do not cointegrate. In this case, the r cointegrating relations are trivial. The situation where there are some $I(1)$ components and the $I(2)$ individual components cointegrate, is also covered by (1), in the case where some (but not all) of the rows of B are identically zero. Note also that there is no loss of generality in the representation (1). If z_t cointegrates with rank r, a trivial extension of Theorem 2 of GBH ensures the existence of a $(p - r)$-dimensional subvector of z_t $(z_{(2)t})$ whose individual components are $I(2)$ and do not cointegrate. These variables represent the common trends of the system. Also, collecting the rest of the observables in $z_{(1)t}$, by the same theorem, there exists an $r \times (p - r)$ matrix B such that $z_{(1)t} + Bz_{(2)t} \sim I(1)$ under the null. Of course, in practice one usually does not know which variables are in $z_{(1)t}$ or $z_{(2)t}$, but we outline in Section 3 below a procedure that allows for inference of both r and $z_{(2)t}$ from the data.

Haldrup (1994) considers the case where there are $r - 1$ $I(1)$ observables, and the $p - r + 1$ $I(2)$ observables cointegrate with rank one, so the cointegrating rank in z_t is r. Thus, in his setting, $z_{(1)t}$ is composed of the $r - 1$ $I(1)$ and one of the $I(2)$ variables (the one which is not part of the common trends). His ADF test is based on the (ordinary least squares) regression of the $I(2)$ variable in $z_{(1)t}$ on the rest of the observables. The null limiting distribution of this statistic is dependent on a vector of both nonintegrated and integrated Brownian motions, the former arising from the $I(1)$ observables and the single cointegrating relation among the $I(2)$ observables, the latter arising from the $I(2)$ common trends $(z_{(2)t})$. Unless these two types of Brownian motions are mutually independent (due for example to a zero coherence at frequency zero between the $I(0)$ error input processes generating the $I(1)$ and $I(2)$ components, respectively), the typical decomposition (see e.g. the proof of Lemma 2 in Haldrup, 1994) leading to standard (and mutually independent) nonintegrated and integrated Brownian motions is not valid. Therefore, in general, the limiting distribution of Haldrup's statistic is not free of nuisance parameters.

Fortunately, a simple correction can be carried out in the regression, so a proper orthogonalization can be achieved in general circumstances. This correction leads to our proposed test statistic, which is based on residuals arising from the regression of z_{1t} on $z_{-1,t}$ and $\Delta z_{(2)t}$, where z_{1t} is the first component of z_t (which obviously coincides with the first component of $z_{(1)t}$) and $z_{-1,t}$ collects the rest of elements of z_t. The inclusion of the additional regressors $\Delta z_{(2)t}$ (first differences of the $I(2)$ common trends) implies that the asymptotic distribution of the statistic is characterized by a vector of nonintegrated Brownian motions (due to the $I(1)$ components and the first differences of the $I(2)$ common trends) and integrated Brownian motions (due to the $I(2)$ common trends). We show in the Appendix (see Proof of Theorem 1) how a proper orthogonalization can be achieved in this case, hence leading to a nuisance parameter free null limiting distribution.

We should comment on several crucial issues here. First, as described in Section 3 below, r and $z_{(2)t}$ can be inferred from data by applying a simple extension of the procedure proposed by GBH. Second, the null limiting distribution of our proposed test statistic is invariant to the choice of left hand side variable on the regression from which the residuals $\widehat{u}_t = (1, -\widehat{\beta}')(z_t', \Delta z_{(2)t}')'$ (where $\widehat{\beta}$ is the ordinary least squares estimator in this regression), are derived, as long as this choice is taken from $z_{(1)t}$. However, as in any residual-based test for cointegration, the choice of left hand side variable in the regression is important for power considerations. Finally, the null limiting distribution of our statistic is invariant to B. The reason is that defining

$$T = \begin{pmatrix} I_r & B & 0 \\ 0 & 0 & I_{p-r} \\ 0 & I_{p-r} & 0 \end{pmatrix},$$

then

$$\widehat{u}_t = (1, -\widehat{\beta}')T^{-1}T(z_t', \Delta z_{(2)t}')' = (1, -\widehat{\theta}')v_t, \tag{2}$$

where $v_t = \left(\left(z_{(1)t} + Bz_{(2)t}\right)', \Delta z_{(2)t}', z_{(2)t}'\right)'$, and $\widehat{\theta}$ is the ordinary least squares estimator of v_{1t} on $v_{-1,t}$ (where v_{1t} is the first component of v_t, and $v_{-1,t}$ collects the rest of elements of v_t). Noting (1), v_t is just a simple transformation of ζ_t which does not depend on B.

Before presenting our main result, we introduce some assumptions which are similar to those in Chang and Park (2002). When applied to matrices, denote by $\|\cdot\|$ the norm $\|A\| = \sup_{\|x\|\leq 1} \|Ax\|$, whereas $\|\cdot\|$ applied to vectors is the usual Euclidean norm. Notice that if a_{ij} denotes the (i,j)-th element of a $p \times p$ matrix A, $\|A\|^2 \leq \Sigma_{i=1}^{p}\Sigma_{j=1}^{p}a_{ij}^2$.

Assumption 1. The process ζ_t in (1) has representation

$$\zeta_t = A(L)\,\varepsilon_t, \text{ where } A(u) = I_p + \sum_{j=1}^{\infty} A_j u^j,$$

and the A_j are $p \times p$ matrices such that:

(i) $\det(A(u)) \neq 0, \quad |u| = 1;$

(ii) $A\left(e^{i\lambda}\right)$ is differentiable in λ with derivative in $Lip(\eta)$, $\eta > 1/2$;

(iii) $(\varepsilon_t, \mathcal{F}_t)$ is a martingale difference sequence with some filtration $(\mathcal{F}_t)$ such that $E(\varepsilon_t) = 0$, $E(\varepsilon_t\varepsilon_t') = \Sigma$, Σ is positive definite, $n^{-1}\Sigma_{t=1}^{n}\varepsilon_t\varepsilon_t' \rightarrow_p \Sigma$, $E\|\varepsilon_t\|^u < K$ with $u \geq 4$, where K is some constant that depends only upon u.

Assumption 1 implies that ζ_t is a fairly general linear process with martingale difference innovations. Notice that (ii) implies the summation condition $\Sigma_{j=1}^{\infty} j\,\|A_j\| < \infty$, so (ii) and (iii) imply that ζ_t is weakly stationary, whereas (iii) holds under suitable mixing conditions. In addition, Assumption 1 enables us to apply the multivariate invariance principle

$$\frac{1}{\sqrt{n}}\sum_{t=1}^{[ns]}\zeta_t \Rightarrow B(s)\,, \tag{3}$$

where $[\cdot]$ denotes integer part and $B(s)$ is a p-vector Brownian motion with covariance matrix $\Omega = A(1)\,\Sigma A(1)'$.

Given the previously defined residuals $\widehat{u}_t$, the standard ADF test statistic is the t-ratio corresponding to the coefficient of $\widehat{u}_{t-1}$ in the regression of $\Delta\widehat{u}_t$ on $\widehat{u}_{t-1}$, $\Delta\widehat{u}_{t-1}, ...\Delta\widehat{u}_{t-q}$. We will denote this t-ratio by t_n, and give its null limiting distribution in Theorem 1 below. This theorem covers the case where no constant is included in the cointegrating regression, but results for alternative specifications including constant and/or deterministic trends might be easily derived by minor modifications of the proof of this theorem.

As is well known (see, e.g., Phillips and Ouliaris, 1990), it is necessary in general to let q increase with n, for which we impose the following condition.

Assumption 2. Let $q \rightarrow \infty$ and $q = o\left(n^{1/3}\right)$ as $n \rightarrow \infty$.

This condition guarantees the consistency of the estimators of autoregressive parameters in a particular autoregressive approximation (see, e.g., Berk, 1974, Chang and

Park, 2002), which is a required step when calculating the null limiting distribution of our test statistic.

Before presenting the main result we introduce some additional notation. For a vector process $G(s)$, $G_1(s)$ denotes its first component and $G_{-1}(s)$ the subvector resulting from omitting this first component. Also, given an arbitrary Brownian motion $G(s)$, define the integrated Brownian motion $\overline{G}(s) = \int_0^s G(l)\,dl$.

Let $W(s)$ be a p-dimensional standard Brownian motion, let $W_{(2)}(s)$ be the subvector made of the last $p-r$ components of $W(s)$ and let $V(s) = (W'(s), \overline{W}'_{(2)}(s))'$. Finally, let $Q(s) = \kappa' V(s)$, where

$$\kappa = \left(1, -\int_0^1 V_1(s)\,V'_{-1}(s)\,ds \left(\int_0^1 V_{-1}(s)\,V'_{-1}(s)\,ds\right)^{-1}\right)'$$

Theorem 1. Let z_t be generated by (1) and Assumptions 1 and 2 hold. Then, as $n \to \infty$, $q \to \infty$,

$$t_n \Rightarrow \Xi(p,r) \equiv \frac{\int_0^1 Q(s)\,dQ(s)}{\left(\int_0^1 Q^2(s)\,ds\right)^{\frac{1}{2}} \left(\kappa' \begin{pmatrix} I_p & 0 \\ 0 & 0 \end{pmatrix} \kappa\right)^{\frac{1}{2}}}. \tag{4}$$

The proof is provided in the Appendix. The distribution of the ADF test is free of nuisance parameters, but it depends on p and r. This test is consistent under the alternative of stationary cointegration, including also any type of multicointegration (see Proposition 1 below). Note that this possibility is not contemplated by Haldrup's (1994) setting. We present in Table 1 the simulated quantiles of the distributions of this residual-based ADF test for series of length $n = \{50, 100, 250, 500, 50,000\}$, 200,000 replications and different (p,r) combinations. In particular, we generated the vector of observables z_t for cases $p = 2, ..., 5$, $r = 1, ..., p-1$, choosing ζ_t to be a p-dimensional zero mean normal innovation with covariance matrix I_p and independent over time. We computed the ADF statistic from the auxiliary regression

$$\Delta\widehat{u}_t = \delta\widehat{u}_{t-1} + \varphi_t,$$

where a constant was included in the cointegrating regression. The estimated distributions of the ADF statistic for alternative specifications of ζ_t and the number of lags in the ADF regression have also been tabulated, and are available from the authors. The small-sample quantiles differ from those in Table 1, but the approximate asymptotic ones ($n = 50,000$) are the same, as Theorem 1 implies. Critical values for cointegrating regressions with a linear and possibly also quadratic trends are also available upon request.

3 Issues regarding the implementation of the test

From a practical point of view, in order to apply our test, the researcher must know both the rank r and the set of $I(2)$ common trends in the system. Given these requirements and the fact that likelihood based procedures for analysis of $I(2)$ systems have been developed (Johansen, 1995b, Paruolo, 1996, and, more recently, Nielsen and Rahbek, 2007), it is warranted that we motivate the usefulness of our proposal. As mentioned in the Introduction, there are at least two contexts in which the current test can be of direct interest. Of course, these contexts are parallel to those in which the regression-based tests of Phillips and Ouliaris (1990) are of interest in $I(1)$ systems.

First, many economic models lead to equilibrium equations which might contain both $I(1)$ or $I(2)$ variables. In particular, some models deliver one single equilibrium condition or several, but one of them is of special interest to the researcher. Examples of these are the analyses of money demand equations (which involve $I(2)$ variables such as nominal money and price indices, and variables with $I(1)$ behavior, such as interest rates or real output: see, e.g., Stock and Watson, 1993, Haldrup, 1994, Bae and DeJong, 2007), purchasing power parity (PPP) models of the exchange rate (which postulate a relationship between domestic and foreign price indices, both $I(2)$, and the exchange rate, typically $I(1)$; see, e.g., Rogoff, 1996, Caner and Kilian, 2001, Pedroni, 2004), or structural models of the exchange rate (which also lead to an expression of the exchange rate as a function of the differentials between domestic and foreign variables: some of these "exchange rate fundamentals" are $I(2)$, such as money or prices, and some are $I(1)$, such as real output or interest rates; see, e.g., Mark and Sul, 2001, Rapach and Wohar, 2002, Rossi, 2006). The empirical researcher may be interested in testing these equilibrium relationships, without necessarily attempting to give a full description of the cointegrating structure of the complete system. Our proposed test is, then, a straightforward way to carry out this analysis. Of course, knowledge of the cointegrating rank among the $I(2)$ components is needed, but usually familiarity with the variables involved and, sometimes, economic theory, provides with this information. For example, in the simplest models of PPP there are only three variables involved, namely the (log)exchange rate between two currencies and the two (log)price indices in the foreign and domestic countries. Log-price indices can be taken to be $I(2)$ (or a simple test for the order of integration would lead to this conclusion), and (log)exchange rates are typically $I(1)$. A simple bivariate test for cointegration among the (log)price indices usually shows that these cointegrate with cointegration vector statistically indistinguishable from (1,-1). This leads to a system with $p = 3$ variables and $r = 2$ and, therefore, if one wants to test for PPP (a stationary relationship between the three variables), this could be carried out by using our proposed test, with one of the price indices or the exchange rate as the left-hand side variable. The other price index would be taken as the common trend and, therefore, it would be included both in levels and first differences in the right-hand side of the cointegrating regression. Incidentally, our theoretical results suggest that some of the tests in the vast empirical literature devoted to PPP may not have been properly designed, since first differences of the common trend were not included as regressors, and the critical values employed were typically those of Phillips and Ouliaris (1990).

Second, our proposed test can be a key tool in order to unveil the whole cointegrating structure of an $I(2)$ system. In an $I(1)$ setting, GBH propose a sequential procedure based on the regression-based ADF tests of Phillips and Ouliaris (1990) which leads to an estimator of the cointegrating rank (r) and to an identification of the common trends. The intuition behind this method is the following. First, if all pairs of observables are cointegrated, then necessarily $r = p - 1$. If not, there is at least a pair of non-cointegrated observables (common trends), and the next step is to test whether trios containing this pair are cointegrated. If they are, $r = p - 2$, while if they are not, we proceed to the next step. The procedure is finalized when all corresponding groups of observables are cointegrated, or, alternatively, when in the last possible step, cointegration among all observables is checked. This tests are carried out by residual-based ADF, and one of the most appealing features of this procedure is that in every step the choice of left-hand side variables in the cointegrating regressions is automatic.

The GBH method can be equally applied to infer the cointegrating rank in $I(2)$ systems. There is an important difference, though, because under the null of no cointegration, the residuals of the different cointegrating regressions are linear combinations of non-cointegrated $I(2)$ variables. Hence, the critical values of Phillips-Ouliaris are not applicable. More importantly, the test based on these residuals is not consistent under the alternative of $I(1)$ cointegration. However, performing the standard ADF test on the first differences of these residuals sorts out this latter problem. It is necessary to modify slightly the proof arguments of Phillips and Ouliaris (1990) in order to find the appropriate null limiting distribution of this ADF statistic (which differs from that in Phillips and Ouliaris, 1990). Thus, this modified GBH procedure leads to an estimator of the rank r and, as a by-product, to the identification of the $p-r$ common trends.

However, in $I(2)$ settings, this might not capture the whole cointegrating structure of the data, which can be also characterized by a possible cointegrating subspace, where particular directions of the cointegrating space lead to stationary linear combinations of the observables (and, possible, also of the first differences of these observables). The ADF tests on differenced residuals are not informative about the $I(1)$ or stationary nature of the cointegrating relationships, so they are not a proper tool in order to infer the dimension of this subspace. Our test, which is specifically designed to distinguish between $I(1)$ and stationary cointegration, becomes the appropriate tool for this second stage of the analysis. Nevertheless, a fully detailed explanation of the precise use of our test within this procedure goes beyond the scope of the present paper.

As mentioned before, the choice of left hand side variable is a critical issue. This problem affects any test for cointegration based on regression methods even in the standard $I(1)$ setting. Specifically, the tests proposed by Phillips and Ouliaris (1990) do not have power if the left-hand side variable does not enter the stationary relation with nonzero coefficient. We postulated this left-hand side variable to be one of the observables in $z_{(1)t}$, noting that if there is stationary cointegration, at least one of the variables in $z_{(1)t}$ must have necessarily a nonzero coefficient in the stationary linear combination of observables (and possibly also first differences of $I(2)$). Note, however that this test only has power with respect to stationary relations in which the chosen left-hand side variable appears with a nonzero coefficient. Again, in our setting, this is not of overriding concern. In the analysis of single equations, there will be typically theoretical reasons which imply that a particular variable must enter the stationary cointegrating relation. Alternatively, if the test is being used with the aim of inferring the rank of a possible cointegrating subspace, a properly designed sequential procedure will select automatically the variable to place in the left-hand side of the equations in every step (as it is the case in the GBH procedure applied to $I(1)$ systems).

A final point of concern is the following: given that we just include first differences of $z_{(2)t}$ in the cointegrating regression, we might wonder whether other multicointegrating relations (apart from those evaluated by the test) are possible. Fortunately, Proposition 1 below rules out the existence of these alternative relations: if there are multicointegrating relations, these must arise from combinations between z_t and $\Delta z_{(2)t}$.

Proposition 1. Let z_t be a p-dimensional cointegrated $I(2)$ vector, with cointegrating rank r, where $0 < r < p$. Define two subspaces R, T of the cointegrating space (C) in the following way:

i. $\theta \in R \subseteq C$ if there exists a p-dimensional vector $\lambda(\theta)$ such that $\theta' z_t + \lambda'(\theta) \Delta z_t \sim I(c)$, $c \leq 0$;

ii. $\phi \in T \subseteq C$ if there exists a $(p-r)$-dimensional vector $\rho(\phi)$ such that $\phi' z_t + \rho'(\theta) \Delta \overline{z}_t \sim I(c)$, $c \leq 0$, where $\overline{z}_t$ is a $(p-r)$-dimensional subvector of z_t with $I(2)$ and not cointegrated individual components.

Then, $\phi \in R$ if and only if $\phi \in T$.

The proof of Proposition 1 is in the Appendix. Incidentally, this result is parallel to that in Johansen (1995b), where multicointegration is tested with first differences of the common trends, which, in his setting, are particular linear combinations of the $I(2)$ observables. In our case, however, we identify these $p-r$ common trends by the $p-r$ dimensional vector of observables $z_{(2)t}$.

4 Monte Carlo evidence

We investigate the finite sample power of our proposed test by means of a simple Monte Carlo experiment. We fix $p = 3$, and in all cases the analysis is based on 10,000 replications of series of lengths $n = 50{,}100{,}250$. We generated ε_t as a Gaussian white noise with $E(\varepsilon_t) = 0$, $Var(\varepsilon_t) = I_3$, and examine three different DGPs for the innovation vector ζ_t: $A(L) = I_3$ (WN), $A(L) = (1 - 0.8L)^{-1} I_3$ (AR), $A(L) = (1 - 0.5L) I_3$ (MA). We generate processes under the alternative using the six possible stationary cointegrating structures in a 3-variable $I(2)$ vector. The simulated models are:

$$\Upsilon z_t = \Delta(L) \{\zeta_t 1(t > 0)\},$$

where

$$\Upsilon = \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix}, \text{Models 1, 2;} = \begin{pmatrix} 1 & -1 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix}, \text{Models 3, 4;}$$
$$= \begin{pmatrix} 1 & -1 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \text{Models 5, 6,}$$

and

$$\Delta(L) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \Delta^{-2} \end{pmatrix}, \text{Model 1;} = \begin{pmatrix} 1 & 0 & \Delta^{-1} \\ 0 & 1 & 0 \\ 0 & 0 & \Delta^{-2} \end{pmatrix}, \text{Model 2;}$$
$$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & \Delta^{-1} & 0 \\ 0 & 0 & \Delta^{-2} \end{pmatrix}, \text{Model 3;} = \begin{pmatrix} 1 & 0 & \Delta^{-1} \\ 0 & \Delta^{-1} & 0 \\ 0 & 0 & \Delta^{-2} \end{pmatrix}, \text{Model 4;}$$
$$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & \Delta^{-2} & 0 \\ 0 & 0 & \Delta^{-2} \end{pmatrix}, \text{Model 5;} = \begin{pmatrix} 1 & \Delta^{-1} & \Delta^{-1} \\ 0 & \Delta^{-2} & 0 \\ 0 & 0 & \Delta^{-2} \end{pmatrix}, \text{Model 6.}$$

In the case of Models 1-4, the cointegrating rank of the $I(2)$ system is $r = 2$, and in models 5, 6, $r = 1$. Also, in Models 1, 2, all directions in the cointegrating space lead to stationarity, whereas in Models 3, 4, there is just one direction leading to stationarity. Multicointegration is present in models 2, 4 and 6. Critical values for tests of nominal size $\alpha = 0.1, 0.05, 0.01$ are taken from Table 1 (cases $p = 3$, $r = 2$ for Models 1-4 and $p = 3$, $r = 1$ for models 5, 6). Our proposal requires the test to be based on the residual from the following cointegrating regressions

$$\begin{aligned} \text{Models 1-4} &: \quad z_{1t} = \alpha_0 + \alpha_1 z_{2t} + \alpha_2 z_{3t} + \alpha_3 \Delta z_{3t} + u_t, \\ \text{Models 5, 6} &: \quad z_{1t} = \alpha_0 + \alpha_1 z_{2t} + \alpha_2 z_{3t} + \alpha_3 \Delta z_{2t} + \alpha_4 \Delta z_{3t} + u_t. \end{aligned} \tag{5}$$

We record the proportions of rejection of the null hypothesis of no stationary cointegration and show them in Table 2. Overall, the performance seems quite satisfactory and rejection proportions behave very similarly for all six models considered. For relatively small sample sizes ($n = 100$), the test rejects the null hypothesis almost with certainty under the white noise and moving average specifications. In the case of the autoregressive process, more observations are needed to achieve high rejection rates, but for a reasonable sample size ($n = 250$) the test already rejects with frequencies close to one.

5 An empirical application: markups and inflation

Banerjee et al. (2001) (BCR, hereafter) analyze a model of the markup of prices for a closed economy. Their main interest is to show that there is a long run negative relationship between the markup of prices over cost and inflation. This implies that the real wage may respond positively to inflation. As a consequence, real activity (and unemployment) would be related in the long run to inflation, thus making the long run Phillips curve not vertical, and, for example, firm's profitability (and stock returns) would be negatively correlated with inflation. In order to justify the empirical analysis, BCR setup a model which delivers a solution for the long-run markup of the form

$$mu \equiv p - \delta ulc - (1-\delta)pm = \omega_0 + \omega_1 x - \omega_2 \Delta p, \tag{6}$$

where δ, ω_0, ω_1, ω_2, are parameters, mu denotes the markup, p, ulc, and pm are prices, unit labor costs and import prices, respectively, and x captures shifts in the bargaining position of labour and firms. In particular, x includes variables that characterize the firm's competitive environment. The relationship (6) expresses a long-run equilibrium among the variables involved. Under certain assumptions, BCR simplify the equation above by assuming that the competitive environment of the firm (variables in x) is constant. Thus, they express the long-run markup as a function of the inflation rate exclusively. The long-run markup equation (6) is then estimated using quarterly Australian data that run from 1970:1 to 1995:2. In these data, the core variables, p_t, ulc_t and pm_t, are defined on a national accounts basis as the private consumption deflator, the Australian Treasury's measure of non-farm unit labour costs and the imports implicit price deflator respectively.

BCR suggest that the three core variables are $I(2)$, so they consider scenarios where the core variables may cointegrate to $I(1)$ or to stationarity or present multicointegration, as implied by the presence of Δp in (6). Thus, the setup of their long-run analysis is an immediate testing ground for the test proposed in the present paper. As mentioned before, BCR assume that the variables in x are all stationary, i.e., they are only present as determinants of short-run deviations from the long-run markup. These variables include the unemployment rate, a measure of tax rates, oil prices and a measure of the number of labor strikes. In fact, there is evidence that the first three of these variables are $I(1)$, and therefore BCR include them in first differences in the analysis. However, there seems to be no theoretical reason to omit the variables in x from the analysis of cointegration, which could in principle allow for a long run relation that involves the six nonstationary variables in the dataset.

We first replicate the BCR analysis by testing for an stationary relationship among the three core variables and, possibly, the inflation rate (Δp_t). BCR characterize p_t, ulc_t and pm_t as $I(2)$ variables that cointegrate with two cointegrating vectors, so $r = 2$. This can be checked by analyzing the relationships in levels and showing that the residuals of the cointegrating regressions of pm_t and p_t and of ulc_t and p_t are of order smaller than $I(2)$. Then, under the hypothesis that there is not a stationary cointegrating relationship, the test for a stationary markup would be performed by running the following regression (where we allow for an intercept, but not a trend in the cointegrating vector)

$$ulc_t = \alpha_0 + \alpha_1 p_t + \alpha_2 pm_t + \alpha_3 \Delta p_t + u_t.$$

From our previous analysis, the necessary correction in the cointegrating regression is the inclusion of the first difference of just one $I(2)$ variable (given that $p = 3$ and $r = 2$), so the above equation, which includes Δp_t and that corresponds exactly to BCR's markup equation, is statistically well specified. Choosing one lag in the ADF regression (based on SIC), the ADF-test yields a value of -2.98, which should be compared with critical values (from Table 1) corresponding to $p = 3$ and $r = 2$, which are -3.83 (10%), -4.35 (5%) and -4.66 (1%). Thus, there is no strong evidence against the null hypothesis of no stationary cointegration.

If we were willing to consider that the competitive environment may not be constant in the long-run, then some of the variables in x may enter the equilibrium relationship. We include oil prices (pet_t), the unemployment rate (ue_t) and a tax rate (tax_t) in the cointegration analysis. If any of these three variables enters a stationary cointegrating relationship, the resulting cointegrating error may be interpreted as the markup net of persistent shocks and inflation or, in BCR's terminology, as the markup that includes the possibility of changes in the competitive environment of the firm. Since there is evidence that these three additional variables are $I(1)$, the system now has $p = 6$ variables and cointegrating rank $r = 5$. Hence, only the first difference of one of the $I(2)$ variables (Δp_t) must be included in the cointegrating regression on which the test is based. This is:

$$ulc_t = \alpha_0 + \alpha_1 p_t + \alpha_2 pm_t + \alpha_3 \Delta p_t + \alpha_4 pet_t + \alpha_5 ue_t + \alpha_6 tax_t + u_t,$$

where we note that ulc_t is one of the variables which constitutes the markup and, therefore, it should necessarily enter the possible cointegrating relation with nonzero coefficient. An ADF test carried out on the residuals (with zero lags as selected by SIC) yields the value -6.09, which should be compared with critical values corresponding to $p = 6$ and $r = 5$ (available from the authors upon request). The critical values are -4.70 (10%), -5.23 (5%) and -5.52 (1%). Thus, the null hypothesis of no-stationary cointegration can be strongly rejected at the 1% level, suggesting that the markup itself is persistent, even accounting for the effect of inflation, but that a markup net of shocks to the competitive environment is indeed stationary: firms set their markup conditional on their competitive environment, and, when this changes, firms adapt their markup behavior. Estimated coefficients in the above regression are given by

$$ulc_t = 2.36 + 0.94 p_t + 0.03 pm_t + 2.99 \Delta p_t - 0.03 pet_t + 0.10 ue_t - 0.23 tax_t + \widehat{u}_t,$$

signs and magnitudes being consistent with those expected by theory.

6 Conclusions

Our main interest in the paper was the analysis of long run relationships that involve $I(2)$ and, possibly, $I(1)$ observables. The objective was to detect linear combinations of these observables that led to stationary cointegrating errors. Cointegrating regressions that combine the $I(2)$ and $I(1)$ observables can be used to test for this possibility, but care has to be exercised to make sure that these regressions are well specified. We show that an adjustment must be made in the cointegrating regression, which consists of including as additional regressors the first differences of a non-cointegrated set of $I(2)$ observables that characterize the common trends in the system. Detection of this set of non-cointegrated $I(2)$ observables must be based either on familiarity with the variables involved in a specific equilibrium relationship or on a previous step, where both the rank and the common trends in an $I(2)$ system are identified.

Once the adjustment has been done, traditional ADF tests can be applied to the residuals of the cointegrating regression in order to test the null hypothesis of no stationary cointegration. We have derived the asymptotic distribution of this ADF test and show that it depends on the number of observables p and on the number of $I(2)$ common trends $(p-r)$ (or, alternatively, on the cointegrating rank of the system, r). We have tabulated the critical values of this distribution for a number of cases, given evidence of the finite sample power of the test and illustrated the use of the test by means of an empirical analysis of markups and inflation.

Appendix

Proof of Theorem 1. Using similar notation to that of Phillips and Ouliaris (1990), the ADF statistic is

$$t_n = \frac{\widehat{U}'_{-1} Q_{X_q} \Delta \widehat{U}}{\left(\widehat{U}'_{-1} Q_{X_q} \widehat{U}_{-1}\right)^{\frac{1}{2}} \widehat{\sigma}}, \tag{7}$$

where $Q_{X_q} = I_{n-q-1} - X_q \left(X_q' X_q\right)^{-1} X_q'$, $X_q = (x_{q,q+2}, ..., x_{q,n})'$, $x_{q,t} = (\Delta\widehat{u}_{t-1}, ..., \Delta\widehat{u}_{t-q})'$, $\widehat{U}_{-1} = (\widehat{u}_{q+1}, ..., \widehat{u}_{n-1})'$, $\Delta\widehat{U} = (\Delta\widehat{u}_{q+2}, ..., \Delta\widehat{u}_n)'$,

$$\widehat{\sigma}^2 = \frac{1}{n-q-1} \sum_t \left(\Delta\widehat{u}_t - \widehat{\alpha}_0 \widehat{u}_{t-1} - \sum_{j=1}^{q} \widehat{\alpha}_j \Delta\widehat{u}_{t-j}\right)^2,$$

where $\Sigma_t = \Sigma_{t=q+2}^n$ and $\widehat{\alpha}_j$, $j = 0, ..., q$, are the ordinary least squares coefficients in the regression of $\Delta\widehat{u}_t$ on $\widehat{u}_{t-1}$, $\Delta\widehat{u}_{t-1}, ..., \Delta\widehat{u}_{t-q}$. First, noting that $\widehat{u}_t = \left(1, -\widehat{\theta}'\right) v_t$, define

$$\widehat{\eta} = \begin{pmatrix} I_p & 0 \\ 0 & nI_{p-r} \end{pmatrix} \begin{pmatrix} 1 \\ -\widehat{\theta} \end{pmatrix}.$$

By (3) and the continuous mapping theorem

$$\widehat{\eta} \Rightarrow \eta \equiv \left(1, -\int_0^1 X_1(s) X'_{-1}(s)\, ds \left(\int_0^1 X_{-1}(s) X'_{-1}(s)\, ds\right)^{-1}\right)', \tag{8}$$

where $X(s) = \left(B'(s), \overline{B}'_{(2)}(s)\right)'$, $B_{(2)}(s)$ being the subvector made of the last $p-r$ components of $B(s)$. We will stress the dependence of the ADF statistic on $\widehat{\eta}$ by defining $t_n(\widehat{\eta}) \equiv t_n$. Theorem 1 follows on showing that, as $n \to \infty$, $q \to \infty$,

$$\begin{aligned} t_n(\widehat{\eta}) - t_n(\eta) &= o_p(1), & (9) \\ t_n(\eta) &\Rightarrow \Xi(p, r), & (10) \end{aligned}$$

where $t_n(\eta)$ is as $t_n(\widehat{\eta})$, just replacing $\widehat{\eta}$ by η. We show (10) first. The proof will be based on the following result. Under our assumptions, $n^{-1/2}\sum_{t=1}^{[nr]}\zeta_t$ is a mixing sequence (see, e.g., Rootzén, 1976, Phillips and Durlauf, 1986, Phillips and Ouliaris, 1990), so $t_n(\widehat{\eta})$ is also mixing. Then, if (9) holds, by Lemma 2.6 of Rootzén (1976) $t_n(\eta)$ is also a mixing sequence, so conditioning on η does not affect the analysis of the limiting distribution of $t_n(\eta)$. Thus, we would act as if η were fixed. Noting that by (2)

$$\Delta\widehat{u}_t = \left(1, -\widehat{\theta}'\right)\begin{pmatrix}\zeta_t \\ \Delta z_{(2)t}\end{pmatrix} = \widehat{\eta}'\begin{pmatrix}\zeta_t \\ n^{-1}\Delta z_{(2)t}\end{pmatrix},$$

define $\overline{u}_t$ and $\overline{x}_{q,t}$ as $\widehat{u}_t$ and $x_{q,t}$, respectively, but replacing $\widehat{\eta}$ by η in these latter expressions. There is a slight abuse of notation here because

$$\Delta\overline{u}_t = \eta'\begin{pmatrix}\zeta_t \\ n^{-1}\Delta z_{(2)t}\end{pmatrix},$$

so, strictly speaking, a more appropriate (but more cumbersome) notation would be $\overline{u}_{t,n}$, given that this is a triangular array.

First, we show that as $q \to \infty$ and $n \to \infty$,

$$\left(\frac{1}{n}\sum_t \overline{x}_{q,t}\overline{x}_{q,t}'\right)^{-1} = O_p(1), \tag{11}$$

$$\frac{1}{n}\sum_t \overline{u}_{t-1}\overline{x}_{q,t} = O_p\left(q^{\frac{1}{2}}\right). \tag{12}$$

Denote by η_1, η_2, the first p and last $p-r$ components of η, respectively, so $\eta = (\eta_1', \eta_2')'$. Then $\overline{x}_{q,t} = a_{q,t} + b_{q,t}$, where

$$a_{q,t} = (\eta_1'\zeta_{t-1}, ..., \eta_1'\zeta_{t-q})', \quad b_{q,t} = (\frac{\eta_2'}{n}\Delta z_{(2)t-1}, ..., \frac{\eta_2'}{n}\Delta z_{(2)t-q})'.$$

In order to show (11), we first prove that

$$\frac{1}{n}\sum_t \overline{x}_{q,t}\overline{x}_{q,t}' = C_n + R_n,$$

where $C_n = (I_q \otimes \eta_1') \Gamma_q (I_q \otimes \eta_1)$, $\otimes$ denotes the Kronecker product,

$$\Gamma_q = \begin{pmatrix} \Gamma(0) & \Gamma(1) & \cdots & \Gamma(q-1) \\ \Gamma(-1) & \Gamma(0) & \cdots & \Gamma(q-2) \\ \vdots & \vdots & \ddots & \vdots \\ \Gamma(1-q) & \Gamma(2-q) & \cdots & \Gamma(0) \end{pmatrix},$$

with $\Gamma(j) = E(\zeta_t \zeta_{t-j}')$, and $\|R_n\| = O_p\left(qn^{-1/2}\right) = o_p(1)$ by Assumption 2. This result follows because it can be shown that under our conditions

$$\begin{aligned} E\left\|\sum_t \zeta_{t-i}\Delta z_{(2)t-j}'\right\|^2 &= O\left(n^2\right), \quad E\left\|\sum_t \Delta z_{(2)t-i}\Delta z_{(2)t-j}'\right\|^2 = O\left(n^4\right), \\ E\left\|\sum_t (\zeta_{t-i}\zeta_{t-j}' - \Gamma(j-i))\right\|^2 &= O(n), \end{aligned}$$

uniformly in i, j, so by the properties of the norm

$$\begin{aligned} E\left\|\frac{1}{n}\sum_t a_{q,t}b_{q,t}'\right\|^2 &= O\left(\frac{q^2}{n^2}\right), \quad E\left\|\frac{1}{n}\sum_t b_{q,t}b_{q,t}'\right\|^2 = O\left(\frac{q^2}{n^2}\right), \\ E\left\|\frac{1}{n}\sum_t a_{q,t}a_{q,t}' - C_n\right\|^2 &= O\left(\frac{q^2}{n}\right). \end{aligned}$$

Next $\|C_n^{-1}\| = O_p(1)$ because Γ_q is positive definite and $I_q \otimes \eta_1$ is a full rank $qp \times q$ matrix. Additionally,

$$\left\|\left(\frac{1}{n}\sum_t \overline{x}_{q,t}\overline{x}_{q,t}'\right)^{-1} - C_n^{-1}\right\| \left(1 - \|R_n\| \left\|C_n^{-1}\right\|\right) \le \|R_n\| \left\|C_n^{-1}\right\|^2.$$

Noting that $\|C_n^{-1}\| = O_p(1)$, $\|R_n\| = o_p(1)$, $1 - \|R_n\| \|C_n^{-1}\| > 0$ with probability approaching one, so that

$$\left\|\left(\frac{1}{n}\sum_t \overline{x}_{q,t}\overline{x}_{q,t}'\right)^{-1} - C_n^{-1}\right\| \le \frac{\|R_n\| \left\|C_n^{-1}\right\|^2}{1 - \|R_n\| \left\|C_n^{-1}\right\|} = O_p\left(\frac{q}{n^{\frac{1}{2}}}\right),$$

to conclude the proof of (11). Next, (12) follows by similar arguments noting that

$$E\left\|\frac{1}{n}\sum_t\begin{pmatrix} v_{(1)t-1} \\ \Delta z_{(2)t-1}\end{pmatrix}\zeta'_{t-j}\right\|^2 = O(1), \quad E\left\|\frac{1}{n^2}\sum_t z_{(2)t-1}\zeta'_{t-j}\right\|^2 = O(1)$$

$$E\left\|\frac{1}{n^2}\sum_t\begin{pmatrix} v_{(1)t-1} \\ \Delta z_{(2)t-1}\end{pmatrix}\Delta z'_{(2)t-j}\right\|^2 = O(1), \quad E\left\|\frac{1}{n^3}\sum_t z_{(2)t-1}\Delta z'_{(2)t-j}\right\|^2 = O(1),$$

uniformly in j, where $v_{(1)t} = z_{(1)t} + Bz_{(2)t}$.

Next we deal with $\overline{U}'_{-1}Q_{\overline{X}_q}\overline{U}_{-1}$, where $\overline{U}_{-1}$, $Q_{\overline{X}_q}$, are defined as $\widehat{U}_{-1}$, Q_{X_q}, replacing $\widehat{u}_t$, $x_{q,t}$, by $\overline{u}_t$, $\overline{x}_{q,t}$, respectively. This is one of the components of the denominator of $t_n(\eta)$ (see (7)), and by (11), (12),

$$\frac{1}{n^2}\overline{U}'_{-1}Q_{\overline{X}_q}\overline{U}_{-1} = \frac{1}{n^2}\sum_t \overline{u}^2_{t-1} + O_p\left(\frac{q}{n}\right). \tag{13}$$

Partitioning

$$\Omega = \begin{pmatrix} \Omega_{11} & \Omega_{12} \\ \Omega_{21} & \Omega_{22}\end{pmatrix}$$

according to $B(s) = \left(B'_{(1)}(s), B'_{(2)}(s)\right)'$, define

$$S = \begin{pmatrix} I_r & -\Omega_{12}\Omega_{22}^{-1} & 0 \\ 0 & I_{p-r} & 0 \\ 0 & 0 & I_{p-r}\end{pmatrix}.$$

Then

$$\frac{1}{n^2}\sum_t \overline{u}^2_{t-1} = \frac{1}{n^2}\eta' S^{-1}\sum_t S\begin{pmatrix} v_{(1)t-1} \\ \Delta z_{(2)t-1} \\ n^{-1}z_{(2)t-1}\end{pmatrix}\left(v'_{(1)t-1}, \Delta z'_{(2)t-1}, \frac{1}{n}z'_{(2)t-1}\right)S'(S')^{-1}\eta.$$

First, note that $S\left(v'_{(1)t}, \Delta z'_{(2)t}, n^{-1}z'_{(2)t}\right)' = \left(w'_t, \Delta z'_{(2)t}, n^{-1}z'_{(2)t}\right)'$, where $w_t = v_{(1)t} - \Omega_{12}\Omega_{22}^{-1}\Delta z_{(2)t}$ is an $I(1)$ process such that the coherence at frequency zero between Δw_t and $\Delta^2 z_{(2)t}$ is zero. Define $Z(s) = \left(B'_{(1.2)}(s), B'_{(2)}(s), \overline{B}'_{(2)}(s)\right)'$, $B_{(1.2)}(s) = B_{(1)}(s) - \Omega_{12}\Omega_{22}^{-1}B_{(2)}(s)$, noting that $B_{(1.2)}(s)$ and $B_{(2)}(s)$ are independent Brownian motions and $B_{(1.2)}(s)$ has covariance matrix $\Phi = \Omega_{11} - \Omega_{12}\Omega_{22}^{-1}\Omega_{21}$. Then, by (3) and the continuous mapping theorem

$$
\begin{aligned}
\frac{1}{n^2}\sum_t \overline{u}_{t-1}^2 &\Rightarrow \eta' S^{-1}\int_0^1 Z(s)\,Z'(s)\,ds\,(S')^{-1}\eta \\
&= \int_0^1 Z_1^2(s)\,ds \\
&\quad -\int_0^1 Z_1(s)\,Z_{-1}'(s)\,ds\left(\int_0^1 Z_{-1}(s)\,Z_{-1}'(s)\,ds\right)^{-1}\int_0^1 Z_1(s)\,Z_{-1}(s)\,ds,
\end{aligned} \tag{14}
$$

because

$$
\eta' S^{-1} = \left(1, -\int_0^1 Z_1(s)\,Z_{-1}'(s)\,ds\left(\int_0^1 Z_{-1}(s)\,Z_{-1}'(s)\,ds\right)^{-1}\right). \tag{15}
$$

As in Phillips and Ouliaris (1990), let

$$
\Phi = \begin{pmatrix} \phi_{11} & \phi_{12} \\ \phi_{21} & \Phi_{22} \end{pmatrix} = L'L, \quad \text{where } L = \begin{pmatrix} l_{11} & 0 \\ l_{21} & L_{22} \end{pmatrix},
$$

where ϕ_{11}, ϕ_{12}, Φ_{22}, are 1×1, $1\times(r-1)$, $(r-1)\times(r-1)$, matrices, respectively, $\phi_{21}=\phi_{12}'$, and $l_{11}=\left(\phi_{11}-\phi_{12}\Phi_{22}^{-1}\phi_{21}\right)^{1/2}$, $l_{21}=\Phi_{22}^{-1/2}\phi_{21}$, $L_{22}=\Phi_{22}^{1/2}$. Thus

$$
Z(s) = \begin{pmatrix} B_{(1.2)}(s) \\ B_{(2)}(s) \\ \overline{B}_{(2)}(s) \end{pmatrix} = \begin{pmatrix} L' & 0 & 0 \\ 0 & \Omega_{22}^{\frac{1}{2}} & 0 \\ 0 & 0 & \Omega_{22}^{\frac{1}{2}} \end{pmatrix} V(s).
$$

Then, by (13), (14) and obvious manipulations

$$
\frac{1}{n^2}\overline{U}_{-1}' Q_{\overline{X}_q}\overline{U}_{-1} \Rightarrow l_{11}^2\int_0^1 Q^2(s)\,ds. \tag{16}
$$

Next

$$
\frac{1}{n}\overline{U}_{-1}' Q_{\overline{X}_q}\Delta\overline{U} = \frac{1}{n}\sum_t \overline{u}_{t-1}\Delta\overline{u}_t - \frac{1}{n}\sum_t \overline{u}_{t-1}\overline{x}_{q,t}'\left(\sum_t \overline{x}_{q,t}\overline{x}_{q,t}'\right)^{-1}\sum_t \Delta\overline{u}_t\overline{x}_{q,t}, \tag{17}
$$

noting that $\Delta \overline{u}_t = \zeta_t' \eta_1 + n^{-1} \Delta z_{(2)t}' \eta_2$. First, by similar arguments to those in the proofs of (11), (12), it is simple to show that

$$\begin{aligned}
\frac{1}{n^2} \sum_t \overline{x}_{q,t} \Delta z_{(2)t}' \eta_2 &= O_p\left(\frac{q^{1/2}}{n}\right), \quad \left(\frac{1}{n} \sum_t \overline{x}_{q,t} \overline{x}_{q,t}'\right)^{-1} - \left(\frac{1}{n} \sum_t a_{q,t} a_{q,t}'\right)^{-1} = o_p(1), \\
\frac{1}{n} \sum_t b_{q,t} \zeta_t' \eta_1 &= O_p\left(\frac{q^{1/2}}{n}\right),
\end{aligned}$$

which implies that (17) equals

$$\begin{aligned}
&\frac{1}{n} \sum_t \overline{u}_{t-1} \left(\zeta_t' \eta_1 - a_{q,t}' \left(\sum_t a_{q,t} a_{q,t}'\right)^{-1} \sum_t a_{q,t} \zeta_t' \eta_1\right) + \frac{1}{n^2} \sum_t \overline{u}_{t-1} \Delta z_{(2)t}' \eta_2 \\
&- \frac{1}{n} \sum_t \overline{u}_{t-1} b_{q,t}' \left(\sum_t a_{q,t} a_{q,t}'\right)^{-1} \sum_t a_{q,t} \zeta_t' \eta_1 + o_p(1). \qquad (18)
\end{aligned}$$

We concentrate on the third term of (18). First, we show that

$$\frac{1}{n} \sum_t \overline{u}_{t-1} b_{q,t}' - \frac{1}{n^2} \sum_t \overline{u}_{t-1} \Delta z_{(2)t}' \eta_2 e_q' = o_p(1), \text{ as } q \to \infty, \, n \to \infty, \qquad (19)$$

where e_q is a q-dimensional vector of ones. The m-th element of the row vector on the left of (19) equals

$$-\frac{1}{n^2} \sum_t \overline{u}_{t-1} \sum_{l=1}^{m} \Delta^2 z_{(2)t-l+1}' \eta_2, \quad \text{for } m = 1, ..., q,$$

which can be easily shown to be $O_p(qn^{-1})$ uniformly in m, so

$$E \left\| \frac{1}{n} \sum_t \overline{u}_{t-1} b_{q,t}' - \frac{1}{n^2} \sum_t \overline{u}_{t-1} \Delta z_{(2)t}' \eta_2 e_q' \right\|^2 = O\left(\frac{q^3}{n^2}\right),$$

to conclude (19). Then the sum of the second and third terms of (18) becomes

$$\frac{1}{n^2} \sum_t \overline{u}_{t-1} \Delta z_{(2)t}' \eta_2 \left(1 - e_q' \left(\sum_t a_{q,t} a_{q,t}'\right)^{-1} \sum_t a_{q,t} \zeta_t' \eta_1\right) + o_p(1). \qquad (20)$$

Next, as in Phillips and Ouliaris (1990), denote $\xi_t = \eta_1' \zeta_t$, which (conditional on η_1)

has an autoregressive representation

$$d(L)\,\xi_t = \gamma_t, \quad d(s) = \sum_{j=0}^{\infty} d_j s^j, \quad d_0 = 1,$$

where the sequence d_j is absolutely summable and γ_t is a zero-mean orthogonal sequence with variance $d^2(1)\,\eta_1'\Omega\eta_1$. Next, note that

$$1 - e_q'\left(\sum_t a_{q,t}a_{q,t}'\right)^{-1}\sum_t a_{q,t}\zeta_t'\eta_1 = 1 + \sum_{j=1}^{q}\widehat{d}_j,$$

where $-\widehat{d}_j$ is the estimated coefficient corresponding to ξ_{t-j}, $j = 1, ..., q$, in the regression of ξ_t on $\xi_{t-1}, ..., \xi_{t-q}$. As in Lemma 3.4 of Chang and Park (2002),

$$1 - e_q'\left(\sum_t a_{q,t}a_{q,t}'\right)^{-1}\sum_t a_{q,t}\zeta_t'\eta_1 \to_p d(1)\,, \text{ as } q \to \infty,\, n \to \infty.$$

Then, by (18), (19), (20),

$$\begin{aligned}\frac{1}{n}\overline{U}_{-1}'Q_{\overline{X}_q}\Delta\overline{U} \quad &= \quad \frac{d(1)}{n}\sum_t \overline{u}_{t-1}\left(\frac{\gamma_t}{d(1)} + \frac{1}{n}\Delta z_{(2)t}'\eta_2\right) + \frac{1}{n}\sum_t \overline{u}_{t-1}\left(\widehat{\gamma}_t - \gamma_t\right) \\ &\quad + \frac{1}{n^2}\sum_t \overline{u}_{t-1}\Delta z_{(2)t}'\eta_2\left(\left(1 - e_q'\left(\sum_t a_{q,t}a_{q,t}'\right)^{-1}\sum_t a_{q,t}\zeta_t'\eta_1\right) - d(1)\right),\end{aligned} \tag{21}$$

where $\widehat{\gamma}_t = \zeta_t'\eta_1 - a_{q,t}'\left(\sum_t a_{q,t}a_{q,t}'\right)^{-1}\sum_t a_{q,t}\zeta_t'\eta_1$. The second and third terms on the right side of (21) can be easily shown to be $o_p(1)$, whereas the first one can be analyzed by identical transformations to those employed in the proof of (16), so that

$$\frac{1}{n}\overline{U}_{-1}'Q_{\overline{X}_q}\Delta\overline{U} \Rightarrow d(1)\,l_{11}^2\int_0^1 Q(s)\,dQ(s)\,.$$

Also, by previous arguments

$$\begin{aligned}\widehat{\sigma}^2 \quad &= \quad \frac{1}{n}\Delta\overline{U}'Q_{\overline{X}_q}\Delta\overline{U} + o_p(1) \\ &= \quad \frac{1}{n}\sum_t \eta_1'\zeta_t\zeta_t'\eta_1 - \frac{1}{n}\sum_t \eta_1'\zeta_t a_{q,t}'\left(\sum_t a_{q,t}a_{q,t}'\right)^{-1}\sum_t \eta_1'\zeta_t a_{q,t} + o_p(1)\,,\end{aligned}$$

so that

$$\widehat{\sigma}^2 \rightarrow_p d^2(1)\, \eta_1' \Omega \eta_1.$$

By (15),

$$\begin{aligned}
\eta_1' \Omega \eta_1 &= \left(1, -\int_0^1 Z_1(s) Z_{-1}'(s)\, ds \left(\int_0^1 Z_{-1}(s) Z_{-1}'(s)\, ds \right)^{-1} \right) \begin{pmatrix} \Phi & 0 & 0 \\ 0 & \Omega_{22} & 0 \\ 0 & 0 & 0 \end{pmatrix} \\
&\times \begin{pmatrix} 1 \\ -\left(\int_0^1 Z_{-1}(s) Z_{-1}'(s)\, ds \right)^{-1} \int_0^1 Z_1(s) Z_{-1}(s)\, ds \end{pmatrix},
\end{aligned}$$

so

$$\eta_1' \Omega \eta_1 = l_{11}^2 \kappa' \begin{pmatrix} I_p & 0 \\ 0 & 0 \end{pmatrix} \kappa,$$

by identical transformations to the ones employed before, to conclude the proof of (10).

Finally, we show (9). Clearly

$$t_n(\widehat{\eta}) - t_n(\eta) = t_n(\widehat{\eta}) - t(\widehat{\eta}) - (t_n(\eta) - t(\eta)) + t(\widehat{\eta}) - t(\eta),$$

where $t(\cdot)$ is like $t_n(\cdot)$, but with the normalized summations replaced by the respective limits in distribution. First, $t(\widehat{\eta}) - t(\eta) = o_p(1)$, by (8) and the continuous mapping theorem. Also, noting that $\widehat{\eta} = O_p(1)$, $t_n(\widehat{\eta}) - t(\widehat{\eta}) = o_p(1)$ by tedious but simple calculations, showing that the difference between the individual components of $t_n(\widehat{\eta})$ with the corresponding ones in $t(\widehat{\eta})$ is $o_p(1)$. For identical reasons, $t_n(\eta) - t(\eta) = o_p(1)$, to conclude (9), and therefore complete the proof of Theorem 1.

Proof of Proposition 1. Given that z_t is cointegrated, there exists a $r \times (p-r)$ matrix A such that $\underline{z}_t + A\overline{z}_t$ is $I(c)$, $c \leq 1$, where $\overline{z}_t$ is a $(p-r)$-dimensional subvector of z_t with $I(2)$ and not cointegrated individual components, and $\underline{z}_t$ collects the remaining r components of z_t. Without loss of generality set $z_t = (\underline{z}_t', \overline{z}_t')'$. If $\underline{z}_t + A\overline{z}_t \sim I(c)$, $c \leq 0$, the theorem holds trivially because $R = T = C$. The proof for the $\underline{z}_t + A\overline{z}_t \sim I(1)$ situation is as follows. Let $\phi \in T$. Then $\phi \in R$, by setting $\lambda(\phi) = (0_r', \rho'(\phi))'$, where 0_r denotes a r-dimensional vector of zeroes. Alternatively, if $\phi \in R$, there exists $\lambda(\phi)$ such that $\phi' z_t + \lambda'(\phi) \Delta z_t \sim I(c)$, $c \leq 0$, or equivalently

$$\phi' z_t + \underline{\lambda}'(\phi) \Delta \underline{z}_t + \overline{\lambda}'(\phi) \Delta \overline{z}_t \sim I(c), \ c \leq 0,$$

where $\lambda(\phi) = (\underline{\lambda}'(\phi), \overline{\lambda}'(\phi))'$ is partitioned according to z_t. From the cointegrating relations

$$\underline{\lambda}'(\phi)\,\Delta\underline{z}_t + \underline{\lambda}'(\phi)\,A\Delta\overline{z}_t \sim I(c)\,,\ c \leq 0,$$

so obviously

$$\phi' z_t + \left(\overline{\lambda}'(\phi) - \underline{\lambda}'(\phi)\,A\right)\Delta\overline{z}_t I(c)\,,\ c \leq 0,$$

and, consequently, $\phi \in T$, to conclude the proof.

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Table 1 Critical values for the cointegration ADF test (intercept included in the cointegration regression)

	p	2				3				4				5			
r	$n \backslash \alpha$	.10	.05	.025	.01	.10	.05	.025	.01	.10	.05	.025	.01	.10	.05	.025	.01
	50	-3.60	-3.94	-4.24	-4.61	-4.40	-4.74	-5.06	-5.45	-5.09	-5.46	-5.80	-6.22	-5.74	-6.13	-6.48	-6.92
	100	-3.55	-3.85	-4.13	-4.47	-4.28	-4.60	-4.88	-5.22	-4.90	-5.23	-5.51	-5.85	-5.47	-5.79	-6.08	-6.43
1	250	-3.51	-3.80	-4.07	-4.38	-4.21	-4.50	-4.76	-5.07	-4.79	-5.09	-5.35	-5.66	-5.31	-5.61	-5.88	-6.20
	500	-3.49	-3.78	-4.03	-4.32	-4.19	-4.47	-4.72	-5.03	-4.76	-5.04	-5.29	-5.58	-5.27	-5.56	-5.80	-6.10
	50,000	-3.49	-3.77	-4.02	-4.31	-4.16	-4.44	-4.70	-5.00	-4.73	-5.02	-5.26	-5.54	-5.21	-5.50	-5.74	-6.02
	50					-3.98	-4.32	-4.63	-5.00	-4.72	-5.08	-5.41	-5.80	-5.39	-5.76	-6.10	-6.53
	100					-3.91	-4.22	-4.50	-4.83	-4.58	-4.90	-5.18	-5.51	-5.17	-5.50	-5.79	-6.16
2	250					-3.86	-4.15	-4.42	-4.71	-4.50	-4.80	-5.06	-5.36	-5.05	-5.34	-5.61	-5.93
	500					-3.85	-4.14	-4.40	-4.70	-4.47	-4.76	-5.01	-5.30	-5.01	-5.29	-5.55	-5.85
	50,000					-3.83	-4.10	-4.35	-4.66	-4.45	-4.74	-4.98	-5.27	-4.97	-5.25	-5.50	-5.79
	50									-4.34	-4.68	-5.00	-5.38	-5.04	-5.40	-5.73	-6.13
	100									-4.24	-4.55	-4.83	-5.18	-4.87	-5.19	-5.48	-5.84
3	250									-4.18	-4.48	-4.74	-5.05	-4.77	-5.07	-5.33	-5.65
	500									-4.16	-4.45	-4.70	-5.01	-4.75	-5.03	-5.29	-5.59
	50,000									-4.15	-4.44	-4.69	4.97	-4.71	-4.99	-5.24	-5.52
	50													-4.68	-5.04	-5.36	-5.75
	100													-4.55	-4.87	-5.15	-5.50
4	250													-4.48	-4.77	-5.04	-5.34
	500													-4.46	-4.75	-5.00	-5.31
	50,000													-4.44	-4.72	-4.96	-5.25

Table 2. Power of the test against stationary alternatives

		$n=50$			$n=100$			$n=250$		
ζ_t	$_{Model}\backslash^{\alpha}$	.10	.05	.01	.10	.05	.01	.10	.05	.01
WN	1	0.912	0.892	0.856	0.994	0.992	0.986	1.00	1.00	1.00
	2	0.900	0.879	0.842	0.995	0.993	0.986	1.00	1.00	1.00
	3	0.911	0.891	0.852	0.998	0.995	0.990	1.00	1.00	1.00
	4	0.907	0.886	0.842	0.995	0.993	0.987	1.00	1.00	1.00
	5	0.891	0.868	0.820	0.993	0.991	0.984	1.00	1.00	1.00
	6	0.886	0.866	0.816	0.994	0.992	0.985	1.00	1.00	1.00
AR	1	0.387	0.271	0.103	0.671	0.520	0.242	0.999	0.995	0.965
	2	0.383	0.272	0.233	0.665	0.514	0.233	0.998	0.996	0.964
	3	0.430	0.310	0.143	0.703	0.547	0.274	0.999	0.997	0.971
	4	0.428	0.308	0.137	0.703	0.548	0.265	0.999	0.997	0.968
	5	0.458	0.341	0.155	0.644	0.493	0.245	0.998	0.992	0.937
	6	0.458	0.335	0.150	0.636	0.490	0.240	0.998	0.992	0.935
MA	1	0.911	0.895	0.865	0.995	0.993	0.989	1.00	1.00	1.00
	2	0.905	0.888	0.855	0.995	0.993	0.988	1.00	1.00	1.00
	3	0.912	0.895	0.865	0.994	0.993	0.987	1.00	1.00	1.00
	4	0.903	0.887	0.854	0.995	0.993	0.988	1.00	1.00	1.00
	5	0.895	0.876	0.839	0.992	0.989	0.983	1.00	1.00	1.00
	6	0.886	0.867	0.833	0.994	0.991	0.985	1.00	1.00	1.00

The cells show the proportion of rejection of the null hypothesis of no-stationary cointegration. 10,000 replications were carried out for each sample size n. Three different significance levels $\alpha = \{.10, .05, .01\}$ were used in the tests. The number of lags in the ADF tests was chosen according to the SIC. ε_t is Gaussian such that $E(\varepsilon_t) = 0$, $Var(\varepsilon_t) = I_3$. The innovation vector ζ_t is generated as: $A(L) = I_3$ (WN), $A(L) = (1-0.8L)^{-1} I_3$ (AR), $A(L) = (1-0.5L) I_3$ (MA). The ADF test is based on the residuals from the cointegrating regressions in (5). Critical values from Table 1, $p=3$, $r=2$ (Models 1-4) and $p=3$, $r=1$ (models 5-6).