Competing Mechanisms in Markets for Lemons*

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Abstract

We study the competitive equilibria in a market with adverse selection and search frictions. Uninformed buyers post general direct mechanisms and informed sellers choose where to direct their search. We demonstrate that there exists a unique equilibrium allocation and characterize its properties: all buyers post the same mechanism and a low quality object is traded whenever such object is present in a meeting. Sellers are thus pooled at the search stage and screened at the mechanism stage. If adverse selection is sufficiently severe, this equilibrium is constrained inefficient. Furthermore, the properties of the equilibrium differ starkly from the case where meetings are restricted to be bilateral, in which case in equilibrium sellers sort across different mechanisms at the search stage. Compared to such sorting equilibria, our equilibrium yields a higher surplus for most, but not all, parameter specifications.

1 Introduction

Since the work of Akerlof (1970) and Rothschild and Stiglitz (1976), the properties of market outcomes in the presence of adverse selection have been the subject of study for many years, both in Walrasian models as well as in models where agents act strategically. In the latter agents compete among themselves over contracts which determine transfers of goods and prices, whereas in the former available contracts specify transfers of goods while prices are taken as given and set so as to clear the markets. Initiated by Gale (1992), Inderst and Mueller (1999) and, more recently, Guerrieri et al. (2010), the use of competitive, directed search models to study markets with adverse selection has generated several interesting insights. In these models the agents who are uninformed act as principals. They post and commit to contracts, across which informed agents then allocate themselves. This framework allows for a richer specification of the terms of contracts available for trade, which - in contrast to Walrasian models - also include the price to be paid. Market clearing is in fact obtained by finding the mass of buyers and sellers wishing to trade each contract;

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their relative mass, together with the search friction, then determines the probability that buyers and sellers are matched and hence trade. Renewed interest in this approach has come from recent developments in financial markets, including the growth of new kinds of market structures as the OTC markets, less centralized and with less transparent trading conditions.[1]

A key underlying assumption in the existing literature on competitive search models with adverse selection is that each principal can meet at most one agent. As a consequence, the contracts which can be posted specify trades and prices that are only contingent on the type reported by the agent but not on the reports of other arriving agents. It was then shown that there always exists a separating equilibrium, where agents of different type search for principals posting different contracts. Principals are in turn indifferent between posting any of the contracts chosen by the different types of agents, implying that in equilibrium there is no cross-subsidization among the types. As a consequence, the equilibrium outcome is inefficient - even taking incentive compatibility constraints into account - when the fraction of higher quality agents is sufficiently large. This result is analogous to the one obtained in Walrasian models (see Dubey and Geanakoplos (2002) and Bisin and Gottardi (2006)), where, in the absence of search frictions, in equilibrium agents of different types choose to trade different contracts and the no cross-subsidization property again holds.[2]

We depart from the existing literature on adverse selection and competitive search by allowing principals to meet multiple agents and to post general direct mechanisms that specify trading probabilities and transfers for agents, contingent not only on their own reported type but also on the number and reports of other agents meeting the same principal - as an illustration we can think of an auction as the mechanism governing trades once meetings occur. In particular, we consider an environment as in Akerlof (1970), with a measure of sellers who are privately informed about the quality of the good they own, and a measure of uninformed buyers. Quality can be either high or low and determines both the seller's valuation and the buyer's valuation. Buyers then act as principals and post mechanisms that specify trading probabilities and transfers as a function of the number of sellers in a meeting and their report. Sellers act as agents, observing all posted mechanisms and choosing where to direct their search.[3] We refer to all buyers posting the same

[1]There are also papers that study adverse selection in economies where meetings and trades can take place over a sequence of periods, both in markets with random search, where offers of contracts are made only after meetings occur, and in markets *á la* Akerlof (1970), where all trades occur at a single price (see for example Blouin and Serrano (2001), Janssen and Roy (2002), Carmargo and Lester (2014), Fuchs and Skrzypacz (2013) and Moreno and Wooders (2015)).

[2]It is interesting to notice that a similar result holds in the models with dynamic trading mentioned in the previous footnote: separation in that case obtains with sellers of different types trading at different prices, at different points in time.

[3]Of course these are just labels. Alternatively we could think of our environment as the labor market where principals are firms and agents are workers who are privately informed about their productivity or as a procurement market where principals are procurers and agents are firms who are privately informed about the quality of their

mechanism and all sellers searching for that mechanism as constituting a submarket. Within each submarket, matching is subject to frictions so that buyers are faced with a distribution over the number of sellers they meet, which depends on the ratio of sellers to buyers in the submarket.

Our main result shows that there exists a unique equilibrium allocation with the following properties: all buyers post the same mechanism and a low quality object is traded whenever such object is present in a meeting. The first property implies that everyone visits a single submarket, while the second property implies that low type sellers receive priority in every meeting. Thus, all sellers are pooled at the search stage, since they all choose the same mechanism, but are screened at the mechanism stage, since their probability of trade varies with their type. It is important to point out that this result does not hinge on any assumption on the relative gains from trade. That is, even when the gains from trade for the low quality object are arbitrarily small and the gains from trade for the high quality object are arbitrarily large, high type sellers only get to trade in meetings where there are no low type sellers. We further demonstrate that all matches between buyers and sellers lead to trade if and only if the gains from trade for the high quality good are sufficiently small compared to those of the low quality good. When this is not the case, high type sellers are rationed in equilibrium, meaning that in meetings where all sellers have a high quality object there is a strictly positive probability that no one trades. In such situations, additional equilibria exist in which sellers partially sort themselves at the search stage: buyers post different mechanisms, attracting different ratios of high versus low type sellers. However, these equilibria yield the same allocation and payoffs as the equilibrium with pooling at the search stage.

We then consider the welfare properties of the search equilibrium. Since the equilibrium features pooling at the search stage, the no cross-subsidization property no longer needs to hold:[4] we show in fact that a buyer's payoff conditional on meeting a low type seller is strictly higher than that conditional on meeting a high type seller. We then demonstrate that whenever the gains from trade of the low quality good exceed those of the high quality good, that is adverse selection is relatively mild, the equilibrium maximizes social surplus. The result follows directly from the fact that pooling at the search stage minimizes search frictions and that the equilibrium mechanism gives priority to the good with the larger gains from trade. On the other hand, if gains from trade of the low quality good are strictly smaller than those of the high quality good, that is adverse selection is more severe, social surplus is no longer maximal in equilibrium. In such case, we further demonstrate that the equilibrium allocation can be Pareto improved, subject to the constraints imposed by incentives and the search friction, if the share of high type sellers is large enough.

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[4]With 'no cross-subsidization' we refer to the property that buyers make the same profits with both types of sellers, but these profits do not have to equal zero.

Finally, we compare our findings with those of the earlier literature on competitive search with adverse selection where meetings are restricted to be bilateral. To do this, we examine the case where, in the environment considered, the mechanisms available to principals are restricted to posted prices. Such restriction on the space of available mechanisms is analogous to a restriction to bilateral meetings, as it implies that the seller with whom a buyer trades is chosen at random among all other arriving sellers.[5] As one would expect, we find that if the set of available mechanisms is restricted to posted prices, the equilibrium is separating, with two active submarkets, each of which is chosen by only one type of seller. The separation of sellers at the search stage can be sustained in equilibrium through different seller-buyer ratios and hence different queue lengths in the two markets: sellers choose between the possibility of receiving a high price in a market with a high seller-buyer ratio and a low price in a market with a low seller-buyer ratio.

We then compare the social surplus at the price posting equilibrium with that at the equilibrium when general mechanisms are available. We show that in the latter social surplus is strictly higher for many, but not all, parameter specifications. In particular, when the gains from trade are higher for the low than for the high quality good, the equilibrium with general mechanisms always yields a higher surplus because, as mentioned above, social surplus is maximal in that case. More surprisingly, the equilibrium with general mechanisms also generates a higher total surplus when it is constrained inefficient and entails rationing, that is when adverse selection is severe. Despite this, we show that there exist parameter specifications where surplus is higher in the price posting equilibrium. In particular, we demonstrate that, provided gains from trade are larger for the high than for the low quality good, this is always case when the ratio of high type sellers to buyers is large enough.

Related Literature: Besides the literature on competitive search and Walrasian equilibria with adverse selection mentioned above, our paper is also closely related to the work on competing mechanisms in independent private value environments. Peters (1997) and Peters and Severinov (1997) assume the same meeting technology as in our paper and show for such environments that there exists an equilibrium where all buyers post the same second-price auction with a reserve price equal to their valuation.[6] Subsequent papers examine the features of the equilibrium and its welfare properties in more general search environments, allowing for example for meeting technologies where buyers face capacity constraints in their ability to meet sellers, while maintaining the

[5]Strictly speaking, the restriction to bilateral meetings is equivalent to restricting the set of available mechanisms to menus such that agents' trading probabilities and transfers only depend on their own report. Given this constraint, it turns out that the restriction to posted prices, i.e. degenerate menus, is without loss of generality.

[6]Note that in this literature labels of buyers and sellers are typically reversed.

independent private value assumption (e.g. Eeckhout and Kircher 2010, Albrecht et al. 2014, Cai et al. 2015). In particular, Eeckout and Kircher (2010) show in this setting the connection between the properties of the meeting technology and the existence of equilibria with ex-ante sorting (at the search stage) versus equilibria with ex-post screening (at the mechanism stage). The environment we consider allows to nest a situation with independent private values as a special case where the valuation of buyers does not depend on the quality of the good. Our characterization of competitive search equilibria shows that, as one moves away from this particular parameter specification, the equilibrium mechanism generally differs from the second-price auction identified in the literature recalled above, while the other features of the equilibrium are robust as long as adverse selection is mild. In contrast, as adverse selection becomes severe, our results show that the search equilibrium features rationing of high type sellers, the possibility of partial sorting at the search stage and that it is constrained inefficient, properties that never arise in independent private value environments.

The paper is organized as follows. The next section presents the economy, the space of mechanisms and defines the notion of competitive search equilibrium that is considered. Section 3 presents the main result, stating the existence of competitive search equilibria and the uniqueness of the equilibrium allocation, and the argument of the proof. The proof is constructive and leads to a characterization of the equilibrium allocations with different properties in different regions of the parameter space. The following section then discusses the properties of the equilibrium, in particular the payoffs attained by buyers and sellers, shows that under some conditions equilibria with partial sorting exist, and analyzes the welfare properties of equilibria. The final section then compares the properties of the equilibrium we found to those of the equilibrium which obtains when the set of available mechanisms is restricted to price postings, showing that welfare is typically, though not always higher in the first one. Proofs are collected in the Appendix.

2 Environment

There is a measure b of uninformed buyers and a measure s of informed sellers. Each seller possesses one unit of an indivisible good with uncertain quality. The good's quality is identically and independently distributed across sellers. Quality can be either high or low and μ denotes the fraction of sellers that possess a high quality good. Let $\overline{\lambda}^p = \mu \frac{s}{b}$ denote the ratio of high type sellers to buyers and let $\underline{\lambda}^p = (1-\mu)\frac{s}{b}$ denote the ratio of low type sellers to buyers. The buyers' and sellers' valuation of the high (low) quality good are denoted by $\overline{v}$ ($\underline{v}$) and $\overline{c}$ ($\underline{c}$), respectively. We assume that both the buyers and the sellers value the high quality good more than the low quality good, i.e. $\overline{v} \geq \underline{v}, \overline{c} > \underline{c}$. For sellers this preference is assumed to be strict, while we allow buyers to have the same valuation for both types of good, i.e. $\overline{v} = \underline{v}$. When $\overline{v}$ is strictly greater than $\underline{v}$, the buyer's valuation depends on the seller's valuation of the object, a situation we refer to as the common value case. This is no longer true when $\overline{v} = \underline{v}$, which we refer to as the private value case. We further assume that there are always positive gains from trade, meaning that for both types of good the buyer's valuation strictly exceeds the seller's valuation, i.e. $\overline{v} > \overline{c}, \underline{v} > \underline{c}$.

Search: Matching between buyers and sellers is subject to frictions and operates as follows. Buyers simultaneously post mechanisms that specify how trade takes place with the sellers with whom they are matched. Sellers observe the posted mechanisms and direct their search to one of the mechanisms they like best. We refer to the collection of buyers posting the same mechanism and the collection of sellers searching for that mechanism as constituting a submarket. We assume that markets are anonymous. Anonymity is captured by the assumption that mechanisms cannot condition on the identity of sellers and that sellers cannot condition their search strategies on the identity of buyers but only on the mechanism they post (see for example Shimer, 2005). More specifically, we adopt the assumption that, in any submarket, a seller visits one of the present buyers at random and that buyers have no capacity constraints, that is they can meet all arriving sellers, no matter how many they are. As a result, the number of sellers that meet a particular buyer follows a Poisson distribution with a mean equal to the seller-buyer ratio in the submarket.[7] According to this meeting technology, referred to as urn-ball matching, sellers are sure to meet a buyer, while buyers may end up with many sellers or with no seller at all. Moreover, a buyer's probability of meeting a certain type of seller is fully determined by the ratio between sellers of that type and buyers in the submarket, while it does not depend on the presence of other types of sellers.[8] This property and the fact that buyers can meet multiple sellers are essential for the following analysis, while most other features of the meeting technology are not.

Under urn-ball matching, a buyer's probability of meeting k sellers in a market with seller-buyer ratio λ is given by

$$P_k(\lambda) = \frac{\lambda^k}{k!} e^{-\lambda}$$

Since the presence of high type sellers does not affect the meeting chances of low type sellers and vice versa, the probability for a buyer to meet L low type sellers and H high type sellers in a market where the ratio between high (low) type sellers and buyers is $\overline{\lambda}$ ($\underline{\lambda}$) is given by

$$P_L(\underline{\lambda}) P_H(\overline{\lambda}) = \frac{(\underline{\lambda})^L}{L!} e^{-\underline{\lambda}} \frac{(\overline{\lambda})^H}{H!} e^{-\overline{\lambda}}$$

[7] This is one of the most commonly used assumptions in the directed search literature, e.g. Peters and Severinov (1997), Albrecht et al. (2006), Kim and Kircher (2015).

[8] The class of meeting technologies that have this property, called 'invariance' (Lester et al., 2015a), includes the urn-ball matching technology as a special case.

Similarly, $P_L(\underline{\lambda})P_H(\overline{\lambda})$ corresponds to the probability for a seller to be in a meeting with other L low type sellers and H high type sellers.

Mechanisms and payoffs: We restrict attention to direct mechanisms that do not condition on mechanisms posted by other buyers. A mechanism m is defined by

$$m : \{(L,H)\}_{L\in\mathbb{N},H\in\mathbb{N}} \to [0,1]^2 \times \mathbb{R}^2,$$

where L is the number of low messages and H is the number of high messages in a meeting. Let $\underline{X}_m(L,H), \overline{X}_m(L,H)$ and $\underline{T}_m(L,H), \overline{T}_m(L,H)$ denote the trading probabilities and transfers specified by mechanism m for sellers reporting, respectively, L and H. We say a mechanisms m is feasible if

$$\underline{X}_m(L,H)L + \overline{X}_m(L,H)H \leq 1, \forall (L,H) \in \mathbb{N}^2 \tag{1}$$

Let M denote the measurable set of feasible mechanisms.

We assume that, when matched with a buyer, a seller does not observe how many other sellers are matched with the same buyer nor their types.[9] Let $\overline{\lambda}$ denote the expected number of H reports and $\underline{\lambda}$ denote the expected number of L reports, which under truthful reporting simply correspond to the respective seller-buyer ratios for mechanism m. The expected trading probabilities for a seller when reporting L and H, respectively, are then given by

$$\begin{aligned} \underline{x}_m(\underline{\lambda},\overline{\lambda}) &= \sum_{L=0}^{+\infty}\sum_{H=0}^{+\infty} P_L(\underline{\lambda})P_H(\overline{\lambda})\underline{X}(L+1,H) \\ \overline{x}_m(\underline{\lambda},\overline{\lambda}) &= \sum_{L=0}^{+\infty}\sum_{H=0}^{+\infty} P_L(\underline{\lambda})P_H(\overline{\lambda})\overline{X}(L,H+1) \end{aligned}$$

Similarly, we can determine expected transfers $\underline{t}_m(\underline{\lambda},\overline{\lambda})$ and $\overline{t}_m(\underline{\lambda},\overline{\lambda})$. The expected payoff for low and high type sellers if they choose mechanism m and reveal their type truthfully is given by

$$\begin{aligned} \underline{u}(m|\underline{\lambda},\overline{\lambda}) &= \underline{t}_m(\underline{\lambda},\overline{\lambda}) - \underline{x}_m(\underline{\lambda},\overline{\lambda})\underline{c} \\ \overline{u}(m|\underline{\lambda},\overline{\lambda}) &= \overline{t}_m(\underline{\lambda},\overline{\lambda}) - \overline{x}_m(\underline{\lambda},\overline{\lambda})\overline{c} \end{aligned}$$

[9]The assumption that a seller cannot observe his competitors' type is standard. The assumption that a seller cannot observe the number of competitors in a meeting facilitates notation considerably but is not essential for any of our results.

Truthful reporting is optimal if the following two inequalities hold

$$\bar{t}_m(\underline{\lambda},\overline{\lambda}) - \overline{x}_m(\underline{\lambda},\overline{\lambda})\underline{c} \leq \underline{t}_m(\underline{\lambda},\overline{\lambda}) - \underline{x}_m(\underline{\lambda},\overline{\lambda})\underline{c} \tag{2}$$

$$\underline{t}_m(\underline{\lambda},\overline{\lambda}) - \underline{x}_m(\underline{\lambda},\overline{\lambda})\overline{c} \leq \bar{t}_m(\underline{\lambda},\overline{\lambda}) - \overline{x}_m(\underline{\lambda},\overline{\lambda})\overline{c} \tag{3}$$

Note that, since incentive compatibility is defined in terms of expected trading probabilities and transfers, whether a given mechanism m is incentive compatible or not depends on the values of $\underline{\lambda}$ and $\overline{\lambda}$. Let $\mathcal{M}^{IC}$ denote the set of tuples $(m, \underline{\lambda}, \overline{\lambda})$ such that $m \in M$ and incentive compatibility with respect to $\underline{\lambda}, \overline{\lambda}$ is satisfied.

Finally, given that sellers report truthfully, the payoff for a buyer posting mechanism m when the expected number of high and low type sellers, respectively, is $\overline{\lambda}$ and $\underline{\lambda}$, is

$$\pi(m|\underline{\lambda},\overline{\lambda}) = \overline{\lambda}[\overline{x}_m(\underline{\lambda},\overline{\lambda})\overline{v} - \bar{t}_m(\underline{\lambda},\overline{\lambda})] + \underline{\lambda}[\underline{x}_m(\underline{\lambda},\overline{\lambda})\underline{v} - \underline{t}_m(\underline{\lambda},\overline{\lambda})]$$

Equilibrium: An allocation in this setting is defined by a measure β over M, where $\beta(m)$ denotes the measure of buyers that post mechanism m, and two maps $\underline{\lambda}, \overline{\lambda} : M \to \mathbb{R}^+ \cup +\infty$ specifying, respectively, the ratio of low and high type sellers directing their search to mechanism m relative to the buyers posting that mechanism. Let M^β denote the support of β. We say an allocation is *feasible* if

$$\int_{M^\beta} d\beta(m) = b, \quad \int_{M^\beta} \underline{\lambda}(m)d\beta(m) = s(1-\mu), \quad \int_{M^\beta} \overline{\lambda}(m)d\beta(m) = s\mu \tag{4}$$

We call an allocation *incentive compatible* if $(m, \underline{\lambda}(m), \overline{\lambda}(m)) \in \mathcal{M}^{IC}$ for all $m \in M^\beta$. We can show[10] that we can restrict our attention to incentive compatible allocations w.l.o.g.: for any non-incentive compatible mechanism, there exists a different incentive compatible mechanisms that yields the same payoff for buyers and sellers as the original mechanism in the reporting equilibrium.

For all $m \notin M^\beta$, the maps $\underline{\lambda}(m)$ and $\overline{\lambda}(m)$ specify the beliefs of buyers over the expected number of low and high type sellers, respectively, that a deviating mechanism attracts. We assume that buyers' beliefs are consistent with seller's optimal choices. More specifically, we assume that a buyer believes to attract some low (high) type sellers if and only if low (high) type seller are indifferent between the deviating mechanism and their equilibrium mechanism, while the other type weakly

[10]For a formal proof see the Online Appendix, available at https://sites.google.com/site/austersarah/

prefers his equilibrium mechanism. This is captured by the set of the following two inequalities[11]

$$\underline{u}(m|\underline{\lambda}(m),\overline{\lambda}(m)) \leq \max_{m' \in M^\beta} \underline{u}(m'|\underline{\lambda}(m'),\overline{\lambda}(m')) \text{ holding with equality if } \overline{\lambda}(m) > 0, \quad (5)$$

$$\overline{u}(m|\underline{\lambda}(m),\overline{\lambda}(m)) \leq \max_{m' \in M^\beta} \overline{u}(m'|\underline{\lambda}(m'),\overline{\lambda}(m')) \text{ holding with equality if } \underline{\lambda}(m) > 0 \quad (6)$$

We then impose the following conditions on out of equilibrium beliefs, $\underline{\lambda}(m), \overline{\lambda}(m), m \notin M^\beta$:

i) if (5,6) admit a unique solution, then $\underline{\lambda}(m)$ and $\overline{\lambda}(m)$ are given by that solution;

ii) if (5,6) admit no solution, we set $\underline{\lambda}(m)$ and/or $\overline{\lambda}(m)$ equal to $+\infty$ and $\pi(m|\underline{\lambda}(m),\overline{\lambda}(m))$ equal to the corresponding limit of $\pi(m|\underline{\lambda}(m),\overline{\lambda}(m))$ as $\underline{\lambda}(m)$ and/or $\overline{\lambda}(m)$ tend to $+\infty$;[12]

iii) if (5,6) admit multiple solutions, then $\underline{\lambda}(m), \overline{\lambda}(m)$ are given by the solution for which the buyer's payoff $\pi(m|\underline{\lambda}(m),\overline{\lambda}(m))$ is the highest.

Condition i) says that whenever there is one belief regarding the seller-buyer ratios $\underline{\lambda}(m), \overline{\lambda}(m)$ for an out of equilibrium mechanism $m \notin M^\beta$ such that conditions (5) and (6) are satisfied, these two inequalities determine buyers' beliefs regarding the deviating mechanism. This implies that, whenever possible, the beliefs regarding the seller-buyer ratios $\underline{\lambda}(m), \overline{\lambda}(m)$ for an out of equilibrium mechanism $m \notin M^\beta$ are set at a positive level such that each type of seller is indifferent between m and a mechanism in the support of M^β or at a level equal to zero if the corresponding type of seller strictly prefers a mechanism in M^β. That is, the seller-buyer ratios are consistent with sellers' optimal choices also for out of equilibrium mechanisms, as if all mechanisms were effectively available to sellers. This is analogous to existing refinements in competitive environments with adverse selection such as Gale (1992), Dubey and Geanakoplos (2002) and Guerrieri et al. (2010), among others.

Since buyers have no capacity constraint and since we allow for arbitrary direct mechanism, including mechanisms where transfers are not contingent on trade, there exist mechanisms for which a solution to (5,6) does not exist.[13] That is, there exist mechanisms such that for any pair $\underline{\lambda}(m), \overline{\lambda}(m)$, there is at least one type of seller that strictly prefers the deviating mechanism over any mechanism in the support of M^β. For example, a mechanism could specify a participation transfer that is paid to a seller independently of whether trade occurs or not. If that participation transfer is large enough, all sellers strictly prefer the deviating mechanism regardless of how many other sellers are expected to be present in a meeting. Condition ii) specifies that in such case the

[11]Equivalent conditions appear in Eeckhout and Kircher (2010) among others.

[12]More precisely, if $\underline{u}(m|\underline{\lambda}(m),\overline{\lambda}(m)) > \max_{m' \in M^\beta} \underline{u}(m'|\underline{\lambda}(m'),\overline{\lambda}(m'))$ and $\overline{u}(m|\underline{\lambda}(m),\overline{\lambda}(m)) > \max_{m' \in M^\beta} \overline{u}(m'|\underline{\lambda}(m'),\overline{\lambda}(m'))$ for all $\underline{\lambda}(m), \overline{\lambda}(m) \in \mathbb{R}^+$, then $(\underline{\lambda}(m),\overline{\lambda}(m)) = (+\infty,+\infty)$. If only one of the two inequalities is violated, say the first one, then $\overline{\lambda}(m) = +\infty$, while $\underline{\lambda}(m)$ is determined by (6).

[13]Note that this situation cannot arise in settings where the meeting technology is restricted to be bilateral or mechanisms are restricted to posted prices.

seller-buyer ratios $\underline{\lambda}(m), \overline{\lambda}(m)$ are set equal to infinity, while a buyer's associated payoff is given by the corresponding limit.

Finally, if a solution to (5,6) exists, it is typically unique. If that should not be the case, we follow McAfee (1993) and others and assume, in condition iii), that buyers are 'optimistic', so that the pair $\underline{\lambda}(m), \overline{\lambda}(m)$ is given by their preferred solution. This specification makes deviations maximally profitable and may thus, in principle, restrict the set of equilibria. We are now ready to define a competitive equilibrium.

Definition 1. *A competitive search equilibrium is a feasible and incentive compatible allocation given by a measure β with support M^β and two maps $\underline{\lambda}, \overline{\lambda}$ such that the following conditions hold:*

- *buyers' optimality: for all $m \in M$ such that $(m, \underline{\lambda}(m), \overline{\lambda}(m)) \in \mathcal{M}^{IC}$,*

$$\pi(m|\underline{\lambda}(m), \overline{\lambda}(m)) \leq \max_{m' \in M^\beta} \pi(m'|\underline{\lambda}(m'), \overline{\lambda}(m')) \text{ holding with equality if } m \in M^\beta$$

- *sellers' optimality: for all $m \in M^\beta$*

$$\begin{aligned} \underline{u}(m|\underline{\lambda}(m), \overline{\lambda}(m)) &\leq \max_{m' \in M^\beta} \underline{u}(m'|\underline{\lambda}(m'), \overline{\lambda}(m')) \text{ holding with equality if } \overline{\lambda}(m) > 0 \\ \overline{u}(m|\underline{\lambda}(m), \overline{\lambda}(m)) &\leq \max_{m' \in M^\beta} \overline{u}(m'|\underline{\lambda}(m'), \overline{\lambda}(m')) \text{ holding with equality if } \underline{\lambda}(m) > 0 \end{aligned}$$

- *beliefs: for all $m \notin M^\beta$, $\underline{\lambda}(m)$ and $\overline{\lambda}(m)$ are determined by conditions i)-iii)*

3 Competitive Search Equilibrium

We state now our main result, which characterizes the competitive search equilibria in the environment described in the previous section.

Theorem 3.1. *There exists a competitive search equilibrium with the following properties:*

- *All buyers post the same mechanism.*
- *Whenever a low type seller is present in a match, a low quality good is traded.*
- *The equilibrium is unique in terms of expected payoffs.*

Theorem 3.1 states that there always exists a search equilibrium in which sellers are pooled at the search stage and screened at the mechanism stage. That is, all buyers post identical mechanisms so that everybody trades in a single market and these mechanisms specify different trading probabilities for different types of sellers. In particular, the equilibrium mechanism always gives priority to low type sellers, meaning that a low quality good is traded whenever there is a low type seller present in a meeting with a buyer. This implies that a low type seller's probability of trade strictly exceeds a high type seller's probability of trade: the probability of trade for a low type seller is strictly larger than his probability of meeting no other low type seller, while the probability of trade for a high type seller is strictly smaller than his probability of meeting no low type seller. It is important to point out that this property of the equilibrium does not depend on the size of the relative gains from trade or the fraction of high type sellers in the population. That is, even when the gains from trade of the low quality good are arbitrarily small and those of the high quality good are arbitrarily large, high type sellers only trade in meetings where there are no low type sellers. Theorem 3.1 also states that the equilibrium is unique in terms of payoffs. In particular, although there may be multiple mechanisms selected in equilibrium, all those mechanisms yield the same expected levels of trade and transfers.

The remainder of this section is devoted to proving the above result. The argument is constructive and proceeds through a series of lemmas and propositions that establish various properties of the equilibrium outcome. As a preliminary step, we show that, in order to characterize equilibrium payoffs, we can conveniently restrict our attention to the space of expected trading probabilities and transfers associated to mechanisms in M, clearly simpler than the original mechanism space. More precisely, the next proposition provides conditions on expected trading probabilities and transfers that any feasible and incentive compatible mechanism satisfies and, viceversa, are generated by some feasible and incentive compatible mechanism.

Proposition 3.2. *For any $(\underline{x}, \overline{x}, \underline{t}, \overline{t}) \in [0,1]^2 \times \mathbb{R}^2$ and $\overline{\lambda}, \underline{\lambda} \in [0, \infty)$, there exists a feasible and incentive compatible mechanism m such that*

$$\underline{x}_m(\underline{\lambda}, \overline{\lambda}) = \underline{x}, \quad \overline{x}_m(\underline{\lambda}, \overline{\lambda}) = \overline{x}, \quad \underline{t}_m(\underline{\lambda}, \overline{\lambda}) = \underline{t}, \quad \overline{t}_m(\underline{\lambda}, \overline{\lambda}) = \overline{t}$$

if and only if

$$\overline{t} - \overline{x}\underline{c} \leq \underline{t} - \underline{x}\underline{c} \tag{7}$$

$$\underline{t} - \underline{x}\overline{c} \leq \overline{t} - \overline{x}\overline{c} \tag{8}$$

$$\overline{\lambda}\overline{x} \leq 1 - e^{-\overline{\lambda}} \tag{9}$$

$$\underline{\lambda}\underline{x} \leq 1 - e^{-\underline{\lambda}} \tag{10}$$

$$\overline{\lambda}\overline{x} + \underline{\lambda}\underline{x} \leq 1 - e^{-(\overline{\lambda} + \underline{\lambda})} \tag{11}$$

Proof See Appendix A.1.

Conditions (7) and (8) are analogous to the sellers' incentive compatibility constraints (2) and

(3). It is immediate to see that these two conditions imply $\underline{x} \geq \overline{x}$, that is the expected trading probability is higher for low than for high type sellers. The remaining three conditions correspond to the properties that the mechanism m associated to $(\underline{x}, \overline{x}, \underline{t}, \overline{t})$ is feasible according to (1) and that meetings take place according to the urn-ball technology. In particular, inequality (9) requires that a buyer's probability of trading with a high type seller is weakly smaller than a buyer's probability of meeting at least one high type seller. The expected probability of trading with a high type seller is given by the product of the expected number of high type sellers in a meeting, $\overline{\lambda}$, and their trading probability, $\overline{x}$, while the probability of meeting at least one high type seller is given by $\sum_{k=1}^{+\infty} P_k(\overline{\lambda}) = 1 - e^{-\overline{\lambda}}$. Similarly, inequality (10) requires that the probability that a buyer trades with a low type seller, $\underline{\lambda x}$, is weakly smaller than the probability that a buyer meets at least one low type seller, $1 - e^{-\underline{\lambda}}$. Finally, inequality (11) requires that a buyer's probability of trading with any seller, $\overline{\lambda x} + \underline{\lambda x}$, is weakly smaller than the probability of meeting at least one seller, $1 - e^{-(\overline{\lambda} + \underline{\lambda})}$. It is useful to point out that condition (9) is redundant: $\underline{x} \geq \overline{x}$ together with condition (11) implies that a buyer's probability of trading a high quality object cannot exceed his probability of meeting a high type seller.[14]

Following Eeckhout and Kircher (2010) and others, we next state an auxiliary optimization problem of a representative buyer who chooses a mechanism m, together with arrival rates $\overline{\lambda}(m)$ and $\underline{\lambda}(m)$, so as to maximize his payoff, taking as given the utility attained by low and high type sellers, denoted by $\underline{U}$ and $\overline{U}$. Given Proposition 3.2, rather than solving for a mechanism in the original mechanism space, we can equivalently solve for the expected values of trading probabilities and transfers associated to the mechanism, $\underline{x}, \overline{x}, \underline{t}, \overline{t}$, as long as they satisfy conditions (7-11). In the auxiliary problem, the choice of arrival rates $\underline{\lambda}, \overline{\lambda}$ associated to the mechanism is constrained by the conditions restricting equilibrium beliefs (5) and (6), which can be viewed as a form of participation constraints.[15] Letting $\underline{U} = \max_{m \in M^\beta} \underline{u}(m|\underline{\lambda}(m), \overline{\lambda}(m)), \underline{U} = \max_{m \in M^\beta} \underline{u}(m|\underline{\lambda}(m), \overline{\lambda}(m))$, this amounts to the optimization problem

$$\max_{\underline{x}, \overline{x}, \underline{t}, \overline{t}, \overline{\lambda}, \underline{\lambda}} \overline{\lambda}(\overline{xv} - \overline{t}) + \underline{\lambda}(\underline{xv} - \underline{t}) \qquad (P^{aux})$$

[14]Formally, $\underline{x} \geq \overline{x}$ together with condition (11) implies $(\overline{\lambda} + \underline{\lambda})\overline{x} \leq 1 - e^{-(\overline{\lambda} + \underline{\lambda})}$. This is equivalent to $\overline{x} \leq \frac{1}{\overline{\lambda} + \underline{\lambda}} \left(1 - e^{-(\overline{\lambda} + \underline{\lambda})}\right) \leq \frac{1}{\overline{\lambda}} \left(1 - e^{-\overline{\lambda}}\right)$. To see the second inequality, note that the function $f(x) = \frac{1}{x}(1 - e^{-x})$ is strictly decreasing in x, for all $x > 0$. That is, $f'(x) = -\frac{1 - e^{-x} - xe^{-x}}{x^2} < 0, \forall x > 0$ (the numerator corresponds to the probability of at least two arrivals given arrival rate x). It should also be noted that $\lim_{x \to 0} f(x) = 1$ and $\lim_{x \to +\infty} = 0$, which implies $f(x) \in (0, 1), \forall x > 0$.

[15]Letting the representative buyer optimize directly over arrival rates implies that in cases where there are multiple solutions to (5,6), the buyer picks the preferred pair, which is consistent with condition iii) of the refinement. The auxiliary optimization problem will not allow the buyer to choose mechanisms for which the set of inequalities (5,6) does not have a solution. This comes without loss of generality because if $\underline{U}, \overline{U} > 0$, attracting infinitely many sellers always yields a strictly negative payoff and thus is never a solution of the auxiliary optimization problem. Lemma 3.3 will show that $\underline{U}, \overline{U} > 0$ is always satisfied.

subject to

$$
\begin{aligned}
\bar{t} - \overline{xc} &\leq \overline{U} \quad \text{holding with equality if } \overline{\lambda} > 0 \\
\underline{t} - \underline{xc} &\leq \underline{U} \quad \text{holding with equality if } \underline{\lambda} > 0 \\
\bar{t} - \overline{x}\underline{c} &\leq \underline{t} - \underline{xc} \\
\underline{t} - \underline{x}\overline{c} &\leq \bar{t} - \overline{xc} \\
\underline{\lambda x} &\leq \left(1 - e^{-\underline{\lambda}}\right) \\
\overline{\lambda x} + \underline{\lambda x} &\leq \left(1 - e^{-\overline{\lambda} - \underline{\lambda}}\right) \\
\overline{\lambda}, \underline{\lambda} &\geq 0
\end{aligned}
$$

If utilities $\underline{U}$ and $\overline{U}$ are such that the solutions of the buyer's auxiliary problem with respect to $\overline{\lambda}$ and $\underline{\lambda}$ are *consistent with the population parameters*, these solutions identify the mechanisms that are offered in equilibrium. By consistent we mean that to any solution $(\underline{x}^*, \overline{x}^*, \underline{t}^*, \overline{t}^*, \underline{\lambda}^*, \overline{\lambda}^*)$ of the buyer's auxiliary problem P^{aux} we can associate a value of β, indicating the measure of buyers posting the associated mechanism, so that the feasibility condition (4) is satisfied. More specifically, if the solution to the auxiliary problem is unique, consistency simply requires that the optimal arrival rates $\underline{\lambda}^*$ and $\overline{\lambda}^*$ coincide with the population parameters $\underline{\lambda}^p$ and $\overline{\lambda}^p$; in such case, there is a pooling equilibrium where all buyers post the same mechanism.[16] If the solution is not unique and the optimal values $\underline{\lambda}^*, \overline{\lambda}^*$ differ across the different solutions, consistency requires that the average value of arrival rates equals the population parameters, with weights equal to the fraction of buyers assigned to each solution; in such case, there is a separating equilibrium where sellers sort according to their type at the search stage.[17]

Provided that the solutions to the buyer's auxiliary optimization problem P^{aux} are indeed consistent with the population parameter, we can find a set of feasible and incentive equilibrium mechanisms M^β such that each mechanism $m \in M^\beta$ corresponds to a solution of P^{aux}. By setting $\lambda(m) = \lambda^*$ for each of those mechanisms, the respective allocation not only satisfies the feasibility condition (4) but also all remaining equilibrium conditions. In particular, the two participation constraints imply that the seller's optimality condition is satisfied for all mechanisms posted in equilibrium and that there is no profitable deviation for buyers: given Proposition 3.2, for any $m \notin M^\beta$, the respective arrival rates, trading probabilities and expected transfers must belong to the constraint set of P^{aux} and thus yield a weakly smaller payoff than a solution of P^{aux}. Similarly,

[16] The same situation arises if we have multiple solutions of the auxiliary problem but for all of them we have the same values of $\underline{\lambda}^*, \overline{\lambda}^*$.

[17] For example, suppose the buyer's auxiliary problem has two solutions with arrival rates, respectively, $\underline{\lambda}_1, \overline{\lambda}_1$ and $\underline{\lambda}_2, \overline{\lambda}_2$. If γ denotes the fraction of buyers posting in market 1, consistency requires $\gamma\underline{\lambda}_1 + (1-\gamma)\underline{\lambda}_2 = \underline{\lambda}^p$ and $\gamma\overline{\lambda}_1 + (1-\gamma)\overline{\lambda}_2 = \overline{\lambda}^p$.

it is easy to see that any competitive search equilibrium, as specified in Definition 1, has to be such that the expected values of trading probabilities, transfers and arrival rates associated to mechanisms $m \in M^\beta$ solve the buyer's auxiliary optimization problem.[18] In the next section, we thus proceed to analyse the solutions of P^{aux}.

3.1 Solving the Buyer's Auxiliary Problem

Before solving the buyer's auxiliary optimization problem, it is useful to derive some conditions on the sellers' market utilities $\underline{U}, \overline{U}$ that need to be satisfied in any equilibrium.

Lemma 3.3. *At a competitive search equilibrium, we have $\underline{U} \in (0, \underline{v} - \underline{c})$, $\overline{U} \in (0, \overline{v} - \overline{c})$, $\underline{U} > \overline{U}$ and $\underline{U} - \overline{U} < \overline{c} - \underline{c}$.*

Proof See Appendix A.2

Condition $\underline{U}, \overline{U} > 0$ implies that both types of sellers make strictly positive payoffs in equilibrium. If market utilities were not strictly positive, buyers would want to attract infinitely many sellers, as additional sellers would come at no cost but increase each buyer's probability of trade. Conditions $\underline{U} < \underline{v} - \underline{c}$ and $\overline{U} < \overline{v} - \overline{c}$ state that a seller's market utility cannot exceed the gains from trade of his good. If that was the case, buyers would make losses from such sellers and consequently prefer not to attract them. Finally, conditions $\underline{U} > \overline{U}$ and $\underline{U} - \overline{U} < \overline{c} - \underline{c}$ assure that in equilibrium, incentive compatibility for low type sellers and high type sellers, respectively, can be satisfied.

Next, we can show that the sellers' participation constraints in P^{aux} can be set binding w.l.o.g. Consider first the possibility that the buyer only wants to attract high-type sellers, i.e. $\underline{\lambda} = 0$. By setting $\underline{x}$ large enough, it is always possible to find a pair $\underline{t}, \underline{x}$ so as to satisfy $\underline{t} - \underline{x}\underline{c} = \underline{U}$ and $\underline{t} - \underline{x}\overline{c} \leq \overline{U}$.[19] That is, the buyer can always find an incentive compatible combination of $\underline{t}, \underline{x}$ that makes the low type seller indifferent. The actual choice of $\underline{x}, \underline{t}$ does not affect the buyer's payoff, as $\underline{x}$ and $\underline{t}$ are multiplied by $\underline{\lambda} = 0$, both in the objective and in the remaining constraints. A symmetric argument can be made for the case of $\overline{\lambda} = 0$. Solving the participation constraints for $\underline{t}, \overline{t}$ and substituting into the objective function and the remaining constraints, the buyer's optimization

[18] Suppose not and let $\underline{U} = \max_{m \in M^\beta} \underline{u}(m|\underline{\lambda}(m), \overline{\lambda}(m)), \underline{U} = \max_{m \in M^\beta} \underline{u}(m|\underline{\lambda}(m), \overline{\lambda}(m))$. Then there exists a tuple $(\underline{x}, \overline{x}, \underline{t}, \overline{t}, \overline{\lambda}, \underline{\lambda})$ that satisfies the constraint set of P^{aux} and yields a strictly higher payoff for the buyer than the associated expected trading probabilities, transfers and arrival rates of any mechanism $m \in M^\beta$. By Proposition 3.2 and the conditions on equilibrium beliefs, we know that there exists a feasible and incentive compatible mechanism $m' \notin M^\beta$ with associated expected trading probabilities, transfers $(\underline{x}, \overline{x}, \underline{t}, \overline{t})$ and beliefs given by $(\overline{\lambda}, \underline{\lambda})$, implying that posting m' is a profitable deviation for a buyer.

[19] Satisfying these two conditions requires $\underline{x} \geq \frac{\underline{U} - \overline{U}}{\overline{c} - \underline{c}}$. By Lemma 3.3, the latter term is strictly smaller than one, implying that there exists some $\underline{x}$ for which this inequality is satisfied.

problem can thus be rewritten in the following simpler form

$$\max_{\underline{x},\overline{x},\underline{t},\bar{t},\overline{\lambda},\underline{\lambda}} \quad \overline{\lambda}\left[\overline{x}(\overline{v}-\overline{c})-\overline{U}\right]+\underline{\lambda}\left[\underline{x}(\underline{v}-\underline{c})-\underline{U}\right] \qquad (P^{aux'})$$

subject to

$$\overline{x}(\overline{c}-\underline{c}) \leq \underline{U}-\overline{U} \qquad (12)$$

$$\underline{x}(\overline{c}-\underline{c}) \geq \underline{U}-\overline{U} \qquad (13)$$

$$\underline{\lambda x} \leq \left(1-e^{-\underline{\lambda}}\right) \qquad (14)$$

$$\overline{\lambda}\overline{x}+\underline{\lambda x} \leq \left(1-e^{-\overline{\lambda}-\underline{\lambda}}\right) \qquad (15)$$

$$\overline{\lambda},\underline{\lambda} \geq 0 \qquad (16)$$

Furthermore, we can show that feasibility condition (14) is satisfied with equality at the optimum: the optimal mechanism is such that the buyer's probability of trading a low quality object equals his probability of meeting some low type seller.

Lemma 3.4. *At a solution of the auxiliary problem $P^{aux'}$, the feasibility constraint (14) is always satisfied with equality.*

Proof See Appendix A.3.

This Lemma implies that whenever $\underline{\lambda}>0$, the optimal trading probability for low type sellers is given by

$$\underline{x}=\frac{1}{\underline{\lambda}}\left(1-e^{-\underline{\lambda}}\right) \qquad (17)$$

Thus, a low quality good is traded whenever such a good is present in a match. It then follows that the buyer's payoff from trading with low type sellers is simply given by $\underline{\lambda}[\underline{x}(\underline{v}-\underline{c})-\underline{U}] = \left(1-e^{-\underline{\lambda}}\right)(\underline{v}-\underline{c})-\underline{\lambda U}$. With probability $1-e^{-\underline{\lambda}}$, the buyer meets some low type seller and gains from trade $\underline{v}-\underline{c}$ are realized. On average the buyer meets $\underline{\lambda}$ low type sellers, to each of whom she needs to guarantee utility $\underline{U}$.

In what follows we will show that the space of admissible values of sellers' utilities $\underline{U},\overline{U}$, characterized in Lemma 3.3, can be partitioned into three subregions where different sets of constraints are binding at a solution of $P^{aux'}$. To this end, we will consider three relaxed problems where, in all of them, the incentive compatibility for the high type seller (13) is ignored. In addition, the first problem also omits the incentive compatibility constraint of the low type seller (12), while the second problem omits the feasibility constraint (15). We analyze in sequence each of these relaxed problems, identifying the sets of values of $\underline{U},\overline{U}$ for which a solution of the relaxed problem solves the buyer's auxiliary optimization problem $P^{aux'}$, i.e. for which it satisfies the omitted constraints. We will show that these sets partition the space of admissible values of $\underline{U}, \overline{U}$ and that the analysis of the relaxed problems provides a convenient way to characterize the solutions of $P^{aux'}$.

Relaxed Problem (A): We begin by considering a relaxed problem with respect to $P^{aux'}$ in which not only the incentive constraint for the high type seller (13) but also the incentive constraint for the low type seller (12) is ignored. It is immediate to see that in this relaxed problem the feasibility constraint (15) needs to be satisfied with equality.[20] Using (17), from condition (15) we obtain

$$\overline{\lambda}\overline{x} = e^{-\underline{\lambda}}\left(1 - e^{-\overline{\lambda}}\right)$$

Substituting these values into the buyer's objective yields the following expression for the relaxed problem under consideration:

$$\max_{\underline{\lambda}, \overline{\lambda}} \quad \left(1 - e^{-\underline{\lambda}}\right)(\underline{v} - \underline{c}) + e^{-\underline{\lambda}}\left(1 - e^{-\overline{\lambda}}\right)(\overline{v} - \overline{c}) - \underline{\lambda}\underline{U} - \overline{\lambda}\overline{U} \tag{A}$$

With probability $1 - e^{-\underline{\lambda}}$, the buyer meets some low type seller and trades the low quality good. With the complementary probability $e^{-\underline{\lambda}}$, the buyer meets no low type seller and trades if and only if he meets some high type seller, which happens with probability $1 - e^{-\overline{\lambda}}$. The high quality good is thus traded with probability $e^{-\underline{\lambda}}\left(1 - e^{-\overline{\lambda}}\right)$.

Lemma 3.5. *The solution of relaxed problem (A) for any admissible pair $\underline{U}, \overline{U}$ is given by*

(i) $\underline{\lambda}^* = \ln\left(\frac{\underline{v}-\underline{c}}{\underline{U}}\right), \overline{\lambda}^* = 0$ *if* $\underline{U} \leq \frac{\underline{v}-\underline{c}}{\overline{v}-\overline{c}}\overline{U}$

(ii) $\underline{\lambda}^* = \ln\left(\frac{(\underline{v}-\underline{c})-(\overline{v}-\overline{c})}{\underline{U}-\overline{U}}\right), \overline{\lambda}^* = \ln\left(\frac{\underline{U}-\overline{U}}{(\underline{v}-\underline{c})-(\overline{v}-\overline{c})}\frac{\overline{v}-\overline{c}}{\overline{U}}\right)$ *if* $\underline{U} \in \left(\frac{\underline{v}-\underline{c}}{\overline{v}-\overline{c}}\overline{U}, (\underline{v}-\underline{c}) - (\overline{v}-\overline{c}) + \overline{U}\right)$

(iii) $\underline{\lambda}^* = 0, \overline{\lambda}^* = \ln\left(\frac{\overline{v}-\overline{c}}{\overline{U}}\right)$ *if* $\underline{U} \geq (\underline{v}-\underline{c}) - (\overline{v}-\overline{c}) + \overline{U}$

Proof See Appendix A.4.

Next, we need to find the values of $\underline{U}$ and $\overline{U}$ for which the solution of problem (A) satisfies the omitted incentive compatibility constraints (12) and (13). These values are characterized in the following Lemma.

Lemma 3.6. *The solution of relaxed problem (A) is also a solution of $P^{aux'}$ if and only if one of the following three conditions is satisfied.*

(i) $\underline{U} \leq \frac{\underline{v}-\underline{c}}{\overline{v}-\overline{c}}\overline{U}$

[20] If $\overline{\lambda} > 0$, the buyer's objective is strictly increasing in $\overline{x}$ and condition (15) provides an upper bound. On the other hand, if $\overline{\lambda} = 0$, (15) reduces to the feasibility constraint (14), which is satisfied with equality by Lemma 3.4.

(ii) $\underline{U} \in \left(\frac{\underline{v}-\underline{c}}{\overline{v}-\overline{c}}\overline{U}, (\underline{v}-\underline{c})-(\overline{v}-\overline{c})+\overline{U}\right)$ *and*
$\left[\ln\left(\frac{\underline{U}-\overline{U}}{(\underline{v}-\underline{c})-(\overline{v}-\overline{c})}\right)-\ln\left(\frac{\overline{U}}{\overline{v}-\overline{c}}\right)\right]^{-1}\left(\frac{\underline{U}-\overline{U}}{(\underline{v}-\underline{c})-(\overline{v}-\overline{c})}-\frac{\overline{U}}{\overline{v}-\overline{c}}\right)\leq\frac{\underline{U}-\overline{U}}{\overline{c}-\underline{c}}$

(iii) $\underline{U} \geq (\underline{v}-\underline{c})-(\overline{v}-\overline{c})+\overline{U}$ *and* $\left[\ln\left(\frac{\overline{v}-\overline{c}}{\overline{U}}\right)\right]^{-1}\left(1-\frac{\overline{U}}{(\overline{v}-\overline{c})}\right)\leq\frac{\underline{U}-\overline{U}}{\overline{c}-\underline{c}}$

Proof See Appendix A.5.

The proof of Lemma 3.6 shows that the incentive compatibility constraint for the high type seller (13) is always satisfied, while the incentive compatibility constraint for the low type seller (12) is satisfied either if $\underline{U}$ is sufficiently small or if $\underline{U}$ is sufficiently large. If $\underline{U}$ is sufficiently small so that it is optimal to only attract low type sellers ($\overline{\lambda}^* = 0$), condition (15) is satisfied for all values of $\overline{x}$. Hence, $\overline{x}$ can be picked freely so as to satisfy the incentive constraint (12). On the other hand, if market utilities are such that it is optimal to also attract high type sellers ($\overline{\lambda}^* > 0$), the value of $\underline{U}$ has to be large enough so that imitating the high type sellers is sufficiently unattractive. This is the case if the second inequalities in conditions (*ii*) and (*iii*) of Lemma 3.6 are satisfied.

Relaxed Problem (B): We consider next an alternative relaxed problem with respect to $P^{aux'}$ where, in addition to the incentive constraint of the high type seller (13), the feasibility constraint (15) is ignored. In this case, the incentive compatibility constraint of the low type seller (12) is the only remaining constraint in which $\overline{x}$ appears. Given that the buyer's objective is strictly increasing in $\overline{x}$ for all $\overline{\lambda} > 0$, (12) must be satisfied with equality whenever $\overline{\lambda} > 0$, i.e.

$$\overline{x} = \frac{\underline{U}-\overline{U}}{\overline{c}-\underline{c}} \tag{18}$$

Substituting this value and (17) into the buyer's objective, the relaxed problem in the present case can be rewritten as the following unconstrained optimization problem:

$$\max_{\underline{\lambda},\overline{\lambda}} \quad \left(1-e^{-\underline{\lambda}}\right)(\underline{v}-\underline{c})+\overline{\lambda}\frac{\underline{U}-\overline{U}}{\overline{c}-\underline{c}}(\overline{v}-\overline{c})-\underline{\lambda}\underline{U}-\overline{\lambda}\overline{U} \tag{B}$$

Lemma 3.7. *Problem (B) has a finite solution for any admissible pair* $\underline{U},\overline{U}$ *such that* $\underline{U}(\overline{v}-\overline{c}) \leq \overline{U}(\overline{v}-\underline{c})$*. The solution is given by*

(i) $\underline{\lambda}^* = \ln\left(\frac{\underline{v}-\underline{c}}{\underline{U}}\right)$, $\overline{\lambda}^* = 0$ *if* $\underline{U}(\overline{v}-\overline{c}) < \overline{U}(\overline{v}-\underline{c})$

(ii) $\underline{\lambda}^* = \ln\left(\frac{\underline{v}-\underline{c}}{\underline{U}}\right)$, $\overline{\lambda}^* \in [0,+\infty)$ *if* $\underline{U}(\overline{v}-\overline{c}) = \overline{U}(\overline{v}-\underline{c})$

The buyer's objective in problem (B) is additively separable in $\overline{\lambda}$ and $\underline{\lambda}$. It is concave in $\underline{\lambda}$,[21] implying that there is a unique optimal value for $\underline{\lambda}$, and it is linear in $\overline{\lambda}$. If $\underline{U}(\overline{v}-\overline{c}) > \overline{U}(\overline{v}-\underline{c})$,

[21] To see this, let $g(\underline{\lambda}) = \left(1-e^{-\underline{\lambda}}\right)(\underline{v}-\underline{c})-\underline{\lambda}\underline{U}$ and note that $g''(\underline{\lambda}) = -e^{-\underline{\lambda}}(\underline{v}-\underline{c}) < 0$.

the buyer's objective is strictly increasing in $\overline{\lambda}$. In this case, a solution does not exist because the buyer would like to attract infinitely many high type sellers. If $\underline{U}(\overline{v}-\overline{c}) < \overline{U}(\overline{v}-\underline{c})$, the buyer's objective is strictly decreasing in $\overline{\lambda}$, in which case it is optimal not to attract any high type sellers, i.e. $\overline{\lambda}^* = 0$. Finally, if $\underline{U}(\overline{v}-\overline{c}) = \overline{U}(\overline{v}-\underline{c})$, the buyer makes zero profits with high type sellers and is thus indifferent between attracting and not attracting them.

We show next that whenever a finite solution of problem (B) exists, a non-empty subset of the set of solutions also satisfies the omitted constraints.[22]

Lemma 3.8. *There exists a solution of relaxed problem (B) that is also a solution of $P^{aux'}$ if and only if $\underline{U}(\overline{v}-\overline{c}) \leq \overline{U}(\overline{v}-\underline{c})$. The set of those solutions is characterized by condition (i) and (ii) of Lemma 3.7 and $\overline{\lambda}^* \leq \overline{\lambda}^{max}$, with $\overline{\lambda}^{max}$ such that*

$$\frac{1}{\overline{\lambda}^{max}}\left(1-e^{-\overline{\lambda}^{max}}\right) = \frac{\underline{v}-\underline{c}}{\overline{v}-\underline{c}}.$$

if $\underline{v} < \overline{v}$ and $\overline{\lambda}^{max} = 0$ if $\underline{v} = \overline{v}$.

Proof See Appendix A.6.

If $\underline{U}(\overline{v}-\overline{c}) < \overline{U}(\overline{v}-\underline{c})$ so that $\overline{\lambda}^* = 0$, the feasibility constraint (15) coincides with (14), which is always satisfied by Lemma 3.4. If $\underline{U}(\overline{v}-\overline{c}) = \overline{U}(\overline{v}-\underline{c})$, (15) is satisfied only for $\overline{\lambda}^*$ not too large, that is only for a strict subset of the values that solve relaxed problem (B). The proof of Lemma 3.8 shows that this set is characterized by $\overline{\lambda}^{max}$. It further shows that whenever the solution of problem (B) is indeed feasible, it also satisfied the incentive compatibility constraint of the high type seller (13).

It is important to point out that at solutions $\overline{\lambda}^* < \overline{\lambda}^{max}$ the feasibility constraint (15) holds as a strict inequality. This implies that whenever there is a meeting in which all sellers are of high type, the object is traded with probability less than one. Such rationing is consistent with individual optimization because buyers make zero profits with high type sellers.

Problem (C): If the values of $\underline{U}$ and $\overline{U}$ are such that neither of the previous two relaxed problems yields a solution that satisfies the original constraint set, the solution of $P^{aux'}$, again ignoring incentive constraint (13), is such that both the incentive compatibility constraint (12) and the feasibility constraint (15) are binding.[23] The trading probabilities are then determined by (17)

[22]Note that the subset of values of $\overline{U}, \underline{U}$ identified in the following lemma includes the set of values identified in condition (i) of Lemma 3.6. In this set, the solution of problem (A) corresponds to the solution of problem (B).

[23]To see this, note that if one of these constraints were slack at a solution of $P^{aux'}$, that solution would be a local maximum of problem (A) or problem (B). However, for the values of $\underline{U}, \overline{U}$ under consideration, the solution to the first order conditions of problem (A) is unique, while a solution to problem (B) does not exists. A contradiction.

and (18) whenever $\underline{\lambda}, \overline{\lambda} > 0$. As shown in the proof of Lemma 3.8, the feasibility condition (15) is sufficient to ensure that these trading probabilities also satisfy incentive compatibility constraint (13). Substituting these values into the buyer's objective, $P^{aux'}$ reduces to the following constrained optimization problem:

$$\max_{\underline{\lambda},\overline{\lambda}} \left(1 - e^{-\underline{\lambda}}\right)(\underline{v} - \underline{c}) + e^{-\underline{\lambda}}\left(1 - e^{-\overline{\lambda}}\right)(\overline{v} - \overline{c}) - \underline{\lambda}\underline{U} - \overline{\lambda}\overline{U} \quad \text{s.t.} \quad e^{-\underline{\lambda}}\left(1 - e^{-\overline{\lambda}}\right) = \overline{\lambda}\frac{\underline{U} - \overline{U}}{\overline{c} - \underline{c}} \tag{C}$$

Lemma 3.9. *The solution of problem (C) is given by:*

(i) $\underline{\lambda}^* = \ln\left(\frac{\underline{v}-\underline{c}}{\underline{U}}\right)$, $\overline{\lambda}^* = 0$ *if* $\underline{U}(\overline{v} - \overline{c}) \le \overline{U}(\overline{v} - \underline{c})$

(ii) $\underline{\lambda}^* = \ln\left(\frac{\overline{\lambda}^*}{1-e^{-\overline{\lambda}^*}}\frac{\overline{c}-\underline{c}}{\underline{U}-\overline{U}}\right)$, $\overline{\lambda}^* = Min\left\{\overline{\lambda}^{int}, \overline{\lambda}^c\right\}$ *if* $\underline{U}(\overline{v} - \overline{c}) \ge \overline{U}(\overline{v} - \underline{c})$, *with* $\overline{\lambda}^{int}$ *such that*

$$(\overline{v} - \underline{c})\frac{\underline{U} - \overline{U}}{\overline{c} - \underline{c}} - e^{\overline{\lambda}^{int}}\frac{e^{\overline{\lambda}^{int}} - \overline{\lambda}^{int} - 1}{(e^{\overline{\lambda}^{int}} - 1)^2}(\underline{v} - \underline{c})\frac{\underline{U} - \overline{U}}{\overline{c} - \underline{c}} + \frac{e^{\overline{\lambda}^{int}} - \overline{\lambda}^{int}e^{\overline{\lambda}^{int}} - 1}{\overline{\lambda}^{int}(e^{\overline{\lambda}^{int}} - 1)}\underline{U} = 0 \tag{19}$$

and $\overline{\lambda}^c$ *such that* $\frac{1}{\overline{\lambda}^c}\left(1 - e^{-\overline{\lambda}^c}\right) = \frac{\underline{U}-\overline{U}}{\overline{c}-\underline{c}}$

Proof See Appendix A.7.

The proof of Lemma 3.9 demonstrates that the correspondence defined by the set of values of $\underline{\lambda}$ satisfying the constraint of problem (C) for any given $\overline{\lambda}$ has a discontinuity at $\overline{\lambda} = 0$. If $\underline{U}(\overline{v} - \overline{c}) < \overline{U}(\overline{v} - \underline{c})$, the solution obtains at this point and is characterized by condition (*i*). If $\underline{U}(\overline{v} - \overline{c}) > \overline{U}(\overline{v} - \underline{c})$, the optimal value of $\overline{\lambda}$ is strictly positive and the solution of problem (C) is characterized by condition (*ii*). Finally, if $\underline{U}(\overline{v} - \overline{c}) = \overline{U}(\overline{v} - \underline{c})$, problem (C) has two solutions, one characterized by condition (i) and the other characterized by condition (ii). Both solutions belong to the set of feasible solutions of relaxed problem (B) and are given by the pairs $\underline{\lambda}^* = \ln\left(\frac{\underline{v}-\underline{c}}{\underline{U}}\right), \overline{\lambda}^* = 0$ and $\underline{\lambda}^* = \ln\left(\frac{\underline{v}-\underline{c}}{\underline{U}}\right), \overline{\lambda}^* = \overline{\lambda}^{max}$, with $\overline{\lambda}^{max}$ as defined in Lemma 3.8.

Taken together, the previous Lemmas partition the space of $\underline{U}, \overline{U}$ into three regions, (I)-(III), illustrated in Figure 1. In region (I), $\overline{U}$ is sufficiently small relative to $\underline{U}$ so that either condition (*ii*) or condition (*iii*) of Lemma 3.6 holds. In this region of $\underline{U}, \overline{U}$, imitating a high type seller is sufficiently unattractive for low type sellers so that the solution of relaxed problem (A) satisfies the omitted incentive constraints and solves $P^{aux'}$. As $\overline{U}$ increases, the incentive compatibility constraint of the low type sellers becomes binding and we enter region (II). Here the solution of $P^{aux'}$ corresponds to the solution of problem (C) where both the incentive compatibility constraint of the low type sellers and the feasibility constraint are binding. If $\overline{U}$ increases further we enter region (III) where the solution of $P^{aux'}$ corresponds to the solution of problem (B). In this region, the buyer needs to ration high type sellers in order to satisfy incentives for the low type sellers, implying that there are some meetings with high type sellers in which there is no trade.[24]

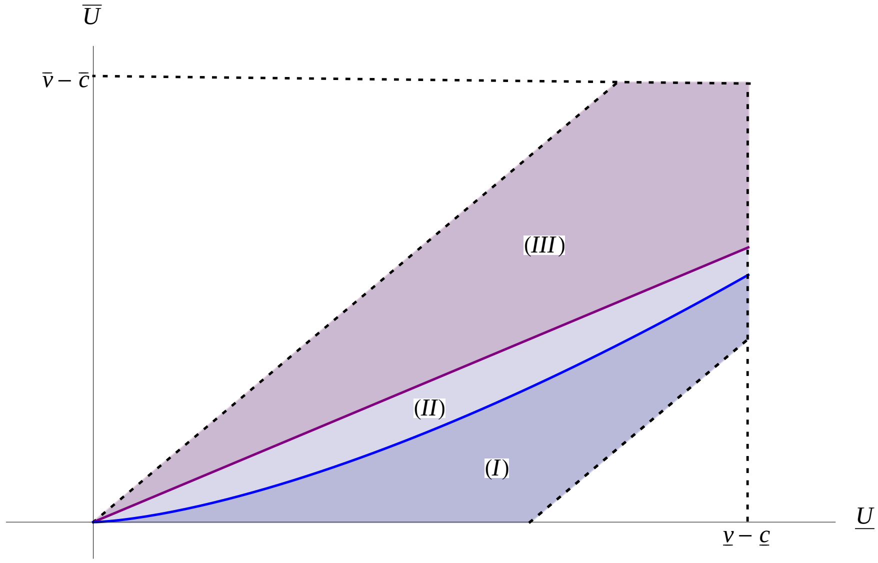

Figure 1: A buyer's auxiliary optimization problem

3.2 Characterization of the Search Equilibrium

We now characterize the competitive equilibria by finding the values of $\underline{U}$, $\overline{U}$ for which the solutions of $P^{aux'}$ are consistent with the population parameters $\overline{\lambda}^p, \underline{\lambda}^p$.

Parameter region (I): We investigate first the case where the equilibrium utility levels of high and low type sellers fall into region (I), that is where the solution of $P^{aux'}$ coincides with the solution of problem (A) described in conditions (*ii*) and (*iii*) of Lemma 3.5. Given that the solution of problem (A) is unique, any equilibrium with utility levels in region (I) necessarily has to be a pooling equilibrium. The solution $\underline{\lambda}^*, \overline{\lambda}^*$ thus needs to coincide with the population parameters $\underline{\lambda}^p, \overline{\lambda}^p$, which implies that $\underline{U}$ and $\overline{U}$ have to be such that the solution of problem (A) is interior. This case is characterized by condition (ii) of Lemma 3.5. Solving the expressions of $\underline{\lambda}^*, \overline{\lambda}^*$ in that

[24]Note that the region characterized in condition (i) of Lemma 3.6 is a subset of region (III). In the intersection of those two regions, problem (A) and problem (B) yield the same solution with $\overline{\lambda} = 0$. In the pure private value case, these two regions coincide, while the remaining admissible values of $\underline{U}, \overline{U}$ fall into region (I). That is, in the private value case, region (II) is empty and the solution of problem (A) always solves $P^{aux'}$.

condition for $\underline{U}$ and $\overline{U}$ and setting $\underline{\lambda}^* = \underline{\lambda}^p, \overline{\lambda}^* = \overline{\lambda}^p$ yields

$$\underline{U} = e^{-\underline{\lambda}^p - \overline{\lambda}^p}(\overline{v} - \overline{c}) + e^{-\underline{\lambda}^p}[(\underline{v} - \underline{c}) - (\overline{v} - \overline{c})] \tag{20}$$

$$\overline{U} = e^{-\underline{\lambda}^p - \overline{\lambda}^p}(\overline{v} - \overline{c}), \tag{21}$$

If (20) and (21) satisfy the requirements of condition (ii) in Lemma 3.6, the solution of problem (A) is indeed interior and satisfies the omitted incentive compatibility constraints. As noticed when commenting on Lemma 3.6, incentive compatibility for the high type seller is always satisfied, while incentive compatibility for the low type sellers requires $\overline{x}^* = e^{-\underline{\lambda}^p}\frac{1}{\overline{\lambda}^p}\left(1 - e^{-\overline{\lambda}^p}\right) \leq \frac{\underline{U} - \overline{U}}{\overline{c} - \underline{c}}$. Substituting (20) and (21) into this inequality yields[25]

$$e^{-\underline{\lambda}^p}\frac{1}{\overline{\lambda}^p}\left(1 - e^{-\overline{\lambda}^p}\right) \leq \frac{e^{-\underline{\lambda}^p}[(\underline{v} - \underline{c}) - (\overline{v} - \overline{c})]}{\overline{c} - \underline{c}}$$

or simply

$$\frac{1}{\overline{\lambda}^p}\left(1 - e^{-\overline{\lambda}^p}\right) \leq 1 - \frac{\overline{v} - \underline{v}}{\overline{c} - \underline{c}} \tag{22}$$

As can be verified, whenever (22) holds, also the conditions for the solution of problem (A) to be interior are satisfied.[26] We have thus proved the following:

Proposition 3.10. *If* $\frac{1}{\overline{\lambda}^p}\left(1 - e^{-\overline{\lambda}^p}\right) \leq 1 - \frac{\overline{v} - \underline{v}}{\overline{c} - \underline{c}}$, *there exists a competitive search equilibrium in which all buyers post the same mechanism, characterized by*

$$\underline{x} = \frac{1}{\underline{\lambda}^p}\left(1 - e^{-\underline{\lambda}^p}\right), \quad \overline{x} = e^{-\underline{\lambda}^p}\frac{1}{\overline{\lambda}^p}\left(1 - e^{-\overline{\lambda}^p}\right), \quad \underline{t} = \underline{xc} + \underline{U}, \quad \overline{t} = \overline{xc} + \overline{U}$$

with $\underline{U}$ *and* $\overline{U}$ *as in (20) and (21).*

Given that $\frac{1}{\overline{\lambda}^p}\left(1 - e^{-\overline{\lambda}^p}\right)$ is decreasing in $\overline{\lambda}^p$ and lies between zero and one for all $\overline{\lambda}^p > 0$,[27] condition (22) is always satisfied in the case of pure private values with $\underline{v} = \overline{v}$. It also holds in the common value case provided that the difference between the buyer's valuation of the high and low quality good is sufficiently small and the ratio of high type sellers to buyers $\overline{\lambda}^p$ is sufficiently large. A necessary condition is that $1 - \frac{\overline{v} - \underline{v}}{\overline{c} - \underline{c}} > 0$ or equivalently $\underline{v} - \underline{c} > \overline{v} - \overline{c}$, that is gains from trade are higher for the low than for the high quality object.

In the private value case the existence of a competitive search equilibrium where all buyers

[25] The same inequality is obtained by directly substituting (20) and (21) into the second inequality of condition (ii) of Lemma 3.6.

[26] The solution to problem (A) is interior if $\frac{\underline{U}}{\underline{v} - \underline{c}} > \frac{\overline{U}}{\overline{v} - \overline{c}}$ and $\underline{U} - \overline{U} < (\underline{v} - \underline{c}) - (\overline{v} - \overline{c})$ (see Lemma 3.6). Substituting (20) and (21) into these conditions, the first inequality becomes $1 > e^{-\overline{\lambda}^p}$ and the second inequality becomes $e^{-\underline{\lambda}^p} < 1$, both of which are always satisfied.

[27] See footnote 14.

post the same mechanism and sellers are screened ex-post was established in earlier work (e.g. Peters, 1997).[28] Proposition 3.10 generalizes this result, in the environment under consideration, to the case of common values where gains from trade for the low quality object remain sufficiently large compared to the high quality good. Peters (1997) also shows that the equilibrium trading probabilities and transfers can be implemented through a second-price auction with a reserve price equal to the buyers' valuation. This property can be seen from the equilibrium characterization in Proposition 3.10: if $\underline{v} = \overline{v} = v$, market utilities of low and high type seller can be rewritten as

$$\underline{U} = e^{-\underline{\lambda}^p} \left(e^{-\overline{\lambda}^p} (v - \underline{c}) + \left(1 - e^{-\overline{\lambda}^p}\right) (\overline{c} - \underline{c}) \right), \quad \overline{U} = e^{-\underline{\lambda}^p - \overline{\lambda}^p} (v - \overline{c}).$$

In a second-price auction with reserve price v, a high type seller has a positive payoff if and only if he is the only seller in a meeting, which happens with probability $e^{-\underline{\lambda}^p - \overline{\lambda}^p}$. In this event, his profit equals the difference between the reserve price v and his valuation $\overline{c}$. Otherwise the seller either looses the auction or pays a price equal to his valuation. A low type seller makes a positive profit if and only if he is the only low type seller in a meeting, which happens with probability $e^{-\underline{\lambda}^p}$. In this event, with probability $e^{-\overline{\lambda}^p}$, he is the only seller and makes a profit equal to $v - \underline{c}$, while with the complimentary probability $1 - e^{-\overline{\lambda}^p}$, there are some high type sellers and he makes a profit equal to $\overline{c} - \underline{c}$. Proposition 3.10 extends this result to the common value case $\underline{v} < \overline{v}$, as long as condition (22) holds. The trading probabilities are identical to those in the private value case, however, the specific mechanism implementing them is generally different from this particular second-price auction.

More precisely, the equilibrium mechanism in this parameter region can be interpreted as a classic auction, where sellers' bids are monotone in their type and the lowest bid always wins the auction. However, if $\underline{v} < \overline{v}$, the equilibrium trading probabilities and transfers cannot be implemented through a standard second-price auction with some reserve price r, potentially coupled with a participation fee or transfer p. To see this, note that the low and high type sellers' payoff associated to such an auction $SPA_{r,p}$ with associated seller-buyer ratios $\underline{\lambda}^p, \overline{\lambda}^p$ are given by

$$\begin{aligned} \underline{u}(SPA_{r,p}|\underline{\lambda}^p, \overline{\lambda}^p) &= e^{-\underline{\lambda}^p} \left(e^{-\overline{\lambda}^p} (r - \underline{c}) + (1 - e^{-\overline{\lambda}^p})(\overline{c} - \underline{c}) \right) + p \\ \underline{u}(SPA_{r,p}|\underline{\lambda}^p, \overline{\lambda}^p) &= e^{-\underline{\lambda}^p - \overline{\lambda}^p} (r - \overline{c}) + p \end{aligned}$$

Setting these two payoffs, respectively, equal to $\underline{U}$ and $\overline{U}$ as in (20) and (21), we obtain a set of two equations that are linearly dependent in r and p. As can be verified, this set of equations has

[28] In this work roles are typically reversed, principals are sellers and agents are buyers. It should also be emphasized that some of this work considers considerably more general environments regarding the type space (e.g. Peters, 1997) and the meeting technology (e.g. Eeckhout and Kircher, 2010).

a solution if and only if $\underline{v} = \overline{v}$.

Parameter region (II): We examine next the case where the equilibrium values of $\underline{U}$ and $\overline{U}$ fall into region (II). In that region, the solution of $P^{aux'}$ coincides with the solution of problem (C). The solution is again unique and hence can only be consistent with the population parameters if it coincides with $\underline{\lambda}^p, \overline{\lambda}^p$.[29] This requires that the solution of problem (C) is interior and satisfies the respective optimality conditions in Lemma 3.9 (*ii*). Setting $\underline{\lambda}^* = \underline{\lambda}^p$ and $\overline{\lambda}^* = \overline{\lambda}^p$ and solving these conditions for $\underline{U}$ and $\overline{U}$ yields

$$\underline{U} = e^{-\underline{\lambda}^p - \overline{\lambda}^p}(\underline{v} - \underline{c}) + e^{-\underline{\lambda}^p} \frac{1}{\overline{\lambda}^p} \left(1 - e^{-\overline{\lambda}^p}\right) \frac{1 - e^{-\overline{\lambda}^p}}{1 - \frac{1}{\overline{\lambda}^p}(1 - e^{-\overline{\lambda}^p})} (\overline{v} - \underline{v}) \tag{23}$$

$$\overline{U} = e^{-\underline{\lambda}^p - \overline{\lambda}^p}(\underline{v} - \underline{c}) + e^{-\underline{\lambda}^p} \frac{1}{\overline{\lambda}^p} \left(1 - e^{-\overline{\lambda}^p}\right) \left[\frac{1 - e^{-\overline{\lambda}^p}}{1 - \frac{1}{\overline{\lambda}^p}(1 - e^{-\overline{\lambda}^p})} (\overline{v} - \underline{v}) - (\overline{c} - \underline{c})\right] \tag{24}$$

The following proposition shows that market utilities (23) and (24) lie in region (II) and thus belong to a competitive search equilibrium whenever $\frac{1}{\overline{\lambda}^p}\left(1 - e^{-\overline{\lambda}^p}\right) \in \left(1 - \frac{\overline{v} - \underline{v}}{\overline{c} - \underline{c}}, \frac{\underline{v} - \underline{c}}{\overline{v} - \underline{c}}\right)$.[30]

Proposition 3.11. *If* $\frac{1}{\overline{\lambda}^p}\left(1 - e^{-\overline{\lambda}^p}\right) \in \left(1 - \frac{\overline{v} - \underline{v}}{\overline{c} - \underline{c}}, \frac{\underline{v} - \underline{c}}{\overline{v} - \underline{c}}\right)$*, there exists a competitive search equilibrium in which all buyers post the same mechanism, characterized by*

$$\underline{x} = \frac{1}{\underline{\lambda}^p}\left(1 - e^{-\underline{\lambda}^p}\right), \quad \overline{x} = e^{-\underline{\lambda}^p} \frac{1}{\overline{\lambda}^p}\left(1 - e^{-\overline{\lambda}^p}\right), \quad \underline{t} = \underline{xc} + \underline{U}, \quad \overline{t} = \overline{xc} + \overline{U}$$

with $\underline{U}$ *and* $\overline{U}$ *as in (23) and (24).*

Proof See Appendix A.8.

Comparing this result with Proposition 3.10, we see that the trading probabilities as a function of $\overline{\lambda}^p, \underline{\lambda}^p$ are equivalent to those in the previous case, while the expressions of the market utilities differ. The condition for the existence of this equilibrium can only be satisfied if $\overline{v} > \underline{v}$, that is if there are common values. The gains from trade for the high quality good can now be larger than those for the low quality good, provided that the ratio of high type sellers to buyers $\overline{\lambda}^p$ is sufficiently large.

Parameter Region (III): It remains to consider the case where the equilibrium utility levels fall into region (III), that is where the solution of $P^{aux'}$ coincides with the solution of the relaxed

[29] Problem (C) only has multiple solutions if $\underline{U}(\overline{v} - \overline{c}) = \overline{U}(\overline{v} - \underline{c})$, which falls into parameter region (III).

[30] Note that $1 - \frac{\overline{v} - \underline{v}}{\overline{c} - \underline{c}} < \frac{\underline{v} - \underline{c}}{\overline{v} - \underline{c}}$, provided that $\underline{v} < \overline{v}$. Suppose not. Then

$$1 - \frac{\overline{v} - \underline{v}}{\overline{c} - \underline{c}} \geq \frac{\underline{v} - \underline{c}}{\overline{v} - \underline{c}} \quad \Leftrightarrow \quad 1 - \frac{\underline{v} - \underline{c}}{\overline{v} - \underline{c}} \geq \frac{\overline{v} - \underline{v}}{\overline{c} - \underline{c}} \quad \Leftrightarrow \quad \frac{\overline{v} - \underline{v}}{\overline{v} - \underline{c}} \geq \frac{\overline{v} - \underline{v}}{\overline{c} - \underline{c}}$$

which is violated by $\overline{v} > \overline{c}$.

problem (B). This solution can only be consistent with the population parameters if it is optimal for a buyer to attract both types of sellers, which is the case if and only if $\underline{U}(\overline{v}-\overline{c}) = \overline{U}(\overline{v}-\underline{c})$.[31] In this case, condition (*ii*) of Lemma 3.7 shows that the solution of problem (B) is not unique. We consider first the possibility of a pooling equilibrium. The population parameters $\overline{\lambda}^p, \underline{\lambda}^p$ must then belong to the set of solutions of problem (B). According to Lemma 3.7, this requires that $\underline{\lambda}^p = \ln\left(\frac{\underline{v}-\underline{c}}{\underline{U}}\right)$. Hence, we have

$$\underline{U} = e^{-\underline{\lambda}^p}(\underline{v}-\underline{c}) \tag{25}$$

Substituting this value into the equality $\underline{U}(\overline{v}-\overline{c}) = \overline{U}(\overline{v}-\underline{c})$ yields

$$\overline{U} = e^{-\underline{\lambda}^p}\frac{(\underline{v}-\underline{c})(\overline{v}-\overline{c})}{\overline{v}-\underline{c}} \tag{26}$$

From Lemma 3.8 we know that the population parameters also belong to the set of feasible solutions of problem (B) if $\lambda^p \leq \overline{\lambda}^{\max}$, or equivalently:

$$\frac{1}{\overline{\lambda}^p}\left(1-e^{-\overline{\lambda}^p}\right) \geq \frac{\underline{v}-\underline{c}}{\overline{v}-\underline{c}}$$

Proposition 3.12. *If* $\frac{1}{\overline{\lambda}^p}\left(1-e^{-\overline{\lambda}^p}\right) \geq \frac{\underline{v}-\underline{c}}{\overline{v}-\underline{c}}$, *there exists a competitive search equilibrium in which all buyers post the same mechanism characterized by*

$$\underline{x} = \frac{1}{\underline{\lambda}^p}\left(1-e^{-\underline{\lambda}^p}\right), \quad \overline{x} = e^{-\underline{\lambda}^p}\frac{\underline{v}-\underline{c}}{\overline{v}-\underline{c}}, \quad \underline{t} = \underline{x}\underline{c}+\underline{U}, \quad \overline{t} = \overline{x}\overline{c}+\overline{U},$$

with $\underline{U}$ *and* $\overline{U}$ *as in (25) and (26).*

The condition under which the above equilibrium exists requires that the ratio of high type sellers to buyers $\overline{\lambda}^p$ is sufficiently small so that $\frac{1}{\overline{\lambda}^p}\left(1-e^{-\overline{\lambda}^p}\right)$ is sufficiently large and that the difference between the buyers' valuation of the high and low quality object is sufficiently large so that $\frac{\underline{v}-\underline{c}}{\overline{v}-\underline{c}}$ is sufficiently small. As before, the condition for equilibrium existence can only be satisfied if $\overline{v} > \underline{v}$. Notably, we see that the expression of the expected trading probability of high type sellers is now different from the previous two cases. More specifically, we see that the probability for a buyer to trade high quality, $\overline{\lambda}^p\overline{x} = \overline{\lambda}^p e^{-\underline{\lambda}^p}\frac{\underline{v}-\underline{c}}{\overline{v}-\underline{c}}$, is smaller than the probability for a buyer to be in a match with high type sellers only, $e^{-\underline{\lambda}^p}(1-e^{-\overline{\lambda}^p})$. The equilibrium mechanism thus not only gives priority to low type sellers but also rations high type sellers in meetings where no low type seller is present. As already noticed when commenting on Lemma 3.8, rationing can be sustained in equilibrium because buyers make no profits when trading with high type sellers, while they make positive profits from low type sellers.

[31] If buyers would only attract low type sellers in equilibrium, we would have $\overline{U} = 0$, contradicting Lemma 3.3.

Since buyers are indifferent between the number of high type sellers they attract, the zero profit conditions also implies that $P^{aux'}$ has multiple solutions with respect to $\overline{\lambda}$. As we show in the next section, the multiplicity of solutions of $P^{aux'}$ implies that other equilibria may exist, where sellers post different mechanisms that attract different ratios of high and low type sellers. However, all such equilibria must be payoff and allocation equivalent to the equilibrium characterized in Proposition 3.12. To see this, note that uniqueness of the solution of $P^{aux'}$ with respect to $\underline{\lambda}$ and the fact that in equilibrium it must be weakly optimal for a buyer to attract both types of sellers implies that market utilities $\underline{U}$ and $\overline{U}$ are always determined by (25) and (26). Given those market utilities, the optimal value of $\underline{x}$ is uniquely pinned down. So is the value $\overline{x}$ for any mechanism that attracts some high type sellers, implying that the trading probabilities and transfers of any equilibrium in the specified parameter region are those specified by Proposition 3.12.

Propositions (3.10)-(3.12) partition the parameter space into three regions. As can be verified, in each of those regions market utilities $\underline{U}$ and $\overline{U}$ are uniquely determined by the exogenous parameters.[32] Together, these two properties imply that the equilibrium characterized in propositions (3.10)-(3.12) is unique in terms of allocation and payoffs. The statement in Theorem 3.1 is thereby established.

4 Properties of the Equilibrium

4.1 Equilibrium Payoffs and Cream Skimming Deviations

In this section we discuss the main properties of the mechanisms that are traded in equilibrium and of the payoffs attained by buyers and sellers. As stated in Theorem 3.1, in equilibrium all buyers post the same mechanism (or mechanisms yielding the same expected trading probabilities and expected transfers). We thus have a pooling outcome where all sellers choose the same market. However, the equilibrium mechanism implies different values of trading probabilities and transfers for low and high type sellers, hence there is screening of sellers within the mechanism: in particular, the expected probability of trade is strictly greater for low than for high type sellers.[33]. Theorem 3.1 further demonstrates that competitive search equilibria exhibit the property that trade of the low quality good is maximal. It is interesting to point out that this feature of the equilibrium holds for all parameter values, in particular no matter how large the gains from trade for the high quality object, $\overline{v} - \overline{c}$, relative to those for the low quality object, $\underline{v} - \underline{c}$, are. To understand why this property holds, the following lemma establishes an important feature of the equilibrium payoff

[32] In all three parameter regions, the conditions pinning down the market utilities are linear in $\underline{U}$ and $\overline{U}$. For the first region, these conditions correspond to (32) and (33), while for the second region they are given by (19) and the constraint of problem (C). The third region was discussed in the previous paragraph.

[33] This property is established formally in the proof of Lemma 4.1

of buyers in all three parameter regions: the payoff of a buyer conditional on meeting a low type seller is strictly higher than his payoff conditional on meeting a high type seller.

Lemma 4.1. *At a competitive equilibrium we always have* $\underline{x}(\underline{v}-\underline{c})-\underline{U}>\overline{x}(\overline{v}-\overline{c})-\overline{U}$.

Proof. See Appendix A.9. □

Thus, in the environment considered, buyers' profits are not equalized across trades with low and high type sellers but are in fact larger with low type sellers. The argument of the proof of Lemma 4.1 shows why it is not possible for buyers to make higher profits with high type sellers in equilibrium: if $\underline{x}(\underline{v}-\underline{c})-\underline{U}<\overline{x}(\overline{v}-\overline{c})-\overline{U}$, there always exists a profitable cream skimming deviation, aiming to replace low type sellers with high type sellers while keeping their trading probabilities and transfers unchanged. This deviation is feasible due to the fact that high type sellers trade with a lower probability than low type sellers. A similar argument applies if buyers make the same profits with high and low type sellers[34]. Hence, in equilibrium buyers must strictly prefer to trade with low type sellers. In such a situation, replacing high type sellers with low type sellers while keeping their trading probabilities and transfers constant would be profitable but violates the feasibility constraint $\underline{x}\leq\frac{1}{\underline{\lambda}}\left(1-e^{-\underline{\lambda}}\right)$. Hence, the property that low type sellers are given priority by the equilibrium mechanism is closely linked to the one regarding buyers' profits established in Lemma 4.1. The lemma also implies that there can be no complete pooling in the competitive search equilibrium. If different types of sellers would trade with identical probabilities, $\underline{x}=\overline{x}$, and receive the same expected transfers, $\underline{t}=\overline{t}$, the difference between a buyer's expected profits with a high and low type seller would be positive and given by $\underline{x}(\overline{v}-\underline{v})$.[35] The cream skimming deviation of replacing low type sellers with high type sellers would thus always be profitable.

The property that buyers must make more profits when they trade with low type sellers, even when the gains from trade of the high quality good are large in relative terms, further implies that most of these gains are appropriated by the sellers. This can be clearly seen from the equilibrium values reported in Propositions 3.10, 3.11, 3.12. Consider an increase in the buyer's valuation of the high quality good $\overline{v}$. If $\overline{v}$ is sufficiently small, we are in parameter region I, where the incentive compatibility constraint of the low type sellers is slack. Due to this property, an increase in $\overline{v}$ benefits buyers and high type sellers but reduces the market utility of low type sellers. If $\overline{v}$ increases sufficiently, we enter parameter region II. Here the incentive compatibility constraint of the low type sellers is binding. An increase in $\overline{v}$ again benefits high type seller but now also raises the market utility of low type seller in order to keep incentives satisfied. The extent to which low

[34]The profitable deviation in this case consists in attracting a slightly larger number of high types, which is again feasible due to $\underline{x}>\overline{x}$

[35]Strict positivity holds only in the common value case $\underline{v}<\overline{v}$. In a pure private value case, a slight modification of the argument is needed.

type sellers can profit from an increase in $\overline{v}$ is bounded above and this bound is reached when we enter parameter region III. Here $\underline{U}$ is invariant with respect to the level of $\overline{v}$, while $\overline{U}$ further increases in $\overline{v}$. Incentive compatibility than requires that the high type sellers' trading probability $\overline{x}$ decreases in $\overline{v}$. The intuition is that since buyers must make lower profits when they trade with high type sellers, most of the gains from trade of the high quality object go to the high type sellers. Given that those gains are relatively large in the third parameter region, the high quality object must be traded at a high price. Incentive compatibility can then only be satisfied if the trading probability of the high type sellers is sufficiently small. As a result, we have rationing of high type sellers in equilibrium and the larger $\overline{v}$ is, the more severe rationing becomes. Note that this feature stands in contrast to a monopolistic auction setting, or one with random rather than directed search, where a larger value of $\overline{v}$ favours pooling offers and thus leads to a weakly larger trading probability of the high quality good.

4.2 Rationing and Partial Sorting

As anticipated at the end of Section 2.2, the fact that in region III buyers make zero profits with the high type sellers has another important implication: since buyers are indifferent between how many high type sellers they attract, the search equilibrium may exhibit partial sorting at the search stage. More precisely, in what follows we show that in region III there exist additional equilibria, where different mechanisms are traded, attracting different ratios of high type sellers to buyers.

Suppose two submarkets are active in equilibrium, labelled 1 and 2, with seller-buyer ratios $\overline{\lambda}_1, \underline{\lambda}_1$ and $\overline{\lambda}_2, \underline{\lambda}_2$, respectively, and assume $\overline{\lambda}_2 = 0$. Let the trading probabilities in the two submarkets be denoted by $\underline{x}_1, \overline{x}_1$ and $\underline{x}_2, \overline{x}_2$ and let γ denote the fraction of buyers posting mechanism 1. In a sorting equilibrium, choosing either submarket has to be optimal for buyers, which according to Lemma 3.7 is the case if

$$\underline{\lambda}_1 = \underline{\lambda}_2 = \ln\left(\frac{\underline{v} - \underline{c}}{\underline{U}}\right) = \underline{\lambda}^p, \quad \text{and} \quad \underline{x}_1 = \underline{x}_2 = \frac{1}{\underline{\lambda}^p}(1 - e^{-\underline{\lambda}^p})$$

As argued in Section 2.2, the market utilities $\underline{U}$ and $\overline{U}$ must be as in (25) and (26), and $\overline{x}_1 = e^{-\underline{\lambda}^p}\frac{\underline{v}-\underline{c}}{\overline{v}-\underline{c}}$, as in the pooling equilibrium, while $\overline{x}_2$ can be chosen freely as long as incentive compatibility for the low type sellers is satisfied.[36] Feasibility is then satisfied if the following condition holds

$$\gamma\overline{\lambda}_1 + \underbrace{(1-\gamma)\overline{\lambda}_2}_{=0} = \overline{\lambda}^p$$

with $\overline{\lambda}_1 \leq \overline{\lambda}^{\max}$. Recalling that $\frac{1}{\overline{\lambda}^{max}}\left(1 - e^{-\overline{\lambda}^{max}}\right) = \frac{\underline{v}-\underline{c}}{\overline{v}-\underline{c}}$, this condition can always be satisfied for

[36] For example, $\overline{x}_2 = \overline{x}_1$.

an interval of values of γ sufficiently close to one as long as we are in the interior of region III, that is if $\frac{1}{\overline{\lambda}^p}\left(1-e^{-\overline{\lambda}^p}\right) > \frac{\underline{v}-\underline{c}}{\overline{v}-\underline{c}}$. The property that γ is large enough corresponds to the requirement that sufficiently many buyers and low type sellers post and search in submarket 1, the market where both types of objects are traded.

In the situation described, two mechanisms coexist in equilibrium. In submarket 2 buyers post a simple mechanism (effectively a price) that only attracts low type sellers, while in submarket 1 buyers post a more complex mechanism (some form of auction) attracting both high and low type sellers. Evidently, one can also construct other sorting equilibria, where two or more mechanisms are traded, attracting different ratios of high type sellers. However, all these equilibria are payoff equivalent to the pooling equilibrium described in Proposition 3.12.

4.3 Welfare Properties of Equilibria

In the economy under consideration the level of total surplus coincides with the realized gains from trade. At an allocation where the trading probabilities are, respectively, $\overline{x}$ and $\underline{x}$ for the high and low type sellers it is then given by

$$b\left[\overline{\lambda}^p\overline{x}(\overline{v}-\overline{c})+\underline{\lambda}^p\underline{x}(\underline{v}-\underline{c})\right]$$

The welfare properties of the search equilibria we characterized depend on the parameter values of the economy. We first establish the following:

Proposition 4.2. *If* $\underline{v}-\underline{c} \geq \overline{v}-\overline{c}$ *and* $\frac{1}{\overline{\lambda}^p}\left(1-e^{-\overline{\lambda}^p}\right) \leq \frac{\underline{v}-\underline{c}}{\overline{v}-\underline{c}}$*, the competitive search equilibrium maximizes total surplus among all feasible allocations.*

The first condition in the statement of Proposition 4.2 says that the gains from trade are higher when trade occurs with the low type than with the high type seller. Under this condition, total surplus $b\left[\overline{\lambda}^p\overline{x}(\overline{v}-\overline{c})+\underline{\lambda}^p\underline{x}(\underline{v}-\underline{c})\right]$ is maximal if, subject to the meeting friction, total trade is maximal and low quality is traded whenever possible. In the competitive search equilibrium the latter property is always satisfied by Lemma 3.4, according to which a buyer's probability of trading a low quality good is equal to a buyer's probability of meeting a low type seller. Turning to the first property, total trade is maximal if the allocation maximizes the total number of meetings subject to the friction and if every meeting leads to trade. The requirement that every meeting leads to trade is satisfied in equilibrium if the feasibility constraint (15) holds with equality. The second condition in Proposition 4.2, $\frac{1}{\overline{\lambda}^p}\left(1-e^{-\overline{\lambda}^p}\right) \leq \frac{\underline{v}-\underline{c}}{\overline{v}-\underline{c}}$, implies that we are in parameter regions (I) or (II), where (15) is indeed satisfied as equality. To complete the proof it remains then to show that also the total number of meetings is maximal. It is well known that under the urn-ball meeting technology the total number of meetings is maximal whenever the expected queue length is the same for all the mechanisms traded in equilibrium[37] This is indeed the case in the competitive search equilibrium, since sellers are pooled at the search stage.

Note that the two conditions of Proposition 4.2 are always satisfied in the pure private value case, $\underline{v} = \overline{v}$. The result that the competitive search equilibrium maximizes social surplus in private value environments is well established in the literature (see for example Eeckhout and Kircher 2010).[38] Proposition 4.2 extends this result to the more general case where the gains from trade are larger for the low quality object, provided the ratio of high type sellers to buyers is not too low. It is interesting to point out that the only constraint that is considered here is the matching friction, while incentive compatibility does not constrain attainable welfare.

Whenever the gains from trade of the low quality object are strictly smaller than those of the high quality object, that is when the common value component of the agents' private information as captured by the difference $\overline{v} - \underline{v}$ is sufficiently large, the allocation of the competitive search equilibrium no longer maximizes total surplus. In this case we would like the high quality good to be traded whenever possible, however incentive constraints clearly limit such trades. We show in the next proposition that, even taking the incentive constraints into account, surplus is no longer maximal at the search equilibrium. Moreover, if the fraction of high type sellers is sufficiently large, there exists an allocation that satisfies the constraints imposed by the matching friction and incentive compatibility and Pareto improves on the allocation of the competitive search equilibrium. In such case, the gains made trading with high type sellers are enough to compensate the possible losses with low type sellers, implying that the competitive search equilibrium is constrained inefficient.

Proposition 4.3. *Suppose that $\underline{v} - \underline{c} < \overline{v} - \overline{c}$. Then there exists a feasible and incentive compatible allocation that attains a higher level of social surplus than at the competitive equilibrium. Moreover, the allocation constitutes a Pareto improvement if one of the two following conditions are satisfied:*

(i) $\frac{1}{\overline{\lambda}^p}\left(1 - e^{-\overline{\lambda}^p}\right) \le \frac{\underline{v}-\underline{c}}{\overline{v}-\underline{c}}$ *and* $\mu \ge \frac{\overline{c}-\underline{c}}{\overline{v}-\underline{v}}$

(ii) $\frac{1}{\overline{\lambda}^p}\left(1 - e^{-\overline{\lambda}^p}\right) > \frac{\underline{v}-\underline{c}}{\overline{v}-\underline{c}}$ *and* $\mu \ge \frac{\overline{c}-\underline{c}}{\overline{v}-\underline{c}}$

Proof See Appendix A.10.

[37] See for example Eeckout and Kircher (2010). For completeness of the argument we provide a formal proof of this property in our environment in the Online Appendix.

[38] In fact Cai et al. (2015) demonstrate for the private value case that pooling at the search stage maximizes social surplus under any type distribution if and only if the meeting technology satisfies a property called 'love for variety'. According to this property, in the binary type case, the probability of meeting at least one low type seller is a concave function of the ratio of low type sellers to buyers and the ratio of high type sellers to buyers. The condition entails that social surplus can be increased by merging any two submarkets, irrespective of their composition.

The proof of the proposition shows that an increase in the trading probability of high type sellers, relative to their level of trade at the search equilibrium, is both feasible and incentive compatible. Such an increase, possibly combined with a suitable reduction in the trading probability of the low type sellers, is in fact always a feasible change of the equilibrium allocation. In addition, the expected transfers to the high and low type sellers can always be suitably adjusted so as to ensure that incentive compatibility is satisfied. This change in the allocation always increases social surplus when the gains from trade are larger for the high quality good and improves sellers' utility. We then show that it also constitutes a Pareto improvement, in the sense that buyers also gain, provided the fraction of high type sellers in the population is sufficiently high. It is interesting to point out that the inefficiency of the search equilibrium obtains not only under condition (ii), when the equilibrium displays rationing, but also under (i), where the feasibility constraint is satisfied with equality in equilibrium and every meeting leads to trade.

The inefficiency of the competitive search equilibrium with private information of the common value type is related to analogous results obtained for different structures of markets (see Gale (1992), Guerrieri et al. (2010) for the case of competitive search equilibria when meetings are restricted to be bilateral[39], Dubey and Geanakoplos (2002) and Bisin and Gottardi (2006) for competitive equilibria in the absence of search frictions). The common feature to all these results is that the equilibrium is separating, with different mechanisms traded by low and high type sellers and buyers being indifferent between trading with any of the two types. In contrast, in the environment considered here, the space of possible mechanisms exploits the richness of the possible meetings between buyers and sellers allowed by the random meeting technology and the equilibrium is pooling with a single mechanism traded in equilibrium, though the implied trading probabilities of high and low type sellers are different. Also, as we noticed earlier, buyers' profits are not equalized across sellers types. However both in the environment considered here and in the work recalled above, the source of the inefficiency is the low trading probability of high type sellers, and a welfare improvement is attained by bringing their probability of trade closer to that of low type sellers.[40]

The reason why at the competitive search equilibrium characterized in Section 2 there is no profitable deviation that allows to capture the additional gains from trade is that such deviation, similarly to the papers above, would attract too many low type sellers in order to be profitable. To see this, notice that, in order to increase the trading probability of high type sellers, a buyer would have to give an additional information rent to low type sellers, which implies that all low type sellers would have strict incentives to switch to the deviating contract, up to the point where

[39] See the next section for further discussion of the relationship with this work.

[40] As recalled above, this stands in contrast to a monopolistic setting, where for μ large enough the equilibrium contract features equal trading probabilities for both types.

their expected payoff from the deviating contract is driven down again to their market utility.[41]

5 Sorting Versus Screening

An important benchmark for our analysis is provided by the results obtained in earlier work on competitive equilibria with directed search in markets with adverse selection (see Gale (1992), Inderst Muller (2002), Guerrieri et al. (2010)). These papers restrict attention to the case where meetings are bilateral, that is where each buyer can meet at most one seller, and show that, under this restriction, the competitive search equilibrium exhibits ex-ante sorting instead of ex-post screening. Restricting the matching technology to bilateral meetings is analogous to restricting the set of available mechanisms to posted prices. In either case, a buyer picks one of the arriving sellers at random. In our environment we can capture the restriction of the set of feasible mechanisms to posted prices by requiring expected trading probabilities and prices to satisfy, respectively, $\overline{x} = \underline{x} = \frac{1}{\underline{\lambda}+\overline{\lambda}}\left(1 - e^{-\underline{\lambda}-\overline{\lambda}}\right)$ and $\overline{t} = \underline{t}$. Under this restriction, the set of mechanisms available to buyers is such that every meeting leads to trade and no type of seller receives priority.[42] The next proposition shows that, given this restriction on admissible mechanisms, the equilibrium outcome, as one should expect, displays ex-ante sorting instead of ex-post screening.

Proposition 5.1. *If the set of available mechanism is restricted to posted prices, the competitive search equilibrium exists and has the following properties:*

- *a fraction $\gamma \in [0, 1)$ of buyers post price p_h and only attract high type sellers;*
- *the remaining fraction $1 - \gamma \in (0, 1]$ of buyers post price $p_l < p_h$ and only attract low type sellers.*

Proof See the Online Appendix.

Under the stated restriction, the auxiliary problem of a buyer simplifies, as feasibility and incentive compatibility always hold, and we are left with the participation constraints only. We then show in the proof that buyers never find it optimal to attract both types of sellers. The proof further demonstrates that, if the gains from trade for the low quality good are considerably larger than

[41]Formally, if a buyer posts a mechanism m that yields a payoff for low type sellers strictly larger than $\underline{U}$ for any pair $\underline{\lambda}, \overline{\lambda}$, the buyer's belief is pinned down by condition (ii) of the refinement, i.e. $\underline{\lambda}(m) = +\infty$. Given this belief, the deviating contract always yields a strictly negative payoff for the buyer. On the other hand, if the buyer post a mechanism for which a solution to (5,6) exists, the pair of seller-buyer ratios is such that the deviating mechanism yields the market utility for low type seller, implying that low type sellers cannot receive an additional information rent.

[42]In principle, we could also allow for general menus. Upon being randomly chosen by a buyer, a seller would then have to chose between two contracts in the menu. For a buyer, such menus are always dominated by posted prices, as shown in Guerrieri et al. (2010).

the gains from trade for the high quality good, in equilibrium buyers strictly prefer to attract low type sellers, while high type sellers are excluded from trade. If this is not the case, a fraction of buyers post prices that only attract high type sellers, while the others post prices that attract low type sellers and, given the implied arrival rates in each market, all buyers are indifferent between posting any of the two prices.

The analysis in Section 3 shows that, once we allow for more general mechanisms, (pure) sorting equilibria as characterized in Proposition 5.1 do not exist. The intuition for why this is the case is simple. Consider a buyer that posts a price which only attracts low type sellers. If the set of available mechanisms were not restricted to posted prices, a buyer could post a more general mechanism, analogous to an auction, that yields the same trading probability and expected transfer for low type sellers as the posted price, but also attracts some high type sellers. Since buyers have no capacity constraints, these additional meetings would not crowd out any meetings with low type sellers. Hence this mechanism would attract the same number of low type sellers, and the buyer would obtain the same expected payoff from them; on top of that, the buyer can make some positive profits from high type sellers.

Given these rather different properties of the allocation at a competitive search equilibrium with price posting and at one with general direct mechanisms, as characterized in our Theorem 3.1, it is then of interest to compare the two in terms of welfare. If the gains from trade for the low quality good exceed the gains from trade for the high quality good, the latter, as we showed in Proposition 4.2, maximizes social surplus. Hence this equilibrium, featuring ex-post screening, always dominates the price posting one with ex-ante sorting in terms of total surplus. On the other hand, if adverse selection is more severe and gains from trade are larger for the high quality good, the equilibrium with general mechanisms can be constrained inefficient, as shown in Proposition 4.3, not only because the mechanism traded in equilibrium gives priority to the good with the lower gains from trade but also because there can be rationing on top of the meeting friction. One might conjecture that the separating equilibrium with price posting may do better in such cases. However, we find that this is typically not the case,[43] as the next numerical example illustrates.

Example 1: *Let* $\underline{\lambda}^p = \overline{\lambda}^p = 1$ *and* $\underline{c} = 0, \overline{c} = 1, \underline{v} = 1, \overline{v} = 3$. *Thus, the gains from trade for the high quality good are twice as large as those for the low quality and there are twice as many buyers as sellers, half of whom have a high quality good. Under this specification, we have* $\frac{1}{\overline{\lambda}^p}\left(1 - e^{-\overline{\lambda}^p}\right) \geq \frac{\underline{v}-\underline{c}}{\overline{v}-\underline{c}}$, *that is we are in parameter region III. The equilibrium with general direct mechanisms is then as characterized in Proposition 3.12 and features rationing. In particular, a*

[43]That is, in all numerical simulations we considered such that parameters fall into region III, the social surplus in the equilibrium with general directed mechanisms exceeds that in the price posting equilibrium.

buyer's probability to trade, respectively, a high and low quality good is given by

$$\begin{aligned}\overline{\lambda}^p\overline{x} &= \overline{\lambda}^p e^{-\underline{\lambda}^p}\frac{\underline{v}-\underline{c}}{\overline{v}-\underline{c}} \approx 0.123 \\ \underline{\lambda}^p\underline{x} &= 1-e^{-\underline{\lambda}^p} \approx 0.632\end{aligned}$$

while a buyer's probability to meet some high type seller without meeting a low type seller is equal to $e^{-\underline{\lambda}^p}(1-e^{-\overline{\lambda}^p}) \approx 0.233$.

In the equilibrium with price posting, a fraction $\gamma \approx 0.120$ *of buyers post the high price, while the remaining fraction of buyers post a low price. A buyer's probability to trade a high and low quality good is now*

$$\begin{aligned}\gamma\overline{\lambda}^p\overline{x} &= \gamma\left(1-e^{-\overline{\lambda}^p}\right) \approx 0.120 \\ (1-\gamma)\underline{\lambda}^p\underline{x} &= (1-\gamma)\left(1-e^{-\underline{\lambda}^p}\right) \approx 0.598\end{aligned}$$

In the above example, a buyer's probability of meeting a seller in the equilibrium with general direct mechanisms is 86.5% while that of meeting only high type sellers is 23.3%. The buyer's probability of trading the high quality good is considerably lower (12.3%), hence in meetings without low type sellers trade occurs only slightly more than half of the time. Comparing these numbers with the corresponding ones in the price posting equilibrium, we see that in the latter the probability that a buyer meets a seller decreases to 71.8%. This is due to the search friction, as allocating buyers and sellers over two submarkets with different seller-buyer ratios implies that there is a higher chance that sellers end up misallocated across buyers. Likewise, the probability of trading a low quality good is lower in the price posting equilibrium since low type sellers distribute themselves only across a fraction of buyers rather than across all of them. In Example 1 also the probability of trading a high quality good decreases slightly from 12.3% to 12.0%. To gain some understanding of why the probability of trading high quality does not increase in the separating equilibrium notice the following features. Incentive compatibility for low type sellers requires that the seller-buyer ratio in the high quality market is higher than in the low quality market. This implies that a buyer's probability of meeting a high type seller strictly exceeds that of meeting a low type seller. Since in equilibrium buyers have to be indifferent between attracting high and low type sellers, it follows that, conditional on meeting a seller, a buyer has to make lower profits with high than with low type sellers. This in turn implies that most of the gains from trade of the high quality good have to go to high type sellers, which can only be incentive compatible if a high type seller's trading probability is sufficiently low, similar to the case of general direct mechanisms.

It is interesting to point out that parameter region III includes as a limiting case the specification of parameter values for which buyers make zero profits in equilibrium. This case is of interest since it corresponds to the situation considered in Rothschild and Stiglitz's (1976) classic model of adverse selection as well as in other models where there is free entry of uninformed traders (or equivalently, each uninformed trader can trade any number of contracts). An analogous situation can be obtained in our environment by letting the measure of buyers tend to $+\infty$. As $b \to +\infty$, the ratio of high type sellers to buyers, $\overline{\lambda}^p$, tends to zero, implying that $\frac{1}{\overline{\lambda}^p}(1 - e^{-\overline{\lambda}^p})$ tends to one. We are thus in the third parameter region of the equilibrium with general direct mechanisms. From the expressions in Proposition 3.12 it can be seen that, as $b \to +\infty$, the transfer to low type sellers conditional on trading converges to $\underline{v}$, while their trading probability converges to one;[44] the transfer to high type sellers conditional on trading, on the other hand, converges to $\overline{v}$, while their trading probability converges to $\frac{\underline{v}-\underline{c}}{\overline{v}-\underline{c}}$. These trading probabilities and transfers precisely correspond to the ones of the separating candidate equilibrium found by Rothschild and Stiglitz (1976).[45] It is interesting to notice that in this case the equilibrium allocation when mechanisms are restricted to posted prices converges to the same limit: here we have two separate markets, one in which buyers post price $\underline{v}$ and sellers trade with probability one, another in which buyers post price $\overline{v}$ and the ratio between sellers and buyers $\frac{\overline{\lambda}^p}{\gamma}$ is such that high type sellers trade with probability $\frac{\underline{v}-\underline{c}}{\overline{v}-\underline{c}}$.[46]

At the same time, we should point out that there are also environments in which the price posting equilibrium yields strictly more surplus than the equilibrium with general mechanisms. It is interesting to observe that this reversal does not arise when the latter equilibrium features rationing (as in the previous example), but rather when parameters fall into region II, where every meeting leads to trade. The next proposition shows that, provided the gains from trade for the high quality good exceed those for the low quality good, as the measure of high type sellers becomes sufficiently large the price posting equilibrium yields a strictly larger social surplus compared to the equilibrium with general mechanisms.

Proposition 5.2. *Assume $\underline{v} - \underline{c} < \overline{v} - \overline{c}$. If the measure of high type sellers tends to $+\infty$, social surplus is strictly greater when buyers are restricted to posted prices compared to when they can post general mechanisms.*

Proof See Appendix A.11.

[44] As $b \to +\infty$, a seller's probability of being alone in a meeting with a buyer converges to one.

[45] Note that in competitive search models, as well as in Walrasian models, the non-existence issue found by Rothschild and Stiglitz (1976) in a strategic setting does not arise.

[46] Formally, this result can be obtained by noting that, as $\overline{\lambda}^p \to 0$, the indifference condition of buyers - condition (12) in the proof of Proposition 5.1 - can only be satisfied if $\gamma \to 0$ so that $\frac{1}{\frac{\overline{\lambda}^p}{\gamma}}\left(1 - e^{-\frac{\overline{\lambda}^p}{\gamma}}\right) \to \frac{\underline{v}-\underline{c}}{\overline{v}-\underline{c}}$.

The above result can be explained as follows. For the equilibrium with general mechanisms, the conditions $\underline{v}-\underline{c}<\overline{v}-\overline{c}$ and $\overline{\lambda}^p \to +\infty$ imply that we are in parameter region II, where the equilibrium features no rationing. The property that low type sellers are given priority in every meeting implies that buyers trade the low quality good with probability $1-e^{-\underline{\lambda}^p}$, the probability with which they meet at least one low type seller. As $\overline{\lambda}^p \to +\infty$, a buyer's probability of meeting some high type seller tends to one, which, given the no rationing property, implies that in the limit buyers trade a high quality good with the residual probability, $e^{-\underline{\lambda}^p}$. Social surplus thus tends to $b[e^{-\underline{\lambda}^p}(\overline{v}-\overline{c})+(1-e^{-\underline{\lambda}^p})(\underline{v}-\underline{c})]$.

In the price posting equilibrium social surplus is strictly higher because, as $\overline{\lambda}^p \to +\infty$, the fraction of buyers attracting high type sellers, γ, tends to one. In this equilibrium the trading probability of sellers in fact converges to zero in both submarkets, but the relative probability of trade in the high quality submarket compared to the low quality submarket is sufficiently small so that the incentive compatibility constraint of low type sellers is satisfied. As a consequence, the probability that a buyer trades tends to one in both submarkets and the measure of buyers posting the high price tends to b. In the limit, social surplus in the price posting equilibrium is thus given by $b(\overline{v}-\overline{c})$, which is equal to the first best level and strictly exceeds social surplus in the equilibrium with general mechanisms. Due to the property that in the latter equilibrium low type sellers are given priority over high type sellers, the difference in social surplus between the two types of equilibria is largest when also the measure of low type sellers is large, illustrated in the following example.

Example 2: *Let $\overline{\lambda}^p = 8$, $\underline{\lambda}^p = 2$ and $\underline{c}=0, \overline{c}<1.5, \underline{v}=2.5, \overline{v}=4$. Again the gains of trade for the high quality good are strictly greater than those for the low quality good. However, compared to Example 1, for every buyer there are eight high type sellers and two low type sellers. Under this specification, we have $\frac{1}{\overline{\lambda}^p}\left(1-e^{-\overline{\lambda}^p}\right) < \frac{\underline{v}-\underline{c}}{\overline{v}-\underline{c}}$, we are thus in parameter region II, where the equilibrium with general mechanisms features no rationing. In this equilibrium, a buyer's probability to trade, respectively, a high and a low quality good is given by*

$$\begin{aligned}\overline{\lambda}^p\overline{x} &= e^{-\underline{\lambda}^p}\left(1-e^{-\overline{\lambda}^p}\right) \approx 0.135\\ \underline{\lambda}^p\underline{x} &= 1-e^{-\underline{\lambda}^p} \approx 0.865\end{aligned}$$

In the equilibrium with price posting, the fraction of buyers posting the high price is $\gamma \approx 0.392$ *and the probability a buyer trades, respectively a high and low quality good is given by*

$$\begin{aligned} \beta\overline{\lambda}\overline{x} &= \beta\left(1 - e^{-\frac{\hat{\mu}}{\beta}\hat{\lambda}}\right) \approx 0.392 \\ (1-\beta)\underline{\lambda}\underline{x} &= (1-\beta)\left(1 - e^{-\frac{1-\hat{\mu}}{1-\beta}\hat{\lambda}}\right) \approx 0.585 \end{aligned}$$

In Example 2, there are ten sellers for every buyer, of whom 80% have a high quality good. Due to the high seller-buyer ratio, in the equilibrium with general direct mechanisms the trading probability for buyers is close to one, but only 13.5% of buyers end up purchasing a high quality good. This is due to the fact that there are twice as many low type sellers as buyers, so that the probability that a buyer meets some low type seller is equal to 86.5%. Thus, although the majority of sellers have a high quality good, the feature that low type sellers are given priority in any match, together with a large seller-buyer ratio, implies that high quality is traded relatively rarely. In the equilibrium with price posting, on the other hand, the probability that a buyer trades is slightly lower (97, 7%), but the probability of trading a high quality good is considerably higher (almost 40%). Whether this leads to an increase in surplus or not depends on the seller's valuation of the high quality good. If $\overline{c}$ is sufficiently small, then the effect of the increased probability of trade of the high quality object outweighs the effect of the reduced overall probability of trade and surplus is larger in the sorting equilibrium. In our numerical example this requires $\overline{c} < 1.28$.

To sum up, this section has demonstrated that the features of the meeting technology and hence of the possible trading mechanisms between buyers and sellers have important and nontrivial implications for the properties of equilibrium allocations, and in particular welfare. While for most parameter specifications, the equilibrium with general mechanisms yields a higher level of social surplus than the equilibrium when mechanisms are restricted to price posting, this is not always the case. Hence, there are some situations in which policies imposing restrictions on the meeting technology or on the set of available mechanisms are beneficial, but in several other situations improving policies are those that encourage meetings without capacity constraints, where buyers can multiple sellers.[47]

[47] Lester et al. (2015b) study a related issue in an environment with random search and imperfect competition. In particular, they examine how the features of the meeting technology affect traders' market power and hence the consequences for the welfare properties of equilibria in the presence of adverse selection.

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A Appendix

A.1 Proof of Proposition 3.2

If: We first show that for any vector $(\underline{x}, \overline{x}, \underline{t}, \overline{t})$ satisfying conditions (7)-(11), there exists a feasible and incentive compatible mechanism m such that $\underline{x}_m(\underline{\lambda}, \overline{\lambda}) = \underline{x}, \overline{x}_m(\underline{\lambda}, \overline{\lambda}) = \overline{x}$ and $\underline{t}_m(\underline{\lambda}, \overline{\lambda}) = \underline{t}, \overline{t}_m(\underline{\lambda}, \overline{\lambda}) = \overline{t}$. Consider the following mechanism

$$\begin{array}{lll} \underline{X}_m(L,H) = & \underline{\rho}\frac{1}{L+\alpha H}, & \underline{T}_m(L,H) = \underline{t}, \quad L \geq 1 \\ \overline{X}_m(L,H) = & \overline{\rho}\frac{\alpha}{L+\alpha H}, & \overline{T}_m(L,H) = \overline{t}, \quad H \geq 1 \end{array}$$

for some $\alpha, \underline{\rho}, \overline{\rho} \in [0,1]$. For the case $\alpha = 0$, let $\overline{X}_m(0,H) = \overline{\rho}\frac{1}{H}$.

This mechanism trivially satisfies $\underline{t}_m(\underline{\lambda}, \overline{\lambda}) = \underline{t}$ and $\overline{t}_m(\underline{\lambda}, \overline{\lambda}) = \overline{t}$. We now show that there always exists some tuple $(\alpha, \underline{\rho}, \overline{\rho})$ such that $\underline{x}_m(\underline{\lambda}, \overline{\lambda}) = \underline{x}$ and $\overline{x}_m(\underline{\lambda}, \overline{\lambda}) = \overline{x}$. Ex-ante trading probabilities are given by

$$\underline{x}_m(\underline{\lambda},\overline{\lambda}) = \underline{\rho}\sum_{H=0}^{+\infty}\sum_{L=0}^{+\infty}P_H(\overline{\lambda})P_L(\underline{\lambda})\frac{1}{L+1+\alpha H}$$
$$\overline{x}_m(\underline{\lambda},\overline{\lambda}) = \overline{\rho}\sum_{H=0}^{+\infty}\sum_{L=0}^{+\infty}P_H(\overline{\lambda})P_L(\underline{\lambda})\frac{\alpha}{L+\alpha(H+1)}$$

Define the function

$$f(\alpha)=\sum_{H=0}^{+\infty}\sum_{L=0}^{+\infty}P_H(\overline{\lambda})P_L(\underline{\lambda})\frac{1}{L+1+\alpha H}$$

Note that $f'(\alpha)<0$. The function's range is given by $\left[\frac{1}{\underline{\lambda}+\overline{\lambda}}\left(1-e^{-\underline{\lambda}-\overline{\lambda}}\right),\frac{1}{\underline{\lambda}}\left(1-e^{-\underline{\lambda}}\right)\right]$. To see this, consider first the case $\alpha=0$:

$$f(0)=\sum_{H=0}^{+\infty}\sum_{L=0}^{+\infty}P_H(\overline{\lambda})P_L(\underline{\lambda})\frac{1}{L+1}=\sum_{L=0}^{+\infty}P_L(\underline{\lambda})\frac{1}{L+1}\underbrace{\sum_{H=0}^{+\infty}P_H(\overline{\lambda})}_{=1}=\frac{1}{\underline{\lambda}}\sum_{L=0}^{+\infty}\frac{\underline{\lambda}^{L+1}}{(L+1)!}e^{-\underline{\lambda}}=\frac{1}{\underline{\lambda}}\left(1-e^{-\underline{\lambda}}\right)$$

Consider next the case $\alpha=1$:

$$f(1)=\sum_{H=0}^{+\infty}\sum_{L=0}^{+\infty}P_H(\overline{\lambda})P_L(\underline{\lambda})\frac{1}{L+1+H}=\sum_{N=0}^{+\infty}P_N(\underline{\lambda}+\overline{\lambda})\frac{1}{N+1}=\frac{1}{\underline{\lambda}+\overline{\lambda}}\left(1-e^{-\underline{\lambda}-\overline{\lambda}}\right)$$

Next, define the function

$$g(\alpha)=\begin{cases}\sum_{H=0}^{+\infty}\sum_{L=0}^{+\infty}P_H(\overline{\lambda})P_L(\underline{\lambda})\frac{\alpha}{L+\alpha(H+1)} & \text{if } \alpha>0\\ \sum_{H=0}^{+\infty}P_H(\overline{\lambda})P_L(\underline{\lambda})\frac{1}{H+1} & \text{if } \alpha=0\end{cases}$$

Note that $g'(\alpha)>0$ and that g is continuous at $\alpha=0$, i.e. $\lim_{\alpha\to 0}g(\alpha)=g(0)$. At $\alpha=1$, g is equal to f. At $\alpha=0$, we have

$$g(0)=\sum_{H=0}^{+\infty}P_H(\overline{\lambda})P_0(\underline{\lambda})\frac{1}{H+1}=e^{-\underline{\lambda}}\sum_{H=0}^{+\infty}\frac{\overline{\lambda}^H}{H!}e^{-\overline{\lambda}}\frac{1}{H+1}=\frac{e^{-\underline{\lambda}}}{\overline{\lambda}}\sum_{H=0}^{+\infty}\frac{\overline{\lambda}^{H+1}}{(H+1)!}e^{-\overline{\lambda}}=e^{-\underline{\lambda}}\frac{1}{\overline{\lambda}}\left(1-e^{-\overline{\lambda}}\right)$$

The range of g is consequently $\left[e^{-\underline{\lambda}}\frac{1}{\overline{\lambda}}\left(1-e^{-\overline{\lambda}}\right),\frac{1}{\underline{\lambda}+\overline{\lambda}}\left(1-e^{-\underline{\lambda}-\overline{\lambda}}\right)\right]$.

With this we can show that for any $\underline{x}$ and $\overline{x}$ satisfying conditions (9)-(11) we can find some $\alpha,\underline{\rho},\overline{\rho}\in[0,1]$ such that $\underline{\rho}f(\alpha)=\underline{x}$ and $\overline{\rho}g(\alpha)=\overline{x}$. Given that $\underline{x},\overline{x}\geq 0$ and and $\underline{\rho},\overline{\rho}\in[0,1]$, this can be satisfied if there exists an α such that $f(\alpha)\geq\underline{x}$ and $g(\alpha)\geq\overline{x}$. The first inequality requires that α is not too large, while the second requires that α is not too small. Consider first the case in which $\underline{x} \leq \frac{1}{\underline{\lambda}+\overline{\lambda}}\left(1-e^{-\underline{\lambda}-\overline{\lambda}}\right)$. Here $f(\alpha) \geq \underline{x}$ is satisfied for all $\alpha \in [0,1]$. Conditions (7),(8) and (11) together imply $\overline{x} \leq \frac{1}{\underline{\lambda}+\overline{\lambda}}\left(1-e^{-\underline{\lambda}-\overline{\lambda}}\right)$, from which it follows that $g(\alpha) \geq \overline{x}$ can be satisfied (e.g. $\alpha = 1$). Consider now the case $\underline{x} \geq \frac{1}{\underline{\lambda}+\overline{\lambda}}\left(1-e^{-\underline{\lambda}-\overline{\lambda}}\right)$ and let $\tilde{\alpha}$ be such that $f(\tilde{\alpha}) = \underline{x}$. We can show

$$\begin{aligned}
\overline{\lambda}g(\tilde{\alpha}) + \underline{\lambda}f(\tilde{\alpha}) &= \overline{\lambda}\sum_{H=0}^{+\infty}\sum_{L=0}^{+\infty} P_H(\overline{\lambda})P_L(\underline{\lambda})\frac{\tilde{\alpha}}{L+\tilde{\alpha}(H+1)} + \underline{\lambda}\sum_{H=0}^{+\infty}\sum_{L=0}^{+\infty} P_H(\overline{\lambda})P_L(\underline{\lambda})\frac{1}{L+1+\tilde{\alpha}H} \\
&= \overline{\lambda}\sum_{H=1}^{+\infty}\sum_{L=0}^{+\infty} \frac{\overline{\lambda}^{H-1}}{(H-1)!}\frac{\underline{\lambda}^{L}}{L!}e^{-\underline{\lambda}-\overline{\lambda}}\frac{\tilde{\alpha}}{L+\tilde{\alpha}H}\frac{H}{H} + \underline{\lambda}\sum_{H=0}^{+\infty}\sum_{L=1}^{+\infty} \frac{\overline{\lambda}^{H}}{H!}\frac{\underline{\lambda}^{L-1}}{(L-1)!}\frac{1}{L+\tilde{\alpha}H}\frac{L}{L} \\
&= \sum_{H=1}^{+\infty}\sum_{L=1}^{+\infty} \frac{\overline{\lambda}^{H}}{H!}\frac{\underline{\lambda}^{L}}{L!}e^{-\underline{\lambda}-\overline{\lambda}}\left(\underbrace{\frac{\tilde{\alpha}H}{L+\tilde{\alpha}H} + \frac{L}{L+\tilde{\alpha}H}}_{=1}\right) + \sum_{H=1}^{+\infty}\frac{\overline{\lambda}^{H}}{H!}e^{-\underline{\lambda}-\overline{\lambda}} + \sum_{L=1}^{+\infty}\frac{\underline{\lambda}^{L}}{L!}e^{-\underline{\lambda}-\overline{\lambda}} \\
&= \left(1-e^{-\overline{\lambda}}-e^{-\underline{\lambda}}+e^{-\underline{\lambda}-\overline{\lambda}}\right) + \left(e^{-\underline{\lambda}}-e^{-\underline{\lambda}-\overline{\lambda}}\right) + \left(e^{-\overline{\lambda}}-e^{-\underline{\lambda}-\overline{\lambda}}\right) \\
&= 1-e^{-\underline{\lambda}-\overline{\lambda}}
\end{aligned}$$

With this,

$$g(\tilde{\alpha}) = \frac{1}{\overline{\lambda}}\left(1-e^{-\underline{\lambda}-\overline{\lambda}}-\underline{\lambda}f(\tilde{\alpha})\right) = \frac{1}{\overline{\lambda}}\left(1-e^{-\underline{\lambda}-\overline{\lambda}}-\underline{\lambda}\underline{x}\right) \geq \overline{x}$$

where the last inequality follows from condition (11). Thus, there exists some $\overline{\rho} \in [0,1]$ such that $\overline{\rho}g(\tilde{\alpha}) = \overline{x}$. Together this implies that for any $\underline{x}$ and $\overline{x}$ satisfying conditions (9)-(11), there exists some $\alpha, \underline{\rho}, \overline{\rho} \in [0,1]$ such that $\underline{x}_m(\underline{\lambda},\overline{\lambda}) = \underline{x}$ and $\overline{x}_m(\underline{\lambda},\overline{\lambda}) = \overline{x}$.

Finally we need to check feasibility and incentive compatibility of the proposed mechanism. Feasibility follows from

$$\underline{x}(L,H)L + \overline{x}(L,H)H = \underline{\rho}\frac{1}{L+\alpha H}L + \overline{\rho}\frac{\alpha}{L+\alpha H}H \leq \frac{1}{L+\alpha H}L + \frac{\alpha}{L+\alpha H}H = 1.$$

Incentive compatibility is trivially satisfied given that $\underline{x}_m(\underline{\lambda},\overline{\lambda}) = \underline{x}, \overline{x}_m(\underline{\lambda},\overline{\lambda}) = \overline{x}$ and $\underline{t}_m(\underline{\lambda},\overline{\lambda}) = \underline{t}, \overline{t}_m(\underline{\lambda},\overline{\lambda}) = \overline{t}$.

Only if: We now want to show that for any feasible and incentive compatible mechanism m, expected trading probabilities and prices satisfy conditions (7)-(11). Let $\underline{x} = \underline{x}_m(\underline{\lambda},\overline{\lambda}), \overline{x} = \overline{x}_m(\underline{\lambda},\overline{\lambda})$ and $\underline{t} = \underline{t}_m(\underline{\lambda},\overline{\lambda}), \overline{t} = \overline{t}_m(\underline{\lambda},\overline{\lambda})$. Incentive compatibility of m then trivially implies (7) and (8). Feasibility will imply the remaining conditions. To see this, note first that $\underline{X}_m(L,H)L +$

$\overline{X}_m(L,H)H \leq 1, \forall L,H$ requires $\underline{X}_m(L,H) \leq \frac{1}{L}$, which in turn implies

$$\underline{x}_m(\underline{\lambda},\overline{\lambda}) = \sum_{H=0}^{+\infty}\sum_{L=0}^{+\infty} P_H(\overline{\lambda})P_L(\underline{\lambda})\underline{X}_m(L+1,H) \leq \sum_{H=0}^{+\infty}\sum_{L=0}^{+\infty} P_H(\overline{\lambda})P_L(\underline{\lambda})\frac{1}{L+1} = \frac{1}{\underline{\lambda}}\left(1-e^{-\underline{\lambda}}\right)$$

Analogously it can be shown that $\overline{X}_m(L,H) \leq \frac{1}{H}$ implies $\overline{x}_m(\underline{\lambda},\overline{\lambda}) \leq \frac{1}{\overline{\lambda}}\left(1-e^{-\overline{\lambda}}\right)$. From the perspective of a buyer the probability of trading a low quality good is given by

$$\sum_{L=1}^{+\infty}\sum_{H=0}^{+\infty}\frac{\underline{\lambda}^L}{L!}e^{-\underline{\lambda}}\frac{\overline{\lambda}^H}{H!}e^{-\overline{\lambda}}\underline{X}_m(L,H)L = \underline{\lambda}\sum_{L=0}^{+\infty}\sum_{H=0}^{+\infty}\frac{\underline{\lambda}^L}{L!}e^{-\underline{\lambda}}\frac{\overline{\lambda}^H}{H!}e^{-\overline{\lambda}}\underline{X}_m(L+1,H) = \underline{\lambda}\underline{x}_m(\underline{\lambda},\overline{\lambda})$$

Similarly, the probability for a buyer to trade a high quality good can be shown to equal $\overline{\lambda}\overline{x}_m(\underline{\lambda},\overline{\lambda})$. Feasibility then implies

$$\begin{aligned}
\overline{\lambda}\overline{x}_m(\underline{\lambda},\overline{\lambda}) + \underline{\lambda}\underline{x}_m(\underline{\lambda},\overline{\lambda}) &= \sum_{H=0}^{+\infty}\sum_{L=0}^{+\infty} P_H(\overline{\lambda})P_L(\underline{\lambda})\left(\underline{X}_m(L,H)L + \overline{X}_m(L,H)H\right) \\
&\leq \sum_{H=0}^{+\infty}\sum_{L=0}^{+\infty} P_H(\overline{\lambda})P_L(\underline{\lambda}) \cdot 1 - P_0(\overline{\lambda})P_0(\underline{\lambda}) \\
&= 1 - e^{-\underline{\lambda}-\overline{\lambda}}
\end{aligned}$$

□

A.2 Proof of Lemma 3.3

- $\underline{U} > \overline{U}$: Let $(\underline{x},\underline{t})$ and $(\overline{x},\overline{t})$ be some expected trading probabilities and transfers at the mechanisms (possibly different) chosen by low and high type sellers at a given equilibrium. Market utilities are then given by $\underline{U} = \underline{t} - \underline{x}\underline{c}$ and $\overline{U} = \overline{t} - \overline{x}\overline{c}$. Incentive compatibility (within or across mechanisms) for the low type seller requires $\underline{t} - \underline{x}\underline{c} \geq \overline{t} - \overline{x}\underline{c}$, which can be rewritten as $\overline{x}(\overline{c} - \underline{c}) \leq \underline{U} - \overline{U}$. Since $\overline{x} \geq 0$, this inequality can only be satisfied if $\underline{U} \geq \overline{U}$. If $\underline{U} = \overline{U}$, then $\overline{x} = 0$. Since buyers cannot make negative profits from any type of seller, this implies $\overline{t} = 0$ and consequently $\overline{U} = \underline{U} = 0$. If $\underline{x} = 0$, then buyers make zero profits. In this case, a buyer can deviate to a posted price $\varepsilon > 0$. This deviation attracts all the sellers and yields a positive profit as long as ε is small enough. If $\underline{x} > 0$, a buyer's equilibrium payoff is smaller or equal than $\left(1 - e^{-\underline{\lambda}^p}\right)(\underline{v} - \underline{c})$. A buyer can then deviate to an alternative mechanism with $\underline{\lambda}' > \underline{\lambda}^p$, $\underline{x}' = \frac{1}{\underline{\lambda}'}\left(1 - e^{-\underline{\lambda}'}\right)$, $\underline{t}' = \underline{x}'\underline{c}$ and $\overline{\lambda}' = \overline{x}' = \overline{t}' = 0$. This mechanism satisfies all the constraints and yields a payoff equal to $\left(1 - e^{-\underline{\lambda}'}\right)(\underline{v} - \underline{c}) > \left(1 - e^{-\underline{\lambda}^p}\right)(\underline{v} - \underline{c})$.
- $\underline{U} - \overline{U} < \overline{c} - \underline{c}$: Here, incentive compatibility (within or across mechanisms) for the high type

seller requires $\bar{t} - \overline{x}\overline{c} \geq \underline{t} - \underline{x}\overline{c}$, which can be rewritten as $\underline{x}(\overline{c} - \underline{c}) \geq \underline{U} - \overline{U}$. Note that $\underline{x} < 1$, which follows directly from the condition $\underline{x} \leq \frac{1}{\underline{\lambda}}\left(1 - e^{-\underline{\lambda}}\right) < 1$, where the last inequality is a general property for all $\underline{\lambda} \in (0, +\infty)$.[48] Hence, incentive compatibility can only be satisfied if $\underline{U} - \overline{U} < \overline{c} - \underline{c}$.

- $\underline{U}, \overline{U} > 0$: Suppose $\overline{U} = 0$. The expected profit a buyer makes trading with high type sellers is given by $\overline{\lambda}[\overline{x}\overline{v} - \bar{t}] = \overline{\lambda}\overline{x}(\overline{v} - \overline{c})$. Consider an increase in $\overline{\lambda}$ together with a reduction in $\overline{x}$ so as to keep $\overline{\lambda}\overline{x}$ unchanged. Adjusting $\bar{t}$ in order to keep the utility of high type sellers constant, this change is incentive compatible. Also, it increases $1 - e^{-\underline{\lambda}-\overline{\lambda}}$, thereby relaxing the last feasibility constraint of P^{aux}. The buyer can then always increase his payoff by increasing the value of $\overline{\lambda}$, while still satisfying all constraints of P^{aux}. In order for P^{aux} to have a solution, we thus need $\overline{U} > 0$. Given that $\underline{U} > \overline{U}$, as established above, this immediately implies $\underline{U} > 0$.

- $\underline{U} < \underline{v} - \underline{c}, \overline{U} < \overline{v} - \overline{c}$: Suppose $\overline{U} \geq \overline{v} - \overline{c}$. Given that $\overline{\lambda} > 0$ implies $\overline{x} < 1$ (see above), a buyer's payoff when meeting a high type seller, given by $\overline{x}(\overline{v} - \overline{c}) - \overline{U}$, is strictly negative when $\overline{\lambda} > 0$. The optimal value of $\overline{\lambda}$ is thus equal to zero. By a perfectly symmetric argument, $\underline{U} \geq \underline{v} - \underline{c}$ implies that the optimal value of $\underline{\lambda}$ is equal to zero. Given that these values are not compatible with an equilibrium, we have $\underline{U} < \underline{v} - \underline{c}, \overline{U} < \overline{v} - \overline{c}$.

□

A.3 Proof of Lemma 3.4

Consider the Lagrange problem

$$\begin{aligned} \mathcal{L} \;=\; & \overline{\lambda}\left[\overline{x}(\overline{v} - \overline{c}) - \overline{U}\right] + \underline{\lambda}\left[\underline{x}(\underline{v} - \underline{c}) - \underline{U}\right] + \gamma_1\left(\frac{\underline{U} - \overline{U}}{\overline{c} - \underline{c}} - \overline{x}\right) + \gamma_2\left(\underline{x} - \frac{\underline{U} + \overline{U}}{\overline{c} - \underline{c}}\right) \\ & + \gamma_3\left(1 - e^{-\underline{\lambda}} - \underline{\lambda}\underline{x}\right) + \gamma_4\left(1 - e^{-\underline{\lambda}-\overline{\lambda}} - \overline{\lambda}\overline{x} - \underline{\lambda}\underline{x}\right) + \gamma_5\overline{\lambda} + \gamma_6\underline{\lambda} \end{aligned}$$

The first-order conditions of the problem with respect $\overline{x}, \underline{x}$ and $\overline{\lambda}, \underline{\lambda}$ are given by

$$\overline{x}: \quad \overline{\lambda}(\overline{v} - \overline{c}) - \gamma_1 - \gamma_4\overline{\lambda} = 0 \tag{27}$$

$$\underline{x}: \quad \underline{\lambda}(\underline{v} - \underline{c}) + \gamma_2 - (\gamma_3 + \gamma_4)\underline{\lambda} = 0 \tag{28}$$

$$\overline{\lambda}: \quad \overline{x}(\overline{v} - \overline{c}) - \overline{U} + \gamma_4\left(e^{-\underline{\lambda}-\overline{\lambda}} - \overline{x}\right) + \gamma_5 = 0 \tag{29}$$

$$\underline{\lambda}: \quad \underline{x}(\underline{v} - \underline{c}) - \underline{U} + \gamma_3\left(e^{-\underline{\lambda}} - \underline{x}\right) + \gamma_4\left(e^{-\underline{\lambda}-\overline{\lambda}} - \underline{x}\right) + \gamma_6 = 0 \tag{30}$$

[48] See footnote 14.

If $\gamma_6 > 0$ such that $\underline{\lambda} = 0$, constraint (14) is always satisfied with equality. We are thus interested in the case $\gamma_6 = 0$ and $\underline{\lambda} > 0$. Assuming $\gamma_3 = 0$ and solving for the remaining Lagrange-multipliers yields

$$\begin{aligned}
\gamma_4 &= \frac{\underline{x}(\underline{v}-\underline{c}) - \underline{U}}{\underline{x} - e^{-\underline{\lambda}-\overline{\lambda}}} \\
\gamma_1 &= \overline{\lambda}\left((\overline{v}-\overline{c}) - \frac{\underline{x}(\underline{v}-\underline{c}) - \underline{U}}{\underline{x} - e^{-\underline{\lambda}-\overline{\lambda}}}\right) \\
\gamma_2 &= \underline{\lambda}\left(\frac{e^{-\underline{\lambda}-\overline{\lambda}}(\underline{v}-\underline{c}) - \underline{U}}{\underline{x} - e^{-\underline{\lambda}-\overline{\lambda}}}\right) \\
\gamma_5 &= \frac{\overline{x} - e^{-\underline{\lambda}-\overline{\lambda}}}{\underline{x} - e^{-\underline{\lambda}-\overline{\lambda}}}(\underline{x}(\underline{v}-\underline{c}) - \underline{U}) - (\overline{x}(\overline{v}-\overline{c}) - \overline{U})
\end{aligned}$$

Suppose first that $\gamma_2 = 0$. This requires $e^{-\underline{\lambda}-\overline{\lambda}}(\underline{v}-\underline{c}) = \underline{U}$, which implies $\gamma_4 = \underline{v}-\underline{c}$. Non-negativity of $\gamma_1 = \overline{\lambda}[(\overline{v}-\overline{c}) - (\underline{v}-\underline{c})]$ then requires either $\overline{\lambda} = 0$ or $\overline{v}-\overline{c} \geq \underline{v}-\underline{c}$ or both. If $\overline{\lambda} = 0$, constraint (14) coincides with constraint (15), which, given that $\gamma_4 = \underline{v}-\underline{c} > 0$, implies that (14) is satisfied with equality. If $\overline{\lambda} > 0$, then $\gamma_5 = 0$ requires

$$\underline{U} - \overline{U} = -\overline{x}[(\overline{v}-\overline{c}) - (\underline{v}-\underline{c})]$$

Given $\overline{v}-\overline{c} \geq \underline{v}-\underline{c}$, this equality can be satisfied only if $\underline{U} = \overline{U}$, ruled out by Lemma 3.3.

Suppose now $\gamma_2 > 0$. Here $\underline{x} = \frac{\underline{U}-\overline{U}}{\overline{c}-\underline{c}}$. Assuming first $\gamma_1 > 0$, we have $\overline{x} = \underline{x} = \frac{\underline{U}-\overline{U}}{\overline{c}-\underline{c}}$, which implies

$$\gamma_5 = [\underline{x}(\underline{v}-\underline{c}) - \underline{U}] - [\overline{x}(\overline{v}-\overline{c}) - \overline{U}] = -\frac{\overline{v}-\underline{v}}{\overline{c}-\underline{c}}(\underline{U}-\overline{U})$$

This term is strictly negative unless $\overline{v} = \underline{v}$. If $\overline{v} = \underline{v}$, then $\gamma_4 \geq 0$ requires $\underline{U}(\overline{v}-\overline{c}) \geq \overline{U}(\overline{v}-\underline{c})$. Substituting the value of $\underline{x}$, the conditions for $\gamma_1 > 0$ and $\gamma_2 > 0$, respectively, can be rewritten as

$$\frac{\overline{U} - e^{-\underline{\lambda}-\overline{\lambda}}(\overline{v}-\overline{c})}{\frac{\underline{U}-\overline{U}}{\overline{c}-\underline{c}} - e^{-\underline{\lambda}-\overline{\lambda}}} \geq 0 \quad \text{and} \quad \frac{\underline{U} - e^{-\underline{\lambda}-\overline{\lambda}}(\overline{v}-\underline{c})}{\frac{\underline{U}-\overline{U}}{\overline{c}-\underline{c}} - e^{-\underline{\lambda}-\overline{\lambda}}} < 0$$

However, given that $\underline{U}(\overline{v}-\overline{c}) \geq \overline{U}(\overline{v}-\underline{c})$, we have

$$\frac{\overline{U} - e^{-\underline{\lambda}-\overline{\lambda}}(\overline{v}-\overline{c})}{\frac{\underline{U}-\overline{U}}{\overline{c}-\underline{c}} - e^{-\underline{\lambda}-\overline{\lambda}}} \leq \frac{\overline{v}-\overline{c}}{\overline{v}-\underline{c}}\frac{\underline{U} - e^{-\underline{\lambda}-\overline{\lambda}}(\overline{v}-\underline{c})}{\frac{\underline{U}-\overline{U}}{\overline{c}-\underline{c}} - e^{-\underline{\lambda}-\overline{\lambda}}} < \frac{\underline{U} - e^{-\underline{\lambda}-\overline{\lambda}}(\overline{v}-\underline{c})}{\frac{\underline{U}-\overline{U}}{\overline{c}-\underline{c}} - e^{-\underline{\lambda}-\overline{\lambda}}}$$

A contradiction. Suppose next that $\gamma_1 = 0$. This implies either $\overline{\lambda} = 0$ or $(\overline{v} - \overline{c}) - \frac{\underline{x}(\underline{v}-\underline{c})-\underline{U}}{\underline{x}-e^{-\underline{\lambda}-\overline{\lambda}}} = 0$ or both. If $\overline{\lambda} = 0$, constraints (14) and (15) again coincide. For condition (14) not to be satisfied with equality, we need $\gamma_4 = 0$. This cannot be solution to the optimization problem since a marginal increase in $\underline{x}$ is both feasible and incentive compatible and increases the buyer's payoff strictly. On the other hand, if $\overline{\lambda} > 0$, then $\gamma_5 = 0$ implies

$$\frac{\underline{x}(\underline{v} - \underline{c}) - \underline{U}}{\underline{x} - e^{-\underline{\lambda}-\overline{\lambda}}} = \frac{\overline{x}(\overline{v} - \overline{c}) - \overline{U}}{\overline{x} - e^{-\underline{\lambda}-\overline{\lambda}}} \tag{31}$$

With this, we have $\gamma_1 = \overline{\lambda}\frac{\overline{U}-e^{-\underline{\lambda}-\overline{\lambda}}(\overline{v}-\overline{c})}{\overline{x}-e^{-\underline{\lambda}-\overline{\lambda}}}$, which, given $\overline{\lambda} > 0$, equals zero if and only if $\overline{U} = e^{-\underline{\lambda}-\overline{\lambda}}(\overline{v} - \overline{c})$. Substituting this value of $\overline{U}$ into condition (31) yields $\frac{\underline{x}(\underline{v}-\underline{c})-\underline{U}}{\underline{x}-e^{-\underline{\lambda}-\overline{\lambda}}} = \overline{v} - \overline{c}$. Using this equality, the second and fourth Lagrange multiplier simplify to $\gamma_2 = \underline{\lambda}[(\overline{v} - \overline{c}) - (\underline{v} - \underline{c})]$ and $\gamma_4 = \overline{v} - \overline{c}$. Strict positivity of γ_2 then requires $\overline{v} - \overline{c} > \underline{v} - \underline{c}$. However, rewriting the condition (31) as $\underline{x}[(\underline{v} - \underline{c}) - (\overline{v} - \overline{c})] = \underline{U} - \overline{U}$ shows that $\underline{U} > \overline{U}$ in fact requires $\underline{v} - \underline{c} > \overline{v} - \overline{c}$. A contradiction. □

A.4 Proof of Lemma 3.5

The Lagrange optimization problem associated to relaxed problem (A) corresponds to the one introduced in proof A.3, where γ_1 and γ_2 are set equal to zero and the trading probabilities are set to $\underline{x} = \frac{1}{\underline{\lambda}}\left(1 - e^{-\underline{\lambda}}\right)$ for $\underline{\lambda} > 0$ and $\overline{x} = e^{-\underline{\lambda}}\frac{1}{\overline{\lambda}}\left(1 - e^{-\overline{\lambda}}\right)$ for $\overline{\lambda} > 0$. Considering the first-order conditions (27)-(30), there are four cases to be distinguished.

- $\underline{\lambda}, \overline{\lambda} = 0$: For this case the set of first-order conditions can always be satisfied. The buyer's payoff associated to this solution is equal to zero.

- $\underline{\lambda} = 0, \overline{\lambda} > 0$: (27) implies $\gamma_4 = \overline{v} - \overline{c}$. $\overline{\lambda} > 0$ requires $\gamma_5 = 0$. (29) then implies $e^{-\overline{\lambda}}(\overline{v} - \overline{c}) = \overline{U}$ or equivalently $\overline{\lambda} = \ln\left(\frac{\overline{v}-\overline{c}}{\overline{U}}\right)$.

- $\underline{\lambda} > 0, \overline{\lambda} = 0$: (28) implies $\gamma_3 + \gamma_4 = \underline{v} - \underline{c}$. $\underline{\lambda} > 0$ requires $\gamma_6 = 0$. (30) then implies $e^{-\underline{\lambda}}(\underline{v} - \underline{c}) = \underline{U}$ or equivalently $\underline{\lambda} = \ln\left(\frac{\underline{v}-\underline{c}}{\underline{U}}\right)$. A necessary condition for positivity of γ_5 is $\underline{U} - \overline{U} < (\underline{v} - \underline{c}) - (\overline{v} - \overline{c})$ (see condition (29)).

- $\underline{\lambda}, \overline{\lambda} > 0$: (27) implies $\gamma_4 = \overline{v} - \overline{c}$ and (28) implies $\gamma_3 = (\underline{v} - \underline{c}) - (\overline{v} - \overline{c})$. $\overline{\lambda} > 0$ requires $\gamma_5 = 0$. (29) then implies

$$e^{-\underline{\lambda}-\overline{\lambda}}(\overline{v} - \overline{c}) = \overline{U} \tag{32}$$

  $\underline{\lambda} > 0$ requires $\gamma_6 = 0$. (30) then requires

$$e^{-\underline{\lambda}}[(\underline{v} - \underline{c}) - (\overline{v} - \overline{c})] + e^{-\underline{\lambda}-\overline{\lambda}}(\overline{v} - \overline{c}) = \underline{U} \tag{33}$$

Substituting (32) into (33) yields

$$e^{-\underline{\lambda}}[(\underline{v}-\underline{c})-(\overline{v}-\overline{c})]=\underline{U}-\overline{U}$$

This equality has a solution if $\underline{U}-\overline{U}\leq(\underline{v}-\underline{c})-(\overline{v}-\overline{c})$, in which case $\underline{\lambda}=\ln\left(\frac{(\underline{v}-\underline{c})-(\overline{v}-\overline{c})}{\underline{U}-\overline{U}}\right)$. $\overline{\lambda}$ is then pinned down by (32):

$$e^{-\overline{\lambda}}\frac{\underline{U}-\overline{U}}{(\underline{v}-\underline{c})-(\overline{v}-\overline{c})}=\frac{\overline{U}}{\overline{v}-\overline{c}}.$$

This equation has a solution if $\frac{\underline{U}-\overline{U}}{(\underline{v}-\underline{c})-(\overline{v}-\overline{c})}\geq\frac{\overline{U}}{\overline{v}-\overline{c}}$ or equivalently $\frac{\underline{U}}{\underline{v}-\underline{c}}\geq\frac{\overline{U}}{\overline{v}-\overline{c}}$, in which case $\overline{\lambda}=\ln\left(\frac{\underline{U}-\overline{U}}{(\underline{v}-\underline{c})-(\overline{v}-\overline{c})}\frac{\overline{v}-\overline{c}}{\overline{U}}\right)$

We can first show that a finite solution exists. To see this, we need to consider $\overline{\lambda}\to+\infty$ and $\underline{\lambda}\to+\infty$. It can be easily seen that the buyer's payoff in both cases tends to $-\infty$, that is

$$\begin{aligned}
&\lim_{\overline{\lambda}\to+\infty}\left[e^{-\underline{\lambda}}\left(1-e^{-\overline{\lambda}}\right)(\overline{v}-\overline{c})+\left(1-e^{-\underline{\lambda}}\right)(\underline{v}-\underline{c})-\overline{\lambda U}-\underline{\lambda U}\right]\\
=&\lim_{\underline{\lambda}\to+\infty}\left[e^{-\underline{\lambda}}\left(1-e^{-\overline{\lambda}}\right)(\overline{v}-\overline{c})+\left(1-e^{-\underline{\lambda}}\right)(\underline{v}-\underline{c})-\overline{\lambda U}-\underline{\lambda U}\right]\\
=&-\infty
\end{aligned}$$

We then need to demonstrate which solution obtains for different values of $\underline{U},\overline{U}$. Consider first the case $\underline{U}-\overline{U}\geq(\underline{v}-\underline{c})-(\overline{v}-\overline{c})$. Here the solutions to first-order conditions (27-30) are given by $\underline{\lambda},\overline{\lambda}=0$ and $\underline{\lambda}=0,\overline{\lambda}=\frac{\overline{v}-\overline{c}}{\overline{U}}$. The latter solves the buyer's optimization problem if it yields a strictly positive payoff. The buyer's payoff $\hat{\pi}(\underline{\lambda},\overline{\lambda})$ associated to the solution $\underline{\lambda}=0,\overline{\lambda}=\frac{\overline{v}-\overline{c}}{\overline{U}}$ is given by

$$\hat{\pi}\left(0,\frac{\overline{v}-\overline{c}}{\overline{U}}\right)=(\overline{v}-\overline{c})-\overline{U}-\ln\left(\frac{\overline{v}-\overline{c}}{\overline{U}}\right)\overline{U}=\overline{U}\left(\underbrace{\frac{\overline{v}-\overline{c}}{\overline{U}}-\ln\left(\frac{\overline{v}-\overline{c}}{\overline{U}}\right)}_{>1}-1\right)>0,$$

where the inequality follows directly from $x-\ln x>1,\forall x>1$. If $\underline{U}-\overline{U}\geq(\underline{v}-\underline{c})-(\overline{v}-\overline{c})$, the solution is thus given by $\underline{\lambda}=0,\overline{\lambda}=\frac{\overline{v}-\overline{c}}{\overline{U}}$.

Consider next the case $\frac{\underline{U}}{\underline{v}-\underline{c}} \leq \frac{\overline{U}}{\overline{v}-\overline{c}}$. Here, the three solutions to the first-order conditions (27-30) are $\underline{\lambda}, \overline{\lambda} = 0$, $\underline{\lambda} = 0, \overline{\lambda} = \frac{\overline{v}-\overline{c}}{\overline{U}}$ and $\underline{\lambda} = \ln\left(\frac{\underline{v}-\underline{c}}{\underline{U}}\right), \overline{\lambda} = 0$. The latter pair yields payoff

$$\hat{\pi}\left(\ln\left(\frac{\underline{v}-\underline{c}}{\underline{U}}\right), 0\right) = \underbrace{(\underline{v}-\underline{c}) - \underline{U} - \ln\left(\frac{\underline{v}-\underline{c}}{\underline{U}}\right)\underline{U}}_{=\left(1-e^{-\underline{\lambda}}-\underline{\lambda}e^{-\underline{\lambda}}\right)(\underline{v}-\underline{c})} > \underbrace{(\overline{v}-\overline{c}) - \overline{U} - \ln\left(\frac{\overline{v}-\overline{c}}{\overline{U}}\right)\overline{U}}_{=\left(1-e^{-\overline{\lambda}'}-\overline{\lambda}'e^{-\overline{\lambda}'}\right)(\overline{v}-\overline{c})} > 0$$

where $\overline{\lambda}' = \ln\left(\frac{\overline{v}-\overline{c}}{\overline{U}}\right)$. The first inequality follows from $\overline{\lambda}' = \ln\left(\frac{\overline{v}-\overline{c}}{\overline{U}}\right) \leq \ln\left(\frac{\underline{v}-\underline{c}}{\underline{U}}\right) = \underline{\lambda}$, the fact that the probability of at least two arrivals $(1 - e^{-x} - xe^{-x})$ strictly increases in the arrival rate x, and that $\frac{\underline{U}}{\underline{v}-\underline{c}} \leq \frac{\overline{U}}{\overline{v}-\overline{c}}$ implies $\underline{v}-\underline{c} > \overline{v}-\overline{c}$. If $\frac{\underline{U}}{\underline{v}-\underline{c}} \leq \frac{\overline{U}}{\overline{v}-\overline{c}}$, the solution to the optimization problem is thus given by $\overline{\lambda} = 0, \underline{\lambda} = \ln\left(\frac{\underline{v}-\underline{c}}{\underline{U}}\right)$.

Finally, consider the case $\underline{U} - \overline{U} < (\underline{v}-\underline{c}) - (\overline{v}-\overline{c})$ and $\frac{\underline{U}}{\underline{v}-\underline{c}} > \frac{\overline{U}}{\overline{v}-\overline{c}}$. Here, we show that the interior solution $\underline{\lambda} = \ln\left(\frac{(\underline{v}-\underline{c})-(\overline{v}-\overline{c})}{\underline{U}-\overline{U}}\right), \overline{\lambda} = \ln\left(\frac{\underline{U}-\overline{U}}{(\underline{v}-\underline{c})-(\overline{v}-\overline{c})}\frac{\overline{v}-\overline{c}}{\overline{U}}\right)$ yields the largest payoff. To see this, suppose instead that the solution to problem (A) is given by $\underline{\lambda} = 0, \overline{\lambda} = \frac{\overline{v}-\overline{c}}{\overline{U}}$ and consider any pair $\underline{U}, \overline{U}$ such that the parameter restrictions are satisfied. Consider now an alternative pair $\underline{U}', \overline{U}$ with $\underline{U}' > \underline{U}$ such that $\underline{U}' - \overline{U} = (\underline{v}-\underline{c}) - (\overline{v}-\overline{c})$. For the pair $\underline{U}', \overline{U}$ the interior solution $\underline{\lambda} = \ln\left(\frac{(\underline{v}-\underline{c})-(\overline{v}-\overline{c})}{\underline{U}'-\overline{U}}\right), \overline{\lambda} = \ln\left(\frac{\underline{U}'-\overline{U}}{(\underline{v}-\underline{c})-(\overline{v}-\overline{c})}\frac{\overline{v}-\overline{c}}{\overline{U}}\right)$ coincides with the corner solution $\underline{\lambda} = 0, \overline{\lambda} = \frac{\overline{v}-\overline{c}}{\overline{U}}$. Noting that $\frac{\partial\hat{\pi}\left(0, \frac{\overline{v}-\overline{c}}{\overline{U}}\right)}{\partial\underline{U}} = 0$ and

$$\frac{\partial\hat{\pi}\left(\ln\left(\frac{(\underline{v}-\underline{c})-(\overline{v}-\overline{c})}{\underline{U}-\overline{U}}\right), \ln\left(\frac{\underline{U}-\overline{U}}{(\underline{v}-\underline{c})-(\overline{v}-\overline{c})}\frac{\overline{v}-\overline{c}}{\overline{U}}\right)\right)}{\partial\underline{U}} = -\ln\left(\frac{(\underline{v}-\underline{c})-(\overline{v}-\overline{c})}{\underline{U}-\overline{U}}\right) < 0$$

the inequality $\underline{U} < \overline{U}'$ implies that for the pair $\underline{U}, \overline{U}$ the solution $\underline{\lambda} = \ln\left(\frac{(\underline{v}-\underline{c})-(\overline{v}-\overline{c})}{\underline{U}-\overline{U}}\right), \overline{\lambda} = \ln\left(\frac{\underline{U}-\overline{U}}{(\underline{v}-\underline{c})-(\overline{v}-\overline{c})}\frac{\overline{v}-\overline{c}}{\overline{U}}\right)$ must yield a strictly larger payoff than $\underline{\lambda} = 0, \overline{\lambda} = \frac{\overline{v}-\overline{c}}{\overline{U}}$.

Analogously it can be shown that the interior solution yields a larger payoff than $\underline{\lambda} = \frac{\underline{v}-\underline{c}}{\underline{U}}, \overline{\lambda} = 0$. Here we consider a pair $\underline{U}, \overline{U}'$, this time with $\overline{U}' > \overline{U}$ such that $\frac{\underline{U}}{\underline{v}-\underline{c}} = \frac{\overline{U}}{\overline{v}-\overline{c}}$. For the pair $\underline{U}, \overline{U}'$ the interior solution coincides with the corner solution $\underline{\lambda} = \frac{\underline{v}-\underline{c}}{\underline{U}}, \overline{\lambda} = 0$. Noting that $\frac{\partial\hat{\pi}\left(\frac{\underline{v}-\underline{c}}{\underline{U}}, 0\right)}{\partial\overline{U}} = 0$ and

$$\frac{\partial\hat{\pi}\left(\ln\left(\frac{(\underline{v}-\underline{c})-(\overline{v}-\overline{c})}{\underline{U}-\overline{U}}\right), \ln\left(\frac{\underline{U}-\overline{U}}{(\underline{v}-\underline{c})-(\overline{v}-\overline{c})}\frac{\overline{v}-\overline{c}}{\overline{U}}\right)\right)}{\partial\underline{U}} = -\ln\left(\frac{\underline{U}-\overline{U}}{(\underline{v}-\underline{c})-(\overline{v}-\overline{c})}\frac{\overline{v}-\overline{c}}{\overline{U}}\right) < 0$$

the inequality $\overline{U} < \overline{U}'$ then implies that for the pair $\underline{U}, \overline{U}$ the solution $\underline{\lambda} = \ln\left(\frac{(\underline{v}-\underline{c})-(\overline{v}-\overline{c})}{\underline{U}-\overline{U}}\right), \overline{\lambda} = \ln\left(\frac{\underline{U}-\overline{U}}{(\underline{v}-\underline{c})-(\overline{v}-\overline{c})}\frac{\overline{v}-\overline{c}}{\overline{U}}\right)$ must yield a strictly larger payoff than $\underline{\lambda} = \frac{\underline{v}-\underline{c}}{\underline{U}}, \overline{\lambda} = 0$. Given that the latter pair yields a strictly positive payoff, we can conclude that if $\underline{U} - \overline{U} < (\underline{v} - \underline{c}) - (\overline{v} - \overline{c})$ and $\frac{\underline{U}}{\underline{v}-\underline{c}} > \frac{\overline{U}}{\overline{v}-\overline{c}}$, the pair $\underline{\lambda} = \ln\left(\frac{(\underline{v}-\underline{c})-(\overline{v}-\overline{c})}{\underline{U}-\overline{U}}\right), \overline{\lambda} = \ln\left(\frac{\underline{U}-\overline{U}}{(\underline{v}-\underline{c})-(\overline{v}-\overline{c})}\frac{\overline{v}-\overline{c}}{\overline{U}}\right)$ indeed solves problem (A). $\square$

A.5 Proof of Lemma 3.6

Consider first the case $\frac{\overline{U}}{\overline{v}-\overline{c}} \geq \frac{\underline{U}}{\underline{v}-\underline{c}}$ such that $\overline{\lambda}^* = 0$. Given that $\underline{U} > \overline{U}$, this inequality implies $\underline{v} - \underline{c} > \overline{v} - \overline{c}$. Incentive compatibility for the low type seller can always be satisfied by setting $\overline{x}$ small enough. Incentive compatibility for the high type seller then requires $\underline{x}^* \geq \frac{\underline{U}-\overline{U}}{\overline{c}-\underline{c}}$. This is always satisfied, as can be seen from the following inequalities:

$$\underline{x}^* = \frac{1}{\ln\left(\frac{\underline{v}-\underline{c}}{\underline{U}}\right)}\left(1 - \frac{\underline{U}}{\underline{v}-\underline{c}}\right) > e^{-\ln\left(\frac{\underline{v}-\underline{c}}{\underline{U}}\right)} = \frac{\underline{U}}{\underline{v}-\underline{c}} > \left(1 - \frac{\overline{v}-\underline{v}}{\overline{c}-\underline{c}}\right)\frac{\underline{U}}{\underline{v}-\underline{c}} = \frac{\underline{U} - \frac{\overline{v}-\overline{c}}{\underline{v}-\underline{c}}\underline{U}}{\overline{c}-\underline{c}} \geq \frac{\underline{U}-\overline{U}}{\overline{c}-\underline{c}}$$

The first inequality comes from the fact that $\frac{1}{x}(1-e^{-x}) > e^{-x} \Leftrightarrow 1 - e^{-x} > xe^{-x}$, where $1 - e^{-x}$ is the probability of at least one arrival given arrival rate x, while xe^{-x} is the probability of exactly one arrival given arrival rate x. The second inequality follows from $\underline{v} - \underline{c} > \overline{v} - \overline{c}$ and the third inequality follows from $\frac{\overline{U}}{\overline{v}-\overline{c}} \geq \frac{\underline{U}}{\underline{v}-\underline{c}}$.

Consider next the case $\underline{U} - \overline{U} \geq (\underline{v} - \underline{c}) - (\overline{v} - \overline{c})$ such that $\underline{\lambda}^* = 0$. Note that the parameter restriction can only be satisfied if $\underline{v} < \overline{v}$ as we required $\underline{U} - \overline{U} < \overline{c} - \underline{c}$. Incentive compatibility for the low type seller is satisfied if $\frac{1}{\overline{\lambda}^*}(1 - e^{-\overline{\lambda}^*}) \leq \frac{\underline{U}-\overline{U}}{\overline{c}-\underline{c}}$. Substituting $\overline{\lambda}^* = \ln\left(\frac{\overline{v}-\overline{c}}{\overline{U}}\right)$, that is

$$\left[\ln\left(\frac{\overline{v}-\overline{c}}{\overline{U}}\right)\right]^{-1}\left(1 - \frac{\overline{U}}{(\overline{v}-\overline{c})}\right) \leq \frac{\underline{U}-\overline{U}}{\overline{c}-\underline{c}}$$

Incentive compatibility for the high type seller can always be satisfied by setting $\underline{x}$ large enough.

Finally, consider the case in which $\overline{\lambda^*} = \ln\left(\frac{\underline{U}-\overline{U}}{(\underline{v}-\underline{c})-(\overline{v}-\overline{c})}\frac{\overline{v}-\overline{c}}{\overline{U}}\right)$ and $\underline{\lambda}^* = \ln\left(\frac{(\underline{v}-\underline{c})-(\overline{v}-\overline{c})}{\underline{U}-\overline{U}}\right)$. Incentive compatibility for the low type seller is satisfied if $e^{-\underline{\lambda}^*}\frac{1}{\overline{\lambda}^*}(1 - e^{-\overline{\lambda}^*}) \leq \frac{\underline{U}-\overline{U}}{\overline{c}-\underline{c}}$. Substituting $\underline{\lambda}^*$ and $\overline{\lambda}^*$, this inequality can be rewritten as

$$\left[\ln\left(\frac{\underline{U}-\overline{U}}{(\underline{v}-\underline{c})-(\overline{v}-\overline{c})}\right) - \ln\left(\frac{\overline{U}}{\overline{v}-\overline{c}}\right)\right]^{-1}\left(\frac{\underline{U}-\overline{U}}{(\underline{v}-\underline{c})-(\overline{v}-\overline{c})} - \frac{\overline{U}}{\overline{v}-\overline{c}}\right) \leq \frac{\underline{U}-\overline{U}}{\overline{c}-\underline{c}}.$$

Incentive compatibility for the high type seller is generally satisfied, which follows from

$$\underline{x}^* = \frac{1}{\ln\left(\frac{(\underline{v}-\underline{c})-(\overline{v}-\overline{c})}{\underline{U}-\overline{U}}\right)}\left(1 - \frac{\underline{U}-\overline{U}}{(\underline{v}-\underline{c})-(\overline{v}-\overline{c})}\right) \geq \frac{1}{\ln\left(\frac{\overline{c}-\underline{c}}{\underline{U}-\overline{U}}\right)}\left(1 - \frac{\underline{U}-\overline{U}}{\overline{c}-\underline{c}}\right) > e^{-\ln\left(\frac{\overline{c}-\underline{c}}{\underline{U}-\overline{U}}\right)} = \frac{\underline{U}-\overline{U}}{\overline{c}-\underline{c}}$$

The first inequality comes from the fact that $\frac{1}{x}(1-e^{-x})$ is strictly decreasing in x and that $\overline{v} \geq \underline{v}$. The second inequality again follows from $\frac{1}{x}(1-e^{-x}) > e^{-x}$. □

A.6 Proof of Lemma 3.8

Suppose $\underline{U}(\overline{v}-\overline{c}) = \overline{U}(\overline{v}-\underline{c})$. Substituting the value of $\underline{x}^*$, constraint (15) can be written as

$$\overline{\lambda}^* \overline{x}^* \leq e^{-\underline{\lambda}^*}\left(1-e^{-\overline{\lambda}^*}\right)$$

Noting that

$$\overline{x}^* = \frac{\underline{U}-\overline{U}}{\overline{c}-\underline{c}} = \frac{\left(1-\frac{\overline{v}-\overline{c}}{\overline{v}-\underline{c}}\right)}{\overline{c}-\underline{c}} \underbrace{e^{-\underline{\lambda}^*}(\underline{v}-\underline{c})}_{=\underline{U}} = e^{-\underline{\lambda}^*}\frac{\underline{v}-\underline{c}}{\overline{v}-\underline{c}}$$

we obtain that feasibility is satisfied if and only if

$$\overline{\lambda}^* \frac{\underline{v}-\underline{c}}{\overline{v}-\underline{c}} \leq \left(1-e^{-\overline{\lambda}^*}\right)$$

Given that $\frac{1}{\overline{\lambda}^*}\left(1-e^{-\overline{\lambda}^*}\right)$ strictly decreases in $\overline{\lambda}^*$[49] with $\lim_{\overline{\lambda}^*\to 0}\frac{1}{\overline{\lambda}^*}\left(1-e^{-\overline{\lambda}^*}\right)=1$, this inequality is satisfied if and only if $\overline{\lambda}^* \leq \overline{\lambda}^{max}$, where λ^{max} is such that $\overline{\lambda}^{max}\frac{\underline{v}-\underline{c}}{\overline{v}-\underline{c}} = \left(1-e^{-\overline{\lambda}^{\max}}\right)$. Note that $\overline{\lambda}^{max}$ is strictly positive if and only if $\underline{v} < \overline{v}$.

If $\overline{\lambda}^* \leq \overline{\lambda}^{max}$, also the incentive compatibility constraint (13) is satisfied. Given that $\overline{x}^* = \frac{\underline{U}-\overline{U}}{\overline{c}-\underline{c}}$, we simply need to show that $\underline{x}^* \geq \overline{x}^*$. This can be seen from the following set of inequalities

$$\underline{x}^* = \frac{1}{\underline{\lambda}^*}\left(1-e^{-\underline{\lambda}^*}\right) > e^{-\underline{\lambda}^*} > e^{-\underline{\lambda}^*}\frac{1}{\overline{\lambda}^*}\left(1-e^{-\overline{\lambda}^*}\right) \geq \overline{x}^*$$

where the first inequality again follows from $1-e^{-x} > xe^{-x}, \forall x > 0$ (see proof A.5). □

A.7 Proof of Lemma 3.9

This problem can be broken up in two parts. Consider first the case $\overline{\lambda}=0$. Here the value of $\underline{\lambda}$ is not pinned down by the constraint of problem (C) but can be picked freely. The optimal value of $\underline{\lambda}$ is given by $\ln\left(\frac{\underline{v}-\underline{c}}{\underline{U}}\right)$.[50] Consider next the case of $\overline{\lambda}>0$. Here the constraint of problem (C) uniquely pins down the value of $\underline{\lambda}$ as a function of $\overline{\lambda}$. That is,

$$e^{-\underline{\lambda}}\frac{1}{\overline{\lambda}}\left(1-e^{-\overline{\lambda}}\right) = \frac{\underline{U}-\overline{U}}{\overline{c}-\underline{c}} \quad \Leftrightarrow \quad \underline{\lambda} = \ln\left(\frac{\overline{\lambda}}{1-e^{-\overline{\lambda}}}\frac{\overline{c}-\underline{c}}{\underline{U}-\overline{U}}\right)$$

[49] See footnote 14.

[50] For $\overline{\lambda}=0$, the buyer's objective is concave in $\underline{\lambda}$ and the first order condition with respect to $\underline{\lambda}$ is given by $e^{-\underline{\lambda}}(\underline{v}-\underline{c}) = \underline{U}$.

for all $\overline{\lambda} \in (0, \overline{\lambda}^c]$, where $\overline{\lambda}^c$ is such that $\frac{1}{\overline{\lambda}^c}\left(1-e^{-\overline{\lambda}^c}\right) = \frac{\underline{U}-\overline{U}}{\overline{c}-\underline{c}}$.[51] For $\overline{\lambda} > \overline{\lambda}^c$, the constraint cannot be satisfied, implying that those values of $\overline{\lambda}$ are not admissible. The buyer's objective as a function of $\overline{\lambda}$ on the subdomain $(0, \overline{\lambda}^c]$ is then given by

$$\tilde{\pi}(\overline{\lambda}) = \overline{\lambda}\frac{\underline{U}-\overline{U}}{\overline{c}-\underline{c}}(\overline{v}-\overline{c}) + \left(1 - \frac{\overline{\lambda}}{1-e^{-\overline{\lambda}}}\frac{\underline{U}-\overline{U}}{\overline{c}-\underline{c}}\right)(\underline{v}-\underline{c}) - \overline{\lambda}\overline{U} - \ln\left(\frac{1-e^{-\overline{\lambda}}}{\overline{\lambda}}\frac{\overline{c}-\underline{c}}{\underline{U}-\overline{U}}\right)\underline{U}$$

The first two derivatives with respect to $\overline{\lambda}$ are

$$\begin{aligned}
\frac{\partial\tilde{\pi}}{\partial\overline{\lambda}} &= (\overline{v}-\underline{c})\frac{\underline{U}-\overline{U}}{\overline{c}-\underline{c}} - e^{\overline{\lambda}}\frac{e^{\overline{\lambda}}-\overline{\lambda}-1}{(e^{\overline{\lambda}}-1)^2}(\underline{v}-\underline{c})\frac{\underline{U}-\overline{U}}{\overline{c}-\underline{c}} + \frac{e^{\overline{\lambda}}-\overline{\lambda}e^{\overline{\lambda}}-1}{\overline{\lambda}(e^{\overline{\lambda}}-1)}\underline{U} \\
\frac{\partial^2\tilde{\pi}}{\partial\overline{\lambda}^2} &= e^{\overline{\lambda}}\frac{2(e^{\overline{\lambda}}-1)-\overline{\lambda}(e^{\overline{\lambda}}+1)}{(e^{\overline{\lambda}}-1)^3}(\underline{v}-\underline{c})\frac{\underline{U}-\overline{U}}{\overline{c}-\underline{c}} + \frac{\overline{\lambda}^2 e^{\overline{\lambda}}-(e^{\overline{\lambda}}-1)^2}{\overline{\lambda}^2(e^{\overline{\lambda}}-1)^2}\underline{U}
\end{aligned}$$

We can show that the second derivative is strictly negative for all $\overline{\lambda} > 0$. This will be done by showing that the numerators of both terms in $\frac{\partial^2\tilde{\pi}}{\partial\overline{\lambda}^2}$ are strictly negative. The numerator of the first term is equal to zero at $\overline{\lambda} = 0$ and strictly decreasing for all $\overline{\lambda} > 0$:

$$\frac{\partial\left(2(e^{\overline{\lambda}}-1)-\overline{\lambda}(e^{\overline{\lambda}}+1)\right)}{\partial\overline{\lambda}} = -\overline{\lambda}e^{\overline{\lambda}}\underbrace{\left(1-\frac{1}{\overline{\lambda}}(1-e^{-\overline{\lambda}})\right)}_{>0} < 0,$$

implying that $2(e^{\overline{\lambda}}-1)-\overline{\lambda}(e^{\overline{\lambda}}+1) < 0, \forall\overline{\lambda} > 0$. To see that also the numerator of the second term is negative, we need to show that

$$(e^{\overline{\lambda}}-1)^2 > \overline{\lambda}^2 e^{\overline{\lambda}} \quad \Leftrightarrow \quad 1-e^{-\overline{\lambda}}-\overline{\lambda}e^{-\frac{\overline{\lambda}}{2}} > 0.$$

Given that $1-e^{-\overline{\lambda}}-\overline{\lambda}e^{-\frac{\overline{\lambda}}{2}}$ is equal to zero at $\overline{\lambda} = 0$, this can be shown by demonstrating that $1-e^{-\overline{\lambda}}-\overline{\lambda}e^{-\frac{\overline{\lambda}}{2}}$ is strictly increasing in $\overline{\lambda}$. The first derivative of this term is given by

$$\frac{\partial\left(1-e^{-\overline{\lambda}}-\overline{\lambda}e^{-\frac{\overline{\lambda}}{2}}\right)}{\partial\overline{\lambda}} = e^{-\overline{\lambda}}\left[\frac{1}{2}\overline{\lambda}-\left(1-e^{-\frac{\overline{\lambda}}{2}}\right)\right]$$

Note that $1-e^{-\frac{\overline{\lambda}}{2}}$ is a strictly increasing, strictly concave function with a function value of zero and a slope of $\frac{1}{2}$ at $\overline{\lambda} = 0$. The linear function $\frac{1}{2}\overline{\lambda}$ is thus tangent to $1-e^{-\frac{\overline{\lambda}}{2}}$ at $\overline{\lambda} = 0$. This implies that the graph of $\frac{1}{2}\overline{\lambda}$ lies strictly above the graph of $1-e^{-\frac{\overline{\lambda}}{2}}$, proving that $e^{-\overline{\lambda}}\left[\frac{1}{2}\overline{\lambda}-\left(1-e^{-\frac{\overline{\lambda}}{2}}\right)\right] > 0$ for all $\overline{\lambda} > 0$.

[51] Note the limit $\lim_{\overline{\lambda}\to 0}\left(\ln\left(\frac{\overline{\lambda}}{1-e^{-\overline{\lambda}}}\frac{\overline{c}-\underline{c}}{\underline{U}-\overline{U}}\right)\right) = \ln\left(\frac{\overline{c}-\underline{c}}{\underline{U}-\overline{U}}\right)$ differs from the optimal value of $\underline{\lambda}$ at $\overline{\lambda} = 0$.

Taken together, this implies that $\frac{\partial^2 \tilde{\pi}}{\partial \overline{\lambda}^2} < 0, \forall \overline{\lambda} > 0$, that is the buyer's objective as a function of $\overline{\lambda}$ is strictly concave on $(0, \overline{\lambda}^c]$. Let $\underline{\lambda}^{int}, \overline{\lambda}^{int}$ denote the pair of values that maximizes the buyer's objective on the subdomain $(0, \overline{\lambda}^c]$.

Lastly, we need to check if the objective of problem (C) attains its maximum at the corner solution $\underline{\lambda}^c = \ln\left(\frac{\underline{v}-\underline{c}}{\underline{U}}\right), \overline{\lambda}^c = 0$ or at the interior solution $\underline{\lambda}^{int}, \overline{\lambda}^{int}$. If $\frac{\overline{U}}{\overline{v}-\overline{c}} > \frac{\underline{U}}{\underline{v}-\underline{c}}$, the unique solution of relaxed problem (B) solves the buyer's auxiliary problem. Given that the corner solution $\underline{\lambda}^c, \overline{\lambda}^c$ coincides with that solution, it must yield a strictly higher payoff than the pair $\underline{\lambda}^{int}, \overline{\lambda}^{int}$. Suppose next $\frac{\overline{U}}{\overline{v}-\overline{c}} \leq \frac{\underline{U}}{\underline{v}-\underline{c}}$. Consider the pair $\underline{\lambda}', \overline{\lambda}'$ with $\underline{\lambda}' = \frac{\underline{v}-\underline{c}}{\underline{U}}$ and $\overline{\lambda}' > 0$ such that

$$e^{-\underline{\lambda}'} \frac{1}{\overline{\lambda}'} \left(1 - e^{-\overline{\lambda}'}\right) = \frac{\underline{U} - \overline{U}}{\overline{c} - \underline{c}}$$

The difference between the buyer's payoff associated to $\underline{\lambda}', \overline{\lambda}'$ and $\underline{\lambda}^c, \overline{\lambda}^c$ is given by the buyer's expected payoff from trading with high type sellers

$$e^{-\underline{\lambda}'} \left(1 - e^{-\overline{\lambda}'}\right)(\overline{v} - \overline{c}) - \overline{\lambda}'\overline{U} = \overline{\lambda}' \left(\frac{\overline{v} - \overline{c}}{\overline{c} - \underline{c}}(\underline{U} - \overline{U}) - \overline{U}\right) = \overline{\lambda}' \left(\frac{\overline{v} - \overline{c}}{\overline{c} - \underline{c}}\underline{U} - \frac{\overline{v} - \underline{c}}{\overline{c} - \underline{c}}\overline{U}\right) \geq 0$$

where the last inequality is strict if $\frac{\overline{U}}{\overline{v}-\overline{c}} < \frac{\underline{U}}{\overline{v}-\underline{c}}$. If that is the case, the pair $\underline{\lambda}', \overline{\lambda}'$ yields a strictly greater payoff than $\underline{\lambda}^c, \overline{\lambda}^c$, implying that also the pair $\underline{\lambda}^{int}, \overline{\lambda}^{int}$ must yield a strictly greater payoff than $\underline{\lambda}^c, \overline{\lambda}^c$.

If $\frac{\overline{U}}{\overline{v}-\overline{c}} = \frac{\underline{U}}{\overline{v}-\underline{c}}$, we have $\overline{\lambda}' = \overline{\lambda}^{max}$, with $\overline{\lambda}^{max}$ as defined in Lemma 3.8. Given that the pair $\underline{\lambda}', \overline{\lambda}'$ solves relaxed problem (B) and satisfies the constraint of problem (C), the pair must also solve problem (C). The interior solution $\underline{\lambda}^{int}, \overline{\lambda}^{int}$ thus coincides with $\underline{\lambda}', \overline{\lambda}'$. Since also the pair $\underline{\lambda}^c, \overline{\lambda}^c$ solves relaxed problem (B), it follows that whenever $\frac{\overline{U}}{\overline{v}-\overline{c}} = \frac{\underline{U}}{\overline{v}-\underline{c}}$, problem (C) has two solutions, $\underline{\lambda}^c = \ln\left(\frac{\underline{v}-\underline{c}}{\underline{U}}\right), \overline{\lambda}^c = 0$ and $\underline{\lambda}^{int} = \ln\left(\frac{\underline{v}-\underline{c}}{\underline{U}}\right), \overline{\lambda}^{int} = \overline{\lambda}^{max}$. □

A.8 Proof of Proposition 3.11

We need to determine the conditions under which the values of $\underline{U}$ and $\overline{U}$ are such that the solution of problem (A) is not incentive compatible and the solution of problem (B) is not feasible (i.e. does not exist).

- It follows from Lemma 3.8 that there is no solution to problem (B) if $\frac{\overline{U}}{\underline{U}} < \frac{\overline{v}-\overline{c}}{\overline{v}-\underline{c}}$. Substituting the values of (23) and (24) into this inequality yields

$$\frac{\overline{U}}{\underline{U}} - \frac{\overline{v}-\overline{c}}{\overline{v}-\underline{c}} = \frac{\left(1-e^{-\overline{\lambda}^p}-\overline{\lambda}^p e^{-\overline{\lambda}^p}\right)(\overline{c}-\underline{c})}{e^{-2\overline{\lambda}^p}\left(1-e^{-\overline{\lambda}^p}+\overline{\lambda}^p e^{-\overline{\lambda}^p}\right)(\underline{v}-\underline{c})+\left(1-e^{-\overline{\lambda}^p}\right)^2(\overline{v}-\underline{v})}\left[\frac{1}{\overline{\lambda}^p}\left(1-e^{-\overline{\lambda}^p}\right)-\frac{\underline{v}-\underline{c}}{\overline{v}-\underline{c}}\right] < 0$$

The first term is strictly positive, implying that this inequality is satisfied if and only if $\frac{1}{\overline{\lambda}^p}\left(1-e^{-\overline{\lambda}^p}\right) < \frac{\underline{v}-\underline{c}}{\overline{v}-\underline{c}}$.

- Provided that $\frac{1}{\overline{\lambda}^p}\left(1-e^{-\overline{\lambda}^p}\right) < \frac{\underline{v}-\underline{c}}{\overline{v}-\underline{c}}$ and consequently $\frac{\overline{U}}{\overline{v}-\overline{c}} < \frac{\underline{U}}{\overline{v}-\underline{c}}$, we know that the solution to problem (A) is either characterized by condition (ii) or by condition (iii) of Lemma 3.6. It is characterized by condition (ii) if and only if $\underline{U}-\overline{U} < (\underline{v}-\underline{c})-(\overline{v}-\overline{c})$. Substituting market utilities (23) and (24) into this inequality yields $e^{-\underline{\lambda}^p}\frac{1}{\overline{\lambda}^p}\left(1-e^{-\overline{\lambda}^p}\right) < 1-\frac{\overline{v}-\underline{v}}{\overline{c}-\underline{c}}$.

  (ii) Suppose first that the inequality $e^{-\underline{\lambda}^p}\frac{1}{\overline{\lambda}^p}\left(1-e^{-\overline{\lambda}^p}\right) < 1-\frac{\overline{v}-\underline{v}}{\overline{c}-\underline{c}}$ is satisfied, i.e. that the solution of problem (A) is interior. According to Lemma 3.6 (ii), this solution is not incentive compatible if

$$\left[\ln\left(\frac{\underline{U}-\overline{U}}{\overline{U}}\frac{\overline{v}-\overline{c}}{(\underline{v}-\underline{c})-(\overline{v}-\overline{c})}\right)\right]^{-1}\left(1-\frac{\overline{U}}{\underline{U}-\overline{U}}\frac{(\underline{v}-\underline{c})-(\overline{v}-\overline{c})}{\overline{v}-\overline{c}}\right) > 1-\frac{\overline{v}-\underline{v}}{\overline{c}-\underline{c}} \quad (34)$$

  Note that the right-hand side of this inequality has the form $\frac{1-z}{\ln(1/z)}$, where $z=\frac{\overline{U}}{\underline{U}-\overline{U}}\frac{(\underline{v}-\underline{c})-(\overline{v}-\overline{c})}{\overline{v}-\overline{c}}$. The term $\frac{1-z}{\ln(1/z)}$ is strictly increasing in z. Substituting market utilities (23) and (24), the ratio $\frac{\overline{U}}{\underline{U}-\overline{U}}$ can be written as

$$\frac{\overline{U}}{\underline{U}-\overline{U}} = \frac{e^{-\overline{\lambda}^p}}{\frac{1}{\overline{\lambda}^p}\left(1-e^{-\overline{\lambda}^p}\right)}\frac{\underline{v}-\underline{c}}{\overline{c}-\underline{c}} + \frac{1-e^{-\overline{\lambda}^p}}{1-\frac{1}{\overline{\lambda}^p}\left(1-e^{-\overline{\lambda}^p}\right)}\frac{\overline{v}-\underline{v}}{\overline{c}-\underline{c}} - 1$$

  This ratio is strictly decreasing $\overline{\lambda}^p$

$$\frac{\partial\left(\frac{\overline{U}}{\underline{U}-\overline{U}}\right)}{\partial\overline{\lambda}^p} = -\left(\frac{\overline{\lambda}^p e^{-\overline{\lambda}^p}\left(1-\frac{1}{\overline{\lambda}^p}(1-e^{-\overline{\lambda}^p})\right)}{(1-e^{-\overline{\lambda}^p})^2}\frac{\underline{v}-\underline{c}}{\overline{c}-\underline{c}} + \frac{(1-e^{-\overline{\lambda}^p})^2-\overline{\lambda}^{p2}e^{-\overline{\lambda}^p}}{(\overline{\lambda}^p-(1-e^{-\overline{\lambda}^p}))^2}\frac{\overline{v}-\underline{v}}{\overline{c}-\underline{c}}\right) < 0$$

  Both the first and the second term inside the parenthesis are strictly positive, which follows from $\frac{1}{\overline{\lambda}^p}(1-e^{-\overline{\lambda}^p}) < 1$ and $(e^{\overline{\lambda}^p}-1)^2 > \overline{\lambda}^{p2}e^{\overline{\lambda}^p} \Leftrightarrow (1-e^{-\overline{\lambda}^p})^2 > \overline{\lambda}^{p2}e^{-\overline{\lambda}^p}$, as was shown in proof A.7. Note further that if $\frac{1}{\overline{\lambda}^p}\left(1-e^{-\overline{\lambda}^p}\right) = 1-\frac{\overline{v}-\underline{v}}{\overline{c}-\underline{c}}$, we have

$$\frac{\overline{U}}{\underline{U}-\overline{U}} = e^{-\overline{\lambda}^p}\frac{\overline{v}-\overline{c}}{(\underline{v}-\underline{c})-(\overline{v}-\overline{c})}$$

such that $z = \frac{\overline{U}}{\underline{U}-\overline{U}} \frac{(\underline{v}-\underline{c})-(\overline{v}-\overline{c})}{\overline{v}-\overline{c}} = e^{-\overline{\lambda}^p}$. In this case the left-hand side of inequality (34) is equal to $\frac{1}{\overline{\lambda}^p}\left(1 - e^{-\overline{\lambda}^p}\right)$, which in turn is equal to the right-hand side. Now consider a decrease in $\overline{\lambda}^p$ so that $\frac{1}{\overline{\lambda}^p}\left(1 - e^{-\overline{\lambda}^p}\right) > 1 - \frac{\overline{v}-\underline{v}}{\overline{c}-\underline{c}}$. This leads to an increase in $\frac{\overline{U}}{\underline{U}-\overline{U}}$, given the property above, and consequently to an increase in the left-hand side of inequality (34), while the right-hand side is unaffected. Together, this implies that inequality (34) is satisfied whenever $\frac{1}{\overline{\lambda}^p}\left(1 - e^{-\overline{\lambda}^p}\right) > 1 - \frac{\overline{v}-\underline{v}}{\overline{c}-\underline{c}}$.

(iii) Suppose now that $e^{-\underline{\lambda}^p}\frac{1}{\overline{\lambda}^p}\left(1 - e^{-\overline{\lambda}^p}\right) \geq 1 - \frac{\overline{v}-\underline{v}}{\overline{c}-\underline{c}}$. Lemma 3.6 (iii) shows that the solution of problem (A) is not incentive compatible if

$$\left[\ln\left(\frac{\overline{v}-\overline{c}}{\overline{U}}\right)\right]^{-1}\left(1 - \frac{\overline{U}}{\overline{v}-\overline{c}}\right) > 1 - \frac{\overline{v}-\underline{v}}{\overline{c}-\underline{c}} \tag{35}$$

Clearly, at $e^{-\underline{\lambda}^p}\frac{1}{\overline{\lambda}^p}\left(1 - e^{-\overline{\lambda}^p}\right) = 1 - \frac{\overline{v}-\underline{v}}{\overline{c}-\underline{c}}$ (or $\underline{U} - \overline{U} = (\underline{v} - \underline{c}) - (\overline{v} - \overline{c})$) inequality (35) coincides with (34). Given that $e^{-\underline{\lambda}^p}\frac{1}{\overline{\lambda}^p}\left(1 - e^{-\overline{\lambda}^p}\right) = 1 - \frac{\overline{v}-\underline{v}}{\overline{c}-\underline{c}}$ implies $\frac{1}{\overline{\lambda}^p}\left(1 - e^{-\overline{\lambda}^p}\right) > 1 - \frac{\overline{v}-\underline{v}}{\overline{c}-\underline{c}}$, (35) is satisfied at this parameter specification. Consider now the case $e^{-\underline{\lambda}^p}\frac{1}{\overline{\lambda}^p}\left(1 - e^{-\overline{\lambda}^p}\right) > 1 - \frac{\overline{v}-\underline{v}}{\overline{c}-\underline{c}}$ so that $\underline{U} - \overline{U} > (\underline{v} - \underline{c}) - (\overline{c} - \underline{c})$. Let $\underline{c}' < \underline{c}$ be such that $e^{-\underline{\lambda}^p}\frac{1}{\overline{\lambda}^p}\left(1 - e^{-\overline{\lambda}^p}\right) = 1 - \frac{\overline{v}-\underline{v}}{\overline{c}-\underline{c}'}$ and let $\overline{U}'$ denote the associated utility according to (24). Note that $\overline{U}$ strictly increases in $\underline{c}$, implying that $\overline{U}' < \overline{U}$.[52] Note further that the left-hand side of (35) is again of the form $\frac{1-z}{\ln(1/z)}$ with $z = \frac{\overline{U}}{\overline{v}-\overline{c}}$, which was strictly increasing in z. This implies

$$\left[\ln\left(\frac{\overline{v}-\overline{c}}{\overline{U}}\right)\right]^{-1}\left(1 - \frac{\overline{U}}{\overline{v}-\overline{c}}\right) > \left[\ln\left(\frac{\overline{v}-\overline{c}}{\overline{U}'}\right)\right]^{-1}\left(1 - \frac{\overline{U}'}{\overline{v}-\overline{c}}\right) > 1 - \frac{\overline{v}-\underline{v}}{\overline{c}-\underline{c}}$$

i.e. for $e^{-\underline{\lambda}^p}\frac{1}{\overline{\lambda}^p}\left(1 - e^{-\overline{\lambda}^p}\right) > 1 - \frac{\overline{v}-\underline{v}}{\overline{c}-\underline{c}}$, inequality (35) is always satisfied.

Taken together, this implies that the solution of problem (C) solves P^{aux} if $1 - \frac{\overline{v}-\underline{v}}{\overline{c}-\underline{c}} < \frac{1}{\overline{\lambda}^p}\left(1 - e^{-\overline{\lambda}^p}\right) < \frac{\underline{v}-\underline{c}}{\overline{v}-\underline{c}}$.

□

A.9 Proof of Lemma 4.1

Note first that $\underline{x} > \overline{x}$ in all parameter regions. This follows directly from the fact that $\overline{x} < e^{-\underline{\lambda}^p}$ and $\underline{x} = \frac{1}{\underline{\lambda}^p}\left(1 - e^{-\underline{\lambda}^p}\right) > e^{-\underline{\lambda}^p}$ (for the latter inequality see proof 3.6). To establish the stated property in Lemma 4.1, we proceed by contradiction. Let $(\underline{x}, \overline{x}, \underline{t}, \overline{t})$ denote the expected trading probabilities and transfers for a mechanism traded in equilibrium and suppose $\underline{x}(\underline{v} - \underline{c}) - \underline{U} \leq \overline{x}(\overline{v} - \overline{c}) - \overline{U}$.

[52] The first derivative of the expression in (24) with respect to $\underline{c}$ is given by $\frac{1}{\overline{\lambda}^p}e^{-\underline{\lambda}^p}\left(1 - e^{-\overline{\lambda}^p} - \overline{\lambda}^p e^{-\overline{\lambda}^p}\right) > 0$.

$(\underline{x}, \overline{x}, \underline{t}, \overline{t}, \underline{\lambda}, \overline{\lambda})$ must then solve the auxiliary optimization problem of the buyers P^{aux}, for $\underline{\lambda}, \overline{\lambda}$ consistent with the population parameters. Consider a deviation in which the buyer replaces every low type seller with a high type seller and adds some more high type sellers, while keeping $\underline{x}, \overline{x}, \underline{t}, \overline{t}$ unchanged. That is, $\underline{\lambda}' = 0, \overline{\lambda}' = \underline{\lambda} + \overline{\lambda} + \varepsilon$ with $\varepsilon > 0$. This deviation clearly satisfies the participation and incentive compatibility constraints of P^{aux} and yields a strictly larger payoff to the buyer, since $\overline{x}(\overline{v} - \overline{c}) - \overline{U} \geq \underline{x}(\underline{v} - \underline{c}) - \underline{U}$ and $\varepsilon > 0$. This deviation is also feasible for ε sufficiently small, as replacing low type with high type sellers implies that the average trading probability decreases (by $\underline{x} > \overline{x}$), while the number of meetings remains constant. Hence, it is possible to attract a few more high type sellers while still satisfying feasibility. We thus have a profitable deviation, a contradiction. □

A.10 Proof of Proposition 4.3

Consider first the case $\frac{1}{\overline{\lambda}^p}\left(1 - e^{-\overline{\lambda}^p}\right) \leq \frac{\underline{v}-\underline{c}}{\overline{v}-\underline{c}}$. From Propositions 3.10 and 3.11 it follows that under this restriction, at a pooling equilibrium the feasibility constraint (11) is binding. Consider an increase in the trading probability of the high type seller by $\Delta\overline{x}$, while adjusting the trading probability of the low type seller so that the feasibility constraint is still satisfied as equality:

$$\overline{\lambda}^p \Delta\overline{x} + \underline{\lambda}^p \Delta\underline{x} = 0 \quad \Leftrightarrow \quad \Delta\underline{x} = -\frac{\mu}{1-\mu}\Delta\overline{x}$$

Let us modify the expected transfer to the high type seller so that his utility remains unchanged

$$\Delta\overline{t} = \Delta\overline{x}\overline{c}$$

and the expected transfer to the low type seller so that his incentive compatibility constraint is satisfied

$$\Delta\underline{t} - \Delta\underline{x}\underline{c} = \Delta\overline{t} - \Delta\overline{x}\underline{c}$$

Substituting the previous equations into the above yields

$$\Delta\underline{t} = \frac{1}{1-\mu}\left((1-\mu)\overline{c} - \underline{c}\right)\Delta\overline{x}$$

Note that these changes make high type sellers indifferent and strictly improve the utility of low type sellers:

$$\Delta\underline{t} - \Delta\underline{x}\underline{c} = \left[\frac{1}{1-\mu}\left((1-\mu)\overline{c} - \underline{c}\right) + \frac{\mu}{1-\mu}\underline{c}\right]\Delta\overline{x} = \overline{c} - \underline{c} > 0$$

It is immediate to verify that the changes considered always allow to increase total surplus (while satisfying incentive compatibility and the feasibility constraints imposed by the matching technology). To verify that they also constitute a Pareto improvement, we need to show that they also make buyers weakly better off. This happens if

$$\overline{\lambda}^p \left[\Delta\overline{x}\overline{v} - \Delta\overline{t}\right] + \underline{\lambda}^p \left[\Delta\underline{x}\underline{v} - \Delta\underline{t}\right] \geq 0$$
$$\Leftrightarrow \quad \frac{\underline{\lambda}^p}{1-\mu}\left[\mu(\overline{v}-\underline{v}) - (\overline{c}-\underline{c})\right]\Delta\overline{x} \geq 0$$

The above inequality is satisfied whenever $\mu \geq \frac{\overline{c}-\underline{c}}{\overline{v}-\underline{v}}$.

Consider now the case $\frac{1}{\overline{\lambda}^p}\left(1-e^{-\overline{\lambda}^p}\right) > \frac{\underline{v}-\underline{c}}{\overline{v}-\underline{c}}$. From Proposition 2.6 it follows that at a competitive equilibrium the feasibility constraint (11) is slack. Consider an increase in the trading probability of the high type seller $\Delta\overline{x}$, small enough that (11) is not violated. Modify then the expected transfer to the high type seller so that his utility is kept constant

$$\Delta\overline{t} = \Delta\overline{x}\overline{c}$$

The trading probability of the low type seller is kept unchanged and the expected transfer to the low type seller is adjusted to ensure that his incentive compatibility constraint is satisfied

$$\Delta\underline{t} - \underbrace{\Delta\underline{x}\underline{c}}_{=0} = \Delta\overline{t} - \Delta\overline{x}\underline{c} \quad \Leftrightarrow \quad \Delta\underline{t} = (\overline{c}-\underline{c})\,\Delta\overline{x}$$

These changes again make the high type sellers indifferent, strictly improve the low type sellers (since $\Delta\underline{t} > 0$), and always increase total surplus. They also make buyers weakly better off and thus constitute a Pareto improvement if

$$\overline{\lambda}^p \left[\Delta\overline{x}\overline{v} - \Delta\overline{t}\right] + \underline{\lambda}^p \left[\Delta\underline{x}\underline{v} - \Delta\underline{t}\right] \geq 0$$
$$\Leftrightarrow \quad \frac{\underline{\lambda}^p}{1-\mu}\left[\mu(\overline{v}-\underline{c}) - (\overline{c}-\underline{c})\right]\Delta\overline{x} \geq 0,$$

which is satisfied whenever $\mu \geq \frac{\overline{c}-\underline{c}}{\overline{v}-\underline{c}}$. □

A.11 Proof of Proposition 5.2

Let W^{GM} and W^{PP} denote social surplus in the equilibrium under general mechanisms and the equilibrium under price posting, respectively. We are interested in the limiting case of $\mu s \to +\infty$, while μs and b are kept finite, implying that $\overline{\lambda}^p$ tends to $+\infty$ and $\underline{\lambda}^p$ is finite.

Consider first the case of general mechanisms. Given the assumption $\underline{v} - \underline{c} < \overline{v} - \overline{c}$, the condition $\frac{1}{\overline{\lambda}^p}\left(1-e^{-\overline{\lambda}^p}\right) \in \left(1 - \frac{\overline{v}-\underline{v}}{\overline{c}-\underline{c}}, \frac{\underline{v}-\underline{c}}{\overline{v}-\underline{c}}\right)$, is always satisfied, meaning that the limiting case falls into parameter region II. To see this, note that $\lim_{\overline{\lambda}^p \to +\infty} \frac{1}{\overline{\lambda}^p}\left(1-e^{-\overline{\lambda}^p}\right) = 0$ and $1 - \frac{\overline{v}-\underline{v}}{\overline{c}-\underline{c}} < 0$, $\frac{\underline{v}-\underline{c}}{\overline{v}-\underline{c}} > 0$ for $\underline{v}-\underline{c} < \overline{v}-\overline{c}$. The limit of social surplus is thus given by

$$\lim_{\overline{\lambda}^p \to +\infty} W^{GM} = \lim_{\overline{\lambda}^p \to +\infty} b\left[\left(1-e^{-\underline{\lambda}^p}\right)(\underline{v}-\underline{c}) + e^{-\underline{\lambda}^p}\left(1-e^{-\overline{\lambda}^p}\right)(\overline{v}-\overline{c})\right] = b\left[\left(1-e^{-\underline{\lambda}^p}\right)(\underline{v}-\underline{c}) + e^{-\underline{\lambda}^p}(\overline{v}-\overline{c})\right]$$

Consider next the case of price posting. We can first show that as $\overline{\lambda}^p \to +\infty$, the equilibrium fraction of buyers going to the high quality market, γ, tends to one. A buyer's profit in the low and high quality market, respectively, is given by

$$\begin{aligned} \underline{\pi} &= \left(1-e^{-\frac{\underline{\lambda}^p}{1-\gamma}} - \frac{\underline{\lambda}^p}{1-\gamma}e^{-\frac{\underline{\lambda}^p}{1-\gamma}}\right)(\underline{v}-\underline{c}) \\ \overline{\pi} &= \left(1-e^{-\frac{\overline{\lambda}^p}{1-\gamma}}\right)(\overline{v}-\underline{c}) - \frac{\overline{\lambda}^p}{\gamma}e^{-\frac{\underline{\lambda}^p}{1-\gamma}}(\underline{v}-\underline{c}) \end{aligned}$$

Suppose γ does not tend to one. Then

$$\lim_{\overline{\lambda}^p \to +\infty}\left(\frac{\overline{\lambda}^p}{\gamma}e^{-\frac{\underline{\lambda}^p}{1-\gamma}}\right) = +\infty \quad \Rightarrow \quad \lim_{\overline{\lambda}^p \to +\infty} \overline{\pi} = -\infty$$

implying that the indifference condition for buyers cannot be satisfied. For the limit of $\frac{\overline{\lambda}^p}{\gamma}e^{-\frac{\underline{\lambda}^p}{1-\gamma}}$ to be finite we thus need γ to be a function of $\overline{\lambda}^p$ with $\lim_{\overline{\lambda}^p \to +\infty} \gamma(\overline{\lambda}^p) = 1$ such that $\lim_{\overline{\lambda}^p \to +\infty}\left(\frac{\overline{\lambda}^p}{\gamma(\overline{\lambda}^p)}e^{-\frac{\underline{\lambda}^p}{1-\gamma(\overline{\lambda}^p)}}\right) = l \in \mathbb{R}$. The indifference condition for buyers then requires $\underline{v}-\underline{c} = \overline{v}-\underline{c}-l \Leftrightarrow l = \overline{v}-\underline{v}$. With this, the limit of social surplus in the price posting equilibrium is given by

$$\lim_{\overline{\lambda}^p \to +\infty} W^{PP} = \lim_{\overline{\lambda}^p \to +\infty} b\left[(1-\gamma(\overline{\lambda}^p))\left(1-e^{-\frac{\underline{\lambda}^p}{1-\gamma(\overline{\lambda}^p)}}\right)(\underline{v}-\underline{c}) + \gamma(\overline{\lambda}^p)\left(1-e^{-\frac{\overline{\lambda}^p}{\gamma(\overline{\lambda}^p)}}\right)(\overline{v}-\overline{c})\right] = b(\overline{v}-\overline{c})$$

which is strictly greater than $\lim_{\overline{\lambda}^p \to +\infty} W^{GM} = b\left[\left(1-e^{-\underline{\lambda}^p}\right)(\underline{v}-\underline{c}) + e^{-\underline{\lambda}^p}(\overline{v}-\overline{c})\right]$. □