Dominant Strategy Implementation with a Convex Product Space of Valuations

Katherine Cuff[1], Sunghoon Hong[2], Jesse A. Schwartz[3], Quan Wen[4], and John A. Weymark[5]

[1] Department of Economics, McMaster University, 1280 Main Street West, Hamilton, Ontario L8S 4M4, Canada.
E-mail: cuffk@mcmaster.ca

[2] Department of Economics, Vanderbilt University, VU Station B #35189, 2301 Vanderbilt Place, Nashville, TN 37215-1819, USA.
E-mail: sunghoon.hong@vanderbilt.edu

[3] Department of Economics, Finance, and Quantitative Analysis, Kennesaw State University, 1000 Chastain Road, Box 0403, Kennesaw, GA 30144, USA.
E-mail: jschwar7@kennesaw.edu

[4] Department of Economics, Vanderbilt University, VU Station B #35189, 2301 Vanderbilt Place, Nashville, TN 37235-1819, USA
E-mail: quan.wen@vanderbilt.edu

[5] (Corresponding Author) Department of Economics, Vanderbilt University, VU Station B #35189, 2301 Vanderbilt Place, Nashville, TN 37235-1819, USA
E-mail: john.weymark@vanderbilt.edu

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Abstract: A necessary and sufficient condition for dominant strategy implementability when preferences are quasilinear is that, for every individual i and every choice of the types of the other individuals, all k-cycles in i's allocation graph have nonnegative length for every integer $k \geq 2$. Saks and Yu (*Proceedings of the 6th ACM Conference on Electronic Commerce (EC'05)*, 2005, 286–293) have shown that when the number of outcomes is finite and i's valuation type space is convex, nonnegativity of the length of all 2-cycles is sufficient for the nonnegativity of the length of all k-cycles. In this article, it is shown that if each individual's valuation type space is a full-dimensional convex product space and a mild domain regularity condition is satisfied, then (i) the nonnegativity of all 2-cycles implies that all k-cycles have zero length and (ii) all 2-cycles having zero length is necessary and sufficient for dominant strategy implementability.

Keywords: 2-cycle condition, dominant strategy implementation, mechanism design, revenue equivalence, Rockafellar–Rochet Theorem, Saks–Yu Theorem.

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1 Introduction

New insights into mechanism design theory, particularly when types are multidimensional, have recently been obtained using graph theory and linear programming. While the literature that uses these techniques focuses on obtaining general results that are not restricted to particular applications of the mechanism design framework, the results that have been obtained can be used in a wide variety of applications, such as auction design and the provision of public goods.

One issue that has attracted considerable attention is the development of necessary and sufficient conditions for dominant strategy implementability of an allocation function that chooses an outcome based on the reported type profile (a list of types, one for each individual) when the type space is restricted. Outcomes may be purely public or they may have private components. The starting point for this literature is a well-known necessary and sufficient condition for dominant strategy implementability for an arbitrary type space when utilities are quasilinear (linear in the payment) due to Rockafellar (1970) and Rochet (1987). Gui, Müller, and Vohra (2004) have provided a graph-theoretic interpretation of this condition: for every individual i and every choice of the types of the other individuals, all cycles with a finite number of arcs in a directed graph defined using the valuations of the outcomes by individual i have nonnegative length. In other words, for every integer $k \geq 2$, any cycle with k arcs (a k-cycle) has nonnegative length.

It may be difficult to verify that this condition is satisfied if there are more than a few possible outcomes. To help overcome this problem, Bikhchandani, Chatterji, Lavi, Mu'alem, Nisan, and Sen (2006) have identified a fairly abstract domain richness condition for which it is sufficient for dominant strategy implementability that all 2-cycles have nonnegative length. For a given individul i and given types of the other individuals, Saks and Yu (2005) have shown that when there are a finite number of outcomes, if i's set of possible valuations for these outcomes is convex and if all 2-cycles in the corresponding graph have nonnegative length, then all cycles with an arbitrary number of arcs also have nonnegative length. Hence, the nonnegativity of all 2-cycles is a sufficient condition for dominant strategy implementability when the Saks–Yu assumptions are satisfied. Extensions and variants of Saks and Yu's results have been established by Archer and Kleinberg (2008), Ashlagi, Braverman, Hassidim, and Monderer (2010), and Berger, Müller, and Naeemi (2009, 2010).

In this article, we strengthen the assumption of Saks and Yu (2005) that the set of individual valuations is convex by requiring that it be a full-dimensional convex product space. With the addition of a mild regularity condition, we show that if all 2-cycles have nonnegative length when our domain restriction is satisfied, then in fact all cycles have zero length. In proving this result, we identify and exploit some geometric properties of this problem. An implication of this result is that all 2-cycles having zero length is necessary and sufficient for dominant strategy implementability given our assumptions.

In order to state our results more precisely, we need to distinguish between the traditional concept of a type, here referred to as a characteristic type, and a valuation type, which we define below. We consider a direct mechanism that consists of an allocation function and a payment function. For each characteristic type profile, these functions determine an outcome and a payment (possibly negative) from each individual. The allocation function is dominant strategy implementable if there exists a payment function such that reporting one's true characteristic type is always a dominant strategy in the resulting direct mechanism.

We assume that the set of possible characteristic type profiles is the Cartesian product of the possible types for each individual and that for a fixed characteristic type, utility is quasilinear. For a given individual i and given types of the other individuals, following Gui, Müller, and Vohra (2004), we define a complete directed graph called the characteristic graph whose nodes are the possible characteristic types of individual i and the length (which could be negative) of the directed arc joining type s^i to t^i is the change in the valuation of the outcome obtained by individual i when he is of type t^i if he truthfully reports t^i instead of s^i. Note that the payments are being ignored in this construction. In terms of characteristic graphs, the Rockafellar–Rochet Theorem shows that an allocation function is dominant strategy implementable if and only if for every individual i and characteristic types of the other individuals, all k-cycles in the corresponding characteristic graph have nonnegative length for every integer $k \geq 2$.

When there are a finite number of outcomes, for a given individual i and given characteristic types of the other individuals, we can equivalently describe i's characteristic type t^i by the vector v^{t^i} in $\mathbb{R}^m$ whose jth component is the value of the jth outcome when he is of type t^i. Here, m is the number of outcomes that are attainable for the possible reported types of individual i given the characteristic types of the other individuals. This vector v^{t^i} is i's valuation type and the set of such types is i's valuation space (which depends on the types of the other individuals). Again following Gui, Müller, and Vohra (2004), this set of valuation types can be used to define a new graph, the allocation graph, whose nodes are the set of attainable outcomes and whose directed arc from a to b is the infimum of the change in valuation for i of having b instead of a over all characteristic types for him for which the allocation function assigns b. The Rockafellar–Rochet Theorem can be restated in terms of the nonnegativity of all k-cycles in these allocation graphs.

The result of Saks and Yu (2005) stated informally above assumes that there are a finite number of outcomes and that, for each individual i, i's valuation type space is convex for any fixed types of the other individuals. With these assumptions, they show that it is sufficient for the nonnegativity of all k-cycles in i's allocation graph that every 2-cycle has nonnegative length. Thus, in view of the Rockafellar–Rochet Theorem, the nonnegativity of all 2-cycles in every allocation graph is necessary and sufficient for dominant strategy implementability. This result is the Saks–Yu Theorem.

We strengthen Saks and Yu's convexity assumption by requiring that, for every individual i and for every fixed types of the other individuals, i's valuation type space is a full-dimensional convex product space; that is, it is the product of nondegenerate intervals of $\mathbb{R}$. We also suppose that i's valuation type space satisfies a regularity condition that ensures that there exists an open set of valuation types for i that results in a being chosen for each outcome a that is attainable given the types of the other individuals. Our assumptions are satisfied if i's valuation type space is unrestricted. With our assumptions, we show that if all 2-cycles in i's allocation graph have nonnegative length, then all k-cycles in this graph have zero length for every integer $k \geq 2$. It then follows from this result and the Rockafellar–Rochet Theorem that an allocation function is dominant strategy implementable if and only if all 2-cycles have zero length in every allocation graph obtained by selecting an individual i and fixing the types of the other individuals, what we call the zero-length 2-cycle condition.

An allocation function is an affine maximizer if it chooses an outcome that maximizes an affine function of the individual valuations, where the weights on the individual valuations are all nonnegative and not all zero. Affine maximizers with strictly positive weights are dominant strategy implementable. When some of the weights are zero, whether an affine maximizer is dominant strategy implementable depends on how ties are broken. Furthermore, for a finite set of three or more outcomes, Roberts' Theorem (see Roberts, 1979) shows that being an affine maximizer is a necessary condition for an allocation function to be dominant strategy implementable when the valuation type spaces are unrestricted and each outcome is chosen for some type profile.[1] The assumptions of Roberts' Theorem imply the assumptions used here. Hence, given Roberts' assumptions, an allocation function is an affine maximizer if and only if it satisfies our zero-length 2-cycle condition.

Dictatorial allocation functions and allocation functions for Vickrey (1961) auctions are affine maximizers. We provide examples of a dictatorial decision procedure and a Vickrey auction with a negative externality that satisfy our assumptions about the valuation type spaces and confirm that our zero-length 2-cycle condition holds. The valuation type spaces for a standard Vickrey auction of an indivisible good are convex product spaces, but not full dimensional. We show that, nevertheless, the zero-length 2-cycle condition is satisfied for such an auction. We also provide an example of a multi-unit Vickrey auction for two units of a homogeneous good in which each valuation type space is convex, but is neither full dimensional nor a product space. In this example, some 2-cycles have positive length.

An allocation function that is dominant strategy implementable satisfies the revenue equivalence property if the payment functions that implement it have the property that for each individual i, given the types of the other indi-

[1] For graph-theoretic proofs of Roberts' Theorem, see Lavi, Mu'alem, and Nisan (2009).

viduals, the implementing payment functions for i only differ by a constant. Revenue equivalence was first analyzed by Myerson (1981) in his study of the design of optimal auctions for a single good when the individual characterstic type spaces are one-dimensional. Heydenreich, Müller, Uetz, and Vohra (2009, Corollary 1) show that revenue equivalence is satisfied by an allocation function that is dominant strategy implementable if and only if all 2-cycles have zero length in every allocation graph. Thus, given our assumptions, our zero-length 2-cycle condition is not only necessary and sufficient for dominant strategy implementability, it is also necessary and sufficient for revenue equivalence.

In order to prove that all 2-cycles in i's allocation graph have nonnegative length, Saks and Yu (2005) use i's allocation graph to define a new graph with the same set of nodes for which the length of the directed arc from a to b is the total change in i's valuation along a particular kind of path in i's valuation type space. They show that under their assumptions, the length of any directed arc in this new graph bounds from below the length of this arc in i's allocation graph and that the length of any k-cycle in this new graph is zero, from which their theorem follows. In our proofs, we do not need to consider this auxillary graph.

The plan of the rest of this article is as follows. In Section 2, we introduce the model, the characteristic and allocation graphs, and the Rockafellar–Rochet Theorem. In Section 3, we consider the Saks–Yu Theorem. Next, in Section 4, we investigate the geometry of the partition of an individual's valuation type space into regions that are allocated the same outcome. In Section 5, we show that dominant strategy implementation is equivalent to our zero-length 2-cycle condition when our assumptions on the valuation type spaces are satisfied. In Section 6, we relate our analysis to the literature on affine maximizers and present our examples. Finally, in Section 7, we offer some concluding remarks.

2 Dominant Strategy Implementability and the Rockafellar–Rochet Theorem

Let N be a finite set of n individuals and Ω be a finite set of outcomes. For each $i \in N$, let T^i denote the *characteristic type space* of individual i with typical element t^i. For now, no assumptions are made about the structure of T^i. The value of t^i is private information. Let $T^{-i} = \times_{j \in N \setminus \{i\}} T^j$ denote the characteristic type space of all individuals other than individual i. A *characteristic type profile* is written as $(t^i, t^{-i}) \in T^i \times T^{-i}$.

For each $i \in N$, let $v^i\colon \Omega \times T^i \to \mathbb{R}$ be the *valuation function* of individual i. This function assigns a value $v^i(a|t^i)$ to each outcome $a \in \Omega$ and characteristic type $t^i \in T^i$. Thus, an individual's valuation of an outcome only depends on his private characteristic type.

A *direct mechanism* (G, P) consists of an *allocation function* $G\colon T^i \times T^{-i} \to \Omega$ and a *payment function* $P\colon T^i \times T^{-i} \to \mathbb{R}^n$. The function P may be written as $P = (P^1, \ldots, P^n)$, where P^i is the payment function for individual i. For each type profile, G determines an outcome in Ω and P^i specifies a payment (which could be negative) from individual i.

An individual need not report his true type. Given the other individuals' reported types $t^{-i} \in T^{-i}$, the *utility* of individual i with characteristic type $t^i \in T^i$ and reported type $s^i \in T^i$ is

$$v^i(G(s^i, t^{-i})|t^i) - P^i(s^i, t^{-i}).$$

Definition. An allocation function G is *dominant strategy implementable* if there exists a payment function P such that for all $i \in N$ and all $t^{-i} \in T^{-i}$,

$$v^i(G(t^i, t^{-i})|t^i) - P^i(t^i, t^{-i}) \geq v^i(G(s^i, t^{-i})|t^i) - P^i(s^i, t^{-i}), \ \ \forall s^i, t^i \in T^i. \tag{1}$$

In other words, an allocation function is dominant strategy implementable if there exists a payment function for which each individual is at least as well off reporting his true type than reporting any other type regardless of what the other individuals report.

Given the allocation function G, for fixed $i \in N$ and $t^{-i} \in T^{-i}$, the *characteristic graph* $T_G(t^{-i})$ is the complete directed graph with nodes T^i and arc length

$$d(s^i, t^i|t^{-i}) = v^i(G(t^i, t^{-i})|t^i) - v^i(G(s^i, t^{-i})|t^i) \tag{2}$$

for the directed arc (s^i, t^i) from s^i to t^i.[2] That is, $d(s^i, t^i|t^{-i})$ is the change in i's valuation if his true characteristic type t^i is reported instead of the characteristic type s^i given the reported characteristic types t^{-i} of the other individuals. This change in valuation is not the overall change in i's utility because the payments have not been taken into account.

For every integer $k \geq 2$, a *k-cycle* in the characteristic graph $T_G(t^{-i})$ is a sequence of k arcs $(t^i_1, t^i_2), \ldots, (t^i_{k-1}, t^i_k), (t^i_k, t^i_1)$ whose *length* is defined to be the sum of the lengths of the arcs in the cycle. That is, the length of the k-cycle is $d(t^i_1, t^i_2|t^{-i}) + \cdots + d(t^i_{k-1}, t^i_k|t^{-i}) + d(t^i_k, t^i_1|t^{-i})$.[3]

Rochet (1987, Theorem 1) uses a theorem about subdifferentials of multidimensional convex functions due to Rockafellar (1970, Theorem 24.8) to provide necessary and sufficient conditions for an allocation function to be dominant strategy implementable. This result is known as the *Rockafellar–Rochet Theorem*. Theorem 1 provides a statement of this theorem in terms of characteristic graphs (see Vohra, 2011, Theorem 4.2.1).[4]

[2] We exclude loops. That is, there are no arcs from a node to itself. The characteristic graph, as well as the allocation graph defined below, were introduced by Gui, Müller, and Vohra (2004). We adopt their terminology in calling (2) a "length" instead of an edge weight even though it may be negative.

[3] Note that because there are no loops, there are no 1-cycles.

[4] When stating this theorem, it is common to assume that everybody has the same characteristic type space, but as Vohra (2011, p. 38) notes, this is not necessary.

Theorem 1. *The allocation function $G\colon T^i \times T^{-i} \to \Omega$ is dominant strategy implementable if and only if for every $i \in N$, $t^{-i} \in T^{-i}$, and integer $k \geq 2$, all k-cycles in the characteristic graph $T_G(t^{-i})$ have nonnegative length.*

Given $t^{-i} \in T^{-i}$, let $A(t^{-i}) = \{a_1, \ldots, a_m\}$ be the finite set of m *attainable outcomes* for the allocation function G. That is,

$$A(t^{-i}) = \{a \in \Omega \mid G(t^i, t^{-i}) = a \text{ for some } t^i \in T^i\}.$$

The value of m may depend on the choice of t^{-i}.

For all $a \in A(t^{-i})$, let

$$R_a(t^{-i}) = \{t^i \in T^i \mid G(t^i, t^{-i}) = a\}$$

be the set of characteristic types for i that induce outcome a using the allocation function G when the other individuals' types are given by t^{-i}. By construction, $R_a(t^{-i})$ is nonempty for all $a \in A(t^{-i})$.

For the characteristic graph $T_G(t^{-i})$, the corresponding *allocation graph* $\Gamma_G(t^{-i})$ is the complete directed graph that has $A(t^{-i})$ as the set of nodes and $\ell(a, b|t^{-i})$ as the *length* of the directed arc from node a to node b, where for all distinct $a, b \in A(t^{-i})$,

$$\begin{aligned}\ell(a, b|t^{-i}) &= \inf_{t^i \in R_b(t^{-i})} \left[v^i(b|t^i) - v^i(a|t^i)\right] \\ &= \inf_{t^i \in R_b(t^{-i})} \left[v^i(G(t^i, t^{-i})|t^i) - v(a|t^i)\right].^5 \end{aligned} \tag{3}$$

In this graph, the length (which could be negative) of the directed arc from a to b is the infimum of the change in i's valuation of having b instead of a over the set of all of his characteristic types for which the outcome function assigns b given t^{-i}.

For any two nodes a and b in the allocation graph $\Gamma_G(t^{-i})$, a *path* from a to b is a sequence of arcs $(a_1, a_2), \ldots, (a_{k-1}, a_k)$ for which $a = a_1$ and $b = a_k$. For every integer $k \geq 2$, a *k-cycle* in the allocation graph $\Gamma_G(t^{-i})$ is a path with k arcs whose endpoints are both the same. That is, it is a sequence of arcs $(a_1, a_2), \ldots, (a_{k-1}, a_k), (a_k, a_1)$. The length of a path or k-cycle is the sum of the lengths of the arcs that comprise it.

The Rockafellar–Rochet Theorem can be restated using allocation graphs by simply substituting the allocation graph $\Gamma_G(t^{-i})$ for the characteristic graph $T_G(t^{-i})$ in the statement of Theorem 1.[6]

In order to analyze dominant strategy implementability, without loss of generality, we can consider a fixed individual $i \in N$ and fixed types $t^{-i} \in T^{-i}$

[5] We adopt the convention that the infimum and supremum are equal to $-\infty$ and ∞, respectively, when they are not finite. Our assumptions do not rule out the possibility that $\ell(a, b|t^{-i}) = -\infty$. As shown by Mishra (2009) and Vohra (2011), $\ell(a, b|t^{-i})$ is finite for all $a, b \in A(t^{-i})$ if (1) is satisfied.

[6] This version of the Rockafellar–Rochet Theorem is stated without proof in Vohra (2011).

of the other individuals. To simplify the notation, we let $v = v^i$, $t = t^i$, $T = T^i$, and suppress the dependence of $A(t^{-i})$, $R_a(t^{-i})$, $d(s^i, t^i|t^{-i})$, and $\ell(a, b|t^{-i})$ on t^{-i}. By fixing i and t^{-i}, (G, P) defines a *single person mechanism* (g, p) with allocation function $g\colon T \to A$ and payment function $p\colon T \to \mathbb{R}$ obtained by setting

$$g(t) = G(t, t^{-i}) \text{ and } p(t) = P^i(t, t^{-i}),\ \forall t \in T.$$

Note that g is surjective. The corresponding characteristic and allocation graphs are denoted by T_g and Γ_g, respectively.

For the mechanism (g, p), the dominant strategy implementability condition (1) simplifies to

$$v(g(t)|t) - p(t) \geq v(g(s)|t) - p(s), \quad \forall s, t \in T. \tag{4}$$

It follows from (4) that if g is dominant strategy implementable and $g(s) = g(t)$, then $p(s) = p(t)$ as well.

For the allocation function g, the Rockafellar–Rochet Theorem can be restated as follows.

Theorem 2. *The following conditions for the allocation function $g\colon T \to A$ are equivalent:*

(i) *g is dominant strategy implementable;*
(ii) *for every integer $k \geq 2$, all k-cycles in the characteristic graph T_g have nonnegative length;*
(iii) *for every integer $k \geq 2$, all k-cycles in the allocation graph Γ_g have nonnegative length.*

Proof. The equivalence of (i) and (ii) follows immediately from Theorem 1 by setting $n = 1$. We now show the equivalence of (i) and (iii).[7]

First, suppose that for every integer $k \geq 2$, all k-cycles in Γ_g have nonnegative length. Because there are a finite number of outcomes in A, there are a finite number of nodes in Γ_g. Hence, because all cycles in Γ_g have nonnegative length, by Vohra (2011, Corollary 3.4.2), for any two nodes $a, b \in A$, there is a minimum-length (i.e., shortest) path from a to b. Fix $a \in A$. Let $\bar{p}\colon T \to \mathbb{R}$ be the length of a shortest path from a to $g(t)$. We shall show that the mechanism $(g, \bar{p})$ satisfies the dominant strategy implementability condition (4).

Consider any two types $s, t \in T$. We have

$$\bar{p}(t) \leq \bar{p}(s) + \ell(g(s), g(t))$$

because the length of a shortest path from a to $g(t)$ cannot exceed the length of a path from a to $g(s)$ to $g(t)$. It follows that

[7] A somewhat different proof of this equivalence may be found in Mishra (2009, Theorem 2).

$$\begin{aligned}\bar{p}(t) - \bar{p}(s) &\leq \ell(g(s), g(t)) \\ &= \inf_{r \in R_{g(t)}} [v(g(t)|r) - v(g(s)|r)] \\ &\leq v(g(t)|t) - v(g(s)|t),\end{aligned}$$

thereby establishing (4).

Second, suppose that there exists a payment function $p \colon T \to A$ such that (g, p) satisfies (4). Rearranging (4), we obtain

$$p(t) - p(s) \leq v(g(t)|t) - v(g(s)|t), \quad \forall s, t \in T. \tag{5}$$

Consider any k-cycle $(a_1, a_2), \ldots, (a_{k-1}, a_k), (a_k, a_1)$ for an arbitrary integer $k \geq 2$. Let $a_{k+1} = a_1$. Then, for all $\varepsilon > 0$ and all $j = \{1, \ldots, k\}$, there exist $s^{\varepsilon,j} \in R_{a_j}$ such that

$$\begin{aligned}\sum_{j=1}^{k} \ell(a_j, a_{j+1}) &= \sum_{j=1}^{k} \inf_{t \in R_{a_{j+1}}} [v(a_{j+1}|t) - v(a_j|t)] \\ &> \sum_{j=1}^{k} \left[v(a_{j+1}|s^{\varepsilon,j+1}) - v(a_j|s^{\varepsilon,j+1}) - \varepsilon \right].\end{aligned} \tag{6}$$

Noting that $s^{\varepsilon,k+1} = s^{\varepsilon,1}$, it follows from (5) that

$$\sum_{j=1}^{k} \left[v(a_{j+1}|s^{\varepsilon,j+1}) - v(a_j|s^{\varepsilon,j+1}) - \varepsilon \right] \geq \sum_{j=1}^{k} \left[p(s^{\varepsilon,j+1}) - p(s^{\varepsilon,j}) - \varepsilon \right]. \tag{7}$$

The sum on the right-hand side of (7) is $-k\varepsilon$. Hence, (6) and (7) imply that

$$\sum_{j=1}^{k} \ell(a_j, a_{j+1}) > -k\varepsilon. \tag{8}$$

Taking the limit as ε goes to 0 in (8) shows that the length of this k-cycle is nonnegative. □

3 The Saks–Yu Theorem

It may be computationally onerous to check that every cycle in either the characteristic graph T_g or in the allocation graph Γ_g has nonnegative length in order to determine if the allocation function g is dominant strategy implementable. It follows from the Rockafellar–Rochet Theorem that a necessary condition for dominant strategy implementability is that all the 2-cycles have nonnegative length. Bikhchandani, Chatterji, Lavi, Mu'alem, Nisan, and Sen

(2006) and Saks and Yu (2005) have identified alternative restrictions on v under which this 2-cycle nonnegativity condition is also sufficient for dominant strategy implementability. Our results build on those of Saks and Yu.

The 2-cycle nonnegativity condition can be defined in either of our two graphs. Below we shall show that these two definitions are equivalent. We begin with the characteristic graph.

Definition. An allocation function g satisfies the *characteristic graph 2-cycle nonnegativity condition* if

$$d(s,t) + d(t,s) \geq 0, \quad \forall s,t \in T, s \neq t. \tag{9}$$

It follows from (2) that (9) is equivalent to

$$v(g(t)|t) - v(g(s)|t) \geq v(g(t)|s) - v(g(s)|s), \quad \forall s,t \in T, s \neq t. \tag{10}$$

That is, the change in valuation obtained by replacing $g(s)$ with $g(t)$ is at least as large for type t as for type s. For this reason, Bikhchandani, Chatterji, Lavi, Mu'alem, Nisan, and Sen (2006) call this condition *weak monotonicity.*

We now define the corresponding condition using the allocation graph.

Definition. An allocation function g satisfies the *allocation graph 2-cycle nonnegativity condition* if

$$\ell(a,b) + \ell(b,a) \geq 0, \quad \forall a,b \in A, a \neq b. \tag{11}$$

Theorem 3 shows that these two 2-cycle nonnegativity conditions are equivalent. In light of this equivalence, we shall simply refer to this condition as the *2-cycle nonnegativity condition.*

Theorem 3. *An allocation function $g\colon T \to A$ satisfies the characteristic graph 2-cycle nonnegativity condition if and only if it satisfies the allocation graph 2-cycle nonnegativity condition.*

Proof. First, suppose the allocation rule g satisfies the characteristic graph 2-cycle nonnegativity condition (9) but that, by way of contradiction, there exist outcomes $\hat{a}$ and $\hat{b}$ in A such that $\ell(\hat{a},\hat{b}) + \ell(\hat{b},\hat{a}) < 0$. Using the definition of ℓ in (3), we can rewrite the last inequality as

$$\inf_{t \in R_{\hat{b}}} [v(\hat{b}|t) - v(\hat{a}|t)] + \inf_{s \in R_{\hat{a}}} [v(\hat{a}|s) - v(\hat{b}|s)] < 0.$$

Thus, there exist characteristic types $\hat{s} \in R_{\hat{a}}$ and $\hat{t} \in R_{\hat{b}}$ such that

$$[v(\hat{b}|\hat{t}) - v(\hat{a}|\hat{t})] + [v(\hat{a}|\hat{s}) - v(\hat{b}|\hat{s})] < 0.$$

This inequality, however, contradicts (10), which is equivalent to the characteristic graph 2-cycle nonnegativity condition (9).

Second, suppose the allocation rule g satisfies the allocation graph 2-cycle nonnegativity condition (11) but that, by way of contradiction, there exist types $\hat{s}$ and $\hat{t}$ in T such that $d(\hat{s}, \hat{t}) + d(\hat{t}, \hat{s}) < 0$ or, equivalently, that

$$[v(\hat{b}|\hat{t}) - v(\hat{a}|\hat{t})] + [v(\hat{a}|\hat{s}) - v(\hat{b}|\hat{s})] < 0,$$

where $\hat{a} = g(\hat{s})$ and $\hat{b} = g(\hat{t})$. From this last inequality, it follows that

$$\ell(\hat{a}, \hat{b}) + \ell(\hat{b}, \hat{a}) = \inf_{t \in R_{\hat{b}}} [v(\hat{b}|t) - v(\hat{a}|t)] + \inf_{s \in R_{\hat{a}}} [v(\hat{a}|s) - v(\hat{b}|s)] < 0,$$

which contradicts the allocation graph 2-cycle nonnegativity condition (11). □

Each characteristic type $t \in T$ has associated with it a corresponding *valuation type* $v^t = (v^t_{a_1}, \ldots, v^t_{a_m}) \in \mathbb{R}^m$, where $v^t_a = v(a|t)$ for all $a \in A$. The jth component of v^t is the value of outcome a_j when individual i is of characteristic type t. Individual i's *valuation type space* (given t^{-i}) is

$$\mathcal{V} = \{v^t \in \mathbb{R}^m \mid t \in T\}.^8$$

If characteristic types s and t have the same associated valuation type v, there is then no loss of generality in identifying them (i.e., treating them as being the same characteristic type). Henceforth, we assume that if $s \neq t$, then $v^s \neq v^t$. With this assumption, there is a unique $t \in T$ associated with each $v \in \mathcal{V}$. Hence, individual i can be equivalently characterized by his characteristic type t or his valuation type v. Let t^v denote the characteristic type associated with v.

Saks and Yu (2005, Theorem 4) show that if the valuation type space $\mathcal{V}$ is convex, then all k-cycles in the allocation graph Γ_g have nonnegative length if Γ_g satisfies the allocation graph 2-cycle nonnegativity condition (11). By Theorem 2, we thus have that the 2-cycle nonnegativity condition is necessary and sufficient for dominant strategy implementability of g when $\mathcal{V}$ is convex. This result, which is Theorem 1 in Saks and Yu (2005), is the *Saks–Yu Theorem.*

Theorem 4. *If $\mathcal{V}$ is convex, then the allocation function $g\colon T \to A$ is dominant strategy implementable if and only if the 2-cycle nonnegativity condition is satisfied.*

Note that the assumption that $\mathcal{V}$ is convex implicitly places restrictions on the characteristic type space T. In particular, T cannot be discrete.

[8] Note that $\mathcal{V}$ is a function of t^{-i} because the set of attainable outcomes A may depend on t^{-i}.

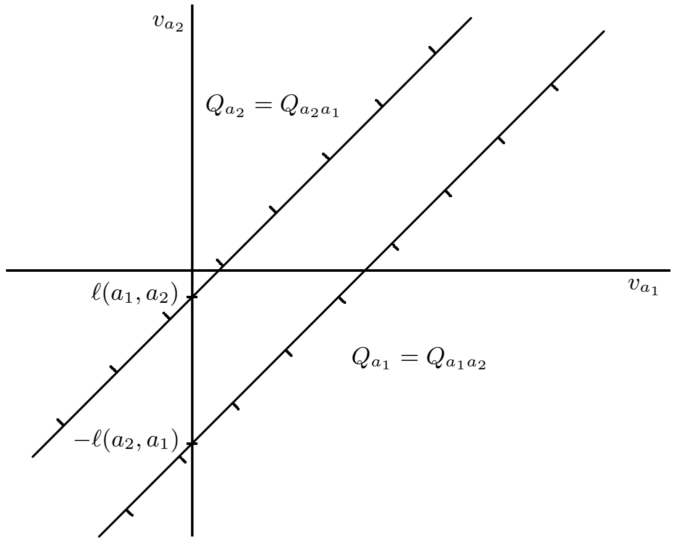

Fig. 1. Difference sets when $m = 2$.

4 Partitioning the Valuation Type Space

Recall that R_a is the set of characteristic types that the allocation function g maps into outcome a. Because there is a bijection between the characteristic type space T and the valuation type space $\mathcal{V}$, the sets R_a for $a \in A$ induce a partition of $\mathcal{V}$ with each cell in the partition associated with the outcome assigned to valuation types in that cell. Our results are obtained by investigating the geometry of this partition.

The valuation type space $\mathcal{V}$ is a subset of $\mathbb{R}^m$. We first define some sets on all of $\mathbb{R}^m$ and then later restrict them to $\mathcal{V}$.

For all $a, b \in A$ with $a \neq b$, the *difference set for* (a, b) is

$$Q_{ab} = \{v \in \mathbb{R}^m \mid v_a - v_b \geq \ell(b, a)\}.$$

Q_{ab} is a closed halfspace. A valuation type v is in Q_{ab} if the change in valuation for individual i of having object a instead of b is at least as large as the infimum of the change in valuation of having b instead of a over the set of all characteristic types for which the outcome function assigns a.

Difference sets are illustrated in Figure 1 for the case in which $A = \{a_1, a_2\}$. In this case, the two difference sets are

$$Q_{a_1 a_2} = \{v \in \mathbb{R}^2 \mid v_{a_2} \leq -\ell(a_2, a_1) + v_{a_1}\}$$

and

$$Q_{a_2 a_1} = \{v \in \mathbb{R}^2 \mid v_{a_2} \geq \ell(a_1, a_2) + v_{a_1}\}.$$

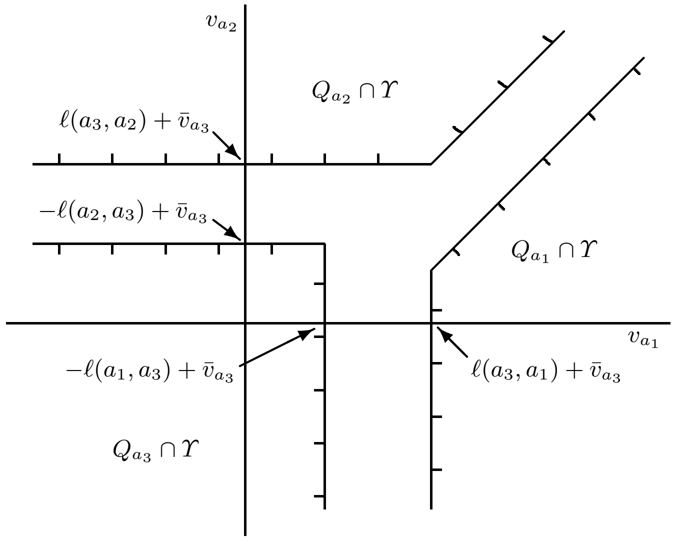

Fig. 2. Difference sets when $m = 3$.

The boundaries of these two sets have slope equal to 1 and their vertical intercepts are $-\ell(a_2, a_1)$ and $\ell(a_1, a_2)$, respectively. The 2-cycle nonnegativity condition $\ell(a_1, a_2) + \ell(a_2, a_1) \geq 0$ holds if and only $\ell(a_1, a_2) \geq -\ell(a_2, a_1)$. Hence, if the 2-cycle nonnegativity condition is satisfied, $Q_{a_2a_1}$ lies above $Q_{a_1a_2}$ and the interiors of these sets have an empty intersection.

For each $a \in A$, the *difference set for a* is

$$Q_a = \bigcap_{b \in A \setminus \{a\}} Q_{ab}.$$

Q_a is the closed convex polyhedron obtained by intersecting the halfspaces Q_{ab} for all outcomes b distinct from a. For the two-outcome case illustrated in Figure 1, $Q_{a_1} = Q_{a_1a_2}$ and $Q_{a_2} = Q_{a_2a_1}$.

For the case in which $A = \{a_1, a_2, a_3\}$, cross sections of the difference sets Q_{a_1}, Q_{a_2}, and Q_{a_3} for a fixed valuation $\bar{v}_{a_3}$ of the third outcome are illustrated in Figure 2. Let

$$\Upsilon = \{v \in \mathbb{R}^3 \mid v_{a_3} = \bar{v}_{a_3}\}.$$

In this diagram, $Q_{a_1} \cap \Upsilon$ is the intersection of

$$Q_{a_1a_2} \cap \Upsilon = \{v \in \Upsilon \mid v_{a_2} \leq -\ell(a_2, a_1) + v_{a_1}\}$$

and

$$Q_{a_1a_3} \cap \Upsilon = \{v \in \Upsilon \mid v_{a_1} \geq \ell(a_3, a_1) + \bar{v}_{a_3}\},$$

$Q_{a_2} \cap \Upsilon$ is the intersection of

$$Q_{a_2a_1} \cap \Upsilon = \{v \in \Upsilon \mid v_{a_2} \geq \ell(a_1, a_2) + v_{a_1}\}$$

and

$$Q_{a_2a_3} \cap \Upsilon = \{v \in \Upsilon \mid v_{a_2} \geq \ell(a_3, a_2) + \bar{v}_{a_3}\},$$

and $Q_{a_3} \cap \Upsilon$ is the intersection of

$$Q_{a_3a_1} \cap \Upsilon = \{v \in \Upsilon \mid v_{a_1} \leq -\ell(a_1, a_3) + \bar{v}_{a_3}\}$$

and

$$Q_{a_3a_2} \cap \Upsilon = \{v \in \Upsilon \mid v_{a_2} \leq -\ell(a_2, a_3) + \bar{v}_{a_3}\}.$$

Let ∂S denote the boundary of set S. The intersections of $Q_{a_1a_2}$ and $Q_{a_2a_1}$ with Υ do not depend on the choice of $\bar{v}_{a_3}$. Hence, for all $\bar{v}_{a_3}$, the upward sloping parts of $\partial Q_{a_1} \cap \Upsilon$ and $\partial Q_{a_2} \cap \Upsilon$ have slope equal to 1 and are contained in lines whose intercepts with the axes do not depend on $\bar{v}_{a_3}$. As $\bar{v}_{a_3}$ increases, the horizontal parts of $\partial Q_{a_2} \cap \Upsilon$ and $\partial Q_{a_3} \cap \Upsilon$ move up and the vertical parts of $\partial Q_{a_1} \cap \Upsilon$ and $\partial Q_{a_3} \cap \Upsilon$ move to the right.

When there are more than three outcomes, the analogue of Figure 2 is obtained by setting $\Upsilon = \{v \in \mathbb{R}^m \mid (v_{a_3}, \ldots, v_{a_m}) = (\bar{v}_{a_3}, \ldots, \bar{v}_{a_m})\}$. In this case, the restrictions of the difference sets Q_{a_1} and Q_{a_2} to Υ have the same shapes as shown in Figure 2. Provided that the 2-cycle nonnegativity condition is satisfied, there is a single outcome a_{d^*} that maximizes both $\ell(a_d, a_1) + \bar{v}_{a_d}$ and $\ell(a_d, a_2) + \bar{v}_{a_d}$ for $d \neq 1, 2$. The points in the vertical boundary of $Q_{a_1} \cap \Upsilon$ all have first coordinate equal to $\ell(a_{d^*}, a_1) + \bar{v}_{a_{d^*}}$ and the points in the horizontal boundary of $Q_{a_2} \cap \Upsilon$ all have second coordinate equal to $\ell(a_{d^*}, a_2) + \bar{v}_{a_{d^*}}$. Generically, the only other difference set that has a nonempty intersection with Υ is $Q_{a_{d^*}}$. It has the same shape as $Q_{a_3} \cap \Upsilon$ in Figure 2, with points in its vertical boundary having first coordinate equal to $-\ell(a_1, a_{d^*}) + \bar{v}_{a_{d^*}}$ and points in its horizontal boundary having second coordinate equal to $-\ell(a_2, a_{d^*}) + \bar{v}_{a_{d^*}}$. It is, however, possible that there is an $a_{\bar{d}} \neq a_{d^*}$ for which $Q_{a_{d^*}} \cap \Upsilon = Q_{a_{\bar{d}}} \cap \Upsilon$. This happens when these sets are common boundary points of $Q_{a_{d^*}}$ and $Q_{a_{\bar{d}}}$.

There is a close connection between the set of characteristic types T_a that are allocated outcome a and the set $Q_a \cap \mathcal{V}$. Except for possibly some of the boundary points of Q_a, the set of characteristic types associated with valuation types in $Q_a \cap \mathcal{V}$ is R_a. More precisely, for every $t \in R_a$, the valuation type v^t is in $Q_a \cap \mathcal{V}$. Moreover, if the allocation function satisfies the 2-cycle nonnegativity condition, then for every $v \in \mathcal{V}$ that is in the interior Q_a° of Q_a, the characteristic type t^v is in R_a. Proofs of these results may be found in Mishra (2009) and Vohra (2011), but for completeness, we include them here.

Theorem 5. *For any allocation function $g \colon T \to A$ and any outcome $a \in A$, (i) for every characteristic type $t \in R_a$, the valuation type v^t is in $Q_a \cap \mathcal{V}$ and (ii) if g satisfies the 2-cycle nonnegativity condition, then for every valuation type $v \in Q_a^\circ \cap \mathcal{V}$, the characteristic type t^v is in R_a.*

Proof. (i) By definition, $g(t) = a$ for any characteristic type $t \in R_a$. Therefore,

$$v(a|t) - v(b|t) \geq \inf_{t \in R_a} [v(a|t) - v(b|t)] = \ell(b, a), \quad \forall b \in A \backslash \{a\}.$$

Hence, by the definition of Q_a, we have $v^t \in Q_a \cap \mathcal{V}$.

(ii) Consider any valuation type $v \in Q_a^\circ \cap \mathcal{V}$. Because $v \in Q_a^\circ$, for the characteristic type t^v, we have

$$v_a - v_b = v(a|t^v) - v(b|t^v) > \ell(b, a), \quad \forall b \in A \backslash \{a\}.$$

Because the allocation rule g satisfies the 2-cycle nonnegativity condition, $\ell(a, b) \geq -\ell(b, a)$ for all $b \in A \backslash \{a\}$. The last two inequalities then imply that

$$v_b - v_a = v(b|t^v) - v(a|t^v) < -\ell(b, a) \leq \ell(a, b), \quad \forall b \in A \backslash \{a\}.$$

Hence, $v \notin Q_b \cap \mathcal{V}$ for any $b \in A \backslash \{a\}$. Therefore, from part (i) it follows that $t^v \notin R_b$ for any $b \in A \backslash \{a\}$. Consequently, t^v must be in R_a. $\square$

An immediate implication of Theorem 5 is that for all $a, b \in A$, $Q_a^\circ \cap Q_b^\circ = \varnothing$ if the 2-cycle nonnegativity condition is satisfied. Furthermore, if $v \in Q_a \cap Q_b \cap \mathcal{V}$, then $v_a - v_b = \ell(b, a) = -\ell(a, b)$.[9]

Our next theorem shows that the allocation function g satisfies a monotonicity property when the 2-cycle nonnegativity condition is satisfied. Specifically, if the valuation of the chosen outcome, say a, increases and the valuation of no other outcome decreases, then no outcome b different from a can be chosen unless b's valuation also increases.

Theorem 6. *If the allocation function $g\colon T \to A$ satisfies the 2-cycle nonnegativity condition, then for every characteristic type $t \in R_a$ and every valuation type $v' \in \mathcal{V}$ with $v' \geq v^t$ for which $v'_a > v^t_a$ and $v'_b = v^t_b$, the characteristic type $t^{v'}$ is not in R_b.*

Proof. Consider any $a \in A$ and $t \in R_a$. By Theorem 5, we have

$$v^t_a - v^t_c \geq \ell(c, a), \quad \forall c \in A \backslash \{a\}.$$

Consider any $b \in A \backslash \{a\}$ and any valuation type $v' \in \mathcal{V}$ with $v' \geq v^t$ for which $v'_a > v^t_a$ and $v'_b = v^t_b$. The preceding inequality then implies that

$$v'_b - v'_a < v^t_b - v^t_a \leq -\ell(b, a).$$

By the 2-cycle nonnegativity condition, we thus have

$$v'_b - v'_a < \ell(a, b).$$

Hence, by Theorem 5, $t^{v'} \notin R_b$. $\square$

[9] See Saks and Yu (2005, Proposition 5).

5 Dominant Strategy Implementation and Zero Length Cycles

In this section, we replace the Saks–Yu assumption that the valuation type space $\mathcal{V}$ is convex with the more restrictive assumption that it is the product of nonegenerate intervals, what we call a *full-dimensional convex product space.*

Definition. The valuation type space $\mathcal{V}$ is a *full-dimensional convex product space* if

$$\mathcal{V} = \prod_{a \in A} \langle L_a, U_a \rangle,$$

where for all $a \in A$, $\langle L_a, U_a \rangle$ is any type of interval of $\mathbb{R}$ with endpoints L_a and U_a for which $L_a < U_a$.[10]

By construction, $Q_a \cap \mathcal{V} \neq \varnothing$ for all $a \in A$. We henceforth assume that $Q_a^\circ \cap \mathcal{V}^\circ \neq \varnothing$ for all $a \in A$. We refer to this restriction as the *interiority assumption.* This condition necessarily holds if $\mathcal{V}$ is open, which is the case if, for example, $\mathcal{V} = \mathbb{R}^m$. If $\mathcal{V}$ is not open, because $\mathcal{V}^\circ \neq \varnothing$, requiring $Q_a^\circ \cap \mathcal{V}^\circ$ to be nonempty for all $a \in A$ is a mild regularity condition.

As we have noted, Saks and Yu (2005) show that if an allocation function $g \colon T \to A$ satisfies the 2-cycle nonnegativity condition and $\mathcal{V}$ is convex, then all k-cycles in the allocation graph Γ_g have nonnegative length. Our main theorem shows that, in fact, all of these k-cycles have zero length if we additionally assume that $\mathcal{V}$ is a full-dimensional convex product space and $Q_a^\circ \cap \mathcal{V}^\circ \neq \varnothing$ for all $a \in A$.

Theorem 7. *Suppose that $|A| \geq 2$. If (i) the allocation function $g \colon T \to A$ satisfies the 2-cycle nonnegativity condition, (ii) the valuation type space $\mathcal{V}$ is a full-dimensional convex product space, and (iii) $Q_a^\circ \cap \mathcal{V}^\circ \neq \varnothing$ for all $a \in A$, then for every integer $k \geq 2$, all k-cycles in the allocation graph Γ_g have zero length.*

We prove this theorem by a sequence of lemmas. Our first lemma shows that our assumptions imply that all 2-cycles in Γ_g have zero length.[11] Note that all 2-cycles in Γ_g have zero length if and only if ℓ is antisymmetric. That is, $\ell(a, b) = -\ell(b, a)$ for all $a, b \in A$.

Lemma 1. *Under the assumptions of Theorem 7, all 2-cycles in the allocation graph Γ_g have zero length.*

[10] We permit L_a to be $-\infty$ and U_a to be ∞.

[11] For the special case in which $\mathcal{V}$ is all of $\mathbb{R}^m$, this lemma has also been established by Lavi, Mu'alem, and Nisan (2009, Claim 8). Their Claim 8 is used to help prove a version of Roberts' Theorem. Although Roberts' Theorem assumes that the allocation function is dominant strategy implementable, this assumption is not used in the proof of this claim.

Proof. Consider any $a, b \in A$. We first prove that $\ell(a,b)+\ell(b,a)=0$ when U_a and U_b are both finite. On the contrary, suppose that the sum of these lengths differs from zero, which by the 2-cycle nonnegativity condition implies that there exists an arbitrarily small $\delta > 0$ such that

$$\ell(a,b) + \ell(b,a) \geq \delta. \tag{12}$$

We assume that

$$U_a - \ell(b,a) \leq U_b. \tag{13}$$

This assumption is without loss of generality because if (13) does not hold, then the 2-cycle nonnegativity condition implies that $U_b - \ell(a,b) \leq U_b + \ell(b,a) < U_a$, and we can reverse the roles of a and b.

Consider any $v \in Q_a^\circ \cap \mathcal{V}^\circ$. By Theorem 5, the characteristic type t^v is in R_a. Because $v \in Q_a^\circ \cap \mathcal{V}^\circ$,

$$v_a - v_b > \ell(b,a) \tag{14}$$

and $v_a < U_a$. The latter inequality and (13) imply that

$$v_a - \ell(b,a) < U_b. \tag{15}$$

Define the valuation type $\tilde{v}$ by setting

$$\tilde{v}_a = v_a + \varepsilon,\ \tilde{v}_b = v_a - \ell(b,a) + 2\varepsilon,\ \text{and}\ \tilde{v}_c = v_c,\ \forall c \in A\backslash\{a,b\}, \tag{16}$$

where $\varepsilon > 0$ is chosen to be sufficiently close to 0 so that both $\tilde{v}$ is in $\mathcal{V}$ and $\delta > \varepsilon$. Because $v_a < U_a$, (15) ensures that such a $\tilde{v}$ exists. Note that (14) and (16) imply that $\tilde{v}_b > v_b$.

It follows from (16) that

$$\tilde{v}_a - \tilde{v}_b = \ell(b,a) - \varepsilon < \ell(b,a).$$

Hence, by Theorem 5, the characteristic type $t^{\tilde{v}}$ cannot be in R_a. From (12), (16), and the assumption that $\delta > \varepsilon$, we have

$$\tilde{v}_b - \tilde{v}_a = \varepsilon - \ell(b,a) \leq \varepsilon + \ell(a,b) - \delta < \ell(a,b),$$

and so by Theorem 5, $t^{\tilde{v}}$ cannot be in R_b. By construction, $\tilde{v} \geq v$, $\tilde{v}_a > v_a$, and $\tilde{v}_c = v_c$ for all $c \in A\backslash\{a,b\}$. Because $t^v \in R_a$, Theorem 6 implies that $t^{\tilde{v}}$ is not in R_c for any $c \in A\backslash\{a,b\}$. We have shown that g does not assign any outcome to $t^{\tilde{v}}$, which is impossible. Therefore, $\ell(a,b)+\ell(b,a)=0$.

If $U_b = \infty$, we do not need to assume (13) in order for (15) to hold, which is all that is needed in order for $\tilde{v}$ to be in $\mathcal{V}$.[12] If U_b is finite, but $U_a = \infty$, an analogous argument with the roles of a and b reversed ensures that the requisite $\tilde{v}$ exists. □

[12] If $U_b = \infty$, (13) is satisfied if U_a is finite.

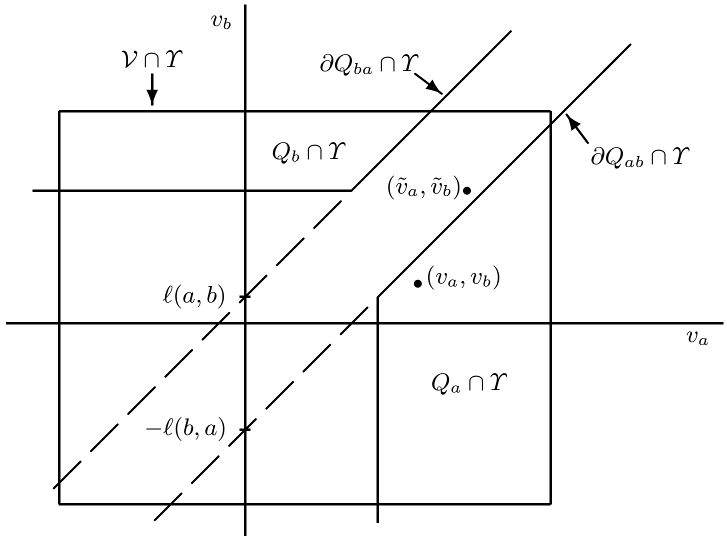

Fig. 3. Illustration of the Proof of Lemma 1.

The proof of Lemma 1 is illustrated in Figure 3 for the case in which $\mathcal{V}$ is compact. In this diagram,

$$\Upsilon = \{\hat{v} \in \mathbb{R}^m \mid \hat{v}_c = v_c, \ \forall c \in A \backslash \{a, b\}\},$$

for some fixed values of v_c for $c \in A \backslash \{a, b\}$ that will be specified later. The sets $\partial Q_{ab} \cap \Upsilon$, $\partial Q_{ba} \cap \Upsilon$, and $\mathcal{V} \cap \Upsilon$ are all independent of the valuations chosen for the outcomes other than a and b. The upward sloping parts of $\partial Q_a \cap \Upsilon$ and $\partial Q_b \cap \Upsilon$ are contained in $\partial Q_{ab} \cap \Upsilon$ and $\partial Q_{ba} \cap \Upsilon$, respectively. By way of contradiction, we suppose that $\ell(a, b) + \ell(b, a) > 0$, which, by the 2-cycle nonnegativity condition, implies that $\partial Q_{ba} \cap \Upsilon$ lies above $\partial Q_{ab} \cap \Upsilon$ when v_a is plotted on the horizontal axis and v_b is plotted on the vertical axis. Our interiority assumption ensures that (i) $\partial Q_{ab} \cap \Upsilon$ intersects the right-hand boundary of $\mathcal{V} \cap \Upsilon$ or (ii) $\partial Q_{ba} \cap \Upsilon$ intersects the upper boundary of $\mathcal{V} \cap \Upsilon$. Without loss of generality, we consider case (i).

We choose v so that it is in $Q_a^\circ \cap \mathcal{V}^\circ$, which is possible by our interiority assumption. The valuation vector $\tilde{v}$ differs from v only in the valuations of outcomes a and b. It is chosen so that $\tilde{v}_a > v_a$ and $\tilde{v}_b > v_b$, and so that $(\tilde{v}_a, \tilde{v}_b)$ is not in either $Q_a \cap \Upsilon$ or $Q_b \cap \Upsilon$, as shown in the diagram. Because $v \in Q_a^\circ$, by Theorem 5, the characteristic type t^v associated with v is allocated a. Because $\tilde{v}$ is in neither Q_a nor Q_b, the same theorem implies that the characteristic type $t^{\tilde{v}}$ associated with $\tilde{v}$ cannot be allocated either a or b. In moving from v to $\tilde{v}$, the valuations of a and b have increased with no change in the valuations of the other outcomes. Hence, by Theorem 6, no outcome other than a or b can be allocated to $t^{\tilde{v}}$. We now have no outcome allocated to $t^{\tilde{v}}$, which is impossible, and so we conclude that $\ell(a,b)+\ell(b,a)=0$. Note that $-\ell(b,a)$ is the vertical intercept of $\partial Q_{ab}\cap \Upsilon$ and $\ell(a,b)$ is the vertical intercept of $\partial Q_{ba}\cap\Upsilon$. When these two values concide, then so do $\partial Q_{ab}\cap\Upsilon$ and $\partial Q_{ba}\cap\Upsilon$.

If there are only two outcomes, say a_1 and a_2, only the 2-cycle nonnegativity condition and convexity of $\mathcal{V}$ are needed to conclude that $\ell(a_1,a_2)+\ell(a_2,a_1)=0$. This can be seen using Figure 1. Because $\mathcal{V}$ must intersect both Q_{a_1} and Q_{a_2}, if, as shown in this diagram, $\ell(a_1,a_2)+\ell(a_2,a_1)>0$, then $\mathcal{V}$ must contain valuation vectors that are in neither of the two difference sets when $\mathcal{V}$ is convex. But then some types are not assigned any outcome, which is impossible.

If $|A|=2$, the proof of Theorem 7 is complete. For $|A|\geq 3$, we now show that if the length of every 2-cycle is zero and the length of every 3-cycle is nonnegative, then all k-cycles have zero length.

Lemma 2. *If all 2-cycles in the allocation graph Γ_g have zero length and all 3-cycles in Γ_g have nonnegative length, then for every integer $k\geq 2$, all k-cycles in Γ_g have zero length.*

Proof. By assumption, any 2-cycle has zero length.

Consider any 3-cycle $(a_1,a_2),(a_2,a_3),(a_3,a_1)$. Because all 3-cycles have nonnegative length,

$$\ell(a_1,a_2)+\ell(a_2,a_3)+\ell(a_3,a_1)\geq 0.$$

Because all 2-cycles have zero length, this inequality is equivalent to

$$-\ell(a_2,a_1)-\ell(a_3,a_2)-\ell(a_1,a_3)\geq 0,$$

or, equivalently,

$$\ell(a_1,a_3)+\ell(a_3,a_2)+\ell(a_2,a_1)\leq 0.$$

Because all 3-cycles have nonnegative length, the preceding inequality implies that the 3-cycle $(a_1,a_3),(a_3,a_2),(a_2,a_1)$ has zero length, which implies that the original 3-cycle $(a_1,a_2),(a_2,a_3),(a_3,a_1)$ also has zero length.

Induction is used to complete the proof. Consider any integer $k\geq 4$ and suppose that any $(k-1)$-cycle has zero length. Consider any k-cycle $(a_1,a_2),\ldots,(a_{k-1},a_k),(a_k,a_1)$. We now insert the 2-cycle $(a_{k-1},a_1),(a_1,a_{k-1})$ before the arc (a_{k-1},a_k). This construction is illustrated in Figure 4 for the case in which $k=4$. The inserted 2-cycle has length zero. Thus,

$$\begin{aligned}\ell(a_1,a_2)+\cdots+\ell(a_{k-1},a_k)+\ell(a_k,a_1)\\ &= [\ell(a_1,a_2)+\cdots+\ell(a_{k-2},a_{k-1})+\ell(a_{k-1},a_1)]\\ &\quad +[\ell(a_1,a_{k-1})+\ell(a_{k-1},a_k)+\ell(a_k,a_1)]\,.\end{aligned}$$

The first (resp. second) term in square brackets on the right-hand side of this equation is the length of a $(k-1)$-cycle (resp. 3-cycle). Both of these lengths are zero. Hence, the length of the original k-cycle is also zero. □

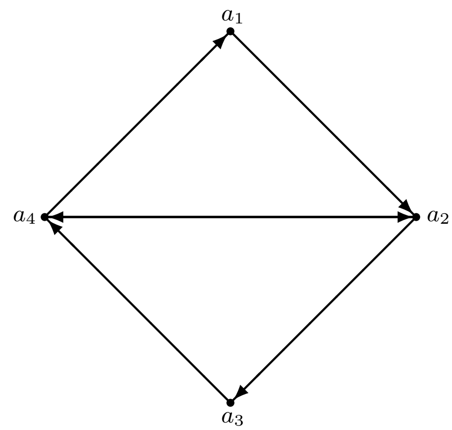

Fig. 4. Inserting a 2-cycle into a 4-cycle.

To complete the proof of Theorem 7, it remains to show that the length of any 3-cycle is nonnegative.[13]

Lemma 3. *Under the assumptions of Theorem 7, all 3-cycles in the allocation graph Γ_g have nonnegative length.*

Proof. Consider any distinct $a, b, c \in A$. We first consider the case in which $L_d \neq -\infty$ for all $d \in A \backslash \{c\}$. Let

$$v_c^* = \max_{d \in A \backslash \{c\}} [L_d + \ell(d, c)] \tag{17}$$

and consider any

$$\bar{d} \in \arg \max_{d \in A \backslash \{c\}} [L_d + \ell(d, c)] . \tag{18}$$

Because the 2-cycle nonnegativity condition is satisfied, $\ell(d, c)$ is finite for all $d \in A \backslash \{c\}$. Thus, our assumptions ensure that such a $\bar{d}$ exists and that v_c^* is finite. Note that for any $\bar{v}_c$ with $\bar{v}_c > v_c^*$, there exist $\bar{v}_d$ for all $d \in A \backslash \{c\}$ arbitrarily close to L_d such that $\bar{v} \in Q_c^\circ$ and, hence by Theorem 5, that $t^{\bar{v}} \in R_c$ if $\bar{v} \in \mathcal{V}$.[14] We must have $v_c^* < U_c$, otherwise $Q_c^\circ \cap \mathcal{V}^\circ = \varnothing$. Furthermore, $L_c < v_c^*$, otherwise $t^v \in R_c$ for all v with $v_c > v_c^*$, which implies that $Q_d^\circ \cap \mathcal{V}^\circ = \varnothing$ for all $d \in A \backslash \{c\}$. Thus, $L_c < v_c^* < U_c$.

We now show that

$$U_a - v_c^* > \ell(c, a) \text{ and } U_b - v_c^* > \ell(c, b). \tag{19}$$

[13] For the special case in which $\mathcal{V}$ is all of $\mathbb{R}^m$, Lavi, Mu'alem, and Nisan (2009, Claim 9) show that a necessary condition for dominant strategy implementability is that all 3-cycles have zero length. Our lemma is concerned with sufficient conditions for dominant strategy implementabilty.

[14] If $L_d \in \langle L_d, U_d \rangle$, $\bar{v}_d$ can be chosen to be L_d.

If the first inequality in (19) is violated, we have $U_a - v_c^* \leq \ell(c,a)$. Because $Q_a^\circ \cap \mathcal{V}^\circ \neq \varnothing$, we must have

$$U_a - L_d > \ell(d,a), \quad \forall d \in A\backslash\{a\}. \tag{20}$$

Thus, by choosing $\varepsilon > 0$ sufficiently small, there exists a valuation type $\tilde{v} \in \mathcal{V}$ defined by setting $\tilde{v}_a = U_a - \varepsilon$, $\tilde{v}_c = v_c^* - \varepsilon/2$, and $\tilde{v}_d = L_d + \varepsilon$ for all $d \in A\backslash\{a,c\}$ such that $U_a - L_d > \ell(d,a) + 2\varepsilon$ for all $d \in A\backslash\{a,c\}$. However, we cannot assign any outcome in A to valuation type $\tilde{v}$ because

$$\tilde{v}_c - \tilde{v}_{\bar{d}} = v_c^* - \varepsilon/2 - [L_{\bar{d}} + \varepsilon] < \ell(\bar{d},c), \tag{21}$$

$$\tilde{v}_a - \tilde{v}_c = U_a - \varepsilon - [v_c^* - \varepsilon/2] = U_a - v_c^* - \varepsilon/2 < U_a - v_c^* \leq \ell(c,a), \tag{22}$$

and

$$\begin{aligned}\tilde{v}_d - \tilde{v}_a = L_d + \varepsilon - [U_a - \varepsilon] &= L_d - U_a + 2\varepsilon \\ &< -\ell(d,a) = \ell(a,d), \quad \forall d \in A\backslash\{a,c\},\end{aligned} \tag{23}$$

where the last inequality in (21) follows from (17), the last inequality in (22) holds by supposition, the inequality in (23) follows from (20), and the last equality in (23) follows because all 2-cycles have zero length. However, $\tilde{v}$ must be assigned an outcome in A, so this contradiction shows that $U_a - v_c^* > \ell(c,a)$. Similarly, we must have $U_b - v_c^* > \ell(c,b)$.

Contrary to what we want to show, now suppose that $\ell(a,b) + \ell(b,c) + \ell(c,a) < 0$. Let $\hat{v}$ be defined by setting

$$\hat{v}_a = v_c^* - \ell(a,c) + 2\delta, \tag{24}$$

$$\hat{v}_b = v_c^* - \ell(b,c) + \xi, \tag{25}$$

$$\hat{v}_c = v_c^* + \delta, \tag{26}$$

and

$$\hat{v}_d = L_d + \delta/2, \quad \forall d \in A\backslash\{a,b,c\}. \tag{27}$$

Because $L_c < v_c^* < U_c$, for $\delta > 0$ sufficiently small, (26) and (27) imply that $L_c < \hat{v}_c < U_c$ and $L_d < \hat{v}_d < U_d$ for all $d \in A\backslash\{a,b,c\}$. Using (17), for $\delta > 0$ and $\xi > 0$ sufficiently small, it follows from (24) and (25) that $L_a < \hat{v}_a$ and $L_b < \hat{v}_b$. Because all 2-cycles have zero length, (24) and (25) also imply that $\hat{v}_a = v_c^* + \ell(c,a) + 2\delta$ and $\hat{v}_b = v_c^* + \ell(c,b) + \xi$. For $\delta > 0$ and $\xi > 0$ sufficiently small, it then follows from (19) that $\hat{v}_a < U_a$ and $\hat{v}_b < U_b$. Hence, by choosing $\delta > 0$ and $\xi > 0$ sufficiently small with $\delta > \xi$, it follows that $\hat{v} \in \mathcal{V}$ and

$$\ell(a,b) + \ell(b,c) + \ell(c,a) + 2\delta - \xi < 0. \tag{28}$$

We have

$$\hat{v}_a - \hat{v}_b = \ell(b,c) - \ell(a,c) + 2\delta - \xi = \ell(b,c) + \ell(c,a) + 2\delta - \xi < -\ell(a,b) = \ell(b,a),$$

where the first equality follows from (24) and (25), the other two equalities follow because 2-cycles have zero length, and the inequality follows from (28). Thus, a cannot be chosen when $v = \hat{v}$.

Because 2-cycles have zero length and $\delta > \xi$, it follows from (25) and (26) that

$$\hat{v}_b - \hat{v}_c = -\ell(b,c) - \delta + \xi < -\ell(b,c) = \ell(c,b).$$

Thus, b cannot be chosen when $v = \hat{v}$.

By (24) and (26),

$$\hat{v}_c - \hat{v}_a = \ell(a,c) - \delta < \ell(a,c).$$

Thus, c cannot be chosen when $v = \hat{v}$.

Finally, because 2-cycles have zero length, (17), (26), and (27) imply that

$$\hat{v}_d - \hat{v}_c = L_d - v_c^* - \delta/2 < L_d - v_c^* \leq -\ell(d,c) = \ell(c,d), \quad \forall d \in A \backslash \{a,b,c\}.$$

Thus, no $d \in A\backslash\{a,b,c\}$ can be chosen when $v = \hat{v}$.

We have shown that no outcome in A can be chosen when $v = \hat{v}$, which is impossible. Thus, our supposition that $\ell(a,b) + \ell(b,c) + \ell(c,a) < 0$ is false. Hence, $\ell(a,b)+\ell(b,c)+\ell(c,a) \geq 0$, which completes the proof when $L_d \neq -\infty$ for all $d \in A\backslash\{c\}$.

If some, but not all, $d \in A\backslash\{c\}$ have $L_d = -\infty$, the argument proceeds as above with a finite value $\bar{L}_d$ used instead of L_d for all $d \in A\backslash\{c\}$ for which $L_d = -\infty$, where $\bar{L}_d$ is chosen to be sufficiently small so that $\bar{d}$ still solves (18) and (20) still holds. If $L_d = -\infty$ for all $d \in A\backslash\{c\}$, we then replace L_d with a finite value $\bar{L}_d$ for all $d \in A\backslash\{c\}$, with $\bar{L}_d$ chosen so that (20) still holds and $v_c^* < U_c$. The proof then proceeds as above. □

The conclusion that the lengths of 3-cycles are zero follows from geometric properties of the difference sets that can be most easily seen when there are only three outcomes, say a_1, a_2, and a_3, and the valuation type space $\mathcal{V}$ is compact. Fix v_3 at its lowest value L_3 and let

$$\Upsilon = \{v \in \mathbb{R}^3 \mid v_3 = L_3\}.$$

Recall that if we increase v_3, the left-hand boundary of Q_{a_1} moves to the right and the lower boundary of Q_{a_2} moves up. Thus, our interiority assumption implies that if we restrict attention to valuation vectors in Υ, then the vertical part of the boundary of $Q_{a_1} \cap \Upsilon$ must lie to the left of the right-hand boundary of $\mathcal{V} \cap \Upsilon$ and the horizontal part of the boundary of $Q_{a_2} \cap \Upsilon$ must lie below the upper boundary of $\mathcal{V} \cap \Upsilon$, as illustrated in Figure 5. Because all 2-cycles have zero length, in Υ, the upward sloping parts of the boundaries of Q_{a_1} and Q_{a_2} lie on a common line with slope equal to 1, the vertical parts of the boundaries of Q_{a_1} and Q_{a_3} lie on a common vertical line, and the horizontal parts of the boundaries of Q_{a_2} and Q_{a_3} lie on a common horizontal line.

Suppose that the kinks on the boundaries of $Q_{a_1} \cap \Upsilon$ and $Q_{a_2} \cap \Upsilon$ do not coincide. Without loss of generality, we can suppose that the kink point for

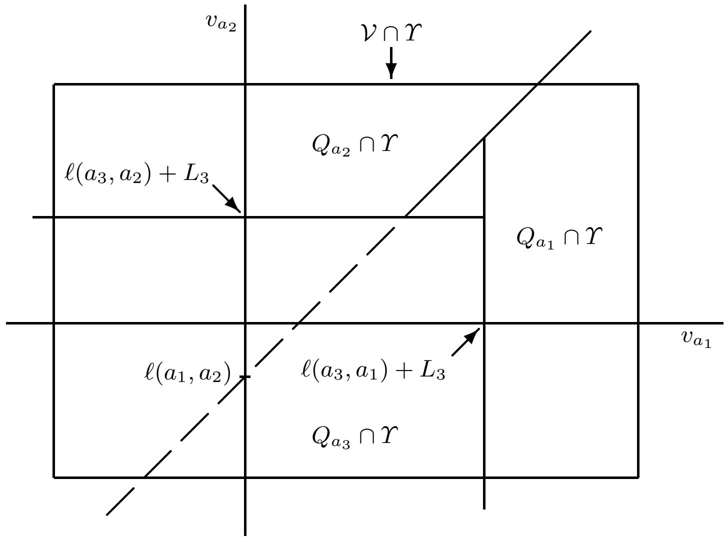

Fig. 5. 3-cycles with nonzero length.

$Q_{a_1} \cap \Upsilon$ lies up and to the right of the kink point for $Q_{a_2} \cap \Upsilon$. As can be seen from the diagram, there is a triangular region in Υ whose interior is not in any of the three difference sets. Furthermore, either (i) the interior of this triangular region has a nonempty intersection with $\mathcal{V}$ or (ii) the interior of this triangular region lies below the lower horizontal boundary of $\mathcal{V} \cap \Upsilon$. In the latter case, we increase the value of v_3 until the interior of this triangular region intersects with the new cross section of $\mathcal{V}$ obtained by increasing v_3 and define Υ using this new value of v_3. Because $Q^\circ_{a_3} \cap \mathcal{V}^\circ \neq \varnothing$, such a value of v_3 must exist. In Figure 5, all of the triangular region lies in $\mathcal{V} \cap \Upsilon$, but this need not be the case. Moreover, it is possible for $Q_{a_3} \cap \mathcal{V} \cap \Upsilon$ to be empty. By Theorem 5, it now follows that the characteristic types that correspond to valuation types in the intersection of the interior of the triangular region and $\mathcal{V}$ are not allocated any outcome, which is impossible. Thus, the kink points on the boundaries of $Q_{a_1} \cap \Upsilon$ and $Q_{a_2} \cap \Upsilon$ coincide, which implies that this common point is also the kink point on the boundary of $Q_{a_3} \cap \Upsilon$.

We now have a situation like that depicted in Figure 6.[15] Because the common boundary of $Q_{a_1} \cap \Upsilon$ and $Q_{a_2} \cap \Upsilon$ has a slope equal to 1,

$$\ell(a_3, a_2) + L_3 = \ell(a_1, a_2) + \ell(a_3, a_1) + L_3. \tag{29}$$

Because all 2-cycles have zero length, ℓ is antisymmetric. Hence, (29) is equivalent to

[15] If case (ii) in the preceding paragraph applies, then L_3 is replaced with the value of v_3 used to ensure that the interior of the triangular region contains valuation vectors in $\mathcal{V}$.

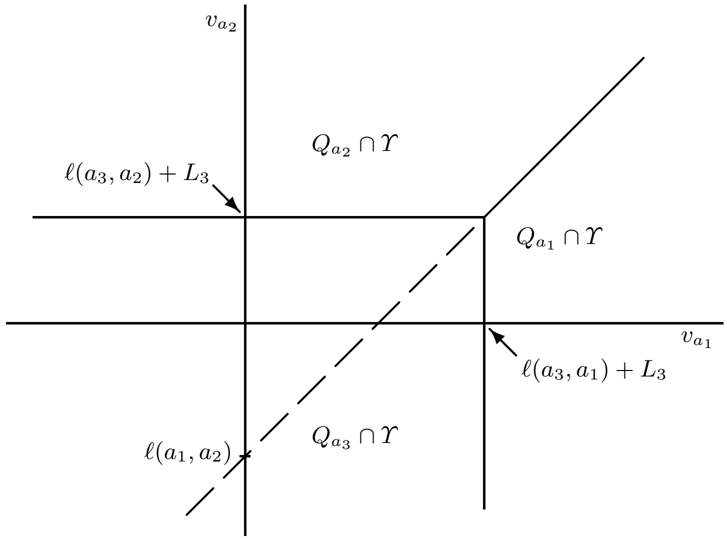

Fig. 6. 3-cycles with zero length.

$$\ell(a_1, a_2) + \ell(a_2, a_3) + \ell(a_3, a_1) = 0.$$

That is, the 3-cycle $(a_1, a_2), (a_2, a_3), (a_3, a_1)$ has zero length. The antisymmetry of ℓ then implies that the only other 3-cycle, $(a_1, a_3), (a_3, a_2), (a_2, a_1)$, also has zero length. Note that for the situation shown in Figure 5 (which we have shown to be inconsistent with our assumptions), the right-hand side of (29) is larger than the left-hand side, from which it follows that the length of the 3-cycle $(a_1, a_2), (a_2, a_3), (a_3, a_1)$ is positive and the length of the 3-cycle $(a_1, a_3), (a_3, a_2), (a_2, a_1)$ is negative.

If $\mathcal{V}$ is a full-dimensional convex product space and our interiority assumption is satisfied, it follows from Theorems 2 and 7 that a necessary condition for dominant strategy implementation of the allocation function g is that all 2-cycles in the allocation graph Γ_g have zero length. By Theorem 4, if all of these 2-cycles have zero length, then g is dominant strategy implementable. Thus, given our structural assumptions, we have identified a new necessary and sufficient condition for dominant strategy implementability of g: all 2-cycles in the allocation graph Γ_g have zero length. We combine this observation with Theorems 2, 3, and 4 in the following equivalence theorem.[16]

Theorem 8. *If (a) $|A| = 1$ or (b) $|A| \geq 2$, the valuation type space $\mathcal{V}$ is a full-dimensional convex product space, and $Q_a^\circ \cap \mathcal{V}^\circ \neq \varnothing$ for all $a \in A$, then the following conditions for the allocation function $g\colon T \to A$ are equivalent:*

(i) *g is dominant strategy implementable;*

[16] Theorem 8 is trivially true if $|A| = 1$.

(ii) *for every integer $k \geq 2$, all k-cycles in the allocation graph Γ_g have nonnegative length;*
(iii) *for every integer $k \geq 2$, all k-cycles in the characteristic graph T_g have nonnegative length;*
(iv) *all 2-cycles in the allocation graph Γ_g have nonnegative length;*
(v) *all 2-cycles in the characteristic graph T_g have nonnegative length;*
(vi) *all 2-cycles in the allocation graph Γ_g have zero length.*

A noteworthy feature of Theorem 8 is that all 2-cycles in the characteristic graph T_g having zero length is not equivalent to the six conditions listed in this theorem. In the characteristic graph, it is possible to have $d(s,t) + d(t,s) > 0$ for some $s,t \in T$ even though $\ell(g(s), g(t)) + \ell(g(t), g(s)) = 0$ because, for example, $d(s,t) = v(g(t)|t) - v(g(s)|t) > 0$, but there is some other type $r \in T$ for which $g(r) = g(t)$ and $d(s,r) = v(g(r)|r) - v(g(s)|r) < d(s,t)$. For the equivalence between conditions (i) and (vi) to hold, it is essential that the length of the directed arc from a to b in the allocation graph Γ_g is found by taking the infimum of the gain in individual i's valuation in going from a to b over *all* characteristic types that result in b being chosen.

When our structural assumptions on the valuation type spaces are satisfied, we can use Theorems 2 and 8 to provide a new necessary and sufficient condition for an allocation function $G\colon T^i \times T^{-i} \to \Omega$ to be dominant strategy implementable, a condition we call the *zero-length 2-cycle condition*.

Definition. An allocation function G satisfies the *zero-length 2-cycle condition* if for every $i \in N$ and $t^{-i} \in T^{-i}$, every 2-cycle in the allocation graph $\Gamma_G(t^{-i})$ has zero length.

To state our result formally, for every $i \in N$, $t^{-i} \in T^{-i}$, and $a \in A(t^{-i})$, let $\mathcal{V}(t^{-i})$ and $Q_a(t^{-i})$ be i's valuation space and the difference set for a conditional on the characteristic types t^{-i} of the other individuals.

Theorem 9. *Suppose that for all $i \in N$ and all $t^{-i} \in T^{-i}$ for which $|A(t^{-i})| \geq 2$, the valuation type space $\mathcal{V}(t^{-i})$ is a full-dimensional convex product space and $Q_a(t^{-i})^\circ \cap \mathcal{V}(t^{-i})^\circ \neq \varnothing$ for all $a \in A(t^{-i})$. Then, the allocation function $G\colon T^i \times T^{-i} \to \Omega$ is dominant strategy implementable if and only if the zero-length 2-cycle condition is satisfied.*

6 Affine Maximizers

If we consider all of the outcomes in Ω, individual i's *valuation type space* is

$$\mathcal{V}^i = \{(v^i(a_1|t^i), \ldots, v^i(a_M|t^i)) \in \mathbb{R}^M \mid t^i \in T^i\}.$$

where $M = |\Omega|$. $\mathcal{V}^i$ is *unrestricted* if $\mathcal{V}^i = \mathbb{R}^M$.

The allocation function $G\colon T^i \times T^{-i} \to \Omega$ is *nonimposed* if $G(T^i \times T^{-i}) = \Omega$ and it is an *affine maximizer* if there exist n nonnegative numbers

$w_1, \ldots, w_n$, not all of them equal to zero, and M numbers K_a, $a \in \Omega$, such that

$$G(t^i, t^{-i}) \in \arg\max_{a \in \Omega} \left[\sum_{j=1}^{n} w_j v^j(a|t^j) + K_a \right], \quad \forall (t^i, t^{-i}) \in T^i \times T^{-i}. \quad (30)$$

The affine maximizer G in (30) is *unresponsive to irrelevant agents* if for all $i \in N$ for which $w_i = 0$, $G(s^i, t^{-i}) = G(t^i, t^{-i})$ for all $s^i, t^i \in T^i$ and all $t^{-i} \in T^{-i}$. That is, any individual with zero weight has no influence on the outcome, even as a tie breaker.

Mishra and Sen (2011) show that if G is an affine maximizer that is unresponsive to irrelevant agents, then it is dominant strategy implementable. *Roberts' Theorem* (Roberts, 1979, Theorem 3.1) shows that being an affine maximizer is necessary for G to be dominant strategy implementable if there are at least three outcomes, each individual's valuation type space is unrestricted, and G is nonimposed. These results are summarized in Theorem 10.[17]

Theorem 10. (a) *If an allocation function $G\colon T^i \times T^{-i} \to \Omega$ is an affine maximizer that is unresponsive to irrelevant agents, then G is dominant strategy implementable.*

(b) *Suppose that there are at least three outcomes in Ω, $\mathcal{V}^i$ is unrestricted for all $i \in N$, and $G\colon T^i \times T^{-i} \to \Omega$ is a nonimposed allocation function. If G is dominant strategy implementable, then G is an affine maximizer.*

If all of the valuation type spaces $\mathcal{V}^i$ are unrestricted, then the assumptions of Theorem 9 are satisfied. Hence, it follows that if the assumptions of part (b) of Theorem 10 are satisfied, then the allocation function G is an affine maximizer if and only if the zero-length 2-cycle condition is satisfied.[18]

[17] Carbajal, McLennan, and Tourky (2011) provide an alternative sufficient condition for an affine maximizer to be dominant strategy implementable to the one used in Theorem 10.(a). They also provide an example showing that an affine maximizer need not be dominant strategy implementable, which is contrary to what is claimed by Roberts (1979, Theorem 3.3). Their example is not unresponsive to irrelevant agents. Extensions of Theorem 10 may be found in Carbajal, McLennan, and Tourky (2011) and Mishra and Sen (2011).

[18] For all $a, b \in \Omega$, all $i \in N$, all $t^i \in T^i$, and all $(t^i, t^{-i}) \in T^i \times T^{-i}$, let $\Delta^i(a, b|t^i) = v^i(a|t^i) - v^i(b|t^i)$ and let $P(a, b)$ be the set of vectors of valuation differences $(\Delta^1(a, b|t^1), \ldots, \Delta^n(a, b|t^n))$ corresponding to the characteristic types (t^i, t^{-i}) for which a is chosen by G. In their proofs of part (b) of Theorem 10, Roberts (1979) and Lavi, Mu'alem, and Nisan (2009) investigate the geometric structure of the $P(a, b)$ sets. Claim 2 in the latter article shows that $P(a, b) \cup -P(b, a) = \mathbb{R}^n$ and that $P(a, b)$ and $-P(b, a)$ have disjoint interiors. Two implications of these observations are: (1) for fixed $t^{-i} \in T^{-i}$, the infimum of $\Delta^i(a, b|t^i)$ in the set of valuation differences $(\Delta^1(a, b|t^1), \ldots, \Delta^n(a, b|t^n))$ that are in $P(a, b)$ is $\ell(a, b|t^{-i})$ and (2) $\ell(a, b|t^{-i}) = -\ell(b, a|t^{-i})$. Thus, with the assumptions of part (b) of Theorem 10, the zero-length 2-cycle condition is necessary for G to be dominant

A dictatorial allocation function is an affine maximizer in which only one weight in (30) is non-zero. Consequently, it is also dominant strategy implementable if it is unresponsive to irrelevant agents. In Example 1, we confirm that such a dictatorial allocation function satisfies our zero-length 2-cycle condition when the assumptions of Theorem 9 are satisfied.

Example 1. A dictator chooses one of his best alternatives from Ω, where $|\Omega| = m \geq 2$. For simplicity, suppose that the cost of each alternative is zero (e.g., the alternatives are candidates for an election). For concreteness, let person 1 be the dictator. To ensure that the allocation function is unresponsive to irrelevant agents, we assume that ties are broken according to a fixed strict ranking of Ω. For all $i \in N$, let $T^i = \mathbb{R}^m$ and assume that $v^i(a|t^i) = t^i_a$ for all $a \in A$ and all $t^i \in T^i$. The allocation function $G\colon T^1 \times T^{-1} \to \Omega$ chooses the alternative a from Ω that maximizes t^1_a, with ties broken as described above. A payment function that implements G is the one in which payments are always set equal to zero.

For any individual $i \neq 1$, for all $t^{-1} \in T^{-1}$, $A(t^{-i})$ only contains a single alternative, the one chosen by the dictator. Therefore, there are no 2-cycles in any of i's allocation graphs. For individual 1, $A(t^{-1}) = \Omega$ for all $t^{-1} \in T^{-1}$. For all $t^{-1} \in T^{-1}$, $\mathcal{V}^1(t^{-1})$ is unrestricted and $Q_a(t^{-1}) = \{v^1 \in \mathbb{R}^m \mid v^1_a \geq v^1_b, \forall b \neq a\}$ for all $a \in A(t^{-1})$. The interiors of $\mathcal{V}^1(t^{-1})$ and $Q_a(t^{-1})$ have a nonempty intersection. Therefore, the assumptions of Theorem 9 are satisfied. Because a is chosen whenever $t^1_a - t^1_b > 0$ for all alternatives b different from a no matter how small the values of $t^1_a - t^1_b$ are, for all $t^{-1} \in T^{-1}$, $\ell(a, b|t^{-1}) = 0$ for all distinct $a, b \in \Omega$ and, hence, all 2-cycles in the allocation graph $\Gamma_G(t^{-1})$ have zero length.

The allocation function for a Vickrey (1961) auction is an affine maximizer in which the weights in (30) are all equal. As a consequence, it is dominant strategy implementable. In Examples 2–4, we consider three different Vickrey auctions. In Example 2, one unit of a good that creates a negative externality is auctioned. In this example, the type spaces satisfy the assumptions of Theorem 9, and we confirm that the zero-length 2-cycle condition is satisfied. While sufficient, the assumptions of Theorem 9 are not necessary for the zero-length 2-cycle condition to hold. This observation is illustrated in Example 3, which considers a standard Vickrey auction of a single unit of an indivisible good. In this example, while the valuation types spaces are convex product spaces, these sets are not full dimensional. Example 4 shows that the zero-length 2-cycle condition is not necessary for dominant strategy implementability if the assumption in Theorem 9 that all of the valuation type spaces are full-dimensional convex product spaces is replaced with the weaker assumption that they are convex. In this example, two units of a good are auctioned to individuals who place a higher value on the first unit received.

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strategy implementable. One of our main contributions has been to show that this condition is also sufficient.

Example 2. There is one unit of an indivisible good to be allocated to one of two individuals. Possession of the good creates a negative externality for the other individual. Let a (resp. b) be the outcome in which individual 1 (resp. 2) gets the good. Thus, $N = \{1, 2\}$ and $\Omega = \{a, b\}$. Let $T^1 = \mathbb{R}_+ \times \mathbb{R}_-$ and $T^2 = \mathbb{R}_- \times \mathbb{R}_+$. For $i = 1, 2$, individual i's characteristic type is $T^i = (t^i_a, t^i_b)$. Define individual 1's valuation function $v^1 \colon \Omega \times T^1 \to \mathbb{R}$ by setting $v^1(a|t^1) = t^1_a$ and $v^1(b|t^1) = t^1_b$ for all $t^1 \in T^1$. Similarly, individual 2's valuation function $v^2 \colon \Omega \times T^2 \to \mathbb{R}$ is given by $v^2(b|t^2) = t^2_b$ and $v^2(a|t^2) = t^2_a$ for all $t^2 \in T^2$. Thus, t^1_a and t^2_b are the benefits of being allocated the good, whereas $|t^1_b|$ and $|t^2_a|$ are the externality costs incurred if the good is allocated to the other individual.

When there are negative externalities, a Vickrey auction mechanism is defined as follows.[19] The allocation function $G \colon T^1 \times T^2 \to \Omega$ is

$$G(t^1, t^2) = \begin{cases} a & \text{if } t^1_a - t^1_b \geq t^2_b - t^2_a \\ b & \text{if } t^1_a - t^1_b < t^2_b - t^2_a \end{cases}$$

and payment function $P \colon T^1 \times T^2 \to \mathbb{R}^2$ is

$$P(t^1, t^2) = \begin{cases} (t^2_b - t^2_a, 0) & \text{if } t^1_a - t^1_b \geq t^2_b - t^2_a \\ (0, t^1_a - t^1_b) & \text{if } t^1_a - t^1_b < t^2_b - t^2_a. \end{cases}$$

Each person has an adjusted value for the good given by $t^1_a - t^1_b$ for person 1 and $t^2_b - t^2_a$ for person 2. Individuals bid their adjusted values and the good is awarded to the highest bidder (with a tie broken in favour of individual 1) with the winner paying the second-highest bid (in this case, the other person's bid) and the loser paying nothing. Note that G chooses the outcome that maximizes the sum of the valuations.

For any $t^2 \in T^2$, $A(t^2) = \{a, b\}$. We have

$$\ell(a, b|t^2) = \inf_{t^1 \in R_b(t^2)} [v(b|t^1) - v(a|t^1)] = \inf_{t^1_b - t^1_a > -[t^2_b - t^2_a]} [t^1_b - t^1_a] = -[t^2_b - t^2_a]$$

because $R_b(t^2) = \{t^1 \in T^1 \mid t^1_a - t^1_b < t^2_b - t^2_a\}$ and we have

$$\ell(b, a|t^2) = \inf_{t^1 \in R_a(t^2)} [v(a|t^1) - v(b|t^1)] = \inf_{t^1_a - t^1_b \geq t^2_b - t^2_a} [t^1_a - t^1_b] = t^2_b - t^2_a$$

because $R_a(t^2) = \{t^1 \in T^1 \mid t^1_a - t^1_b \geq t^2_b - t^2_a\}$. Thus, the only 2-cycle in the allocation graph $\Gamma_G(t^2)$ has zero length. Similarly, for any $t^1 \in T^1$, $A(t^1) = \{a, b\}$ and the only 2-cycle in $\Gamma_G(t^1)$ has zero length.

For all $t^2 \in T^2$, $\mathcal{V}^1(t^2) = \mathbb{R}_+ \times \mathbb{R}_-$, $Q_a(t^2) = \{(v^1_a, v^1_b) \in \mathbb{R}^2 \mid v^1_a - v^1_b \geq t^2_b - t^2_a\}$, and $Q_b(t^2) = \{(v^1_a, v^1_b) \in \mathbb{R}^2 \mid v^1_a - v^1_b \leq t^2_b - t^2_a\}$. $\mathcal{V}^1(t^2)$ is a full-dimensional convex product space. Furthermore, the interiors of $Q_a(t^2)$

[19] See Jehiel, Moldovanu, and Stacchetti (1999) for an analysis of auctions when there are externalities.

and $Q_b(t^2)$ both have a nonempty intersection with the interior of $\mathcal{V}^1(t^2)$. Similar computations show that for all $t^1 \in T^1$, $\mathcal{V}^2(t^1)$ is a full-dimensional convex product space and that the interiors of $Q_a(t^1)$ and $Q_b(t^1)$ both have a nonempty intersection with the interior of $\mathcal{V}^2(t^1)$.

Example 3. We now consider a standard Vickrey auction of a single unit of an indivisible good with two bidders and no externalities. The sets N and Ω are defined as in Example 2. Let $T^1 = T^2 = \mathbb{R}_{++}$ (the set of positive numbers). Define individual 1's valuation function $v^1 \colon \Omega \times T^1 \to \mathbb{R}$ by setting $v^1(a|t^1) = t^1$ and $v^1(b|t^1) = 0$ for all $t^1 \in T^1$. Similarly, individual 2's valuation function $v^2 \colon \Omega \times T^2 \to \mathbb{R}$ is given by $v^2(b|t^2) = t^2$ and $v^2(a|t^2) = 0$ for all $t^2 \in T^2$.

The allocation and payment functions and the arc lengths for the Vickrey auction when there are no externalities are obtained by replacing $t^1_a - t^1_b$ with t^1 and $t^2_b - t^2_a$ with t^2 in Example 2.[20] Because $\ell(a, b|t^2) = -t^2$ and $\ell(b, a|t^2) = t^2$, the only 2-cycle in the allocation graph $\Gamma_G(t^2)$ has zero length. Similarly, for any $t^1 \in T^1$, the only 2-cycle in $\Gamma_G(t^1)$ has zero length.

For all $t^2 \in T^2$, $\mathcal{V}^1(t^2) = \mathbb{R}_{++} \times \{0\}$ and for all $t^1 \in T^1$, $\mathcal{V}^2(t^1) = \{0\} \times \mathbb{R}_{++}$. While these valuation type spaces are convex product spaces, they are not full dimensional. Although the type spaces do not satisfy the assumptions of Theorem 9, nevertheless, the zero-length 2-cycle condition is satisfied.[21]

Example 4. There are two units of an indivisible good to be allocated to two individuals. Let a be the outcome in which individual 1 gets both units, b be the outcome in which each individual gets one unit, and c be the outcome in which individual 2 gets both units. Thus, $N = \{1, 2\}$ and $\Omega = \{a, b, c\}$. For $i \in N$, let $T^i = \{t^i \in \mathbb{R}^2_{++} \mid t^i_1 > t^i_2\}$. Individual 1's valuation function $v^1 \colon \Omega \times T^1 \to \mathbb{R}$ is defined by setting $v^1(a|t^1) = t^1_1 + t^1_2$, $v^1(b|t^1) = t^1_1$, and $v^1(c|t^1) = 0$ for all $t^1 \in T^1$. Similarly, individual 2's valuation function $v^2 \colon \Omega \times T^2 \to \mathbb{R}$ is defined by setting $v^2(a|t^2) = 0$, $v^2(b|t^2) = t^2_1$, and $v^2(c|t^2) = t^2_1 + t^2_2$ for all $t^2 \in T^2$. Thus, each individual places a lower marginal valuation on receiving a second unit of the good than on the first.

When there are two units of a good to be allocated, a multi-unit Vickrey auction is defined as follows.[22] The allocation function $G \colon T^1 \times T^2 \to \Omega$ is

[20] Because the good always has positive value, for every $i \in N$ and every $t^i \in T^i$, $A(t^i) = \{a, b\}$. If we permit individuals to assign a zero value to winning the good, then $A(t^2) = \{a\}$ when $t^2 = 0$ because individual 1 wins the object no matter what his valuation is. To simplify the discussion, we have ruled this possibility out by assuming that the object always has value to either individual.

[21] It is straightforward to generalize Examples 2 and 3 to more than two individuals. When there are n individuals, there are n alternatives in Ω because the good can be given to anyone. However, assuming that ties are broken by a given priority order of the individuals, for any fixed t^{-i}, there are only two outcomes in $A(t^{-i})$, the one in which individual i wins the good and the one in which the winner is the person with the highest (adjusted) valuation in t^{-i}.

[22] See Krishna (2002, Section 12.1) for a detailed discussion of multi-unit Vickrey auctions.

$$G(t^1,t^2)=\begin{cases} a & \text{if } t^1_2 \geq t^2_1 \\ b & \text{if } t^1_1 \geq t^2_2 \text{ and } t^1_2 < t^2_1 \\ c & \text{if } t^1_1 < t^2_2 \end{cases}$$

and the payment function $P\colon T^1 \times T^2 \to \mathbb{R}^2$ is

$$P(t^1,t^2)=\begin{cases} (t^2_1+t^2_2,0) & \text{if } t^1_2 \geq t^2_1 \\ (t^2_2,t^1_2) & \text{if } t^1_1 \geq t^2_2 \text{ and } t^1_2 < t^2_1 \\ (0,t^1_1+t^1_2) & \text{if } t^1_1 < t^2_2. \end{cases}$$

As in Examples 2 and 3, ties are broken in favour of individual 1. In this auction, individual i submits two bids, t^i_1 and t^i_2, with t^i_1 indicating how much he is willing to pay for one unit of the good and t^i_2 indicating how much more he is willing to pay for a second unit. His highest bid must exceed the other individual's lowest bid to win a first unit and his lowest bid must exceed the other individual's highest bid to win a second unit, with ties broken as described above. In effect, each individual faces two prices in this multi-unit Vickrey auction, a price equal to the other individual's lowest bid for the first unit and a price equal to the other individual's highest bid for the second.

For concreteness, consider individual 1 and any $t^2 \in T^2$. First, note that $A(t^2)=\{a,b,c\}$. We have

$$\ell(c,a|t^2)=\inf_{t^1\in R_a(t^2)}\left[v^1(a|t^1)-v^1(c|t^1)\right]=\inf_{t^1\in R_a(t^2)}\left[t^1_1+t^1_2-0\right]=2t^2_1$$

because the lowest that t^1_2 can be and still have individual 1 win both units is t^2_1 and t^1_1 can be chosen to be arbitrarily close to t^1_2. We also have

$$\ell(a,c|t^2)=\inf_{t^1\in R_c(t^2)}\left[v^1(c|t^1)-v^1(a|t^1)\right]=\inf_{t^1\in R_c(t^2)}\left[0-t^1_1-t^1_2\right]=-2t^2_2$$

because individual 2 wins both units whenever t^2_2 exceeds both t^1_1 and t^1_2, which can be arbitrarily close to each other. Thus,

$$\ell(c,a|t^2)+\ell(a,c|t^2)=2(t^2_1-t^2_2)>0.$$

Similar reasoning shows that

$$\ell(b,a|t^2)=t^2_1,\ \ell(a,b|t^2)=-t^2_1,\ \text{and } \ell(b,a|t^2)+\ell(a,b|t^2)=0$$

and

$$\ell(b,c|t^2)=-t^2_2,\ \ell(c,b|t^2)=t^2_2,\ \text{and } \ell(b,c|t^2)+\ell(a,c|t^2)=0$$

Therefore, one 2-cycle in the allocation graph $\Gamma_G(t^2)$ has positive length and the other 2-cycles have zero length.

For all $t^2 \in T^2$, $\mathcal{V}^1(t^2)=\{v^1\in\mathbb{R}^3 \mid 2v^1_b > v^1_a > v^1_b > 0 \text{ and } v^1_c=0\}$. While $\mathcal{V}^1(t^2)$ is convex, it is not a product set, nor is it full dimensional, so Theorem 9 does not apply.

7 Concluding Remarks

For given types of the other individuals, we have shown that by requiring an individual's valuation type space to be a full-dimensional convex product space and by adopting a mild domain regularity condition, the 2-cycle nonnegativity condition is sufficient for all k-cycles in his allocation graph to have zero length, not just to have nonnegative length, as is the case in Saks and Yu (2005). Furthermore, given our assumptions, the zero-length 2-cycle condition is necessary and sufficient for dominant strategy implementability of the allocation function G. As noted in Section 1, it follows from the analysis in Heydenreich, Müller, Uetz, and Vohra (2009) that this condition is also necessary and sufficient for the revenue equivalence property to be satisfied.

The full dimensionality, convexity, and product space structure of valuation type spaces are used in a number of steps in our proof that the 2-cycle nonnegativity condition is sufficient for all k-cycles to have zero length. However, as our discussion of the standard Vickrey auction in Example 3 shows, these assumptions are not necessary for this condition to hold. While it is an open question to what extent our assumptions can be relaxed and still have the 2-cycle nonnegativity condition imply that all k-cycles have zero length, as the multi-unit Vickrey auction in Example 4 demonstrates, this implication need not hold if valuation type spaces are only assumed to be convex, as in Saks and Yu (2005).

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