A limit result on bargaining sets

Javier Hervés-Estévez
Universidad de Vigo. RGEA.
e-mail: javiherves@uvigo.es

Emma Moreno-García
Universidad de Salamanca. IME.
e-mail: emmam@usal.es

Abstract. We introduce a notion of bargaining set for finite economies and show its convergence to the set of Walrasian allocations.

Keywords: Bargaining set, coalitions, core, veto mechanism, justified objections.

JEL Classification: D51, D11, D00.

* We are grateful to Carlos Hervés-Beloso for his helpful comments and suggestions. We are also indebted to Joao Correia-da-Silva and Marta Faias, the participants in the XXIII and XXIV European Workshop on General Equilibrium Theory and in the SAET 2015 Conference. We specially thank the comments and suggestions of an anonymous referee that helped to improve the final version of this work.

** This work is partially supported by Research Grants ECO2016-75712-P (AEI/FEDER, UE), RGEA-ECOBAS (AGRUP2015/08 Xunta de Galicia) and SA072U16 (Junta de Castilla y León).

1 Introduction

The core of an economy is defined as the set of allocations which cannot be blocked by any coalition. Thus, the veto mechanism that defines the core implicitly assumes that individuals are not forward-looking. However, one may ask whether an objection or veto is credible or, on the contrary, not consistent enough so other agents in the economy may react to it and propose an alternative or counter-objection.

The first outcome of this two-step conception of the veto mechanism was the work by Aumann and Maschler (1964), who introduced the concept of bargaining set of a cooperative game.[1] The main idea is to inject a sense of credibility and stability to the veto mechanism, hence permitting the implementation of some allocations which otherwise would be formally blocked, although in a non-credible way. Thus, only objections without counter-objections are considered as credible or justified, and consequently, the core is a subset of the bargaining set since blocking an allocation becomes more difficult.

The original concept of bargaining set was later adapted to atomless economies by Mas-Colell (1989) who, under conditions of generality similar to those required in Aumann's (1964) core-Walras equivalence theorem, showed that the bargaining set and the competitive allocations coincide. These equivalence results give foundations to the Walrasian market equilibrium and, at the same time, bring up the question of whether there are analogies in economies with a large, but finite number of agents. A classical contribution in this direction is the one by Debreu and Scarf (1963), who stated a first formalization of Edgeworth's (1881) conjecture, showing that the core and the set of Walrasian allocations become arbitrarily close whenever a finite economy is replicated a sufficiently large number of times.

In contrast to the Debreu-Scarf core convergence theorem, Anderson, Trockel and Zhou (1997), ATZ from now on, adapted Mas-Colell's bargaining set to finite economies showing that it does not shrink to the set of Walrasian allocations in a sequence of replicated economies as the core does. This fact has been used as an argument against the continuum framework as the proper idealization of a "large" economy (see also Anderson, 1998). Actually, in ATZ's work one reads that *the discrepancy between the behavior of the Mas-Colell bargaining set in the continuum and its behavior in sequences of large finite economies gives reason to be cautious in accepting the continuum as the proper idealization of a "large" economy.*

In this paper, we highlight the significance of the two-step procedure of the veto mechanism applied to sequences of replicated economies when identical consumers are involved in the objection and counter-objection process at the same time. In fact, our contribution consists on providing a natural way to introduce credibility and stability in the veto mechanism,

[1]Maschler (1976) discussed the advantages that the bargaining set has over the core.

as the Mas-Colell's bargaining set, for a sequence of replicated economies. More precisely, the notion we define for an objection leads us to show an asymptotic result that avoids the inconvenient discrepancy between the behavior of the bargaining set in an atomless and in a large but finite economy framework.

According to this end, we consider that each of the n agents of a finite economy behaves as a representative of a large enough number of identical individuals. Following this way of enlarging asymptotically the economy, a typical coalition S^* is formed by a_i members identical to each agent i in a non-empty set $S \subseteq \{1, \ldots, n\}$. Observe that $S^* = (a_i, i \in S)$ is actually a coalition in any r-replica of the original n-agents economy, for every $r \geq \max\{a_i, i \in S\}$. Within this framework, we say that an objection is justified* if it is not counter-objected by any coalition S^*.

The ATZ's non-convergence example relies crucially on the stringent requirements that are inherent to a justified objection for the case of continuum economies with a finite number of types. However, in contrast to Mas-Collel's notion for an atomless n-type economy, in our discrete approach, if a consumer belongs to an objecting coalition, then no *clone* of this individual may be worse off in the counter-objection than her representative agent in the objection.

The adaptation of the bargaining set we propose allows us to strengthen Debreu and Scarf's (1963) limit result, which states that any non-Walrasian allocation is objected in some replicated economy. To be precise, we show that a Walrasian allocation either has no objection in any replicated economy or, if there is an objection, it is counter-objected in some replicated economy. Consequently, we provide a reformulation of the bargaining set and show its convergence to the set of Walrasian allocations when the economy is replicated. Our result is obtained under the assumption that the Walrasian correspondence is continuous on a domain that includes the replica sequence of the original n-agents economy, and we state an example that establishes the impossibility of dropping this continuity hypothesis.

Although we find conditions on the primitives of the original finite economy that ensure the required continuity property holds, the continuity of the equilibrium correspondence imposes a limitation to our convergence result. Nevertheless, this assumption has also been required to show the non-manipulability of the Walrasian mechanism for increasing sequences of finite economies (see Roberts and Postlewaite, 1976). Thus, our result supports the intuition that a continuum economy constitutes a proper approximation to a sequence of large finite economies whenever some continuity property holds.

The rest of the work is structured as follows. In Section 2, we collect notations and preliminaries. In Section 3, we introduce a notion of bargaining set and point out the main differences with the definition that ATZ adapt from Mas-Colell's (1989) one. In Section 4, we analyze convergence properties of our bargaining set. In order to facilitate the reading of the paper, the proofs of the results are contained in a final Appendix.

2 Preliminaries and notations

Let $\mathcal{E}$ be an exchange economy with a finite set of agents $N = \{1, \ldots, n\}$, who trade a finite number ℓ of commodities. Each consumer i has a preference relation $\succsim_i$ on the consumption set $\mathbb{R}^\ell_+$, with the properties of continuity, convexity[2] and strict monotonicity[3]. Then, preferences are represented by utility functions $U_i, i \in N$. Let $\omega_i \in \mathbb{R}^\ell_{++}$ be the endowments[4] of consumer i. So the economy is $\mathcal{E} = (\mathbb{R}^\ell_+, \succsim_i, \omega_i, i \in N)$.

An allocation x is a consumption bundle $x_i \in \mathbb{R}^\ell_+$ for each agent $i \in N$. The allocation x is feasible in the economy $\mathcal{E}$ if $\sum_{i=1}^n x_i \leq \sum_{i=1}^n \omega_i$. A price system is an element of the $(\ell-1)$-dimensional simplex of $\mathbb{R}^\ell_+$. A Walrasian equilibrium is a pair (p, x), where p is a price system and x is a feasible allocation such that, for every agent i, the bundle x_i maximizes U_i in the budget set $B_i(p) = \{y \in \mathbb{R}^\ell_+ \text{ such that } p \cdot y \leq p \cdot \omega_i\}$. We denote by $W(\mathcal{E})$ the set of Walrasian allocations for the economy $\mathcal{E}$.

A coalition is a non-empty set of consumers. An allocation y is said to be attainable or feasible for the coalition S if $\sum_{i \in S} y_i \leq \sum_{i \in S} \omega_i$. The coalition S blocks or objects the allocation x in the economy $\mathcal{E}$ if there exists an allocation y which is feasible for S, such that $y_i \succsim_i x_i$ for every $i \in S$ and $y_j \succ_j x_j$ for some $j \in S$. When S objects x via y we say that (S, y) is an objection to x. A feasible allocation is efficient if it is not objected by the grand coalition, formed by all the agents. The core of the economy $\mathcal{E}$, denoted by $C(\mathcal{E})$, is the set of feasible allocations which are not objected by any coalition of agents. It is known that, under the hypotheses above, the economy $\mathcal{E}$ has Walrasian equilibria and that any Walrasian allocation belongs to the core (in particular, it is efficient).

Along this paper, we will refer to sequences of replicated economies. For each positive integer r, the r-fold replica economy $r\mathcal{E}$ of $\mathcal{E}$ is a new economy with rn agents indexed by ij, $i = 1, \ldots, n$; $j = 1, \ldots, r$, such that each consumer ij has a preference relation $\succsim_{ij} = \succsim_i$ and endowments $\omega_{ij} = \omega_i$. That is, $r\mathcal{E}$ is a pure exchange economy with r agents of type i for every $i \in N$. Given a feasible allocation x in $\mathcal{E}$ let rx denote the corresponding equal treatment allocation in $r\mathcal{E}$, which is given by $(rx)_{ij} = x_i$ for every $j \in \{1, \ldots, r\}$ and $i \in N$.

[2] The convexity of preferences we require is the following: If a consumption bundle z is strictly preferred to $\hat{z}$ so is the convex combination $\lambda z + (1-\lambda)\hat{z}$ for any $\lambda \in (0,1)$. This convexity property is weaker than strict convexity and it holds, for instance, when the utility functions are concave.

[3] The preference relation $\succsim_i$ is strictly monotone if $x \geq y$, $x \neq y$ implies that $x \succ_i y$.

[4] We wrote $\mathbb{R}^\ell_{++}$ for sake of simplicity; it is actually enough that $\omega_i \in \mathbb{R}^\ell_+$ as long as $\omega = \sum_{i=1}^n \omega_i >> 0$.

In addition, we will use the fact that a finite economy $\mathcal{E}$ with n consumers can be associated to a continuum economy $\mathcal{E}_c$ with n-types of agents as we specify next. Given the finite economy $\mathcal{E}$, let $\mathcal{E}_c$ be the associated continuum economy, where the set of agents is $I = [0,1] = \bigcup_{i=1}^{n} I_i,$ with $I_i = \left[\frac{i-1}{n}, \frac{i}{n}\right)$ if $i \neq n$; $I_n = \left[\frac{n-1}{n}, 1\right]$; and all the agents in the subinterval I_i are of the same type i. In this case, $x = (x_1, \ldots, x_n)$ is a Walrasian allocation in $\mathcal{E}$ if and only if the step function f_x (defined by $f_x(t) = x_i$ for every $t \in I_i$) is a competitive allocation in $\mathcal{E}_c$.[5]

3 Bargaining sets for finite economies

Mas-Colell's (1989) bargaining set, stated for continuum economies, is well defined for the economy $\mathcal{E}$ with a finite number of traders as follows:

An objection (S, y) to the allocation x has a counter-objection in the economy $\mathcal{E}$ if there exists a coalition $T \subset N$ and an allocation z such that

(i) $\sum_{i \in T} z_i \leq \sum_{i \in T} \omega_i$ and

(ii) $z_i \succsim_i y_i$ for every $i \in T \cap S$ and $z_i \succsim_i x_i$ for every $i \in T \setminus S$, with a strict preference for some individual $i \in T$.

An objection which cannot be counter-objected is said to be justified. *$\mathcal{B}(\mathcal{E})$ is the set of all the feasible allocations in the economy $\mathcal{E}$ which, if they are objected, could also be counter-objected. Therefore $\mathcal{B}(\mathcal{E})$ denotes Mas-Colell's bargaining set for the economy $\mathcal{E}$.*

The argument that objections might be met with counter-objections leads to bargaining set notions that depend on the way justified objections are defined. In fact, since the original bargaining set was introduced by Aumann and Maschler (1964) and Davis and Maschler (1963) for cooperative games, several versions have been defined and studied.[6] The way in which justified objections are defined becomes particularly relevant in sequences of replicated economies, where identical agents may participate simultaneously in the objecting and counter-objecting mechanisms.

Next, we present both the notion of bargaining set we provide in this paper and the one that ATZ used for replicas of a finite economy $\mathcal{E}$, highlighting the main differences between the one and the other.

[5]See García-Cutrín and Hervés-Beloso (1993) for further details.

[6]See Geanakoplos (1978), Mas-Colell (1989), Dutta et al. (1989), Zhou (1994) and Anderson (1998).

Our bargaining set

An objection (S, y) to the allocation x in the initial economy $\mathcal{E}$ is counter-objected in the replicated economy $r\mathcal{E}$ if there exist $T^* = (a_i, i \in T)$, with $a_i \in \mathbb{N} \cup \{0\}$ and $a_i \leq r$, for every $i \in T \subset N$, and an equal treatment allocation $(z_i, i \in T)$ such that

(i) $\sum_{i \in T} a_i z_i \leq \sum_{i \in T} a_i \omega_i$ and

(ii) $z_i \succsim_i y_i$ for every $i \in T \cap S$ and $z_i \succsim_i x_i$ for every $i \in T \setminus S$, with a strict preference for some $i \in T$.

Condition (i) above states that the counter-objecting coalition is formed by a_i agents identical to consumer i with $i \in T$ and is able to get the bundle z_i for each member of type $i \in T$; and (ii) guarantees that any common type of the two coalitions T and S will be as well off as with the objection, no representative of agents becomes worse off, and some one strictly improves.

We say that an objection is justified* if it is not counter-objected in any replicated economy. *A feasible allocation belongs to the bargaining set, $\mathcal{B}^*(\mathcal{E})$, if it has no justified* objection.*[7]

To simplify, in the sequence of replicated economies we restrict the objecting mechanism to equal treatment allocations. We remark that restricting the objection process to equal treatment allocations makes more difficult to have justified objections and then the convergence of the bargaining set we define implies the convergence when objections are not required to be equal treatment allocations.

To be precise, an objection to rx in the replicated economy $r\mathcal{E}$ is justified* if it has the equal treatment property and it is not counter-objected in any replicated economy. Thus, a potential justified* objection in every economy $r\mathcal{E}$ is given by a coalition $S^* = (a_i^*, i \in S)$, with $S \subseteq \{1, \ldots, n\}$, and a commodity bundle y_i for each $i \in S$.

Bargaining set as in ATZ

Let $\mathcal{S}$, $\mathcal{T} \subset \{ij$, such that $i \in \{1, \ldots, n\}$ and $j \in \{1, \ldots r\}\}$ be coalitions of agents in the replica $r\mathcal{E}$. An objection $(\mathcal{S}, y)$ to the allocation rx has a counter-objection in the replicated economy $r\mathcal{E}$ if there exists a coalition $\mathcal{T}$ and an allocation z such that

(i) $\sum_{ij \in \mathcal{T}} z_{ij} \leq \sum_{ij \in \mathcal{T}} \omega_{ij}$ and

(ii) $z_{ij} \succsim_i y_{ij}$ for every $ij \in \mathcal{T} \cap \mathcal{S}$ and $z_{ij} \succsim_i x_i$ for every $ij \in \mathcal{T} \setminus \mathcal{S}$, with a strict preference for some individual $ij \in \mathcal{T}$.

An objection which cannot be counter-objected in $r\mathcal{E}$ is said to be justified.

[7]Note that this notion can be applied to any cooperative game. Note also that the bargaining set is non-empty since it contains the core.

$\mathcal{B}(\mathcal{E})$ vs. $\mathcal{B}^*(\mathcal{E})$: a comparison

Note that the only difference is the way a justified objection is specified when the Mas-Colell's bargaining set is adapted to a sequence of replicated economies.

In ATZ' bargaining set of $r\mathcal{E}$, if the agent ik does not participate in an objection to the allocation rx, but is a member of a counter-objecting coalition, then it suffices that ik improves with respect to x_i, regardless of whether another identical agent ij has previously participated or not in the objection. However, in our definition, each agent is considered as a representative of a large enough number of identical consumers. Thus, if ij belongs to an objecting coalition and ik is involved in a counter-objection, then it is required that ik improves with respect to the objecting allocation.

Due to continuity and strict monotonicity of preferences, the definition of counter-objection, in both bargaining sets $\mathcal{B}(\mathcal{E})$ and $\mathcal{B}^*(\mathcal{E})$, can be strengthened by requiring strict preference for every individual in the counter-objection coalition. As Mas-Colell's (1989) Remarks 1 and 6 pointed out, weak preference cannot simply be replaced by strict preference in the objection in $\mathcal{B}(\mathcal{E})$. This is crucial to obtain the non-convergence result in ATZ (1997).[8] However, from the proof of our convergence result one deduces that, if an allocation x is not Walrasian then, for r large enough, rx has a justified* objection in the r-replicated economy in which every agent becomes better off.

Let (S, y) be an objection to an allocation x in $\mathcal{E}$. Observe that, if the objection (S, y) has a counter-objection in the initial economy $\mathcal{E}$ (i.e, $r = 1$), then it is counter-objected in any replicated economy. But, there may be an r so that (S, y) has a counter-objection in $(r+1)\mathcal{E}$ even though it has no counter-objection in $r\mathcal{E}$. To see this, consider an economy $\mathcal{E}$ with two commodities and two consumers with the same utility function $U(a, b) = ab$ and endowed with $\omega_1 = (1, 9)$ and $\omega_2 = (9, 1)$, respectively. The big coalition $\{1, 2\}$ blocks ω via the allocation y that assigns $y_1 = (4, 4)$ to consumer 1 and $y_2 = (6, 6)$ to consumer 2. Since y is individually rational and efficient, it is in the core of $\mathcal{E}$. Therefore, this objection has no counter-objection in $\mathcal{E}$. However, y is counter-objected in the economy $2\mathcal{E}$. For instance, the coalition formed by two consumers of type 1 and one consumer of type 2 counter-objects via the allocation that gives $(5/2, 13/2)$ to both consumers of type 1 and $(6, 6)$ to the consumer of type 2. Actually, we can state the following more general remark: any objection defined by a non-Walrasian allocation that is the core of the original economy has no counter-objection in $\mathcal{E}$ but, applying Debreu and Scarf's (1963) limit theorem on the core, it is counter-objected in some replicated economy.

[8] Actually this asymmetry is responsible for Lemma 3.2 in ATZ (1997).

Let $\hat{\mathcal{B}}(r\mathcal{E})$ be the set of allocations x in $\mathcal{E}$ such that $rx \in \mathcal{B}^*(r\mathcal{E})$. If rx has a justified* objection in $r\mathcal{E}$, then the same objection is also justified* for $\bar{r}x$ in $\bar{r}\mathcal{E}$ for any $\bar{r} \geq r$. Thus, as it happens with the core, our bargaining set shrinks under replication, i.e., $\hat{\mathcal{B}}((r+1)\mathcal{E}) \subseteq \hat{\mathcal{B}}(r\mathcal{E})$ for any natural number r. This is not the case for the bargaining set considered by ATZ.[9] The fact that our bargaining set becomes smaller when the economy is enlarged allows us to prove our convergence result that provides an extension of the Debreu-Scarf core-convergence to bargaining sets.

As in Mas-Colell (1989), we consider a special class of objections that are generated by prices. To be precise, an objection (S, y) to the allocation x in the economy $\mathcal{E}$ is said to be Walrasian if there exists a price system p such that (i) $p \cdot v \geq p \cdot \omega_i$ if $v \succsim_i y_i$, $i \in S$ and (ii) $p \cdot v \geq p \cdot \omega_i$ if $v \succsim_i x_i$, $i \notin S$. Next we characterize justified* objections as Walrasian objections.

Proposition 3.1 *Let x be a feasible allocation in the finite economy $\mathcal{E}$. Then, any objection to x is justified* if and only if it is a Walrasian objection.*

The proof of this result also shows that what does become important is the set of types which are involved in the objection rather than the number of members of each type that form the objecting coalition.[10] Furthermore, from the above characterization we can also deduce that when the objection $(S^* = (a_i, i \in S), y = (y_i, i \in S))$ is such that $S = N$, then it is justified* if and only if y is a competitive allocation in the economy restricted to S^*. However, in general, being a Walrasian objection is much more demanding. In fact, a coalition S^* that does not involve all the types and blocks via a Walrasian allocation for the economy restricted to S^*, does not necessarily define a justified* objection.

In addition, when an equal treatment justified objection includes all the types, then it is also a justified* objection. However, the converse is not true. These facts are illustrated and exploited in the examples and in the proof of our convergence result. Moreover, from the characterization of justified* objections as Walrasian objections we can deduce that the fact that (S, y) is a justified* objection to rx in $r\mathcal{E}$ and $y_i \succ_i x_i$ do not imply that all the agents of type $i \in S$ are members of the objecting coalition. This is in contrast to both Mas-Colell's notion for continuum economies and the adaptation to finite economies by ATZ, for which

[9]Note that a justified objection as in ATZ in the economy $r\mathcal{E}$ is not necessarily justified in the economy $(r+1)\mathcal{E}$.

[10]As it is stressed by Mas-Collel, a Walrasian objection requires a price system p, and is based on a self selection property. Within our approach, types that participate in a coalition in a Walrasian objection against an allocation are those who would trade at the price vector p rather than get the consumption bundle they receive by such an allocation.

if a coalition with a justified objection includes only part of some type of agents then it is not possible for these agents to strictly improve at the objection. This fact, which makes the concept of justified objection too stringent, becomes crucial in ATZ's work and constitutes, roughly speaking, the reason why they prove non-convergence.

Finally, we point out that Dutta et al. (1989) introduced a notion of consistency for a bargaining set, in which each objection in a "chain" of objections is tested in precisely the same way as its predecessor. However, this property would be achieved whenever the bargaining set shrinks to the set of Walrasian allocations. Thus, consistency of our bargaining set can be obtained as a consequence of Theorem 4.1.

ATZ's non-convergence example revisited

Next, we consider ATZ's example of a well behaved economy. We aim to illustrate why this economy allows ATZ to show their non-convergence result while pointing out why our bargaining set converges. For this, we parameterize a domain of economies, which includes the sequence of replicas, where the Walrasian equilibrium correspondence is continuous.

Consider an economy with two commodities and two consumers who have the same utility function $U(a,b) = \sqrt{ab}$ and endowments $\omega_1 = (3,1)$ and $\omega_2 = (1,3)$. ATZ showed that the measure of the set of individually rational Pareto optimal equal treatment allocations that have a justified objection tends to zero as the economy is replicated. In what follows we state an alternative non-convergence proof. For each $\tau = r_1/r_2 \in \mathbb{R}_+$, let $\mathcal{E}_{|\tau}$ be the economy restricted to r_i agents of type $i = 1,2$. Let us consider the unique Walrasian allocation for $\mathcal{E}_{|\tau}$ which assigns $x_1(\tau)$ and $x_2(\tau)$ to agents of type 1 and 2, respectively. Let $V_i(\tau) = (U(x_i(\tau)))^2$, for $i = 1,2$. The function V_1 is decreasing and convex whereas V_2 is increasing and concave.

Let $\hat{x}$ be the non-Walrasian allocation given by $\hat{x}_1 = (4,4) - x_2(\sqrt{2})$ and $\hat{x}_2 = x_2(\sqrt{2})$. We find a unique positive number $\hat{\tau}$ such that $(U(\hat{x}_1))^2 = V_1(\hat{\tau})$. Consider the two types associated economy where agents of type 1 are represented by the interval [0,1] and agents of type 2 by (1,2]. Since V_1 is decreasing and $\hat{x}$ is individually rational, the set of all potential justified objections (in the sense of Mas-Colell, 1989) is given by the interval $[\sqrt{2}, \hat{\tau}]$ (see figure below). However, the only coalitions able to make a justified objection are those with measure $1 + 1/\sqrt{2}$.[11] In other words, although every $\tau \in [\sqrt{2}, \hat{\tau}]$ defines an objection to $f_{\hat{x}}$, the unique which is (Mas-Colell) justified is given by $\tau = \sqrt{2}$, which does not correspond to any replica. Thus we conclude that there is no justified objection in any replicated economy, that is, $r\hat{x}$ belongs to $\mathcal{B}(r\mathcal{E})$ for every r, which proves the non-convergence.

[11] This is so because if a coalition with a Mas-Colell justified objection includes only part of some type of agents, then it is not possible for these agents to strictly improve with the objection.

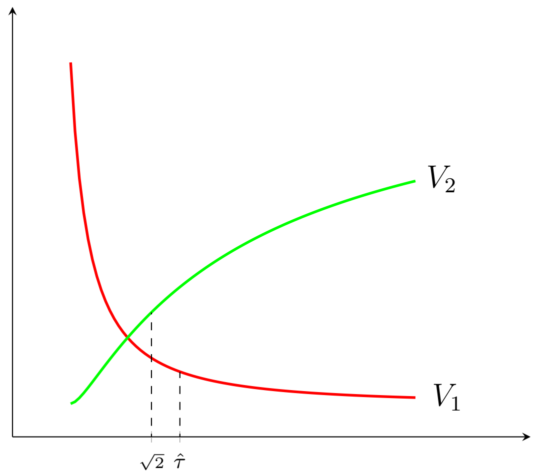

Fig 1: $(U(\hat{x}_1))^2 = V_1(\hat{\tau})$ and $(U(\hat{x}_2))^2 = V_2(\sqrt{2})$.

Let us now analyze the previous example under our notion of bargaining set. For it, we remark that any rational number $\tau \in \left[\sqrt{2}, \hat{\tau}\right]$ leads to a justified* objection for the allocation $r\hat{x}$ for some replicated economy $r\mathcal{E}$. This implies that $r\hat{x}$ does not belong to our bargaining set for any large enough replicated economy.

Furthermore, for each $\alpha \in (\sqrt{3}, 2) \cup (2, 4-\sqrt{3})$, there exist τ_α and τ^α such that $V_1(\tau_\alpha) = \alpha^2$ and $V_2(\tau^\alpha) = (4-\alpha)^2$ and $\tau^\alpha < \tau_\alpha$.[12] Then, $V_1(\tau) > \alpha^2$ and $V_2(\tau) > (4-\alpha)^2$, for any $\tau \in (\tau^\alpha, \tau_\alpha)$. For each rational number $\tau \in (\tau^\alpha, \tau_\alpha)$, let $r_1(\tau), r_2(\tau)$ be natural numbers such that $\tau = r_1(\tau)/r_2(\tau)$. Note that the coalition formed by $r_i(\tau)$ consumers of type $i = 1, 2$ with the allocation $x(\tau)$ is a Walrasian objection to the allocation that gives (α, α) to agents of type 1 and $(4-\alpha, 4-\alpha)$ to agents of type 2 for any replicated economy $r\mathcal{E}$ with $r \geq \max\{r_1(\tau), r_2(\tau)\}$. By Proposition 3.1, the objection we have constructed is justified*. Therefore, we conclude that the counterexample by ATZ does not lead to a non-convergence result for the notion of bargaining set we have proposed. Actually, since we deal with a set of economies where the equilibrium is unique, the convergence result we state in the next section guarantees that our bargaining set shrinks to the Walrasian allocation.

4 A convergence result

In this section we analyze convergence properties of our bargaining set. First we show that under a continuity property of the equilibrium price correspondence, the Walrasian allocations of a finite economy are characterized as allocations that belong to the bargaining set of every replicated economy. Then, we state an example showing that such a continuity is a necessary condition.

[12]Note that $\alpha = 2$ defines the Walrasian allocation and $V_1(1) = V_2(1) = 4$.

Starting from the finite economy $\mathcal{E}$, we construct auxiliary continuum economies with a finite number of types and use the following notation. Consider a vector $\alpha = (\alpha_i, i \in N) \in [0,1]^n$ such that $\sum_{i \in N} \alpha_i = 1$. Let $N_\alpha = \{i \in N | \alpha_i > 0\}$, n_α denotes the cardinality of N_α and $m_\alpha = \max\{i | i \in N_\alpha\}$. For each $i \in N_\alpha$, let $I_i(\alpha) = [\bar{\alpha}_{i-1}, \bar{\alpha}_i)$ if $i \neq m_\alpha$ and $I_i(\alpha) = [\bar{\alpha}_{m_\alpha - 1}, 1]$ if $i = m_\alpha$, where $\bar{\alpha}_i = \sum_{h=0}^{i-1} \alpha_h$, with $\alpha_0 = 0$. Finally, $\mathcal{E}_c(\alpha)$ denotes the continuum economy with n_α types of agents, where consumers in the subinterval $I_i(\alpha)$ are of type i (i.e, have endowments ω_i and preferences $\succsim_i$). We will use the following continuity assumption.[13]

(C) The equilibrium correspondence, that associates to each α the equilibrium prices of the auxiliary continuum economy $\mathcal{E}_c(\alpha)$ with a finite number of types, is continuous.

Theorem 4.1 *Assume that the continuity property (C) holds. Then, an allocation x is Walrasian in the finite economy $\mathcal{E}$ if and only if, for every r, the allocation rx belongs to the bargaining set of the replicated economy $r\mathcal{E}$. That is, $W(\mathcal{E}) = \bigcap_{r \in \mathbf{N}} \hat{\mathcal{B}}(r\mathcal{E})$.*

Since the Walrasian correspondence is upper semicontinuous (see Hildenbrand 1972), uniqueness of equilibrium guarantees the continuity requirement for our convergence result. Different works have provided conditions[14] on preferences and endowments that yield individual demand functions with the gross substitute property which ensures that the equilibrium is unique.[15] Thus, since each auxiliary atomless economy $\mathcal{E}_c(\alpha)$ includes no more types of consumers than $\mathcal{E}$, we can conclude that if these conditions on the primitives of the original economy $\mathcal{E}$ are verified, then the equilibrium is unique not only for $\mathcal{E}$ but also for all the economies $\mathcal{E}_c(\alpha)$. This implies that condition (C) holds and therefore we have convergence of our bargaining set.

In the proof of Theorem 4.1 we exploit the fact that objections that prevent an allocation from belonging to our bargaining set are those generated by means of prices. This characterization of justified* objections as Walrasian objections states reasons for our continuity requirement to be in accordance with the related literature on the non-manipulability of the Walrasian mechanism, where it is also assumed that the correspondence that assigns each

[13]Recall that the set of economies on which the equilibrium correspondence is continuous is open and dense (see Hildenbrand 1972 or Dierker 1973).

[14]These conditions refer basically to differentiability of the utility functions, degrees of risk aversion, elasticity of substitution for commodities and collinearity of endowments. See, for instance, Varian (1985), Mityushin and Polterovich (1978), Fisher (1972) and Mas-Colell (1991). See also Arrow and Hahn (1971), for additional details on uniqueness of equilibrium.

[15]The standard example is a demand that comes from the maximization of a Cobb-Douglas utility function subject to a budget constraint with strictly positive endowments. A generalization is the utility function $U(x) = \prod_{h=1}^{\ell} (x_h - \beta_h)^{\gamma_h}$, with $\beta_h \leq 0$, $\gamma_h > 0$ and $\sum_{h=1}^{\ell} \gamma_h = 1$.

economy its set of market-clearing prices is continuous (see, for instance, Roberts and Postlewaite 1976). In fact, our convergence result adds to those on the asymptotic properties of the core and the non-manipulability analysis of the Walrasian process pointing out that in order to justify the competitive assumption that consumers will adopt price-taking behavior it is necessary to limit attention to large economies.

Next, we state an example that illustrates why the continuity assumption is required and shows the impossibility of obtaining a convergence result if we allow for discontinuities of the equilibrium correspondence.

Counterexample. Let $\mathcal{E}$ be an exchange economy with two commodities and two agents, endowed with $\omega_1 = (2,1)$ and $\omega_2 = (1,2)$ respectively, who have the same utility function[16]:

$$U(x,y) = \begin{cases} \frac{1}{2^{1/4}}\sqrt{x} + \sqrt{y} & \text{if } \ x > \sqrt{2}\, y, \text{ and} \\ \sqrt{x} + (2 - 2^{1/4})\sqrt{y} & \text{if } \ x \leq \sqrt{2}\, y. \end{cases}$$

Let x be the numeraire, p denote the price of y and $d_i(p)$ be the demand function for each agent i. The equilibrium price for this economy is $p^* = 2 - 2^{1/4}$.

Consider r_i agents of type $i = 1, 2$ and let $\tau = r_1/r_2$. The Walrasian equilibrium price $p(\tau)$ for the restricted replicated economy, $\mathcal{E}_{|\tau}$, and by extension for $\tau \in \mathbb{R}_+$, is unique except when $\tau^* = 1 + \frac{3}{2}\sqrt{2}$. Note that there is a continuum of equilibria for $\mathcal{E}_{|\tau^*}$ given by the interval of prices $[\underline{p}, \overline{p}]$ with $\underline{p} = 2^{1/4}(2 - 2^{1/4})$ and $\overline{p} = \sqrt{2}$. For each $\tau \in \mathbb{R}_+$, the utility levels which can be attained for each type of consumers at a Walrasian allocation of the economy $\mathcal{E}_{|\tau}$ are given by the mappings $V_i(\tau) = U(d_i(p(\tau)))$, $i = 1, 2$, whose graphical representations are shown in the following figure, where $\alpha_i = \min\{V_i(\tau^*)\}$ and $\beta_i = \max\{V_i(\tau^*)\}$:

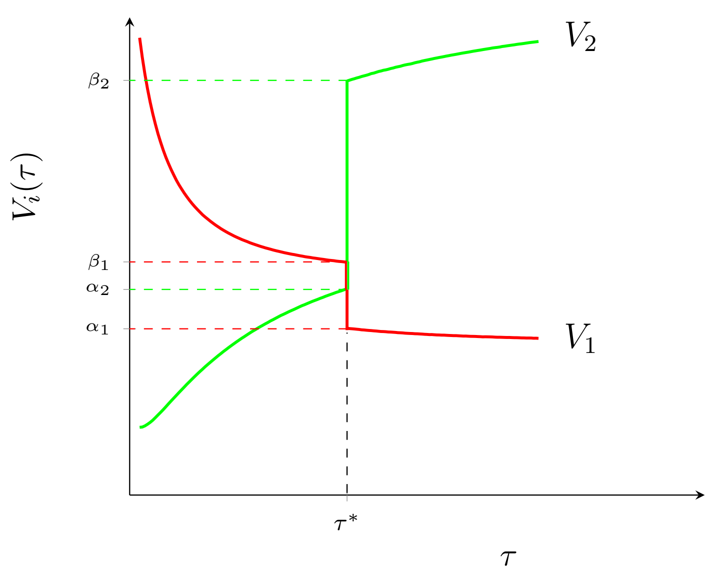

Fig. 2: V_1 and V_2 are not lower semicontinuous at τ^*.

[16]Note that this utility function is not differentiable.

Consider a feasible allocation $h = (h_1, h_2)$ such that $U(h_i) \in (\alpha_i, \beta_i)$.[17] Since h is individually rational, in order to block it in a replicated economy, both types need to be present. In addition, there is no justified* objection for h whenever $\tau > \tau^*$ or $\tau < \tau^*$. It is possible, though, to find justified* objections in $\mathcal{E}_{|\tau^*}$. Let p_i be the equilibrium price for $\mathcal{E}_{|\tau^*}$ such that $U(d_i(p_i)) = U(h_i)$. As illustrated in the figure below, any price in $[p_2, p_1] \subset \left[\underline{p}, \overline{p}\right]$ leads to a justified* objection. However, since τ^* is an irrational number, such set of justified* objections cannot be attained in any replicated economy, which proves the non-convergence.

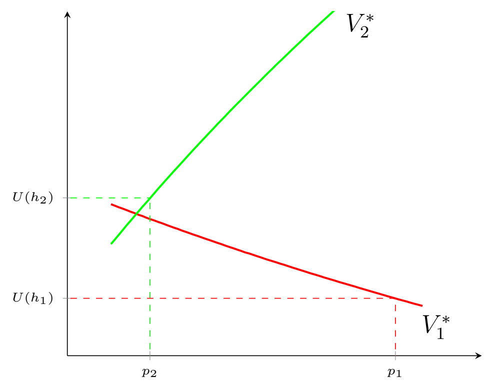

Fig. 3: We get $V_i^*(p) = U(d_i(p))$, with $p \in p(\tau^*)$ by "zooming in" on the Fig. 2 when $\tau = \tau^*$.

Appendix

Proof of Proposition 3.1. Let (S, y) be a Walrasian objection to x. Assume that it is counter-objected in some replicated economy $r\mathcal{E}$. That is, there exist $T \subseteq N$ and natural numbers $r_i \leq r$ for each $i \in T$, such that: $\sum_{i\in T} r_i z_i \leq \sum_{i \in T} r_i \omega_i$; $z_i \succ_i y_i$ for every $i \in T \cap S$ and $z_i \succ_i x_i$ for every $i \in T \setminus S$. Since (S, y) is a Walrasian objection at prices p we have that $p \cdot z_i > p \cdot \omega_i$, for every $i \in T \cap S$ and $p \cdot z_i > p \cdot \omega_i$, for every $i \in T \setminus S$. This implies $p \cdot \sum_{i \in T} r_i z_i > p \cdot \sum_{i\in T} r_i \omega_i$, which is a contradiction. Thus, we conclude that (S, y) is a justified* objection.

To show the converse, let (S, y) be a justified* objection to x and let $a = (a_1, \dots, a_n)$ be an allocation such that $a_i = y_i$ if $i \in S$ and $a_i = x_i$ if $i \notin S$. For every i define $\Gamma_i = \{z \in \mathbb{R}^\ell | z + \omega_i \succsim_i a_i\} \bigcup \{0\}$ and let Γ be the convex hull of the union of the sets $\Gamma_i, i \in N$. A similar proof to the limit theorem on the core by Debreu and Scarf (1963) shows that $\Gamma \bigcap (-\mathbb{R}^\ell_{++})$ is empty, which implies that 0 is a frontier point of Γ. Then, there exists a price system p such that $p \cdot z \geq 0$ for every $z \in \Gamma$. Therefore, we conclude that (S, y) is a Walrasian objection. Q.E.D.

[17]For instance, we can take $h_1 = \left(\frac{11^2}{5^2 (3 - 2^{1/4})^2}, \frac{11^2}{5^2(3-2^{1/4})^2} \right)$ and $h_2 = (3,3) - h_1$.

To prove Theorem 4.1 we show the following lemma.

Lemma. *Let x be a non-Walrasian feasible allocation in the economy $\mathcal{E}$. Then, the following statements hold:*

(i) For each i, there exist a sequence of rational numbers $r_i^k \in (0,1]$ converging to 1 and a sequence of allocations $(x^k, k \in \mathbb{N})$ that converges to x such that: (a) $\sum_{i=1}^n r_i^k x_i^k \leq \sum_{i=1}^n r_i^k \omega_i$, (b) $x_i^k \succ_i x_i$ for every i, and (c) $x_i^k \succ_i x_i^{k+1}$ for every k and every i.

Let $r^k = \sum_{i\in N} r_i^k$ and $\alpha^k = (r_i^k/r^k, i \in N) \in (0,1]^n$. Let f^k be the step function given by $f^k(t) = x_i^k$ for every $t \in I_i(\alpha^k)$ in the continuum economy $\mathcal{E}_c(\alpha^k)$.

(ii) If rx belongs to $\mathcal{B}^*(r\mathcal{E})$ for every replicated economy, then for every k, there is a justified objection (S^k, g^k), in the sense of Mas-Colell, to f^k in the economy $\mathcal{E}_c(\alpha^k)$.

Proof of (i). Observe that if x^k converges to x and $x_i^k \succ_i x_i$, for every i and k, then, under continuity of preferences, condition (c) holds by taking a subsequence if necessary.

If x is a feasible allocation that is not efficient, then, for every i, there exists y_i such that $\sum_{i=1}^n y_i \leq \sum_{i=1}^n \omega_i$ and $y_i \succ_i x_i$. The sequence given by $x_i^k = \frac{1}{k} y_i + (1-\frac{1}{k})x_i$ fulfills the requirements in (a) with $r_i^k = 1$ for all i and k.

Let x be a non-Walrasian feasible allocation which is efficient. Then, there exist rational numbers $a_i \in (0,1]$ (with $a_j < 1$ for some j) and bundles y_i for all $i = 1, \ldots, n$, such that $\sum_{i=1}^n a_i(y_i - \omega_i) = -\delta$, with $\delta \in \mathbb{R}_{++}^\ell$ and $y_i \succ_i x_i$, for every i (see Hervés-Beloso and Moreno-García 2001, for details). Let $a = \sum_{i=1}^n a_i$. Given $\varepsilon \in (0,1]$, let $y_i^\varepsilon = \varepsilon y_i + (1-\varepsilon)x_i$. By convexity of preferences, $y_i^\varepsilon \succ_i x_i$ for every i. Consider $x_i^\varepsilon = x_i + \frac{\varepsilon\delta}{a^\varepsilon}$, where $a^\varepsilon = (1-\varepsilon)(n-a)$. By monotonicity, $x_i^\varepsilon \succ_i x_i$ for every i. Take a sequence of rational numbers ε_k converging to zero and, for each k and i, let $a_i^k = (1-\varepsilon_k)(1-a_i)$, $r_i^k = a_i + a_i^k \in (0,1]$, and define $x_i^k = \frac{a_i}{r_i^k} y_i^{\varepsilon_k} + \frac{a_i^k}{r_i^k} x_i^{\varepsilon_k}$. By construction, the sequences r_i^k and x_i^k ($i = 1, \ldots, n$ and $k \in \mathbb{N}$) verify the required properties.

Proof of (ii). Let q^k be a natural number such that $r_i^k = b_i^k/q^k$, with $b_i^k \in \mathbb{N}$ for each i. Since $x \in \bigcap_{r\in\mathbb{N}} \hat{\mathcal{B}}(r\mathcal{E})$, x^k cannot be a Walrasian allocation for the economy formed by b_i^k agents of type i; otherwise, the coalition formed by b_i^k members of each type i joint with x^k would define a justified* objection in the q^k-replicated economy.[18] Then, f^k cannot be a competitive allocation in $\mathcal{E}_c(\alpha^k)$. By Mas-Colell's (1989) equivalence result, f^k is blocked by a justified objection (S^k, g^k) in $\mathcal{E}_c(\alpha^k)$. By convexity of preferences, we can consider without loss of generality that g^k is an equal treatment allocation.

Q.E.D.

Proof of Theorem 4.1 Since $W(\mathcal{E}) \subseteq C(r\mathcal{E})$, it is immediate that $W(\mathcal{E}) \subseteq \bigcap_{r\in\mathbb{N}} \hat{\mathcal{B}}(r\mathcal{E})$.

To show the converse, assume that x is a non-Walrasian allocation that belongs to $\bigcap_{r\in\mathbb{N}} \hat{\mathcal{B}}(r\mathcal{E})$. Consider the sequence of justified objections (S^k, g^k) to f^k in the economy $\mathcal{E}_c(\alpha^k)$ as constructed in the previous lemma. Let $\gamma^k = \big(\gamma_i^k = \mu(S^k \cap I_i(\alpha^k))/\mu(S^k), i \in N\big) \in [0,1]^n$. Since the number

[18] We remark that any objecting coalition involving all types along with a Walrasian allocation for such a coalition defines a justified* objection. This is not the case for the corresponding Mas-Colell's notion.

of types of consumers is finite, without loss of generality we can consider, taking a subsequence if necessary, that $N_{\gamma^k} = \{i \in N | \gamma_i^k > 0\} = T$ for every k. We use the same notation for such a subsequence and write γ_i^k converges to γ_i for every $i \in T$ and $\sum_{i \in T} \gamma_i = 1$. Consider the sequence of economies $\mathcal{E}_c(\gamma^k)$ and the limit vector γ.

Then, by the previous lemma, for each natural number k, there is a subset T of types and a competitive equilibrium (p^k, g^k) in $\mathcal{E}_c(\gamma^k)$ such that:

(i) $g_i^k \succsim_i x_i^k$ for every $i \in T$, with $g_j^k \succ_j x_j^k$ for some $j \in T$, and

(ii) $g_i^k \in d_i(p^k)$ for every $i \in T$, and $x_i^k \succsim_i d_i(p^k)$ for every $i \in N \setminus T$.[19]

Let $A_k = \left\{i \notin T | x_i \succsim_i d_i(p^k)\right\}$, $B_k = \left\{i \notin T | x_i \prec_i d_i(p^k)\right\}$. Since the number of types is finite, without loss of generality we can consider, taking a subsequence if it is necessary, that $A_k = A$ and $B_k = B$ for every k.

Let us choose a sequence of numbers $\delta_k \in (0,1)$ converging to 1 and let $\varepsilon^k = 1 - \delta_k$, which converges to zero. For each $i \in B$ take $\varepsilon_i^k > 0$ such that $\varepsilon^k = \sum_{i \in B} \varepsilon_i^k$. Let $T_1 = T \cup B$ and for each $i \in T_1$ define $\widetilde{\gamma}_i^k \in (0,1)$ as follows:

$$\widetilde{\gamma}_i^k = \begin{cases} \delta_k \gamma_i^k & \text{if } i \in T \\ \varepsilon_i^k & \text{if } i \in B \end{cases}$$

Note that $\sum_{i \in T_1} \widetilde{\gamma}_i^k = 1$. Moreover, $\lim_{k \to \infty} \widetilde{\gamma}_i^k = \lim_{k \to \infty} \gamma_i^k = \gamma_i$ for every $i \in T$ and $\widetilde{\gamma}_i^k$ goes to zero as k increases for every $i \in B$. Then, the economy $\mathcal{E}_c(\widetilde{\gamma}^k)$ differs from $\mathcal{E}_c(\gamma^k)$ only in at most a finite set of types of agents whose measure goes to zero when k increases. Now, for each k and for each $i \in T_1 = T \cup B$, take a sequence of positive rational numbers γ_i^{km} converging to $\widetilde{\gamma}_i^k$ when m increases and such that $\sum_{i \in T_1} \gamma_i^{km} = 1$ for every m. In this way, for each k, let us consider the sequence of continuum economies $\mathcal{E}_c(\gamma^{km})$. To simplify notation, let $\mathcal{E}_c^{kk} = \mathcal{E}_c(\gamma^{kk})$. Note that $\lim_{k \to \infty} \gamma_i^{kk} = \lim_{k \to \infty} \gamma_i^k$ for every $i \in T$ and $\lim_{k \to \infty} \gamma_i^{kk} = 0$ for every $i \in B$. Then, the sequence γ^{kk} that describes the diagonal sequence of economies $\mathcal{E}_c^{kk}$ converges to γ as well.

Then, by the continuity of the equilibrium correspondence at γ and the continuity of preferences, we deduce that for every k large enough there is an equilibrium price $\widetilde{p}_1^k$ for the economy $\mathcal{E}_c^{kk}$ such that $d_i(\widetilde{p}_1^k) \succ_i x_i$ for every $i \in T_1$. If $x_i \succsim_i d_i(\widetilde{p}_1^k)$ for every $i \in A$, we have found a Walrasian objection to x in a replicated economy, which is in contradiction to the fact that x belongs to $\bigcap_{r \in \mathbf{N}} \hat{\mathcal{B}}(r\mathcal{E})$. Otherwise, let $\widetilde{A}_k = \left\{i \notin T_1 | x_i \succsim_i d_i(\widetilde{p}_1^k)\right\}$, $\widetilde{B}_k = \left\{i \notin T_1 | x_i \prec_i d_i(\widetilde{p}_1^k)\right\}$. As before, without loss of generality, taking a subsequence if it is necessary, we can consider $\widetilde{A}_k = \widetilde{A}$ and $\widetilde{B}_k = \widetilde{B}$ for every k. Let $T_2 = T_1 \cup \widetilde{B}$ and repeat the analogous argument. In this way, after a finite number h of iterations, we have either *(i)* $T_h = N = \{1, \ldots, n\}$ or *(ii)* $N \setminus T_h \neq \emptyset$ but $\left\{i \notin T_h | x_i \prec_i d_i(\widetilde{p}_h^k)\right\} = \emptyset$. If *(i)* occurs we find a justified* objection to x in a replicated economy which involves all the types of agents. If *(ii)* is the case, there is also a justified* objection to x in a replicated economy but involving only a strict subset of types. In both cases we obtain a contradiction.

Q.E.D.

[19] Note that, given a price vector p, all the bundles in $d_i(p)$ are indifferent; thus, when we write $z \succsim_i d_i(p)$ it means $z \succsim_i d$ for every $d \in d_i(p)$.

References

Anderson, R. M., Trockel, W., Zhou, L.: Nonconvergence of the Mas-Colell and Zhou bargaining sets. Econometrica 65, 1227–1239 (1997)

Anderson, R.M.: Convergence of the Aumann-Davis-Maschler and Geanakoplos Bargaining Sets. Econ. Theory 11, 1–37 (1998)

Arrow, K., Hahn, F.: General Competitive Analysis. Holden-Day, San Francisco (1971)

Aumann, R.J.: Markets with a continuum of traders. Econometrica 32, 39–50 (1964)

Aumann, R., Maschler, M.: The bargaining set for cooperative games. In: Dresher, M., Shapley, L. S., Tucker, A. W. (eds.) Advances in game theory, pp. 443–476, Princeton University Press, Princeton (1964)

Davis, M., Maschler, M.,: Existence of stable payoff configurations for cooperative games. Bull. Amer. Math. Soc. 69, 106–108 (1963)

Debreu, G., Scarf, H.: A limit theorem on the core of an economy. Int. Econ. Rev. 4, 235–246 (1963)

Dierker, E.: Topological methods in Walrasian Economics. Berlin, Heidelberg and New York Springer-Verlag (1973)

Dutta, B., Ray, D., Sengupta, K., Vohra, R.: A consistent bargaining set. J. Econ. Theory 49, 93-112 (1989)

Edgeworth, F.Y.: Mathematical Psychics. Paul Kegan, London (1881)

Fisher, F.: Gross substitutes and the utility function. J. Econ. Theory 4, 82-87 (1972)

García-Cutrín, J., Hervés-Beloso, C.: A discrete approach to continuum economies. Econ. Theory 3, 577-584 (1993)

Geanakoplos, J.: The bargaining set and nonstandard analysis. Chapter 3 of Ph.D. Dissertation, Department of Economics, Harvard University, Cambridge, MA. (1978)

Hervés-Beloso, C., Moreno-García, E.: The veto mechanism revisited. In: Marc Lasonde (ed.) Approximation, Optimization and Mathematical Economics, pp. 147-159. Physica Verlag, Heidelberg, New York (2001)

Hildenbrand, W.: Continuity of the equilibrium-set correspondence. J. Econ. Theory 5, 152-162 (1972)

Mas-Colell, A.: An equivalence theorem for a bargaining set. J. Math. Econ. 18, 129-139 (1989)

Mas-Colell, A.: On the uniqueness once again. In: Barnett, W., Cornet, B., d' Aspremont, J., Gabszewicz, J., Mas-Colell, A. (eds.) Equilibrium Theory and Applications, pp. 275-296, Cambridge University Press (1991)

Maschler, M.: An advantage of the bargaining set over the core. J. Econ. Theory 13, 124-192 (1976)

Mityushin, L. G., Polterovich, V. W.: Criteria for monotonicity of demand function (in Russian). Ekonomika i Matematicheskie Metody 14, 122-128 (1978)

Roberts, D. J., Postlewaite, A.: The incentives for price-taking behavior in large exchange economies. Econometrica 44, 115-127 (1976)

Varian, H.: Additive utility and gross substitutes. Mimeo, University of Michigan, Ann Arbor (1985)

Zhou, L.: A new bargaining set of an n-person game and endogenous coalition formation. Games Econ. Behav. 6, 512-526 (1994)