# Job Market Paper: Sovereign Default under Model Uncertainty*

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**Abstract**

Model uncertainty regarding the true process governing economic fundamentals can account for both the level of sovereign debt returns and the structure of the cross-section of returns across countries. I introduce misspecification doubts and ambiguity-averse investors into a model of sovereign default. In this framework, where default can be an optimal outcome and default decisions are more likely to occur in low income scenarios, model uncertainty generates a risk premium without the need for correlation between foreign investors' consumption and default. Keeping default probabilities at historical levels, the introduction of ambiguity-averse investors can explain the level of returns observed in the data. Investors charge a premium on risky debt in order to guard themselves against possible specification errors in the estimated income process. I quantify the premium, and characterize its main component, the market price of model uncertainty. While risk premiums are increasing in debt, the market price of uncertainty is decreasing, partially offsetting the higher risk from default. A calibrated model captures these effects, and shows the role of overall uncertainty in the premium. Emerging economies, for which the model that governs the data is usually harder to identify, are prone to have higher levels of model uncertainty, and hence higher risk premiums.

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# 1 Introduction

I construct a general equilibrium model of sovereign debt default where uncertainty regarding the true stochastic process for government's revenue generates a price adjustment term capable of explaining both observed spreads and historical probabilities of default. In the model, a small country facing period-by-period payments makes both next period debt and default decisions taking into account the price of new debt. Markets are incomplete, since the small economy can only access one-period non-contingent bonds. The country chooses when to default optimally, by comparing the value of default with that of repayment.

Foreign investors buying the bond are ambiguity-averse and guard themselves against possible misspecification errors in their estimated model for revenue by choosing the worst from a set of hard-to-distinguish income processes. They give a price to the bond accordingly, by calculating default probabilities under this distorted process. Since default is more likely in low-income scenarios, investors distort the distribution of revenue towards low-income states, reducing the mean of the stochastic process. This distortion in investors' beliefs has an impact on the decision of the government, which now faces harsher conditions for issuing new debt. The equilibrium outcome is then composed of a price function and the country's optimal policy. These functions determine equilibrium prices and interest rates, as well as the level of debt in the economy. I solve this fixed-point problem numerically and characterize the behavior of the investors, showing how the density is distorted and how the risk premium is affected as a result.

The model generates an adjustment term in the equation that determines the price and a higher spread on risky debt than previous models. It makes a contribution to the recent literature on default risk, which, while able to capture the main components of the cycle associated with default risk, utterly fails to reproduce the level of spreads. In the traditional literature, the low covariance across foreign investors' consumption and default generates a risk adjustment term insufficient to explain high observed spreads: the correlation between foreign investors' consumption and default is not large enough.

Figure 1 displays spreads on a bond index for a selected group of countries. Even countries with no recent history of default show spreads above 350 basis points, indicating larger risk adjustments. Historical default probabilities are not enough to generate such spreads. In this paper, higher price adjustments come from a positive market price of uncertainty. Fears of misspecification errors in the estimated model for income generate a premium on risky debt. This premium explains the high level of spreads, even when default probabilities under the estimated model are kept at low levels. Spreads are also positive for lower levels of debt than under risk neutrality, increasing the set of optimal contracts with positive spreads relative to the set obtained under the risk-neutral framework.

The introduction of model uncertainty also contributes to explaining another feature of the data in Figure 1: the small differences in spreads across countries. Leaving periods under default aside, all countries exhibit spreads in the same order of magnitude, even those with recent default history. For example, Argentina's spread six months before default was 200 basis points higher than the average for the index, and only 120 points higher than Brazil. By then, Argentina was seen as a riskier country but the difference between its spread and those of other countries was rather small. The defaults of Russia and Ecuador illustrate the same problem: emerging countries with apparently much higher probabilities of default offer returns not substantially higher than those from other emerging economies. Default risk alone would require a greater difference between high-risk countries and the rest of the emerging economies.

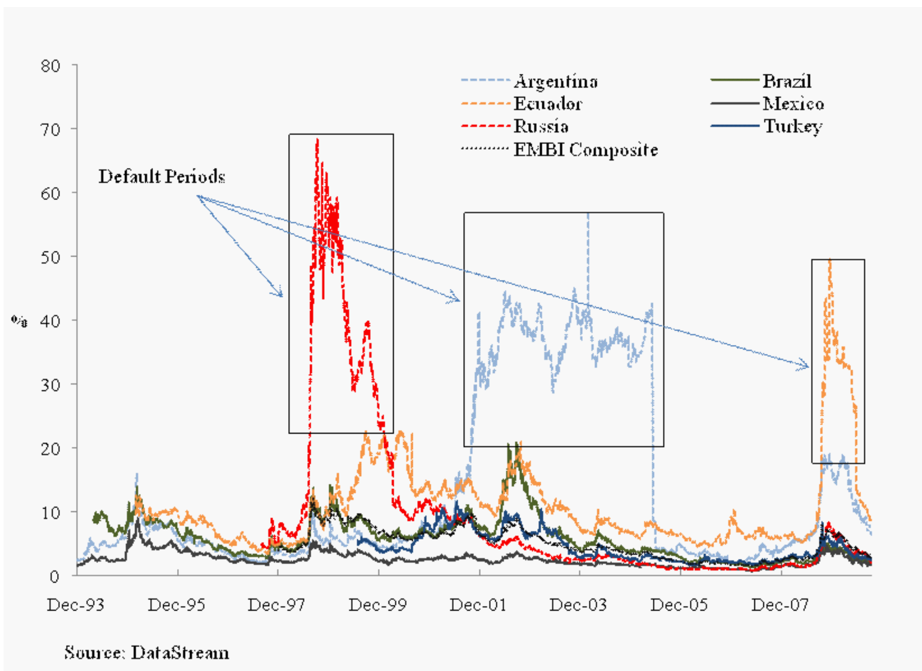



Figure 1. Sovereign bond spreads for selected countries: EMBI+

The price adjustment term can account for differences in spreads across riskier countries. In particular, I show that, conditional on a level of uncertainty, the market price of model uncertainty is decreasing in debt. While the price adjustment term is increasing in debt, premiums on riskier countries increase at a decreasing rate due to the offsetting effect of the market price of model uncertainty. The market price of model uncertainty is decreasing because countries with small probabilities of default are hard to tell apart: empirically, two countries with probabilities of default of 1% and 2% are harder to distinguish than countries with probabilities of default of 5% and 10%. This result contributes to explaining the small differences across spreads for different countries, even before default episodes: countries close to default are easier to understand than those with very low probabilities of default, and the risk premium reflects that ease.

## 1.1 Why Model Uncertainty?

Emerging economies exhibit highly volatile cycles and experience economic crisis more frequently than developed economies. They go through structural changes often as a consequence of recurrent modifications to their fiscal and monetary policies. Analysts have difficulty understanding the main forces driving the fundamentals of the economy, and specification errors behind the processes used to model economic variables are a common concern. This means that it is not only risk, which is usually associated with the volatility of a random variable, but model uncertainty that represents investors' main concern. Risk and uncertainty are different concepts and hence imply different risk adjustments. Knight (1921) reserves the term "risk" for payoffs whose probability is known. Probabilities of returns for many investments are not known, so Knight uses the term uncertainty to refer to those payoffs whose outcomes are unknown. Default in emerging economies is triggered by macroeconomic variables governed by stochastic processes prone to specification errors. Hence, the valuation of risky debt has an uncertainty component that should be priced when analyzing investors' behavior.

I use the concept of uncertainty in the spirit of Knight, and apply the tools developed by Hansen and Sargent (2008) in their work on robustness. They understand uncertainty as a situation where the true model governing the data is unknown to the investor, but an "approximating" model is known. Ambiguity-averse investors realize that this model is just an approximation of an unknown process and surround the model with other models, under a constraint on the distance across models. In this way, they obtain a set of densities with similar likelihood in the data. The constraint that determines this set is formally expressed using an entropy condition, a measure of distance across densities.

The main parameter measuring the above distance is calibrated in an intuitive form: I build time series for income and debt from both the approximating model and the worst model (in terms of expected payoffs) in the set. With these results, I calculate, under the nulls that each model is the true model, the probability of making a mistake and choosing the wrong model from the data. This probability is a measure of how difficult investors find it to distinguish the approximating model from other models in the set (since they take the worst density as a comparison, mistakes can only be less painful under the remaining models in the set). Hansen and Sargent consider it reasonable to expect investors to have an error detection probability of at least 20% and calibrate the constraint measuring the distance across densities with this in mind. If the error detection probability were closer to zero, the two models would be too far apart, and the worst density would be easily discarded in the data, being an overcautious choice. On the other hand, if error detection probabilities were close to 50%, then the models would be too similar, and the worst model would be vulnerable to specification errors. Section 6.1 explains these procedure in detail.

### 1.2 Related Literature

The literature on default risk is vast, with main contributions by Eaton and Gersovitz (1981) and Bulow and Rogoff (1989). In the context of general equilibrium models of default, Arellano (2008) and Arellano and Ramanarayanan (2008) develop default models similar to the one presented here, but with risk neutral investors. The first paper presents a model where investors have complete knowledge of the process followed by the economy, but markets are incomplete so default is a likely outcome. Its main contribution is to show that the model can match the main qualitative aspects of macroeconomic variables for default episodes, although it fails to offer a reasonable measure for the spread observed in the data. The second paper extends the first idea to different debt structures, and analyzes both short and long-term bonds. Aguiar and Gopinath (2005) propose different types of shocks in a model with risk neutral investors. Lizarazo (2006) introduces risk-averse investors in a very similar framework.

This paper builds on those contributions by incorporating the ideas of model uncertainty and ambiguity-averse investors. These concepts have been analyzed in different frameworks. Hansen and Sargent (2008) provide an extensive treatment of the topic. Hansen, Sargent and Tallarini (1999) develop a particular class of model misspecification in the context of a permanent income model. Boyarchenko (2008) modifies Duffie and Lando (1999) to include model misspecification, offering an alternative explanation to high spreads for short maturities. Trojani and Vanini (2004) as well as Trojani et al (2005) analyze the effect of ambiguity aversion on asset prices in a more general context.

## 2 The Basic Model of Default

This section develops the basic model of default risk, calibrated in section 6 where the effects of uncertainty on spreads are presented. Consider an economy composed of two types of players: a small open economy, entitled with a stochastic stream of income, and a group of atomistic foreign investors. The small open economy is represented by a central government, capable of trading non-contingent debt with foreign investors to both smooth consumption and transfer future income. In this economy contracts cannot be enforced and default can be an optimal outcome. In that case, the small economy suffers a small output loss and is excluded from international capital markets for a random number of periods. If there is no default, the country maintains access to international credit markets, where the price for debt is determined.

The representative household in the small country has preferences defined by

$$E_0 \sum_{t=0}^{\infty} \beta^t u\left(c_t\right) \tag{1}$$

where $0 < \beta < 1$ is the discount factor, $c$ is consumption, and $u(.)$ is an increasing and strictly concave utility function. Each period, households receive a stochastic endowment of a tradable good $y$, which I will simply refer to as output. The output shock follows a Markov process. The density of the Markov process is unknown, but the government believes the density function $f(y' \mid y)$ to be a good approximation to a true, unknown density $\widehat{f}(y' \mid y)$, and acts as if it was the true distribution. The central government is altruistic and maximizes the utility of households. Each period, it has the possibility to issue one period discount bonds $B'$, at a price $q(B', y)$, whose proceeds are rebated to consumers in a lump sum fashion.

Issuing a bond implies a non-enforceable commitment from the government to pay $B'$ units of the consumption good in the next period, receiving $q(B', y) B'$ units in this period. While debt can be used to transfer wealth across periods, markets are incomplete, so the idiosyncratic shock on $y$ cannot be insured completely. As a consequence, default can come up as an optimal outcome, increasing consumption in states where rolling over debt is expensive. In addition, the government cannot roll over debt indefinitely: Ponzi schemes are prevented by imposing a debt limit $Z$, such that, for any period, $B' \leq Z$. While not binding in equilibrium, this constraint will influence default probabilities, by conditioning the set of contracts available to roll over debt. If the government decides to save, so $B' < 0$, the interest rate on savings is the risk free rate and the price is simply $q = 1/(1+r)$ for any level of savings.

The government's default decision comes at the beginning of the period. The indicator variable $\delta$ is used to refer to repayment states, with $\delta = 0$ in case of default and $\delta = 1$ otherwise. When the country decides to honor its debts it can also access international capital markets and the resource constraint for the economy is simply:

$$c = y + q(B', y) B' - B \tag{2}$$

In case of default, however, the country is excluded from international capital markets and consumption equals the reduced level of output, $y^{def}$, due to the loss from default:

$$c_t = y^{def} \tag{3}$$

I assume $y^{def} = h(y) \leq y$, where $h(y)$ is an increasing function.

Foreign investors are the only suppliers of funds: there is a continuum with measure one of identical, infinitely lived international bankers, with access to international credit markets, where they can borrow or lend at the risk free interest rate $r$. A risk neutral investor would give a value to the bond by simply discounting the expected payoff using the risk free rate. The expectation would be taken under the "approximating" density $f(y' \mid y)$, the same density used by the government. In fact, if there was no correlation across foreign consumption and default, the same result would hold for risk averse investors. For those cases, and assuming no arbitrage opportunities, prices would simply be

$$q^{RN}(B', y) = \int_0^{+\infty} \frac{\delta(B', y')}{1+r} f(y' \mid y)\, dy' = \frac{\left[1 - p^D(B', y)\right]}{1+r}$$

But investors in this model fear misspecification errors and, unlike the local government, would like to guard themselves against those possible errors in the estimated process. They look for decisions that are robust to mistakes, by surrounding the approximating density with other densities offering a similar fit in the data and choosing the one that gives them the lowest value for the security. In that sense, investors are ambiguity-averse: they give a price to the bond under a distorted probability of default, slanting probabilities towards worse scenarios. More formally, the distorted density minimizes the expected value of the security, subject to an entropy constraint.

The entropy constraint imposes a maximum distance $\eta$ across different densities such that the set of distributions considered are not too different from each other. The entropy constraint is a simple way to measure that distance, and is chosen due to that simplicity. The distance $\eta$ is calibrated so that the distorted and the approximating densities are hard to distinguish empirically. If the distance was too large, the two densities would be too different and the data would easily reject one of them. If they were too close to each other, then the densities would be too similar, making them hard to distinguish in the data. The choice would not be cautious enough and the decision would be vulnerable to misspecification errors. Section 6.1 explains the calibration procedure in detail. The following minimization problem summarizes the problem faced by investors:

$$q(B', y) = \min_{\widehat{f}(y' \mid y)} \int_0^{+\infty} \frac{1}{1+r} \delta(B', y') \widehat{f}(y' \mid y)\, dy' \tag{4}$$

$$s.t. \frac{1}{1+r} \int_0^{+\infty} \widehat{f}(y' \mid y) \log \frac{\widehat{f}(y' \mid y)}{f(y' \mid y)} dy' \leq \eta \tag{5}$$

$$\int_0^{+\infty} \widehat{f}(y' \mid y)\, dy' = 1 \tag{6}$$

In the above setup, the payoff is the random variable $\delta(B', y')$ whose distribution is Bernoulli with probability $\left(1 - p^D(B', y')\right)$, where $p^D(B', y')$ is the probability of default. The first part of the problem minimizes the expected value of a unit payment next period by choosing a different density, while constraint (5) is the entropy constraint, that sets a limit to the densities distance with respect to the approximating model. The last constraint,(6), imposes the new measure to be a proper probability density function.

# 3 Optimal Policy

I now describe the policy functions for both the government and the representative investor. Foreign creditors and the government act sequentially. The latter takes the price function for debt as given and makes a decision on next period debt and default. Foreign creditors take next period debt as given and give a value to the bond taking into account the corresponding repayment states. Timing of decisions within the period is as follows: the government starts the period with an initial level of debt $B$, observes the income shock $y$ and decides whether to repay debt or not. If the optimal decision is to repay the debt, then taking as given the price function for new debt $q(B', y)$ it chooses the next period level of debt $B'$. Then creditors, taking $q(B', y)$ as given, choose $B'$. Finally, the level of consumption implied by next period debt $B'$ takes place. In case of default, the level of consumption in the small country is exogenously given. Otherwise, consumption (and hence next period debt) is optimally chosen. The solution is characterized by a set of state variables where default is optimal. The probability of default originated from this set determines the price given to the bond by foreign investors, though the density used in calculating this probability is the distorted one. Ambiguity-averse investors' discount factor multiplies the ordinary discount factor by the ratio of densities of the endogenous worst-case distorted model relative to the approximating model.

## 3.1 The Government

The government has the option to default, and takes into account the value of this option in the maximization problem. I define $V^O(B, y)$ as the value function for the government that has the option to default and starts the period with debt $B$ and endowment $y$. The function satisfies:

$$V^O(B, y) = \max_{\delta(B,y)\in\{0,1\}} \left\{(1-\delta) V^D(y) + \delta V^C(B, y)\right\} \tag{7}$$

where $V^C(B, y)$ is the value function associated with no default, and $V^D(y)$ the value function associated with default:

$$V^C(B, y) = \max_{B'<Z} \left\{ u\left(y + q(B') B' - B\right) + \beta \int_{y'} V^O(B', y') f(y' \mid y)\, dy' \right\} \tag{8}$$

$$V^D(y) = u(h(y)) + \beta \left[ (1-\gamma) \int_{y'} V^D(y') f(y' \mid y)\, dy' + \gamma \int_{y'} V^O(0, y') f(y' \mid y)\, dy' \right] \tag{9}$$

After default, the government is excluded from international markets for a random number of periods and access is recovered with probability $\gamma$. In that case, the initial level of debt is reduced to zero. The optimal default policy is characterized by repayment sets and default sets. Formally, $C(B')$ is defined as the set of next period income values $y'$ for which it is optimal to repay, while $D(B')$ is defined as the set of next period income values $y'$ such that default is the optimal choice:

$$\begin{aligned} C(B') &= \left\{ y' \in Y : V^C(B', y') \geq V^D(y') \right\} \\ D(B') &= \left\{ y' \in Y : V^C(B', y') < V^D(y') \right\} \end{aligned} \tag{10}$$

When there is no default, next period debt is determined by the government's policy function, defined as $B' = \widetilde{B}(B, y)$. Consumption is determined, using the default indicator variable, by:

$$c(B, y) = \begin{cases} y - B + q\left(\widetilde{B}(B, y)\right)\widetilde{B}(B, y) & if \ \delta(B, y) = 1 \\ y^{def} & if \ \delta(B, y) = 0 \end{cases}$$

.

## 3.2 Foreign Investors

I reexpress problem (4) for foreign investors, following Hansen and Sargent (2008), by introducing the entropy constraint (5) in the objective function, attaching $\alpha(B')$ as the associated Lagrange multiplier, conditional on a particular level of debt in the next period:

$$q(B', y) = \min_{\widehat{f}(y' \mid y; B')} \int_0^{+\infty} \frac{1}{1+r} \left( \delta(B', y') + \alpha(B') \log \frac{\widehat{f}(y' \mid y; B')}{f(y' \mid y)} \right) \widehat{f}(y' \mid y; B')\, dy' \tag{11}$$

$$s.t. \int_0^{+\infty} \widehat{f}(y' \mid y; B')\, dy' = 1 \tag{12}$$

The value of the Lagrange multiplier $\alpha(B')$ depends (for a given level of $\eta$, the entropy distance) on the level of debt in the next period. The solution of the above minimization problem gives the distorted density as a function of the Lagrange multiplier function, $\alpha(B')$, the level of next-period debt $B'$, the default decision policy and the approximating density:

$$\widehat{f}^*(y' \mid y; B') = f(y' \mid y) \frac{\exp\left(-\frac{\delta(B', y')}{\alpha(B')}\right)}{E\left[\exp\left(-\frac{\delta(B', y')}{\alpha(B')}\right)\right]} \tag{13}$$

The value of the bond will be determined using this distorted density instead of the approximating:

$$q\left(B',y\right)=\frac{1}{1+r}\int_{0}^{+\infty}\left[\frac{\widehat{f}^{*}\left(y'\mid y;B'\right)}{f\left(y'\mid y\right)}\right]\delta\left(B',y'\right)f\left(y'\mid y\right)dy'=\frac{1}{1+r}\int_{C(B')}\widehat{f}^{*}\left(y'\mid y;B'\right)dy' \tag{14}$$

In the above expression, the term $m\left(y';B'\right)=\left[\frac{1}{1+r}\frac{\widehat{f}^{*}(y'|y;B')}{f(y'|y)}\right]$ plays the role of the discount factor. It is the modified stochastic discount factor for ambiguity-averse investors. This modified stochastic discount factor can be expressed, using (13) as:

$$m\left(B',y'\right)=\left(\frac{1}{1+r}\right)\frac{e^{-\frac{\delta(B',y')}{\alpha(B')}}}{\Psi\left(B'\right)} \tag{15}$$

where $\Psi\left(B'\right)=E\left\{\exp\left[-\delta\left(B',y'\right)/\alpha\left(B'\left(B,y\right)\right)\right]\right\}$. Without model uncertainty, the discount factor for risk-neutral investors would simply be $(1+r)^{-1}$, since both the distorted density and the approximating density coincide. However, in this paper the stochastic discount factor changes with the level of debt, due to the term $\exp\left[-\delta\left(B',y'\right)/\alpha\left(B'\left(B,y\right)\right)\right]$ in both the numerator and the denominator on the second part of the right hand side. The new stochastic discount factor depends on two elements: the payoff term $\delta\left(B',y'\right)$, highly correlated with income, and the penalty function $\alpha\left(B'\right)$. Under continuous discounting, this would be equivalent to directly adding an additional discount factor to the risk free rate, given by the term $\left[\frac{\delta(B',y')}{\alpha(B'(B,y))}-\log\left\{E\left[\exp\left(-\frac{\delta(B',y')}{\alpha(B')}\right)\right]\right\}\right]$. In this framework, a lower level of uncertainty is characterized by a higher $\alpha\left(B'\right)$ for all levels of next-period debt. As uncertainty regarding the model disappears, and $\alpha$ tends to infinite, the distorted density converges to the approximating density and the risk-neutral discount factor $(1+r)^{-1}$ is recovered.

# 4 The Model Misspecification Function $\alpha\left(B'\right)$

In the literature, the penalty function $\alpha\left(B'\right)$ is also associated with a risk aversion measure. But in this paper, the penalty function measures the degree of robustness required by the investor, since it determines how much extra weight default states receive. To see the connection between the two concepts, the investor's problem (11) can also be seen as the problem of a (risk-neutral) ambiguity-averse consumer (with value function $W\left(B'\right)$ and the entropy constraint attached through a Lagrange multiplier $\alpha\left(B'\right)$), investing $q$ units today with a random payoff $\delta\left(B',y'\right)$ tomorrow:

$$W(B') = -q + \frac{1}{1+r} \min_{\widehat{f}(y';B')} \int_0^{+\infty} \left( \delta(B', y') + \alpha(B') \log \frac{\widehat{f}(y' \mid y)}{f(y' \mid y)} \right) \widehat{f}(y' \mid y)\, dy' \tag{16}$$

$$s.t. (12)$$

Replacing the distorted density in the value function (16), it simplifies to:

$$W(B') = -q - \frac{1}{1+r} \alpha(B') \log E \exp\left( -\frac{\delta(B', y')}{\alpha(B')} \right) \tag{17}$$

This is a simple version, for a two period problem, of the risk-sensitive recursion of Hansen and Sargent (1995). Tallarini (2000) obtains a similar recursion using recursive preferences, but his interpretation of $\alpha(B')$ is quite different. In this framework, $\alpha(B')$ is a penalty parameter, originally attached to the entropy constraint, that measures the size of the set of models the representative investor has trouble differentiating. In Tallarini's framework, the parameter $\alpha$ is matched to the coefficient of risk aversion $\gamma$ by $\alpha = \frac{-(1+r)}{r} \frac{1}{(1-\gamma)}$, so a lower level of $\alpha$ implies a larger level of risk aversion and is independent of the level of debt.

While the representative consumer from Tallarini's recursive utility and the setup here are observationally equivalent conditioning on a particular level of debt, the motivation behind each parameter is very different. Risk aversion is related to gambles with known probability distributions, while in this paper investor concerns are about unknown probability distributions. Here, the calibration of the penalty function will be completely different, generating a set of model misspecification parameters as a function of next period debt, $B'$. As a result, there will not be a unique parameter, as in Tallarini's exponential utility framework, but a function of next period debt $B'$, that will vary with the level of uncertainty, determined by $\eta$, and will also depend on the stochastic process for income under the approximating model. Exponential utility alone is unable to capture this function and has completely different implications for spreads. In the model presented here, the relationship between the penalty parameter and the level of debt is the main force behind the risk premium.

## 4.1 Characterizing the Function

I now characterize the function $\alpha(B')$ that acts as a penalty in the representative investor value function. This function measures the degree of misspecification accepted by investors, and as such is the main component of preferences. The entropy constraint, by setting a limit to the distance across different densities, imposes different properties on the function $\alpha(B')$. Since the likelihood ratio can take only two values conditioning on next period debt $B'$, the entropy constraint can be expressed in terms of default probabilities under the approximating model:

$$\frac{\exp\left(-\alpha\left(B'\right)^{-1}\right)}{\Psi\left(B'\right)}\log\left(\frac{\exp\left(-\alpha\left(B'\right)^{-1}\right)}{\Psi\left(B'\right)}\right)\left(1-p^{D}\right)+\frac{1}{\Psi\left(B'\right)}\log\left(\frac{1}{\Psi\left(B'\right)}\right)p^{D}\leq\eta\left(1+r\right)$$

From this constraint, an implicit function $F\left(\alpha,p^{D}\right)$ can be found and $\alpha$ can be numerically characterized as a function of default probabilities.

**Condition 1.** $p^{D}\left(B',y\right)<e^{-\eta(1+r)}\equiv\overline{p}$

**Proposition 1**. *The function* $\alpha\left(p^{D}\right)$ *is well define only if condition 1 holds*: $\alpha\left(p^{D}\right):\left(0,\overline{p}\right)\rightarrow\left(0,+\infty\right).$

Proof: See appendix.

For the distorted density to be far enough from the approximating density, the probability of default under the approximating model cannot be too large. In fact, once a density for the approximating model is chosen, proposition 1 gives us the domain of $\alpha\left(p^{D}\right)$. When default gets close to a sure thing, even pushing $\alpha$ towards zero is not enough to create a density satisfying the entropy constraint with equality. In other words, the worst scenario is just not bad enough. In that case, the entropy condition does not bind, the distorted density degenerates, and the probability of default under the distorted density is one for any value of $p^{D}$ above $\overline{p}$.

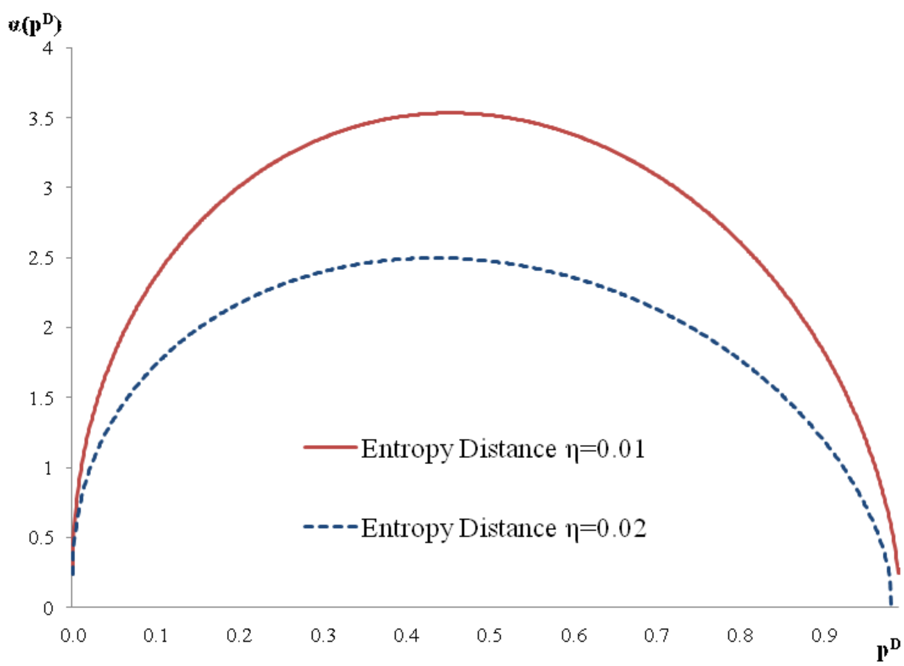



Figure 2: Penalty $\alpha$ as a function of default probabilities. In this framework, the penalty is represented as a function instead of a single parameter.

For the interval determined by condition 1, the function $\alpha\left(p^D\right)$ has an inverted U-shape, reaching a maximum at a point $p^*$. The model penalty function $\alpha(B')$ is increasing in the probability of default under the approximating model, $p^D(B',y)$, for $p^D \in (0,p^*]$ and decreasing in the interval $p^D \in (p^*,\overline{p})$. Figure 2 illustrates this for different values of $\eta$.

## 4.2 The Market Price of Model Uncertainty

The model misspecification function is intimately related to the market price of model uncertainty. Hansen and Sargent (2008) define the market price of model uncertainty, $\rho^u$, as the standard deviation of the ratio of densities:

$$\rho_t^u = std_t\left(\frac{\widehat{f}^*(y_t)}{f(y_t)}\right) \tag{18}$$

Using (15) the market price of model uncertainty can be expressed as a function of the probability of default under the approximating model:

$$\rho^u(B',y) = \lambda(B',y)\,\sigma_{p^D}(B',y) > 0 \tag{19}$$

where

$$\lambda(B',y) = \left[\frac{1-e^{-\frac{1}{\alpha(p^D(B',y))}}}{e^{-\frac{1}{\alpha(p^D(B',y))}} + \left(1-e^{-\frac{1}{\alpha(p^D(B',y))}}\right)p^D(B',y)}\right]$$

and $\sigma_{p^D}(B',y) = \sqrt{(1-p^D(B',y))\,p^D(B',y)}$ is the standard deviation of the payoff under the approximating model. The function $\lambda(B',y)$ is a measure of distance between the risk neutral situation, where $\alpha \to \infty$ (in which case $\exp\left(-1/\alpha\left(p^D(B',y)\right)\right) = 1$) and the current framework, where $\alpha$ is finite (so $0 < \exp\left(-1/\alpha\left(p^D(B',y)\right)\right) < 1$) and the numerator is positive. That distance is rescaled in terms of the expected value of $\exp\left(-1/\alpha\left(p^D(B',y)\right)\right)$, the term in the denominator.

As $\alpha$ tends to infinite, the market price of model uncertainty tends to zero since both the approximating and the distorted density coincide. But when $\alpha$ is finite, it increases by a factor determined by the measure $\lambda(B',y)$ and the standard deviation of the payoff, $\sigma_{p^D}$. In that case, uncertainty will be priced and the risk premium will reflect it. The adjustment due to uncertainty can be seen decomposing the price of the bond. Using the stochastic discount factor $m(B',y')$, the price of the one-period risky bond can be expressed as:

$$q(B',y) = E\left[m(B',y')\,\delta(B',y')\right] = \frac{1-p^D(B',y)}{1+r} + Cov\left[m(B',y')\,\delta(B',y')\right]$$

The last term gives the usual price adjustment, due to the correlation between the stochastic discount factor and the payoff. But unlike the usual consumption model, where the stochastic discount factor depends on foreign investors' consumption, the discount factor here is highly correlated with the payoff since it depends on the process for income. Hence, the low correlation across the payoff and foreign investors consumption observed empirically is irrelevant. The last term, the covariance, can be expressed as

$$Cov\left[g\left(B',y'\right)\delta\left(B',y'\right)\right] = -\sigma_{p^D}\left(B',y\right)\rho^u\left(B',y\right) < 0$$

The market price of model uncertainty is simply the price of an extra unit of default risk under the approximating model. The price of the bond is then composed of two parts: the usual term, taking into account the risk free rate and the probability of default, and a price adjustment, determined by the market price of model uncertainty and the risk of the payoff under the approximating model. Since the price adjustment term is always positive, it will always be the case that prices are lower under ambiguity-averse investors, i.e. $q\left(B',y\right) < q^{RN}\left(B',y\right)$:

$$q\left(B',y\right) = \underbrace{\frac{1}{1+r} - \frac{p^D\left(B',y\right)}{1+r}}_{q^{RN}\left(B',y\right)} - \frac{\sigma_{p^D}\left(B',y\right)\rho^u\left(B',y\right)}{1+r} = \frac{1-\widehat{p}^D\left(B',y\right)}{1+r} \tag{20}$$

**Proposition 2**. *The bond price function is decreasing in the entropy distance, $\eta$.*

Since a higher level of $\eta$ will imply a lower level of $\alpha\left(p^D\right)$ for any $p^D$ (this is directly implied by the entropy constraint), the price of the bond will be lower, for any combination of income and next period debt, $\left(B',y\right)$.

Figure 3 presents the price adjustment term and its components, for two different values of $\eta$, the entropy distance. The market price of risk is decreasing in the probability of default. This result would be absent under Tallarini's exponential utility, since it is caused by the behavior of penalty function $\alpha\left(p^D\left(B',y\right)\right)$. It implies that, as a function of the probability of default, the price adjustment term increases less than proportionally with the risk in the payoff, $\sigma_{p^D}\left(B',y\right)$. The decreasing market price of uncertainty partially counterbalances the effect of higher risk, generating a lower, yet still increasing price adjustment term[1]. Densities are harder to differentiate for low probabilities of default, forcing investors to reduce the penalty term $\alpha$ in order to satisfy the entropy constraint with equality.

[1]The payoff risk term, $\sigma_{p^D}\left(B',y\right)$, is decreasing after $p^D = 0.5$, generating a decreasing price adjustment term. But that section of the curve will not be relevant in the model, as countries will endogenously choose debt levels with low probabilities of default, due to the high cost of not doing so.

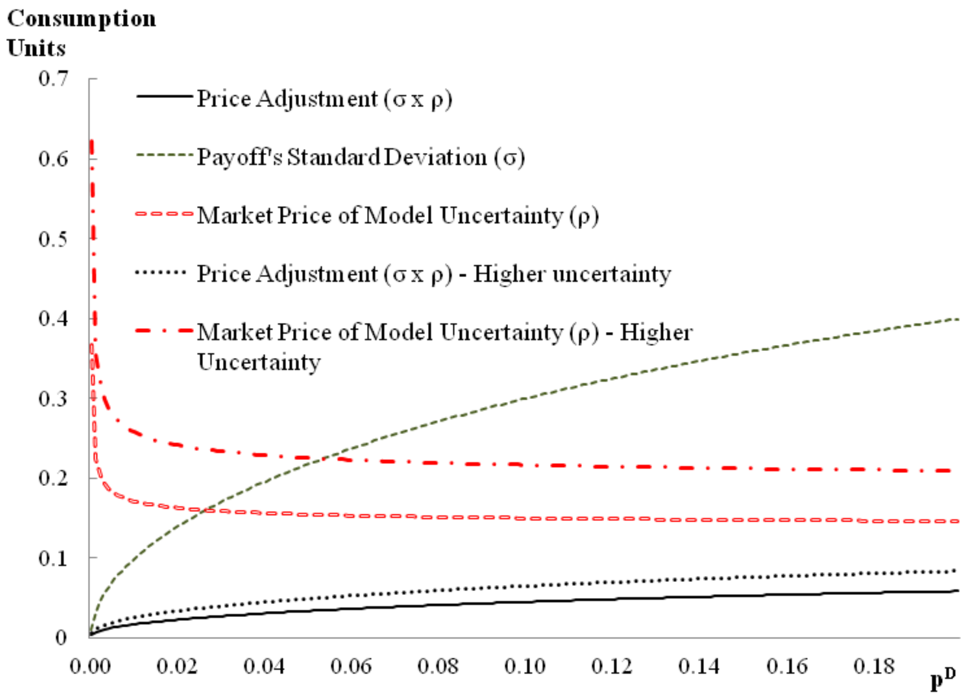



Figure 3. Components of the price adjustment term due to model uncertainty: market price of uncertainty and payoff volatility.

# 5 Recursive Equilibrium

The recursive equilibrium in this economy is determined, given an aggregate state $s = (B, y)$, by a policy function for government's debt, $\widetilde{B}(s)$, repayments sets $C(B')$, a price function for bonds $q(B', y)$, a distorted density $\widehat{f}(y' \mid y; B')$ and a policy function for consumption.

**Definition 1**. The recursive equilibrium is defined as a set of policy functions for $(i)$ consumption $c(s)$, $(ii)$ government's asset holdings $\widetilde{B}(s)$, repayment sets $C(B')$ and default sets $D(B')$, and $(iii)$ the price function for bonds $q(B')$, as well as the distorted density $\widehat{f}(y' \mid y; B')$ used for obtaining that price, such that:

1. Taking as given government policies, households' consumption $c(s)$ satisfies the resource constraint of the economy.

2. Taking as given the bond price function $q(B', y)$, the government's policy function $\widetilde{B}(B, y)$ and repayment sets $C(B'), D(B')$ satisfy the government optimization problem.

3. Bond prices $q(B', y)$ reflect government default probabilities under the distorted density $\widehat{f}(y' \mid y; B')$, and are consistent with expected zero profits for creditors under the distorted distribution.

In this equilibrium, local households are passive: they limit themselves to consume their endowment plus government transfers (that could be negative) from international capital markets. Default probabilities are obtained integrating on the default set, but the densities used by the government and investors differ. While the government uses the approximating model to calculate the probabilities of default $p^D$, investors calculate them using the distorted density $\widehat{f}(y' \mid y)$:

$$p^D(B', y) = \int\limits_{D(B^*)} f(y' \mid y)\, dy'$$

$$\widehat{p}^D(B', y) = \int\limits_{D(B^*)} \widehat{f}^*(y' \mid y; B')\, dy'$$

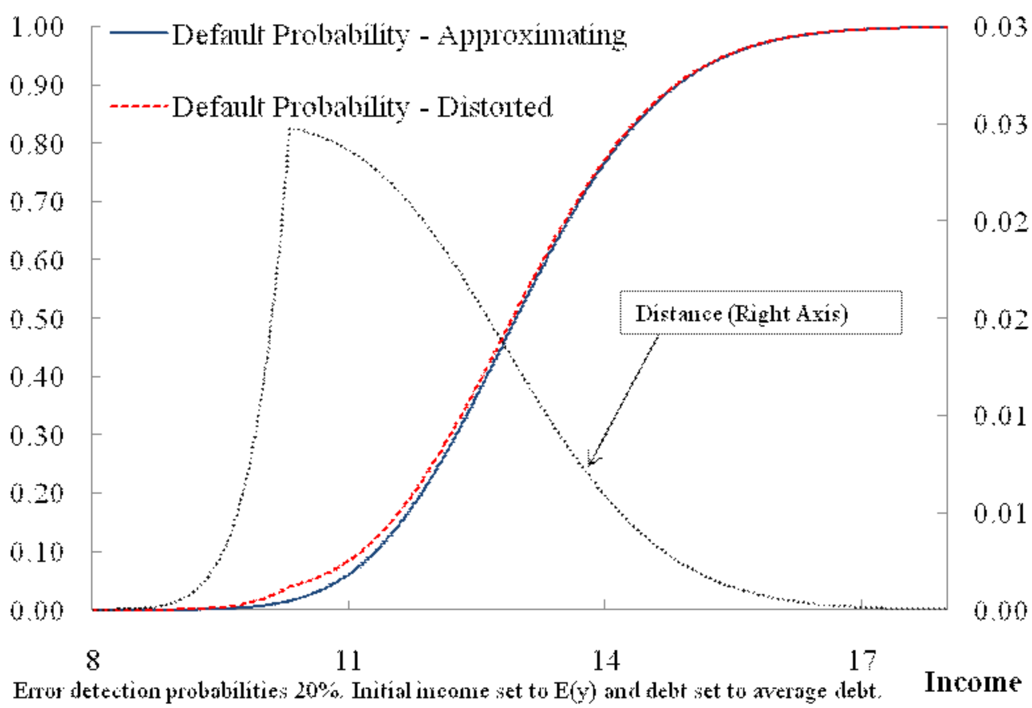



Figure 4. Probabilities of default under distorted and approximating models

Finally, the government's optimality condition determines the level of next period debt

$B^*$, in case of repayment:

$$\frac{\partial\left[q\left(B^*,y\right)B^*\right]}{\partial B'}=\beta\int_{C(B^*)}\frac{u_c\left(y'+q\left(\widetilde{B}\left(B^*,y'\right)\right)\widetilde{B}\left(B^*,y'\right)-B^*\right)}{u_c\left(y+q\left(B^*,y\right)B^*-B\right)}f\left(y'\mid y\right)dy' \tag{21}$$

Condition (21) also imposes a restriction on the level of debt chosen in equilibrium. The change in revenue from issuing debt, the term on the left, must be positive. This sets an upper bound $\overline{B}\left(y\right)$ on the level of debt such that for values of next period debt $B'$ higher than $\overline{B}\left(y\right)$, the term $\partial\left[q\left(B^*\right)B^*\right]/\partial B^*$ will be negative, and first order conditions will not hold, since the utility function is increasing. Since the term $q\left(B^*\right)B^*$ represents the revenue from bond issuance, a negative derivative would imply that higher revenue is possible under lower debt, a situation that would contradict optimality. This result is stated in proposition 5, using results from proposition 3 and 4. A higher level of debt can only reduce welfare by increasing the burden of future payments, so default is more tempting for high levels of debt, a result due to the concavity of the utility function. Thus, debt has a lower price for high debt states, as default probabilities are then higher. These properties are summarized as follows.

**Proposition 3**. *$V^C\left(B,y\right)$ is decreasing in $B$.*

This is a result of the Envelope theorem, since $\frac{\partial V^C(B,y)}{\partial B}=-u'\left(c\right)<0$.

**Corollary 1.** The continuation value is also decreasing in debt: $\frac{\partial V^O(B,y)}{\partial B}\leq 0$.

**Proposition 4.** *Default incentives are stronger the higher the level of debt: $\frac{\partial\widetilde{p}^D(B',y)}{\partial B'}\geq 0$. For all $B^1>B^2$, if default is optimal for $B^2$, in some states $y$, then default is optimal for $B^1$ for the same states $y$. That is $D\left(B^2\right)\subseteq D\left(B^1\right)$.*

This proposition is also a result of the envelope theorem. The probability of default is calculated integrating on the default set under the approximating measure. By the Envelope theorem, the derivative of the continuation value function with respect to debt is negative. Hence, an increment in next period debt can only reduce the continuation value under repayment, with no effect on the continuation value under default. And, as a result, the probability of default can only be increasing.

**Corollary 2**: The price function is decreasing in debt: $\frac{\partial q(B',y)}{\partial B'}\leq 0$.

A corollary of proposition 4 and the numerical result found in part 2 is that the market price of uncertainty will be decreasing in debt. This is one of the key results of the paper. It suggests that, while the price adjustment term will be increasing in debt, its main component, the market price of uncertainty, will partially offset the higher risk. As a result, countries with high risk will benefit from a lower market price of uncertainty, and the spread on their debt will increase less than predicted under a constant market price of risk.

**Corollary 3.** *The market price of uncertainty is decreasing in debt.*

The corollary above will hold for a particular level of uncertainty measured by the entropy distance $\eta$. This means that the calibration of the parameter $\alpha(B')$ will depend on the particular process estimated for income. A country with higher volatility will have an income process harder to identify from other similar processes, and hence the distance $\eta$ will be allowed to be higher, implying a larger distance across densities, and a downwards shift on the function $\alpha(B')$. As a consequence, the market price of uncertainty will be higher for all levels of debt, shifting $\rho_u(B')$ up (see Figure 3). Section 6 calibrates the model to a particular case, and shows these results.

**Proposition 5.** *The policy function $\widetilde{B}(B, y)$ is bounded above.* Proof: See appendix.

The ability of the model to generate primary deficits and interest rates fluctuations similar to those observed empirically is due to two main features - the behavior of the policy function and the price adjustment due to uncertainty. The latter is a result of proposition 4 and corollary 3. Government expenditure, on the other side, is affected by debt prices: while low income states would suggest higher levels of debt in order to smooth consumption, those states also imply lower prices for debt, reducing incentives for higher indebtment. The model replicates this feature present in the data. The introduction of ambiguity-averse investors can only reduce the price of debt, generating lower debt levels in bad states, making the primary deficit more volatile. Propositions 6 and corollary 4 summarize those results.

**Proposition 6.** *The country's debt policy function is decreasing in the price of the bond.* Proof: See appendix.

**Corollary 4.** *The country's debt policy function is decreasing in $\eta$ : $\frac{\partial \widetilde{B}(y,B)}{\partial \eta} < 0$.*

Given a pair of period-t income and debt $(y, B)$, an increment in the entropy distance reduce the optimal level of next-period debt. This is just a result of propositions 2 and 6.

## 5.1 The Case of i.i.d. Shocks

The case in which endowment shocks are i.i.d. is useful to understand the mechanics of the model. Moreover, I assume that the i.i.d. shock has a compact support, determined by a set $Y \in R^{+}$. For this case, bond prices will be a function of next period debt only, $q(B')$, and default probabilities will be determined by an income cutoff point $\overline{y}(B')$ above which default will not occur and below which default will occur with probability one. In addition, the compact support implies that there will be a level of debt $\overline{B}$ such that default will occur with probability one for levels of debt above that threshold. In the same way, repayment will be a sure thing when debt is below some level $\underline{B}$. The following definition summarizes that property.

**Definition 2.** Denote $\underline{B}$ as the lower bound on debt for which default sets are empty is debt is below that value, and $\overline{B}$ as the upper bound on debt such that the default set constitutes the entire set, where $\overline{B} \geq \underline{B} \geq 0$ by proposition 4.

$$\overline{B} = \min \{B : D(B) = Y\}$$
$$\underline{B} = \sup \{B : D(B) = \varnothing\}$$

Propositions 7 and 8 replicate Arellano (2008) arguments, which apply under i.i.d. shocks:

**Proposition 7.** *If for some $B$ the default set is non-empty so that $D(B) \neq \varnothing$, then there are no contracts available $\{q(B'), B'\}$ such that the economy can experience capital inflows: $q(B')B' - B < 0$.*

Default arises only if the borrower does not have access to a contract that allows him to roll over debt due. If he could roll over the debt, then he would increase consumption today and default tomorrow.

**Proposition 8.** *Default incentives are stronger the lower the endowment. For all $y_1 \leq y_2$, if $y_2 \in D(B)$, then $y_1 \in D(B)$.*

The concavity in the utility function makes debt repayment more costly when income is low. Hence, given an endowment level where default is optimal, any level of endowment lower than that will also imply default as the optimal choice. Income shocks have two opposing effects on the default decision. While a higher level of income increases the value of default, hence increasing default incentives, it also increases the value of repayment. Under i.i.d. shocks and incomplete markets, the latter effect dominates, and the default set is simply determined by an upper bound level of income $\overline{y}(B')$, such that if the endowment is below that value, default will be the optimal choice. Repayment and default sets are:

$$C(B') = \{y' \in Y : y' \geq \overline{y}(B')\}$$
$$D(B') = \{y' \in Y : y' < \overline{y}(B')\}$$

When there is no default, the optimal new debt policy function is defined by $B' = \widetilde{B}(B, y)$.

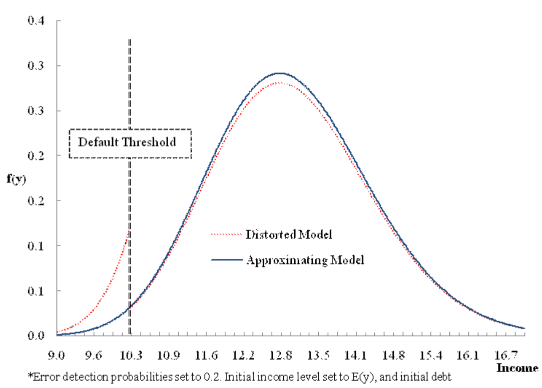
Foreign investors can now determine default by considering only income realizations above the threshold, so the price for the bond (14) simplifies to:

$$q\left(B'\right)=\frac{1}{1+r}\int\limits_{\overline{y}\left(B'\right)}^{\infty}\widehat{f}^{*}\left(y'\right)dy'=\frac{1}{1+r}\left[1-\widehat{F}^{*}\left(\overline{y}\left(B'\right)\right)\right] \tag{22}$$

The distorted density takes a simple form:

$$\widehat{f}^{*}\left(y',B'\right)=f\left(y'\right)\frac{1\left(y<\overline{y}\left(B'\right)\right)+\exp\left(-\frac{1}{\alpha(B')}\right)1\left(y>\overline{y}\left(B'\right)\right)}{E\left[\exp\left(-\frac{\delta}{\alpha(B')}\right)\right]} \tag{23}$$

Where $1\left(y<\overline{y}\left(B'\right)\right)$ is an indicator variable that takes the value of 1 if the condition in the argument holds. The behavior of investors can be seen in Figures 5a and 5b, where the approximating and distorted processes are compared in terms of density functions and income processes. The distorted density redistributes weight towards worse scenarios, generating a lower mean for income in the distorted process. In Figure 5, states above the threshold have a lower probability density under the distorted density, and the mass corresponding to those states is shifted towards states below the threshold, obtaining a mixed distribution as a result. Since expectations are based on this density, investors make decisions as if the distorted process was the true one, introducing a price adjustment on risky debt.



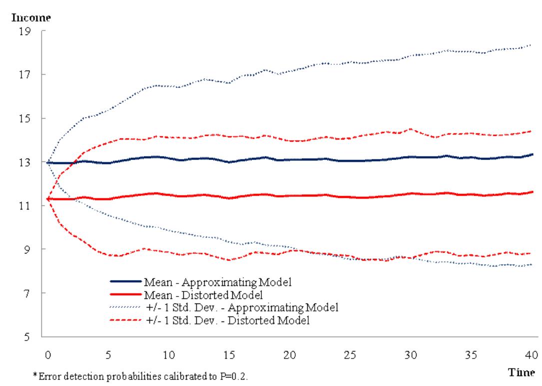




Figure 5a. Probability Density Functions under the Distorted and Approximating Model.
Figure 5b. Income process under each model.

The set of contracts available to the government is determined by the price function. Each contract $\{q\left(B'\right),B'\}$ changes consumption today by the amount $q\left(B'\right)B'$, the revenue from the transaction, creating a non-enforceable commitment to pay $B'$ in the next period. The solution is characterized by the issuance revenue term

$$q\left(B'\right)B'=\frac{1}{1+r}\left[1-\widehat{F}^{*}\left(\overline{y}\left(B'\right)\right)\right]B'$$

By proposition 5, the level of debt in equilibrium is bounded above by

$$\overline{\overline{B}} = \min_{B'} \left( \frac{1}{1+r} \left[ 1 - \widehat{F}^* \left( \overline{y} \left( B' \right) \right) \right] B' \right)$$

Figure 5c illustrates the set of contracts, showing the limit $\overline{\overline{B}}$. The figure shows the level of revenue from debt issuance as a function of next period debt. For levels of debt below $\underline{B}$ the spread over the risk-free rate vanishes, while for levels of debt above $\overline{B}$ prices are zero. But for debt levels in the interval $\left(\underline{B}, \overline{B}\right)$ bond prices are decreasing in debt, while the the revenue from issuance, $q\left(B'\right) B'$, is increasing first and then decreasing, since bond prices converge to zero. The figure illustrates this sort of 'Laffer Curve' for borrowing. Levels of debt above $\overline{\overline{B}}$ are never optimal, as alternative contracts can offer higher revenue with a lower level of next-period debt.

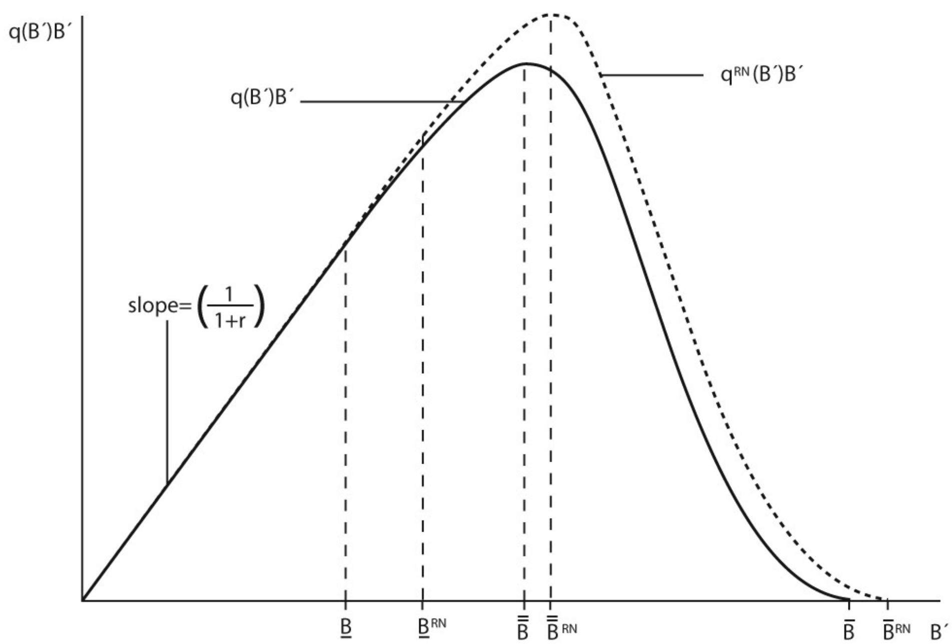



Figure 5c. Debt Issuance Revenue 'Laffer Curve'

The figure also shows the corresponding function and bounds for the risk neutral case. The lower bound $\underline{B}$ is lower relative to the risk neutral case, a consequence of proposition 2. Under model uncertainty and ambiguity aversion, positive spreads can be observed for lower levels of debt than under risk neutrality, a feature observed empirically: countries with relatively low levels of debt exhibit positive spreads.

# 6 Quantitative Analysis

This section presents the results of a calibration exercise, where the general equilibrium model is solved as a fixed point problem. I first discuss the calibration method in detail, and later on give the results. The parameter measuring the distance across densities, $\eta$, is calibrated using error detection probabilities, a method proposed by Hansen and Sargent (2008), presented in the section that follows.

## 6.1 Error Detection Probabilities

A key component in calibrating the model is the penalty function $\alpha(B')$. Its values are found calibrating the entropy distance $\eta$ using error detection probabilities. The calibration starts with an initial pair of income and debt, to then generate pairs of next period income and debt $(y', b')$, obtaining income levels from both the distorted density and the approximating density, and next-period debts from the policy function for each case. I generate time series for these variables for a period of 10 years (40 quarters), and find the likelihood ratio for the time series under both the null hypothesis of the approximating (AM) being the true model and the distorted (DM) being the true model. The process is repeated 5000 times, in order to obtain the probabilities of model detection error under each model, by counting the number of times the likelihood ratio gives the wrong model as the one with higher likelihood.

Defining $L_{AM}$ and $L_{DM}$ as the likelihood for the approximating and distorted model (that depends on $\eta$, and hence on the function $\alpha(B')$), the probabilities of detecting an error under each model are:

$$p_{AM} = \text{Prob}\left(\log \frac{L_{AM}}{L_{DM}} < 0 \mid AM\right), \; p_{DM} = \text{Prob}\left(\log \frac{L_{AM}}{L_{DM}} > 0 \mid DM\right)$$

Finally, I follow a bayesian approach, and average those two error probabilities to get the error detection probabilities $p(\eta) = \frac{1}{2}(p_{AM} + p_{DM})$. Figure 6 illustrates this procedure in a diagram where, as an example, an approximating density, denoted by $f(y)$, is surrounded by a discrete set of densities, denoted by $\widehat{f}^{(i)}(y)$. A distorted density $(\widehat{f}^{*}(y))$ is chosen from that set at a distance $\eta$ from the approximating one, leaving some densities inside the set and others outside. A number N of T-length time series from each density are built (denoted in the figure as $y$ and $\widehat{y}$) and, with these time series, the likelihood under each model is calculated. Error detection probabilities under each model are obtained by calculating the fraction of times the likelihood ratio test chooses the wrong model and the final error detection is obtained in a bayesian form, averaging the two probabilities. Since error detection probabilities implied by the distance $\eta$ must match a particular level, the initial distance must be appropriately calibrated.

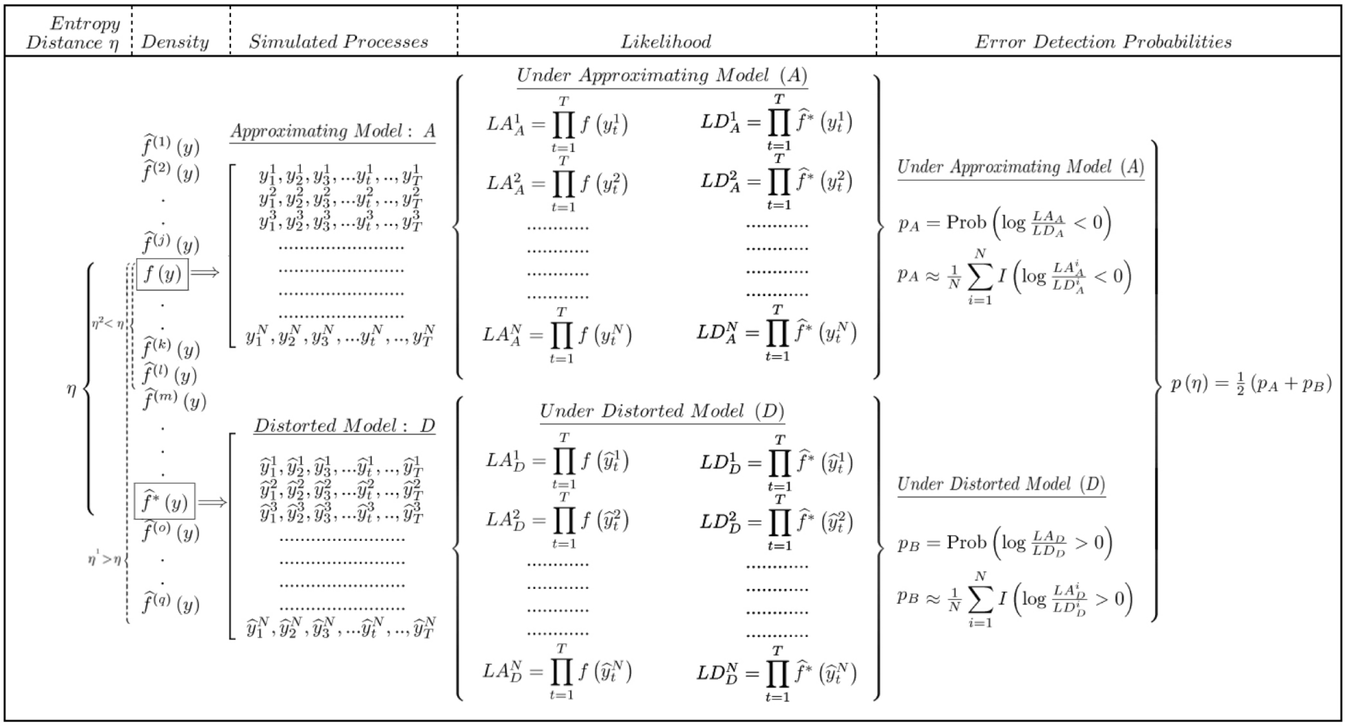



Figure 6. The distorted density is chosen by selecting the worst density (in terms of expected payoffs) from a set of densities determined by a level of error detection probabilities.

Hansen and Sargent (2008) propose fixing $p(\eta)$ to a reasonable number, then invert it to find $\eta$. I fix $p(\eta)$ to 0.2, and choose the value of $\eta$ that keeps error detection probabilities at that level. To do that, I start with some $\eta^0$,and build the function $\alpha(B')$ for that level. I then find the value functions and policy function numerically, and use the results to simulate sequences of income and debt, starting with observed levels of debt and income, under both the approximating model and the distorted model. For each pair of $(B', y')$ I find a corresponding $\alpha$ through the function $\alpha(B')$. Once the sequences are obtained, I compute $p(\eta)$. If the probability is below 0.2 (to a tolerance level), then $\eta$ is reduced to $\eta^0 - \xi$, where $\xi$ is some small number. If it is above 0.2, then $\eta$ is increased to $\eta^0 + \xi$. The process continues until a value of $\eta$ satisfying an error detection probability of 0.2 is found.

## 6.2 Calibration and Functional Forms

I assume that the output process is a log-normal autoregressive process with unconditional mean $\mu$:

$$log(y_{t+1}) = (1-\rho)\mu + \rho \log(y_t) + \varepsilon_{t+1}, \tag{24}$$

with $E(\varepsilon) = 0$, $E(\varepsilon^2) = \sigma_\varepsilon^2$. The government utility function has the CRRA form: $u(c) = \frac{c^{1-\psi}-1}{1-\psi}$.

The function that determines income in case of default, $h(y)$, is:

$$h(y) = \left\{ \begin{array}{c} y = \widehat{y} \; if \; y > \widehat{y} \\ y = y \; if \; y \leq \widehat{y} \end{array} \right\}$$

This function penalizes default more heavily during high income states, and keeps the country at the same level of income in low income states. I calibrate the model to quarterly fiscal data from Argentina, for the period from 1994 to 2001. Argentina declared default on more than \$100 billion dollars of federal government debt in December 2001, almost 40% of its GDP. This makes it an interesting candidate for the calibration, with spreads over the period going from average levels in emerging economies to levels only observed during default episodes.

Central's government revenue and debt services are used as a measure of income and debt. In addition to the level of error detection probability mentioned before, I match three moments from fiscal variables: a 3% default probability, primary deficit volatility of 0.77 and a 1.66% debt service to revenue. The first moment is based on historical evidence found in previous literature (Arellano (2008)): Argentina's government defaulted on its debt three times in the last 100 years, giving an estimate for the historical default probability. The other two moments, primary deficit volatility and debt to revenue ratio, are obtained from Argentina's data for the period 1994-2001. These moments are used to identify three parameters: the discount factor $\beta$, output costs in case of default $\widehat{y}$ and the probability of re-entry $\gamma$. Using data for the period 1994-2001, I set values for the risk free rate at 1.17% (quarterly rate for 10-year treasury bonds), the relative risk aversion of the government at 2 (as usual in the literature) and I estimate the persistence in the growth process of revenue at 0.99 and its variance at 0.10622, a calculation based on government revenue data obtained from Argentina's finance department (Mecon) for the period 1994-2001.

| In billions of AR\$, 1994 prices | Government Revenue | Government Expenditure | Debt Payments | Debt/Revenue Ratio | Primary Deficit |
|---|---|---|---|---|---|
| **Mean** | **12.67** | **12.43** | **1.66** | **0.13** | **(0.24)** |
| **Std Dev** | **1.30** | **0.96** | **0.68** | **0.05** | **0.77** |

Table 1. Main Fiscal Indicators

| Parameters | Values | |
|---|---|---|
| Risk-free Rate | r=1.17% | U.S. 5 year bond quarterly yield |
| Risk-Aversion | ψ=2 | |
| Revenue Process | μ=12.96 ρ=0.99 σ=0.10 | Argentina's Revenue process |

Table 2. Parameter Values

The calibrated parameters are consistent with the evidence on default, and their levels are almost identical to the ones obtained in the previous literature. The probability of re-entering credit markets is 0.17, in line with the evidence collected by Benjamin and Wright (2009), who find default episodes are time-consuming to resolve, taking almost eight years on average. Calibrated output costs are also consistent with the data, which shows a reduction of more than 10% in revenue during the quarters after default. Table 2 summarizes the values found in the calibration.

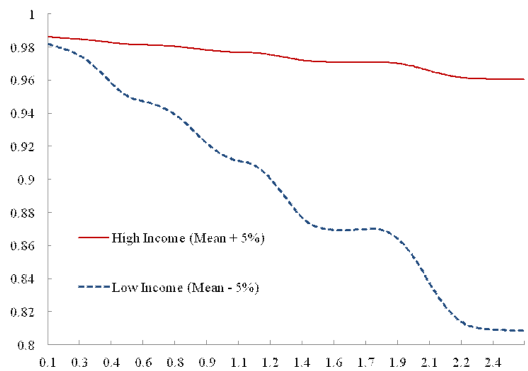
| Calibration | Values | Calibrated Moments |
|---|---|---|
| Discount Factor | β=0.97 | 3% Default Probability |
| Probability of re-entry | γ=0.17 | Primary Deficit Volatility 0.68 |
| Entropy Parameter | η=0.01 | Error Detection Probability P(η)= 0.2 |
| Revenue Cost | 0.9 E(y) | 13% Debt Services to Revenue |

Table 3. Calibration Moments and Values

## 6.3 Results

I now present the numerical results, using the calibration from the previous section, in order to give some insight on the mechanics and the implications of the model in terms of default probabilities, interest rates and debt levels. I use Argentina's data as a benchmark, and do some comparative statics to show the effect of changes in uncertainty.

The bond price function is presented in Figure 7. Larger debt levels imply lower prices, but higher income levels shift the function up, since default is less likely for high income scenarios and the process is very persistent. As a consequence, booms are associated with more benevolent debt contracts, generating a higher incentive to issue debt due to the low spreads. This feature replicates business cycles in emerging economies where, unlike what occurs in developed economies, crises are associated with high interest rates and peaks with low interest rates. It was certainly the case of Argentina, whose debt increased substantially during the boom, encouraged by low interest rates. Along the same lines, the model is also able to predict Argentina's default, a consequence of the low revenue, heavy debt burden and high interest rates observed during the last quarter of 2001. By then, spreads were above 4000 basis points. The model captures these high spreads through high default probabilities under the distorted density.



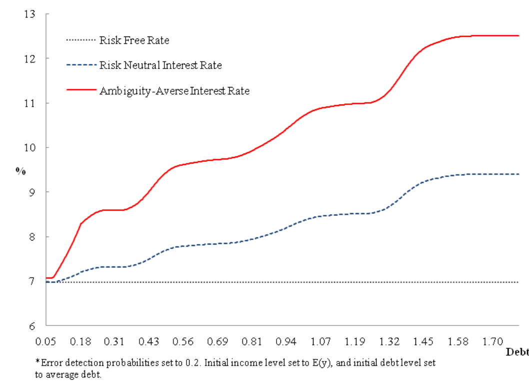




Figure 7. Bond Price as a Function of Debt. Figure 8. Interest rates as a function of debt

While the countercyclical behavior of interest rates is matched reasonably well in the literature, the size of spreads observed in the data is not explained. The first contribution of this model is to generate high returns on debt while keeping default probabilities at historical levels. This has been a problem for recent models of default, unable to generate high risk adjustments. In this paper, not only the level of debt and revenue, but also the level of uncertainty in the economy (that feeds into higher uncertainty regarding the true process) generate higher interest rates: the difficulty of distinguishing different income processes in the data produces a price adjustment term, as agents guard themselves against misspecification errors. This is captured by Figure 8, that compares the interest rate function under risk neutral and ambiguity averse investors.

The spread increases considerably when investors are ambiguity-averse, even for low levels of debt, a feature observed empirically. Table 4 illustrates these results, as well as other moments generated from the model. While the empirical mean of interest rates is well captured, the model has difficulties matching the volatility: in the model, spreads are higher than in the data during crises and lower during booms, producing a higher standard deviation. This tendency of bond prices to overshoot is the main anomaly of the model, a feature also present in the rest of the literature.

The main parameter in the calibration behind these results is $\eta$, the distance across densities as measured by the entropy constraint. Since error detection probabilities are calibrated to a particular value, $\eta$ is determined by that value, given some approximating density. If the approximating model's variance is higher, then the same level of error detection probabilities is going to imply a higher $\eta$, generating an overall higher price adjustment. This suggests that countries with a more volatile cycle will pay higher interest rates, not only due to the higher likelihood of default, but because of the difficulty to identify the approximating process from the data.

Table 4 also shows the results from an identical exercise with a 10 percent increment in volatility. Detection errors probabilities are kept fixed to 0.2, so the entropy distance $\eta$ is re-calibrated to a higher value. The benchmark distance for $\eta$ was initially calibrated at 0.014, and the new distance is 0.02. I use the values found in the benchmark for the rest of the parameters. The goal of this alternative assumption is to capture the effect on spreads of higher volatility. While investors' preferences are not affected by this change in volatility, primary deficit, default probabilities and the level of debt should be affected. For that reason, only error detection probabilities are kept at benchmark values, and moments related to fiscal variables are allowed to re-adjust.

Spreads are now higher than before, due to two factors: the increment in probability of default and the higher level of uncertainty, reflected in a higher $\eta$. Spreads increase significantly more for the ambiguity-averse case, since the last effect is not present in the risk-neutral framework. This is consistent with the high spreads observed during financial turmoil periods, when model uncertainty is particularly high.

| Business Cycle | Model | | Model (High Std Dev) | | Data | |
|---|---|---|---|---|---|---|
| | Mean | Std Dev | Mean | Std Dev | Mean | Std Dev |
| Expenditure | 12.62 | 0.76 | 12.58 | 0.93 | 12.83 | 1.61 |
| Debt | 1.69 | 0.67 | 1.58 | 0.72 | 1.10 | 0.70 |
| Spread (Ambiguity-Averse) | 5.52 | 5.50 | 7.72 | 6.94 | 5.49 | 2.35 |
| Spread (Risk-Neutral) | 1.81 | 2.42 | 2.51 | 2.86 | | |

Table 4. Main Business Cycle Moments

Digging deeper into the components of the price adjustment, the model has also implications in terms of the market price of model uncertainty. The larger difficulty in distinguishing densities for low probabilities of default implies that the market price of model uncertainty is decreasing in debt. This leads to a partial offset in price adjustments for high debt countries. The reduction in the market price of uncertainty implies that, keeping the rest of the variables constant, riskier countries are less heavily penalized in terms of market price of risk, a result previously absent in the literature. Empirically, it would imply a low standard deviation in the cross section of sovereign bond yields, a feature observed in the data. The figure below shows the off-setting effect of this result on the price adjustment term. The adjustment term increases at a lower rate than risk as debt gets larger, giving an explanation to the small differences in yields across countries.

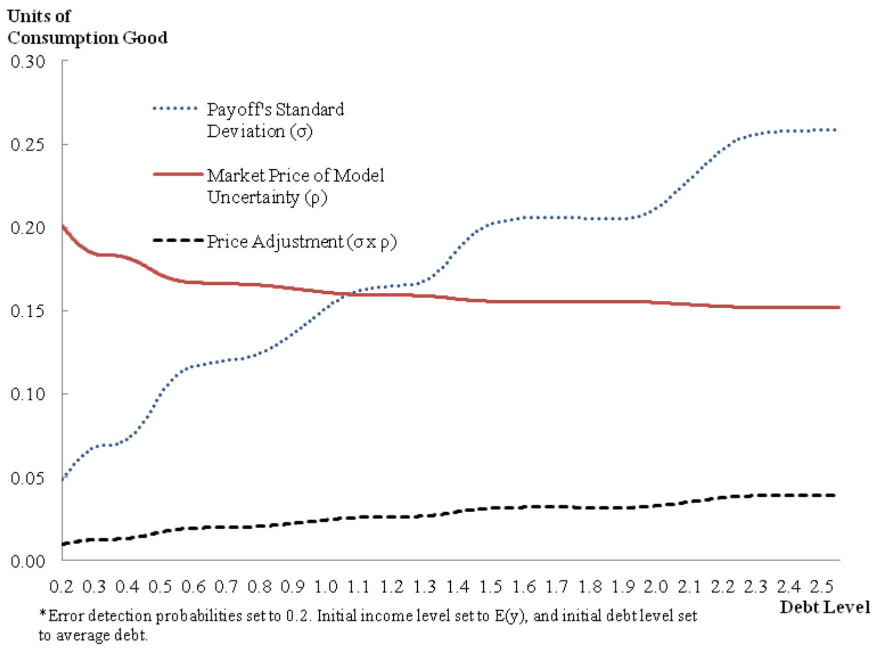



Figure 9. Price Adjustment Components as a function of debt

A final question for this calibration exercise is whether default is more common under this framework compared to the risk-neutral case. The following proposition addresses that question. I define the "stopping time" function $\tau$ as the first time period after t such that the default occurs:

$$\tau^* = \min_{\tau} \tau \text{ s.t. } V^D\left(\widetilde{B}\left(B_{t+\tau-1}, y_{t+\tau-1}\right), y_{t+\tau}\right) > V^C\left(\widetilde{B}\left(B_{t+\tau-1}, y_{t+\tau-1}\right), y_{t+\tau}\right) \tag{25}$$

The average default time, or average stopping time, is defined as the expected value of $\tau$ conditional on information up to t, $E\left[\tau \mid y_t\right]$. Since the stopping time $\tau\left(B_t, y_t\right)$ depends on the endowment process for both the distorted and approximating model, it is a random variable whose distribution can be found numerically. Its expected value converges to:

$$\overline{\tau} = E\left[\tau^* \mid y_t\right] = \sum_{i=1}^{\infty} (t+i)\, p^D\left(s_{t+i}\right) \prod_{j=1}^{i} \left[1 - p^D\left(s_{t+j}\right)\right] \tag{26}$$

**Proposition 9**. *The average default time is decreasing in* $\eta$. Proof: See appendix.

This result implies that the frequency of defaults observed when investors are ambiguity averse is higher than in the case of risk neutral investors (a case where $\eta = 0$, since both densities are identical). The higher premiums on new debt can only make the country worst off, reducing the incentive to remain in the contract, since default value functions are less affected by the price function. However, the increment in prices also reduces the actual level of debt issued by the country, partially offsetting the incentive to default.

To analyze these effects I estimate the average default time for different initial values, by running 5000 time series simulations of the model. For the average level of debt for Argentina during the period, the average default time $\overline{\tau}$ under the distorted model is 13.3 years. Under the approximating model it is 18.9, more than 40 percent higher. However, for levels of debt around initial values, where spreads were lower, the difference is proportionally smaller: 35.72 years under the approximating model and 31.06 under the distorted - only 15 percent higher. This result allows the calibration to match historical probabilities of default, and still find parameters for output costs $\widehat{y}$ and the probability of recovering access to credit markets $\gamma$ on levels extremely similar to those in the previous literature, found under risk neutrality. Since the values obtained for these parameters are consistent with empirical observations, the higher spread is not a result of a different calibration. It is truly a consequence of investors attitude towards uncertainty.

# 7 Concluding remarks

Model uncertainty can reconcile historical default probabilities with observed market returns on risky debt. By introducing ambiguity-averse investors, the model can generate risk premia in the levels observed in the data. Investors charge a premium on risky debt by slanting probabilities towards worse but plausible scenarios, determined by an entropy constraint. The entropy constraint limits those scenarios according to their likelihood in the data, allowing for processes similar enough to the approximating model of the economy but hard to distinguish empirically. This generates a price adjustment term, composed of two parts: a pure risk part, determined by the volatility of the expected payoff, and the market price of model uncertainty. The latter is the main component of the risk premium.

The market price of model uncertainty is decreasing in debt. This result implies that higher risk is partially offset in the risk premium by a reduction in the market price of model uncertainty. Riskier countries then have risk premia that increase less than proportionally with risk, explaining the small differences in the spreads across countries. The calibrated model also shows the role of uncertainty in the risk premium. Since the premium is due to the difficulty in distinguishing different models from the data, countries with higher volatilities in their income will also have higher premiums. In those countries, investors will find it more difficult to understand the process behind the variable that determines default and hence will punish debt more severely through risk premia.

The calibrated model shows that these results are a genuine consequence of investors' ambiguity aversion. The calibration matches historical default probabilities, the primary deficit volatility, the average debt/revenue ratio and error detection probabilities (set at 0.2) for Argentina's process. While the calibrated parameters are almost identical to the ones obtained by the previous literature, spreads are significantly higher, even for low debt values. These results are consistent with the empirical evidence on spreads and can explain the small differences across countries, a consequence of a decreasing market price of uncertainty that partially counterbalances higher risk.

# 8 Appendix

## 8.1 Computational Algorithm

The value function for the debtor is approximated using a collocation method based on splines for each value function, and expectations are calculated by numerical integration, using a 20-node Gaussian quadrature procedure. Then, the worst case density for the investor is found and the price function is constructed, also through splines. The numerical procedure implies the following steps:

1. Start with guess for parameters, a bond price function, and a function that determines the default decision, as well as first and second derivatives of the those functions.
2. Use spline polynomial approximation, selecting the nodes, basis functions and the initial guesses for the coefficients on the functions from 1., as well as the initial guess for the coefficients on the value functions in case of default and repayment.
3. Given the price function, solve for government policy function and repayment sets by using Newton's method on the value function.
4. Using default and repayment sets compute new price function. Come back to 2) if price function different than initial, updating with Gauss-Seidel.
5. Compute statistics from N samples of data and adjust parameters until moments and error detection probabilities are matched to 0.2.

## 8.2 Propositions

**Proposition 1**. A solution for $\alpha\left(B'\right)$ exists if and only if $p^D\left(B',y\right) < e^{-\eta}$.

Proof: Define, for a given level of B': $F\left(\alpha,p^D\right) = \frac{1}{\Psi(\alpha)}\log\left(\frac{1}{\Psi(\alpha)}\right)p^D + \frac{e^{-\frac{1}{\alpha}}}{\Psi(\alpha)}\log\left(\frac{e^{-\frac{1}{\alpha}}}{\Psi(\alpha)}\right)\left(1-p^D\right) - \eta\left(1+r\right)$

$F\left(\alpha,p^D\right)$ is continuous in $\alpha$., and $\underset{\alpha\to\infty}{lim} F\left(\alpha,p^D\right) = -\eta\left(1+r\right) < 0$

Since $\frac{\partial F\left(\alpha,p^D\right)}{\partial\alpha} = -\frac{1}{\alpha^3}\frac{e^{-\frac{1}{\alpha}}}{\Psi^2(\alpha)}p^D\left(1-p^D\right) < 0.$, and $F\left(\alpha,p^D\right) > 0$ if and only if $\underset{\alpha\to 0}{lim} F\left(\alpha\right) = \log\left(\frac{1}{p^D}\right) - \eta\left(1+r\right) > 0$,there is a solution if and only if $p^D < e^{-\eta(1+r)} \equiv \overline{p}$

**Proposition 5.** *The policy function* $\widetilde{B}\left(B,y\right)$ *is bounded above.*

To see this, notice that the term $\frac{\partial[q(B^*,y)B^*]}{\partial B^*}$ must be positive for the first order condition to hold in an interior solution. Suppose $\frac{\partial[q(B^*)B^*]}{\partial B^*} = \frac{\partial q(B',y)}{\partial B'}B' + q(B',y) < 0$. In that case, the country could reduce the level of debt and increase welfare, since consumption in the current period would increase, and the continuation value would be higher due to proposition 3. Hence, that inequality cannot hold in equilibrium. In order to have an interior solution, we must have $\frac{\partial[q(B^*)B^*]}{\partial B^*} = \frac{\partial q(B',y)}{\partial B'}B' + q(B',y) > 0$.

By proposition 4, the price function is decreasing in debt, so the first term is negative. Since Z si arbitrarily chosen, we can make $q(Z,y) = \xi$, for $\xi$ small enough, and there will be a $\overline{B}(y)$ such that $\frac{\partial q\left(\overline{B}(y),y\right)}{\partial \overline{B}(y)}\overline{B}(y) + q\left(\overline{B}(y),y\right) = 0$, for all $y$. And the level of debt will always be below that threshold.

**Proposition 8.** *The country's debt policy function is decreasing in the price of the bond.*

Consider $q^1(B',y) > q^2(B',y)$ for one period. The claim of the proposition is that then B$^1 > B^2$. By contradiction, suppose $B^2 \geq B^1$.In that case, two things can happen:

Case 1) $q^2(B',y^1)B^2 < q^1(B',y^1)B^1$

$u(y - B + q^2(B',y^1)B^2) + \beta EV^o(B^2,y') < u(y - B + q^2(B',y^1)B^2) + \beta EV^o(B^1,y') < u(y - B + q^1(B',y^1)B^2) + \beta EV^o(B^1,y') < u(y - B + q^1(B',y^1)B^1) + \beta EV^o(B^1,y') \Rightarrow B^1 > B^2$. And a contradiction is reached..

Case 2) $q^2(B',y^1)B^2 > q^1(B',y^1)B^1 \Rightarrow \frac{\partial qB}{B} > 0$

$u(y - B + q^2(B',y^1)B^2) + \beta EV^o(B^2,y') < u(y - B + q^1(B',y^1)B^2) + \beta EV^o(B^2,y') < u(y - B + q^1(B',y^1)B^2) + \beta EV^o(B^1,y') < u(y - B + q^1(B',y^1)B^1) + \beta EV^o(B^1,y')$

This is a contradiction to $B^1 < B^2$.

**Proposition 10**. *The average default time is decreasing in $\alpha$.*

This is a corollary of propositions 8 and 2.

## 8.3 Value Functions and Income Process

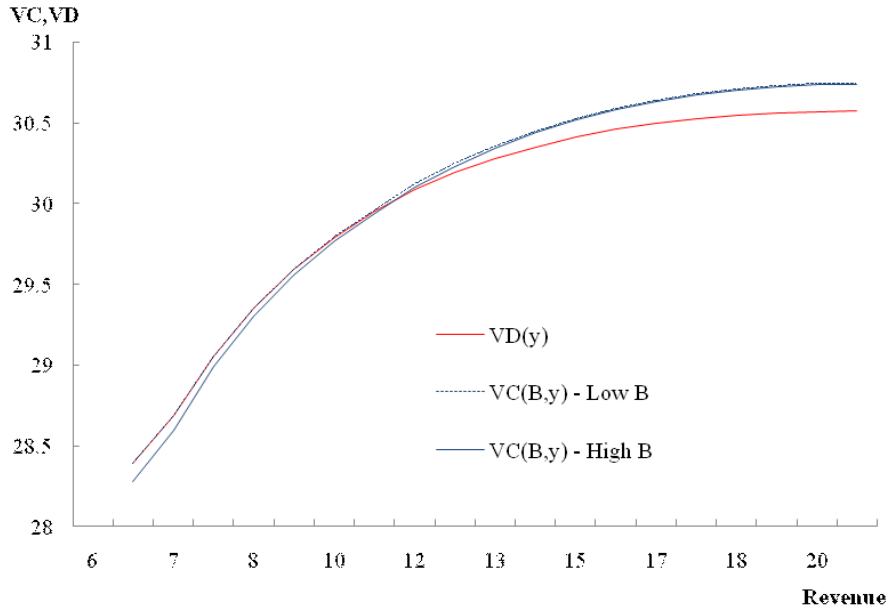



Figure 10. Value Function in terms of revenue.

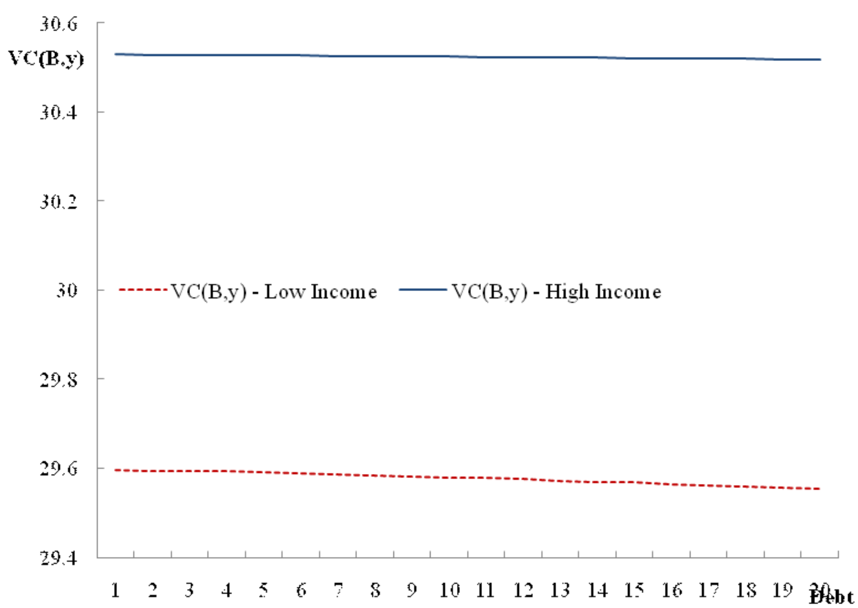



Figure 11. Value Function in case of continuation, in terms of debt.

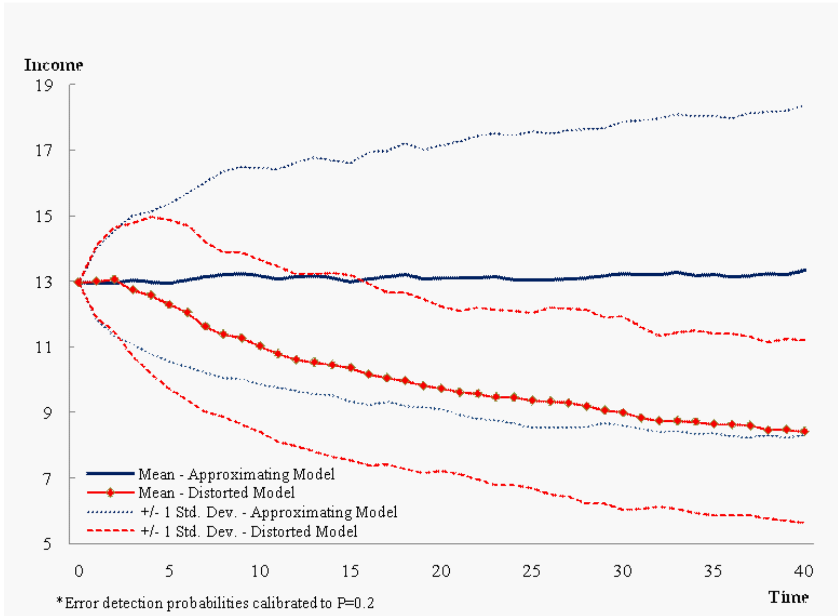



Figure 12. Income process under each model.

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