Pure public goods and incomplete information. Multiplicity and implementation

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Abstract

The paper analyzes the implementation of the Condorcet Winner. It is shown that for a non negligible set of probability distributions exact implementation of Condorcet consistent social choice functions cannot be obtained. A trade-off between the robustness of the Condorcet Winner and Bayesian monotonicity appears. A positive result is provided for uniform priors.

1 Introduction

The aim of this paper is to investigate the implementation of social choice rules under incomplete information in environments with pure public goods. It is a case in which the classical results on Bayesian implementation do no hold. Typically such environments are not economic (in the sense of Jackson (1991)) so the condition of Bayesian Monotonicity need not to be sufficient for a social choice set for being implementable in Bayesian equilibrium.

We start from the case in which agents have single peaked preferences on a finite set of allocations and we focus on the Condorcet Winner correspondence. As a reference point in Social Choice and Political Science is of interest in itself. But it seems particularly suited to the scope because it is Pareto optimal and strategy-proof. It is implementable in dominant strategies by the simple mechanism which asks players to vote their favorite option. It is then ordinally incentive compatible (Majumdar and Sen (2004)) on the domain of single peaked preferences. It means that it is incentive compatible for all informational structures and for any utility representation. But voting mechanisms produce a multiplicity of equilibria and this is not an exception. Even if the Condorcet Winner is implementable in Nash Equilibrium when information
is complete (it is Maskin Monotonic and satisfies No-veto Power) the implementing mechanism is much more complex than voting to eliminate multiple equilibria.

And apart of incentive compatibility, multiplicity is the other main obstacle to implementation. Laffont and Maskin (1982) observed that the incentive compatible mechanism proposed by d’Aspremont and Gerard-Varet (1979) for a class of public goods problems may generate multiple suboptimal Bayesian equilibria. And the difference between the so-called mechanism design and “exact” Bayesian implementation stays exactly here: the objective of mechanism design is to build game forms whose equilibrium outcomes include the desired social choice function, in Bayesian implementation the equilibrium outcome must coincide with the desired social choice function (or set).

The literature studying exact implementation under incomplete information has devoted most of its attention to environments either containing only private goods (Palfrey and Srivastava (1989)) or characterized by a transferable good. Not only the transferrable utility assumption makes such environments economic. It results to be crucial in many positive results in order to eliminate undesired equilibria by separation arguments. In environment with transferable goods, Palfrey (1992) and Palfrey and Srivastava (1993) prove that, under appropriate assumptions on priors (independence and a consistency condition on beliefs), incentive compatibility is not only necessary but also sufficient for a social choice function to be implementable in Bayesian equilibrium. Arya et al. (2000) show that in a principal-agent model of adverse selection, the single crossing property and a strong incentive compatibility property are sufficient and almost necessary for exact Bayesian implementation.

In our framework things change dramatically even if incentive compatibility is satisfied for any prior. Bayesian Monotonicity is shown to be sufficient for Bayesian implementation even if the environment is not economic in the sense defined by Jackson (1991). We prove that the existence of a state of the world in which each of the players is “redundant” in the sense that changing her type would not affect the outcome, make implementation quite complex. These are states of the world in which the Condorcet Winner beats all other candidates by at least two votes. The first finding is that if one of these point have high enough probability to occur the multiple equilibrium problem cannot be solved. If all players believe that their probability of influencing the final outcome is too low, there is no way to eliminate “bad equilibria”, without allowing profitable deviations from the desired allocations. The threshold probability depends on the utility cardinalization., then on agents’ risk attitudes. It is lower the higher the perceived difference among candidates. It is
a straightforward consequence that, if all these states have null probability to occur, the Condorcet Winner is Bayesian monotonic and then implementable in Bayesian equilibrium. The result can be extended to a more general framework. More precisely whenever a social choice function make all agents not pivotal at some state of the world, then for an open set of beliefs and utility representations such social choice function is not Bayesian monotonic. The result suggests a general trade-off between incentive compatibility and Bayesian Monotonicity. The existence of such “redundant states” makes incentive compatibility easier to be satisfied. On the other hand if they are too likely to occur the make impossible to eliminate bad equilibria. As corollary it follows that all Condorcet consistent social choice functions are not implementable in Bayesian Nash equilibrium for an open set of beliefs. The Condorcet Winner is implementable in dominant strategies, Condorcet consistent rules aren’t, in general.

When the information is complete the Condorcet Winner is implementable in Bayesian equilibrium. But we are able to provide a different positive example If at least one agent has uniform beliefs on other players’ peaks the Condorcet Winner is Bayesian monotonic. In this case it is possible to “transfer utility” from one state to another one as a sufficient number of states are equiprobable. So implementation is possible for two informational opposite cases: almost complete information and almost complete ignorance. In Triossi (2004) is shown that when a a social choice function is Bayesian Monotonic at some profile of beliefs then it is Bayesian Monotonic in a neighborhood of such profile, the both examples are non generic.

The structure of the paper is the following. In the next section we introduce the framework of Bayesian implementation and of single peaked environments. We show directly that Bayesian Monotonicity is sufficient for implementing the Condorcet Winner in Bayesian Nash equilibrium. In the third section we prove the impossibility result and we extend it to more general environments. In the fourth section we provide a positive result for the case of uniform priors. The fifth section draws the conclusions and possible directions for future investigation on the topic.

2 Main definitions

Let \( N = \{1, ..., n\} \) be the set of agents. \( A \) denotes the (finite) set of possible alternatives: \( A = \{a_1, ..., a_p\} \). For each \( i \in N \), the set \( S^i \) describes the possible, types or attributes of agent \( i \). We consider only

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3In this case Bayesian Implementation and Nash Implementation are equivalent.
finite type spaces $S^i$, $|S^i| < \infty$. Let $S = \prod_{i=1}^{n} S^i$ be the states space.

At each $s \in S$, agent $i$’s preferences on $A$ are represented by a utility function $U^i(\cdot \mid s) : A \rightarrow \mathbb{R}^2$. For each $i \in N$ set $S^{-i} = \prod_{j\neq i} S^j$. For all $B \subset S$ let $B^{-i} = \{s^{-i} \in S^{-i} : \exists s^i \in S^i, (s^{-i}, s^i) \in B\}$, be the projection of $B$ on $S^{-i}$ and let $B^i = \{s^i \in S^i : \exists s^{-i} \in S^{-i}, (s^{-i}, s^i) \in B\}$ be the projection of $B$ on $S^i$.

Each agent $i$’s information is summarized by a collection of conditional distribution functions on $S^{-i}$. For each $s^i \in S^i$, $q^i(\cdot \mid s^i)$ describes agent $i$’s beliefs. If $B \subset S^{-i}$ let $q^i(B \mid s^i)$ be the probability that $s^i$ assigns to $B$: $q^i(B \mid s^i) = \sum_{(s^{-i}, s^i) \in B} q^i(s^{-i} \mid s^i)$. We assume that agents beliefs are consistent, which is for all $i, j \in N$, for each $s \in S^i$, $q^i(s^{-i} \mid s^i) > 0$ if and only if $q^j(s^{-j} \mid s^j) > 0$. We denote by $T = \{s \in S : \exists i \in N, \exists s^i \in S^i, q^i(s^{-i} \mid s^i) > 0\}$ the set of states occurring with positive probability.

A social choice function (SCF) is a function $x : S \rightarrow A$. $X$ denotes the set of all SCF. A social choice set (SCS) is a subset of $X$.

Agents’ preferences on social choice functions are specified as follows. Let $x, y$ be SCF and let $s^i \in S^i$. Then

$$xR^i(s^i)y \iff \sum_{s^{-i} \in S^{-i}} q^i(s^{-i} \mid s^i)U^i(x(s) \mid s) \geq \sum_{s^{-i} \in S^{-i}} q^i(s^{-i} \mid s^i)U^i(x(s) \mid s)$$

Two SCF, $x, x' \in X$ are equivalent if they agree on $T$, which if $x(t) = x'(t)$ for all $t \in T$. In this case we will write $x \sim x'$. Two social choice sets $F, F' \subset X$ are equivalent and we will write $F \sim F'$, if for all $x \in F$, there exists $x' \in F'$ such that $x \sim x'$ and for all $x' \in F'$ there exists $x \in F$ such that $x \sim x'$.

An environment is a family $[N, S, A, \{q^i\}_{i \in I}, \{U^i(\cdot \mid s)\}_{i \in N, s \in S}]$. The structure of the environment is assumed to be known by the agents and by the planner.

### 2.1 Implementation

A mechanism is a pair $(M, g)$ where $M = \prod_{i=1}^{n} M^i$ is the action or message space and $g$ is a function from $M$ to the outcome space $A$, $g : M \rightarrow A$. A strategy for player $i$ is a map $\sigma^i : S^i \rightarrow M^i$. Let $\Sigma^i = (M^i)^{S^i}$ be player $i$’s strategy space and let $\Sigma = \prod_{i=1}^{n} \Sigma^i$ be the strategy space.

\(^2\)All along the paper we will assume that utility functions depends only on individual agents' states and we will write simply $U^i(\cdot \mid s^i)$. 
Let $\sigma = (\sigma^1, ..., \sigma^n)$ be a vector of strategy and let $s = (s^1, ..., s^N)$ be a vector of types, then $\sigma(s)$ denotes the vector of actions $\sigma(s) = (\sigma^1(s^1), ..., \sigma^n(s^n))$. $\sigma \in \Sigma$ constitutes a Bayesian-Nash equilibrium (BNE) for the game induced by the mechanism $(M, g)$ if for all players $i$, $g(\sigma) R^i(s^i) g(\sigma^{-i}, \tau_i)$ for all $s^i$ in $S^i$, $\tau_i \in \Sigma^i$.

**Definition 1** A mechanism $(M, g)$ implements a SCS $F$ in Bayesian-Nash equilibrium if there exists $F^* \sim F$ such that i) for each $x \in F^*$, there exists a BNE $\sigma$, induced by $(M, g)$ such that $g(\sigma) = x$ ii) for each $\sigma$, BNE induced by $(M, g)$, $g(\sigma) \in F^*$.

**Definition 2** $F$ is implementable in Bayesian-Nash equilibrium, if there exists a mechanism which implements $F$ in Bayesian-Nash equilibrium.

We next present necessary conditions for a social choice set to be implementable in Bayesian equilibrium.

**Definition 3** A social choice function is incentive compatible if for each $i \in \{1, N\}$, $x(s^i, \cdot) R^i(t^i) x(t^i, s^{-i})$ for all $s^i, t^i \in T^i$. $x \in F$.

**Definition 4** A social choice set $F$, is incentive compatible if $x$ is incentive compatible for all $x \in F$.

Incentive compatibility assures that true-telling is a Bayesian equilibrium of the revelation game $(S, x)$ for all $x \in F$. Incentive compatibility depends on players’ believes and on utility cardinalization. A stronger conditions which guarantees incentive compatibility independently on the priors and on utility cardinalization is strategy-proofness.

**Definition 5** A social choice function $x$, is strategy-proof if for all $i \in \{1, N\}$, $U^i(x(s^i, s^{-i}) \mid s^i) \geq U^i(x(t^i, s^{-i}) \mid s^i)$ for all $s^i, t^i \in S^i$ and for all $s^{-i} \in S^{-i}$.

If the preference domain is the whole set of preferences on $A$, then any strategy-proof SCF must be dictatorial (Gibbard (1973), Satterthwaite (1975)). For restricted domain of preferences non dictatorial strategy-proof schemes exist. Majumdar and Sen (2004) show that any ordinal Bayesian incentive compatible social choice function is dictatorial. Priors are supposed to be independent and satisfying an additional property.

A function $\alpha^i : S^i \rightarrow S^i$ is said to be a an individual deception. A profile $\alpha = (\alpha^1, ..., \alpha^n)$ will be called a (joint) deception. $\mathbf{1}$ will denote the identity function. $\mathbf{D}^i$ will denote the set of agent $i$'s individual
deceptions and $D = \prod_{i=1}^{n} D^i$ will denote the set of joint deceptions.

Let $x : S \to A$ be a SCF in $X$ and let $\alpha$ be a joint deception. We
will use $x_{\alpha}$ to denote $x \circ \alpha$, the composition of $x$ and $\alpha$. If $\alpha^i$ is an
individual deception $x_{\alpha^i}$ will denote the $x \circ (1-i, \alpha^i)$, defined by $x_{\alpha^i}(s) =
(x(s^i, \alpha^i(s^i)))$ for all $s \in S$. Let $B \subset X$ be a social choice set
and let $\alpha$ is be joint deception, $B_\alpha$ will denote the set of the SCF that are
composition of members of $B$ and $\alpha$, formally $B_\alpha = \{x_{\alpha} ; x \in B\}$.

Definition 6 Let $F$ be a social choice set and let $\alpha$ be a joint deception.
$F$ satisfies $\alpha$-Bayesian Monotonicity if for all $x \in F$ such that $x_{\alpha} \notin F$ there exist
$i = i(\alpha) \in N$, $s^i \in T^i$ such that $y_{\alpha_i} P^n(s^i)x_{\alpha}$
and $x(R_s^i(t^i)y_{\alpha_i}(s^i))$ for all $t^i \in T^i$.

Definition 7 $F$ is Bayesian monotonic if it satisfies $\alpha$-Bayesian
monotonicity for all joint deceptions, $\alpha$.

Bayesian monotonicity provide to the planner the instruments to
eliminate unwanted equilibrium. Assume all players deceive, using de-
ception $\alpha$. If $x_{\alpha} \notin F$ and $F$ is $\alpha$-Bayesian monotonic then the planner
can offer to player $i$ at state $s^i$ a profitable deviation by offering her $y$ as $y_{\alpha_i} P^n(s^i)x_{\alpha}$.
At the same time such offer does not eliminate the true
telling equilibrium yielding $x$: no type $t^i$ can profit from asserting of
being of type $\alpha^i(s^i)$ because $x(R_s^i(t^i)y_{\alpha_i}(s^i))$ for all $t^i \in T^i$.

It is a well known result (Postlewaite and Schmeidler (1986), Pal-
frey and Srivastava (1989a)) that incentive compatibility and Bayesian
Monotonicity are necessary for a social choice set to be implementable in
Bayesian equilibrium). In economic environments (Jackson (1991)) in-
centive compatibility and Bayesian monotonicity are sufficient as well\footnote{A stronger result (Palfrey (1992)) can be obtained if agents’ utility is transferable and there are no consistent deceptions (which means that any deception induces a probability distribution which is different than true-telling). In such a case incentive compatibility alone is sufficient for implementation.}
Bayesian Monotonicity is needed to eliminated unwanted multiple equi-
libria under incomplete information in the same way Maskin Monotonic-
ity is needed to eliminate multiple Nash equilibria under complete inform-
ation.

2.2 Single peaked preferences and the Condorcet Winner

A preference relation $P$, on $A$, is single peaked, if there exists an
alternative $a^*(P) = a_k \in A$, called the peak of $P$ such that, for all
$s, t = 1, ..., q$, $a_t P a_s$ whenever $s < t < k$ or $k < t < s^i$. An agent has

\footnote{We do not allow for indifference on peaks, but the results can be easily generalized to include this case.}
single peaked preferences, if for any pair of alternatives that are on the same side with respect to the peak, she prefers the one which is nearer to the peak. In an alternative way, a preference relation is single peaked if $P$ has no interior local minima. Let $SP(A)$ be the set of single peaked preferences on $A$. Let $a, b \in A$ and let $P = (P^1, \ldots, P^n)$ be a profile of preferences on $A$.

All along the paper, if not otherwise specified we consider only single peaked environments where, for each $s^i \in S$, $U^i(\cdot | s^i)$ represents a profile of single peaked preferences.

Let $U^S$ be the set which contains the representation of utility functions for one of such environments having $S$ as type space. From now on we consider it endowed with the euclidean topology. For all $P = (P^1, \ldots, P^n) \in SP(A)$ set $S(P) = \{s \in S : U^i(\cdot | s^i) \text{ represents } P\}$. $S^i(P^i) = \{s^i \in S^i : U^i(\cdot | s^i) \text{ represents } P^i\}$.

Let $P(a) = \{P \in SP(A) : a^*(P) = a\}$ be the set of single peaked preferences having $a$ as peak. $S^i(a)$ will denote $S^i(P(a))$. For all $s^i \in S^i$ let $P^a_s$ denote the preferences represented by $U^i(\cdot | s^i)$. Let $P_s$ denote the preference profile at state $s \in S$.

$N(a, b, P)$ denotes the number of agents preferring $a$ to $b$ under $P$:

$$N(a, b, P) = |\{i \in N : a^P_i b\}|.$$ 

**Definition 8** Let $P = (P^1, \ldots, P^n)$ be a profile of preferences on $A$. An outcome $a \in A$ is a **Strong Condorcet Winner at $P$** if, for all $b \in A$ if $N(a, b, P) > N(b, a, P)$ for all $b \neq a$.

If $N(a, b, P) \geq N(b, a, P)$ for all $b \neq a$, a is a **(Weak) Condorcet Winner**.

The following Remark summarizes some well known properties of the Condorcet Winner (see, for instance Moulin (1980))

**Remark 1** Let $P = (P^1, \ldots, P^n) \in SP(A)$ and assume that $k_1 \leq \ldots \leq k_n$, where $a_{k_i} = a^*(P^i)$ for $i = 1, \ldots, n$.

If $n$ is odd a unique Condorcet Winner exists and it is the median of agents’ peaks, namely $a^*(P_{\frac{n+1}{2}})$.

If $n$ is even then the set of weak Condorcet Winners is not empty and consists of all outcomes between $a^*(P_{\frac{n}{2}})$ and $a^*(P_{\frac{n+1}{2}})$.

All selections from the Condorcet Winner Correspondence is individually and group strategy-proof on $SP(A)$. 

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Let \( CW(P^1, \ldots, P^n) \) to denote the Condorcet Winner Correspondence evaluated at \((P^1, \ldots, P^n) \in SP(A)^n \). \( CW(s) \) will denote \( CW(P_s) \). With abuse of language we use Condorcet Winner to indicate the SCS which contains all selections from the Condorcet Winner Correspondence. Single-peakedness and Condorcet Winner are purely ordinal concept. The informational requirements to compute the Condorcet Winner in single peaked domain are very low. It is simply needed that each agent to communicate her peaks.

Bayesian Monotonicity is sufficient for the Condorcet Winner to be implementable in Bayesian-Nash equilibrium. We provide a complete proof. The result is not implied by the known characterizations (see Jackson (1991)) as the environment is not economic.

**Proposition 1** The Condorcet Winner is Implementable in Bayesian Nash equilibrium if and only if it is Bayesian monotonic.

**Proof.** As already pointed out Bayesian Monotonicity is necessary to Bayesian implementation. Let’s proof sufficiency. Define the following message space for agent \( i \), \( M^i = S^i \times A^{S^i-1} \times D \times N^{S^i-1} \). \( A^{S^i-1} \) denotes the set of functions from \( S^{-i} \) to \( A \) and \( N^{S^i-1} \) denotes the set of functions from \( S^{-1} \) to \( N = \{0, 1, 2, 3, \ldots \} \), the set of integer numbers. Let \( m^i = (m^i_1, m^i_2, m^i_3, m^i_4) \) to denote a generic message for agent \( i \), where \( m^i_1 \in S^i \), \( m^i_2 \in A^{S^{-i}} \), \( m^i_3 \in D \), \( m^i_4 \in N^{S^{-i}} \). For \( h = 1, 2, 3, 4 \) \( m_h \) will denote the message profile \((m^1_h, \ldots, m^n_h)\) 

Set \( D_1 = \{ m : m^i_3 = 1 \text{ for all } i \} \cap \{ m : m^i_2(m_1^{-i}) \in CW(m_1) \text{ for all } i \} \)

\( D_2 = \{ m : m^i_3 = 1 \text{ for all } i \neq j \} \cap \{ m : m^j_2(m_1^{-i}) \in CW(m_1) \text{ for all } i \neq j \} \) \( \setminus D_1 \), for \( j = 1, \ldots, n \)

\( D_3 = \bigcup_{i=1}^n D_2 \)

\( D_3 = M \setminus (D_1 \cup D_2) \)

For all \( m \in M \), set \( j^*(m) = \min \arg \max_{i=1, \ldots, n} \{ m_4(m_1^{-i}) \} \). For all \( \alpha \in D \) let \( j(\alpha) \) and \( y(\alpha) \) \( s^j \) as in Definition 6 Consider the following mechanism:

\[
g(m) = m^i_2(m^{-1}) \text{ if } m \in D_1 \text{ and } \]
\[
g(m) = CW(m_1) \text{ if } m \in D_2 \text{ and } j \neq j(m^i_3) \]
\[
g(m) = y(m^i_3(m_1)) \text{ if } m \in D_2 \text{ and } j = j(m^i_3) \]
\[
g(m) = m_j^{r(m)}(m_1^{-i}) \text{ if } m \in D_3 \]

Let \( x \) be a Condorcet Winner Selection. From Bayesian Monotonicity and Strategy-proofness it follows that \( \sigma^i(s^i) = (s^i, x(s^i, \cdot), 1, \ldots) \) is a Bayesian equilibrium yielding \( x \) at each state of the world.

Let \( \sigma^* \) be a Bayesian Nash equilibrium.

Let \( S_i = \{ s : \sigma^*(s) \in D_i \} \) for \( i = 1, 2, 3 \).

If \( s \in S_2 \cup S_3 \) then at least \( n - 1 \) players get their peak so \( \sigma^*(s) \) is the Condorcet Winner at \( s \).
For some $\alpha \in D$, $\sigma^*(\alpha(s)) = x(\alpha(s))$ for all $s$ for some Condorcet Winner selection $x$. By contradiction let $x(\alpha(s))$ be not a Condorcet Winner at $s$. Let $s^i$ as in the definition. For $j = j(\alpha)$, set $\sigma^i_1(s^j) = \alpha(s^j)$, $\sigma^i_2(s^j)(s^{-i}) = y(\alpha)(\alpha^{-i}(s^{-i}))$ for all $s^{-j} \in S^{-j}$, $\sigma^i_3(s^j) = \alpha$ and $\sigma^i_4(s^j)(s^{-i}) > (\sigma^*)_4(s^j)(s^{-i})$ for all $i \neq j$, $s^{-j} \in S^{-j}$. Let $\sigma^i(t^j) = (\sigma^*)^i(t^j)$ if $t^j \neq s^j$. Then $g(\sigma^i(s^j)), (\sigma^*)^{-i}(s^{-i}) = y(\alpha)(s^j, s^{-j})$ for all $s^{-j} \in S^{-j}$ and $g(\sigma^i, (\sigma^*)^{-i}) P^i(s^j) g(\sigma^*)$, which contradicts the hypothesis of $\sigma^*$ being a Bayesian Nash equilibrium.

Set $S^* = \{s \in S : \forall s^i \in S^i, CW(s^i) = CW(s^i)\}$. $S^*$ coincides with the the set of states in which the Condorcet Winner has a supermajority $\{s \in S : \forall s^i \in S^i, N(a, b, P_a) \geq N(b, a, P_a) + 2$ for all $a \in CW(P_a)\}$ so no agent is pivotal, as the reader can easily check. For instance, if $n = 3$ $S^*$ includes all state of the world in which agents agree on their favorite peak. For $n$ odd, if $s^* \in S^*$ at least two agents must agree on their favorite peak.

In order to check Bayesian Monotonicity of the Condorcet winner it suffices to consider deceptions $\alpha$ such that $\alpha(s) \in S^*$ for all $s$ and such that $CW(\alpha(s)) = a$ for all $s$, for some $a \in A$.

**Lemma 1** Let $\alpha$ be deception such that the Condorcet Winner $\alpha$-Bayesian Monotonicity (Definition 6) is not satisfied. Then $\alpha(s) \in S^*$ for all $s \in T$ and there exists $a \in A$ such that $CW(\alpha(s)) = a$ for all $s \in T$.

**Proof.** In all the other cases one can find an $i \in N$ and some $s^* = (s^{*i}, s^{*i}) \in S$ such that $CW(\alpha(s^*)) \neq CW(\alpha(s^{*i}), s^{*i})$. From strategy-proofness of the Condorcet winner it follows that $U^i(CW(\alpha(s^{*i}), s^{*i}) | s^{*i}) > U^i(CW(\alpha(s^*) | s^{*i})$. Then by setting $y_\alpha\sigma(s^*) = CW(\alpha(s^{*i}), s^{*i})$ and $y(s^{*i}, s^*) = CW(s^{*i}, s^*)$. We have $y_\alpha P^i(s^{*i}) CW(\alpha, s^{*i})$ and $CW_\alpha R^i(s^{*i}) y_{\alpha s^{*i}}$ for all $s^i \in S^i$.

As a first implication if no event in $S^*$ is likely to occur then $CW$ is Bayesian monotonic then implementable in Nash equilibrium.

**Corollary 1** If there exists $i$ such that for each $s^{*i} \in S^{*i}, q^i(s^{*i} | s^i) = 0$ for all $s^i \in T^i$, then the Condorcet Winner is Bayesian monotonic then implementable in Bayesian Equilibrium.

It suffices to check a unique $i$ because beliefs are assumed to be consistent. In other world the Corollary says that if each player is pivotal with probability one then the CW is Bayesian monotonic.

### 3 Impossibility

In this section we show that there exists a not negligible set of beliefs and of utility representation under which the Condorcet Winner is not Bayesian monotonic.
The following example introduces the main argument for the result.

**Example 1 (Non Bayesian Monotonicity of Condorcet Winner).** Let \( n = 3 \), \( A = \{a_1, a_2, a_3\} \). Let \( S^1 = S^2 = S^3 = \{s^1, s^2, s^3\} \). For \( i = 1, 2, 3 \) let \( U^i(\cdot \mid s^i) = U(\cdot \mid s^i) \), where:

\[
\begin{align*}
U(a_1, s^1, s^{-1}) &= U^3, \quad U(a_2, s^1, s^{-1}) = U^2, \quad U(a_3, s^1, s^{-1}) = U^1 \\
U(a_1, s^2, s^{-2}) &= U^1, \quad U(a_2, s^2, s^{-2}) = U^3, \quad U(a_3, s^2, s^{-2}) = U^2 \\
U(a_1, s^3, s^{-3}) &= U^3, \quad U^i(a_2, s^3, s^{-3}) = U^2, \quad U(a_3, s^3, s^{-3}) = U^3.
\end{align*}
\]

Here \( U^1, U^2, U^3 \) are real numbers such that \( 0 < U^1 < U^2 < U^3 \). Assume players’ priors are derived from the same probability distribution, \( p \) with independent types. Let \( p_i \geq 0 \) for \( i = 1, 2, 3 \) with \( p_1 + p_2 + p_3 = 1 \) such that \( p(s^i, s^j, s^k) = p^i p^j p^k \) for all \( i, j, k = 1, 2, 3 \). Let \( p^2 \geq p^* \), where \( p^* = \frac{U^3 - U^1}{2U^3 - U^1} \in \left[\frac{1}{4}, 1\right] \).

Consider the following individual constant deceptions \( \alpha^i(s^i) = s_2 \) for \( i = 1, 2, 3 \). Then \( CW \circ \alpha(s) = a_2 \) for all \( s \).

If \( CW \) is monotonic there exists a SCF \( y, i \) and \( s^i \) such that

(i) \( E[U(y(\alpha^i(\cdot, s^i) \mid s^i)] > E[U(CW(\alpha^i(\cdot, s^i) \mid s^i)] \) and

(ii) \( E[U(CW(\cdot, s^j) \mid s^j)] \geq E[U(y_{\alpha(s)}(\cdot, s^j) \mid s^j)] \) for \( j = 1, 2, 3 \).

As type \( s^2 \) gets its favorite allocation with probability one from \( CW \circ \alpha \), it must be the case that \( y(s_2, s_2, s_2) \in \{a_1, a_3\} \) and \( s^i \in \{s^1, s^3\} \).

Let \( i \in \{1, 3\} \) let \( y \in X \) such that \( y(s_2, s_2, s_2) = a_i \). From \( p^2 \geq p^* \) it follows that \( E[U(CW(\cdot, s^i) \mid s^i)] \leq p^2 U^2 + (1 - p^2 U^2) U^3 \leq p^2 U^3 + (1 - p^2 U^2) U^1 < E[U(y_{\alpha(s)}(\cdot, s^i) \mid s^i)] \) contradicting (ii) above. Then \( CW \) is not Bayesian Monotonic.

At state \( (s^2, s^2, s^2) \) no agent is pivotal. Each agent, independently on her type, believes she is not pivotal with high probability. Any deviation the planner can offer to any agent would make her profitable to deviate from the equilibrium strategy when all the other agents reveal their type truthfully.

Observe that the example is non generic in \( (U^1, U^2, U^3) \) And independence is not necessary for the impossibility result to hold. What matters is the belief some player has about being not pivotal.

**Proposition 2** (i) Given \( \{U^i(\cdot \mid s^i)\}_{i \in N, s \in S^i} \in U^S \) there exist \( q^1, \ldots, q^n \) \((\frac{1}{2}, 1)^n\) such that if, for some \( s^* \in S^* \) \( q^i(s^{*i} \mid s^i) > q^i \) for all \( s^i \in S^i \), then the Condorcet Winner is not Bayesian Monotonic.

(ii) Let \( s \in S^* \) and let \( q^i(s_{-i} \mid s^i) > \frac{1}{2} \) for all \( s^i \in S^i \). Then there exists an open set \( B \subset U^S \), such that, for all environments having q as priors and utility representations in \( B \), the Condorcet Winner is not Bayesian Monotonic.

**Proof.** For simplicity in the notations we prove the claim in the case in which \( T^i \) can be identified with \( SP\{A\} \). We assume that for each \( i \),
for each $P \in SP(A)$ there exists exactly one $s^i$ such that $P^i_s = P^i$. We take, with abuse of notation $T^i = SP(A)$. For each $P^i$ and for each $k = 1, ..., p$, set $U^i(P^i) = U^i(a_k \mid P^i)$, the utility $i$ derives from alternative $a_k$ when her preferences are $P^i$. Set $U^i(P^i) = (U^i_1(P^i), ..., U^i_p(P^i))$ such that $U^i_1(P^i) \leq ... < U^i_p(P^i)$.

Let $P^i \in S^i$ and consider the following individual deceptions $\alpha^i(P^i) = P^i$ for all $P^i \in SP(A)$. Then $E[U^i(CW \circ \alpha \mid P^i)] = U^i(CW(P^i) \mid P^i)$ for all $P^i$. From the definition of $S^* CW(P_{\alpha}^{i-1}, P^i) = CW(P^i)$ whatever is $P^i$. So if $E[u(CW \circ \alpha \mid P^i)] < E[u(y \circ \alpha \mid P^i)] = U^i(y(P^i) \mid P^i)$ then $y(P^i) \neq CW(P_{\alpha}^{i-1}, P^i)$, in particular $y \neq CW$. Furthermore the inequality holds also for any $P^i$ having peak in $y(P^i)$.

We have $E[U^i(y_{\alpha}(P^i)) \mid P^i]] \geq q^i(P_{\alpha}^{i-1} \mid P^i)U^i_p(P^i) + (1 - q^i(P_{\alpha}^{i-1} \mid P^i))U^i_1(P^i)$

And $E[U^i(CW) \mid P^i]) \leq q^i(P_{\alpha}^{i-1} \mid P^i)U^i_p(P^i)+ (1-q^i(P_{\alpha}^{i-1} \mid P^i))U^i_1(P^i)$.

The first claim follows setting $q^{i*} = \max_{P^i \in SP(A)} \frac{U^i_p(P^i)-U^i_1(P^i)}{2U^i_p(P^i)-U^i_1(P^i)-U^i_{p-1}(P^i)} \in \left[\frac{1}{2}, 1\right]$. The second claim follows observing that the function $(U_1, ..., U_p) \mapsto \frac{U_{p-1} - U_1}{2U_p-U_{p-1}}$ is continuous in $\{(U_1, ..., U_p) \in \mathbb{R}^p : U_1 \leq ... < U_p\}$.

The proof of Proposition 2 relies on the non-emptiness of $S^*$. Let $[N, S, A, (U^i(\cdot \mid s))_{i \in N, s \in S}]$ be any environment (not necessarily single peaked), in which no agents is indifferent among all alternatives. If $x$ is a social choice function set $S^*(x) = \{s \in S : \forall s^i \in S^i, x(s^i, s^{-i}) = x(s^i, s^{-i})\}$. $S^*(x)$ is the set of the states of the world in which no agent is pivotal. If $S^*(x) \neq \emptyset$ the argument used in Proposition 2 leads to prove the following facts.

Remark 2 (i) There exist $(q^{*1}, ..., q^{*n}) \in \left[\frac{1}{2}, 1\right]^n$ such that if, for some $s^* \in S^*$, $q^i(s^{*-i} \mid s^i) > q^i_*$ for all $s^i \in S^i$ then $x$ is not Bayesian Monotonic.
(ii)Let $s \in S^*(x)$ and assume $q^i(s_{-i} \mid s^i) > \frac{1}{2}$ for all $s^i \in S^i$ then there exists an open subset of $U^S$, $B$ for all environments having $q$ as priors and and utility representations in $B$, $x$ is not Bayesian Monotonic. In particular all Condorcet consistent SCF are not Bayesian Monotonic for an open set of beliefs.

4 Uniform priors

In this section we, show that if at least one agent has sufficiently symmetric information on peaks then the Condorcet Winner is Bayesian
monotonic, in particular it is implementable. The result extends to any strategy-proof SCF which depends only on peaks (tops-only).\footnote{For the intimate relation between strategy proofness and the tops-only property the interested reader can refer at Weymark (2004) or Massó and Neme (2002)}

**Definition 9** Let \( i \in N \). \( i \)'s priors are **uniform on peaks** if, for all \( s^i \in S^i \) \( q^i(\prod_{j=1}^{n-1} P(a_j) \times SP(A) \mid s^i) = q^i(\prod_{j=1}^{n-1} P(b_j) \times SP(A) \mid s^i) \) for all \( a_1, \ldots, a_{n-1}, b_1, \ldots, b_{n-1} \in A \).

An agent’s priors that are independent on peak if she assigns the same probability to any configuration of other agents’ peak.

**Proposition 3** If information is complete or if there exists an agent \( i \in N \), having uniform priors on peaks then the Condorcet Winner is implementable in Bayesian Nash equilibrium. In particular the Condorcet Winner is implementable in Bayesian equilibrium for an open set of beliefs.

**Proof.** If information is complete Bayesian Monotonicity and Maskin Monotonicity are equivalent. Then the Claim follows from Proposition 1 as the CW is Maskin Monotonic. For the case of uniform priors, let \( \alpha \) such that there exists \( s^* \in S^* \) with \( CW(\alpha(s)) = a_k \) for some \( k \), for all \( s \). For all \( j \neq i \), let \( a_j = a^*(s^*) \) and let \( b_j = a_{k+1} \). Set \( y(s^{-i}, s^i) = CW(P^{s^{-i}}, P^i(s^i)) = a_k \) if \( s^{-i} \in \prod_{j=1}^{n-1} P(a_j), \) set \( y(s^{-i}, s^i) = x(P^{s^{-i}}, P^i(s^i)) = a_{k+1} \) if \( s^{-i} \in \prod_{j=1}^{n-1} P(b_j) \).

Set \( y(s) = x(s) \) otherwise. Then we have \( E[U^i(y(\cdot, t^i) \mid s^i] = E[U^i(CW(\cdot, t^i) \mid s^i) for all s^i, t^i \in S^i, \) because \( i \)'s priors are uniform on peaks. In particular \( E[U^i(y(\cdot, \alpha^i(s^i) \mid s^i] = E[U^i(CW(\cdot, \alpha^i(s^i)) \mid s^i] for all s^i \in S^i.Let s^i having peak in \( a_{k+1} \) then \( E[U^i(y(\cdot, s^i) \mid s^i] > U^i(a_{k+1}, s^i) = E[U^i(CW(\cdot, s^i) \mid s^i]. \)

The last claim follows from Triossi (2004), where it is shown that, if a social choice correspondence is Bayesian Monotonic for some profile of beliefs, then it is Bayesian Monotonic in a neighborhood of such profile.

\[\square\]

5 Conclusions

In the paper we presented a first step in the investigation of the implementation of social decision under incomplete information in pure public good environment. We devoted most of our attention to the Condorcet Winner. We underline the trade-off between strategy-proofness
and Bayesian Monotonicity. It suggests that the problem of multiplicity might be hard to tackle. We offer a positive results too.

The mechanism implementing the Condorcet Winner in dominant strategies requires players only to indicate their favorite alternative. The mechanism implementing the Condorcet Winner in Bayesian Nash equilibrium is much more complicated and it requires the planner a detailed information about the informational environment in order to exactly implement it even when it is possible. It would be interesting to compare the performance of the different mechanisms under costly participation, when the voting for the favorite alternative is no longer a dominant strategy and to see if the positive results presented here survive. Under the light of the classical and most recent results on costly voting and electoral turnout (see for instance Ledyard (1984) Palfrey and Rosenthal (1983), (1985), Börgers (2004) and Goeree and Großer (2004)) we can conjecture that also for our mechanism some (if not all) pure strategy equilibria would not survive and mixed-strategy equilibria would have the main part.

Future research should also clarify that if it is possible to build some implementing mechanism which is (at least locally) independent on the underlying probability distribution. It is to say a mechanism which make the work for some open set of probability distributions (see also Duggan and Roberts (2001) and Bergemann and Morris (2005) for possible approaches and further discussion).
References


