Trade Dynamics in the Market for Federal Funds*

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Abstract

We develop a model of the market for federal funds that accounts for the two distinctive features of this market: banks have to search for a suitable counterparty, and once they have met, both parties negotiate the size of the loan and the repayment. The theory is used to answer a number of positive and normative questions: What are the determinants of the fed funds rate? How does the market reallocate funds? Is the market able to achieve an efficient reallocation of funds? We also use the model for theoretical and quantitative analyses of several policy-relevant issues.

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1 Introduction

In the United States, financial institutions keep reserve balances at the Federal Reserve Banks to meet requirements, earn interest, or to clear financial transactions. The *market for federal funds* is an interbank over-the-counter market for unsecured, mostly overnight loans of dollar reserves held at Federal Reserve Banks. This market allows institutions with excess reserve balances to lend reserves to institutions with reserve deficiencies. An average measure of the market interest rate on these loans, is commonly referred to as the *fed funds rate*.

The fed funds market is primarily a mechanism that reallocates reserves among banks. As such, it is a crucial market from the standpoint of the economics of payments, and the branch of banking theory that studies the role of interbank markets in helping banks manage reserves and offset liquidity or payment shocks.[1] The fed funds market is the setting where the interest rate on the shortest maturity, most liquid instrument in the term structure is determined. This makes it an important market from the standpoint of Finance. The fed funds rate affects commercial bank decisions concerning loans to businesses and individuals, and has important implications for the loan and investment policies of financial institutions more generally. This makes the fed funds market critical to macroeconomists. The fed funds market is the epicenter of monetary policy implementation: The Federal Open Market Committee (FOMC) communicates monetary policy by choosing the fed funds rate it wishes to prevail in this market, and implements monetary policy by instructing the trading desk at the Federal Reserve Bank of New York to "create conditions in reserve markets" that will encourage fed funds to trade at the target level. As such, the fed funds market is of first-order importance for economists interested in monetary theory and policy. For these reasons, we feel it is important to pry into the micro mechanics of trade in the market for federal funds, in order to understand the mechanism by which this market reallocates liquidity among banks, and the determination of the market price for this liquidity provision—the fed funds rate.

To this end, we develop a dynamic equilibrium model of trade in the fed funds market that explicitly accounts for the two distinctive features of the over-the-counter structure of the actual fed funds market: search for counterparties, and bilateral negotiations. In the theory, banks are required to hold a certain level of end-of-day reserve balances and participate in the fed funds market to achieve this target. We model the fed funds market as an over-the-counter market

[1] The recent financial crisis has underscored the importance of having well-functioning interbank markets. See Acharya and Merrouche (2009) and Afonso, Kovner and Schoar (2010).

in which banks randomly contact other banks, and once they meet, bargain over the terms of the loans. We first put the theory to work by providing theoretical answers to a number of elementary normative and positive questions: What are the determinants of the fed funds rate? What accounts for the dispersion in observed rates? How does the market reallocate funds? Is the over-the-counter market structure able to achieve an efficient reallocation of funds?

We also use the model for theoretical and quantitative analyses of several policy-relevant issues. We explore the ability of the theory to account for the most salient empirical regularities of the market, such as the dependence of the interest rate on the time of the day when it was negotiated, on the borrower and the lender's reserve balances at the time of the transaction, and the degree of activity of the borrower and the lender in the fed funds market. We use the theory to identify the determinants of trade volume and trading delays. We describe the equilibrium dynamics of the fed fund balances of individual banks, and propose theory-based measures of the importance of bank-provided intermediation in the process of reallocation of fed funds among banks. The baseline model has banks that only differ in their initial holdings of reserve balances. We also develop extensions that allow for ex-ante heterogeneity in bank types. Each extension is motivated by a particular aspect of the fed funds market that our baseline model has abstracted from. One extension allows banks to differ in their bargaining strengths. Another allows for heterogeneity in the rate at which banks contact potential trading partners. A third extension allows for the fact that policy may induce heterogeneity in the fed fund participants' payoffs from holding end-of-day balances. For example, the Federal Reserve remunerates the reserve balances of some participants, e.g., depository institutions, but not others, e.g., Government Sponsored Enterprises (GSEs).

This paper is related to the early theoretical research on the federal funds market which includes the micro model of Ho and Saunders (1985) and the stochastic general equilibrium model of Coleman, Christian and Labadie (1996). Relative to the existing literature on the fed funds market, our contribution is to model the intraday allocation and pricing of overnight loans of federal funds using a dynamic equilibrium search-theoretic framework that captures the salient features of the decentralized interbank market in which these loans are traded. Recently, the search-theoretic techniques introduced in labor economics by Diamond (1982a, 1982b), Mortensen (1982) and Pissarides (1985) have been extended and applied to other fields. Our work is related to a young literature that studies search and bargaining frictions in financial markets. To date, this literature consists of two subfields: one that deals with macro issues, and another that focuses on micro considerations in the market microstructure tradition.

On the macro side, for instance, Lagos (2010a, 2010b, 2010c) uses versions of the Lagos and Wright (2005) search-based model of exchange to study the effect of liquidity and monetary policy on asset prices. On the micro side, the influential work of Duffie, Gârleanu and Pedersen (2005) was the first to use search-theoretic techniques to model the trading frictions characteristic of real-world over-the-counter markets. Their work has been extended by Lagos and Rocheteau (2007, 2009) to allow for general preferences and unrestricted long positions, and by Vayanos and Wang (2007) and Weill (2008) to allow investors to trade multiple assets. Duffie, Gârleanu and Pedersen (2007) incorporate risk aversion and risk limits, and Afonso (2010) endogenizes investors' entry decision to the market. Relative to this particular micro branch of the literature, our contribution is twofold. First, our model of the fed funds market provides a theoretical framework to interpret and rationalize the findings of existing empirical investigations of this market, such as Furfine (1999), Ashcraft and Duffie (2007), Bech and Atalay (2008), and Afonso, Kovner and Schoar (2010). Our second contribution is methodological: we offer the first analytically tractable formulation of a search-based model of an over-the-counter market in which all trade is bilateral, and agents can hold essentially unrestricted asset positions.[2]

2 The model

There is a large population of agents that we refer to as *banks*, each represented by a point in the interval $[0,1]$. Banks hold integer amounts of an asset that we interpret as *reserve balances*, and can negotiate these balances during a trading session set in continuous time that starts at time 0 and ends at time T. Let τ denote the time remaining until the end of the trading session, so $\tau = T - t$ if the current time is $t \in [0,T]$. The reserve balance that a bank holds (e.g., at its Federal Reserve account) at time $T-\tau$ is denoted by $k(\tau) \in \mathbb{K}$, with $\mathbb{K} = \{0, 1, ..., K\}$, where $K \in \mathbb{Z}$ and $1 \leq K$. The measure of banks with balance k at time $T-\tau$ is denoted $n_k(\tau)$. A bank starts the trading session with some balance $k(T) \in \mathbb{K}$. The initial distribution of balances, $\{n_k(T)\}_{k\in\mathbb{K}}$, is given. Let $u_k \in \mathbb{R}$ denote the flow payoff to a bank from holding k balances during the trading session, and let $U_k \in \mathbb{R}$ be the payoff from holding k balances at the end of the trading session. All banks discount payoffs at rate r.

[2]In contrast, the tractability of the model of Lagos and Rocheteau (2009) (the only other tractable formulation of a search-based over-the-counter market with unrestricted asset holdings) relies on the assumption that all trade among investors is intermediated by dealers who have continuous access to a competitive interdealer market. While there are several instances of such pure dealer markets, the market for federal funds is not one of them.

Banks can trade balances with each other in an over-the-counter market where trading opportunities are bilateral and random, and represented by a Poisson process with arrival rate $\alpha > 0$. We model these bilateral transactions as loans of reserve balances. Once two banks have made contact, they bargain over the size of the loan and the quantity of reserve balances to be repaid by the borrower. After the terms of the transaction have been agreed upon, the banks part ways. We assume that (signed) loan sizes are elements of the set $\bar{\mathbb{K}} = \mathbb{K} \cup \{-K, ..., -1\}$, and that every loan gets repayed at time $T+\Delta$ in the following trading day, where $\Delta \in \mathbb{R}_+$. Let $x \in \mathbb{R}$ denote the net credit position (of federal funds due at $T+\Delta$) that has resulted from some history of trades. We assume that the payoff to a bank with a net credit position x who makes a new loan at time $T - \tau$ with repayment R at time $T + \Delta$, is equal to the post-transaction discounted net credit position, $e^{-r(\tau+\Delta)}(x + R)$.

3 Institutional features of the market for federal funds

The market for federal funds is a market for unsecured loans of reserve balances at the Federal Reserve Banks, that allows participants with excess reserve balances to lend balances (or *sell funds*) to those with reserve balance shortages. These unsecured loans, commonly referred to as *fed(eral) funds*, are delivered on the same day and their duration is typically overnight.[3] The interest rate on these loans is known as the *fed funds rate*. Participants include commercial banks, thrift institutions, agencies and branches of foreign banks in the United States, government securities dealers, government agencies such as federal or state governments, and GSEs (e.g., Freddie Mac, Fannie Mae, and Federal Home Loan Banks). The market for fed funds is an over-the-counter market: in order to trade, a financial institution must first find a willing counterparty, and then bilaterally negotiate the size and rate of the loan. We use a search-based model to capture the over-the-counter nature of this market.[4]

In practice, there are two ways of trading federal funds. Two participants can contact each other directly and negotiate the terms of a loan, or they can be matched by a fed funds

[3]There is a "term" federal funds market where maturities range from a few days to more than a year, with most loans having a maturity of no more than six months. The amount of term federal funds outstanding has been estimated to be on the order of one-tenth of the amount of overnight loans traded on a given day (see Meulendyke, 1998).

[4]There is a growing search-theoretic literature on financial markets which includes Afonso (2010), Duffie, Gârleanu, and Pedersen (2005, 2007), Gârleanu (2009), Lagos and Rocheteau (2007, 2009), Lagos, Rocheteau, and Weill (2010), Miao (2006), Rust and Hall (2003), Spulber (1996), Vayanos and Wang (2007), Vayanos and Weill (2008), and Weill (2007, 2008), just to name a few. See Ashcraft and Duffie (2007) for more on the over-the-counter nature of the fed funds market.

broker. Non-brokered transactions represent the bulk of the volume of fed funds loans, so we abstract from brokers in our baseline model.[5] Most Fed funds loans are settled through *Fedwire Funds Services*, a large-value real-time gross settlement system operated by the Federal Reserve Banks. More than 7,000 Fedwire participants can lend and borrow in the fed funds market including commercial banks, thrift institutions, government securities dealers, federal agencies, and agencies and branches of foreign banks in the United States. In 2008, the average daily number of borrowers and lenders was 164 and 255, respectively.[6]

Fedwire operates 21.5 hours each business day, from 9.00 pm Eastern Time (ET) on the preceding calendar day to 6.30 pm ET. On a typical day, institutions receive the repayments corresponding to the fed funds loans sold the previous day, before they send out the new loans. In 2006, the average value-weighted time of repayment was 3.09 pm $\pm$ 9 minutes, while the average time of delivery was 4.30 pm $\pm$ 7 minutes. The average duration of a loan was 22 hours and 39 minutes.[7] For simplicity, in our theory we take as given that every loan gets repaid after the end of the operating day, at a fixed time $T + \Delta$.

Fed funds activity is concentrated in the last two hours of the operating day. For a typical bank, for example, until mid afternoon transactions reflect its primary business activities. Later in the day, the trading and payment activity is orchestrated by the fed funds trading desk and aimed at achieving a target balance of fed funds. In 2008, more than 75 percent of the value of fed funds traded among banks was traded after 4:00 pm.[8] By this time, each bank has a balance of reserves resulting from previous activities which is taken as given by the bank's fed funds trading desk.[9] We think of $t = 0$ as standing in for 4:00 pm and model the distribution

[5] Ashcraft and Duffie (2007) report that non-brokered transactions represented 73 percent of the volume of federal funds traded in 2005. Federal fund brokers do not take positions themselves; they only act as matchmakers, bringing buyers and sellers together.

[6] See Afonso, Kovner and Schoar (2010).

[7] This is documented in Bech and Atalay (2008).

[8] In line with this observation, Bartolini et al. (2005) and Bech and Atalay (2008) report very high fed funds loan activity during the latter part of the trading session. (See, for example, the illustrations of intraday loan networks for each half hour in a trading day in their Figure 6.)

[9] For example, as reported by Ashcraft and Duffie (2007), at some large banks, federal funds traders responsible for managing the bank's fed funds balance ask other profit centers of their bank to avoid large unscheduled transactions (e.g., currency trades) near the end of the day. Toward the end of the trading session, once the fed funds trading desk has a good estimate of the send and receive transactions pending until the end of the day, it begins adjusting its trading negotiations to push the bank's balances in the desired direction. Also in line with this observation, Bartolini et al. (2005) attribute the late afternoon rise in fed funds trading activity to the clustering of institutional deadlines, e.g., the settlement of securities transactions ends at 15:00, causing some institutions to defer much of their money market trading until after that time, once their security-related balance sheet position becomes certain. Uncertainty about client transactions and other payment flows diminishes in the hour or two before Fedwire closes at 18:30, which also contributes to the concentration of fed funds trading

of actual reserve balances given to the bank's fed funds trading desk at this time, with the initial condition $\{n_k(T)\}_{k\in\mathbb{K}}$. Fed funds transactions are usually made in round lots of over \$1 million.[10] In 2008, the average loan size was \$148.5 million while the median loan was \$50 million. The most common loan sizes were \$50 million, \$100 million and \$25 million. To keep the analytics tractable, we assume discrete loan sizes in our model.

The motives for trading federal funds may vary across participants and their specific circumstances on any given day. In general, however, there are two main reasons why institutions borrow and lend federal funds. First, some institutions such as commercial banks use the fed funds market to offset the effects on their fed funds balances of transactions (either initiated by their clients or by profit centers within the banks themselves) that would otherwise leave them with a reserve position that does not meet Federal Reserve regulations. Also, some participants regard fed funds loans as an investment vehicle; an interest-yielding asset that can be used to "park" balances overnight. In our model, all payoff-relevant policy and regulatory considerations are captured by the intraday and end-of-day payoffs, $\{u_k, U_k\}_{k\in\mathbb{K}}$.

4 Equilibrium

Let $J_k(x,\tau)$ be the maximum attainable payoff to a bank that holds k units of reserve balances and whose net credit position is x, when the time until the end of the trading session is τ. Let $\boldsymbol{s} = (k,x) \in \mathbb{K}\times\mathbb{R}$ denote the bank's individual state, then

$$J_k(x,\tau) = \mathbb{E}\left\{\int_0^{\min(\tau_\alpha,\tau)} e^{-rz}u_k dz + \mathbb{I}_{\{\tau_\alpha>\tau\}}e^{-r\tau}\left(U_k + e^{-r\Delta}x\right)\right. \tag{1}$$
$$\left. + \mathbb{I}_{\{\tau_\alpha\leq\tau\}}e^{-r\tau_\alpha}\int J_{k-b_{\boldsymbol{s}\boldsymbol{s}'}(\tau-\tau_\alpha)}\left(x + R_{\boldsymbol{s}'\boldsymbol{s}}(\tau-\tau_\alpha), \tau-\tau_\alpha\right)\mu\left(d\boldsymbol{s}', \tau-\tau_\alpha\right)\right\},$$

where $\mathbb{E}$ is an expectation operator over the exponentially distributed random time until the next trading opportunity, τ_α, and $\mathbb{I}_{\{\tau_\alpha\leq\tau\}}$ is an indicator function that equals 1 if $\tau_\alpha \leq \tau$ and 0 otherwise. For each time $\tau \in [0,T]$ until the end of the trading session, $\mu(\cdot,\tau)$ is a probability measure (on the Borel σ-field of the subsets of $\mathbb{K}\times\mathbb{R}$) that describes the heterogeneity of potential trading partners over individual states, $\boldsymbol{s}' = (k',x')$. The pair $(b_{\boldsymbol{s}\boldsymbol{s}'}(\tau-\tau_\alpha), R_{\boldsymbol{s}'\boldsymbol{s}}(\tau-\tau_\alpha))$ denotes the bilateral terms of trade between a bank with state $\boldsymbol{s}$ and a (randomly drawn) bank with state $\boldsymbol{s}'$, when the remaining time is $\tau-\tau_\alpha$. That is, $b_{\boldsymbol{s}\boldsymbol{s}'}(\tau-\tau_\alpha)$ is the amount of balances

activity late in the day.

[10] See Furfine (1999) and Stigum (1990).

that the bank with state $\boldsymbol{s}$ lends to the bank with state $\boldsymbol{s}'$, and $R_{\boldsymbol{s}'\boldsymbol{s}}(\tau-\tau_\alpha)$ is the amount of balances that the latter commits to repay at time $T+\Delta$.

For all $\tau \in [0,T]$ and any $(\boldsymbol{s},\boldsymbol{s}')$ with $\boldsymbol{s},\boldsymbol{s}' \in \mathbb{K}\times\mathbb{R}$, we take $(b_{\boldsymbol{s}\boldsymbol{s}'}(\tau), R_{\boldsymbol{s}'\boldsymbol{s}}(\tau))$ to be the outcome corresponding to the symmetric Nash solution to a bargaining problem.[11] For all $(k,k') \in \mathbb{K}\times\mathbb{K}$, the set

$$\Pi\left(k,k'\right)=\left\{\left(k+k'-y,y\right)\in\mathbb{K}\times\mathbb{K}:y\in\left\{0,1,\ldots,k+k'\right\}\right\}$$

contains all feasible pairs of post-trade balances that could result from the bilateral bargaining between two banks with balances k and k'. This set embeds the restriction that an increase in one bank's balance must correspond to an equal decrease in the other bank's balance, and that no bank can transfer more balances than it currently holds. For every pair of banks that hold $(k,k') \in \mathbb{K}\times\mathbb{K}$, the set $\Pi(k,k')$ induces the set of all feasible (signed) loan sizes,

$$\Gamma\left(k,k'\right)=\left\{b\in\bar{\mathbb{K}}:\left(k-b,k'+b\right)\in\Pi\left(k,k'\right)\right\}.$$

Notice that $\Pi(k,k')=\Pi(k',k)$, and $\Gamma(k,k')=-\Gamma(k',k)$ for all $k,k'\in\mathbb{K}$. The bargaining outcome, $(b_{\boldsymbol{s}\boldsymbol{s}'}(\tau), R_{\boldsymbol{s}'\boldsymbol{s}}(\tau))$, is the pair (b,R) that solves

$$\max_{b\in\Gamma(k,k'),R\in\mathbb{R}}\left[J_{k-b}\left(x+R,\tau\right)-J_k\left(x,\tau\right)\right]^{\frac{1}{2}}\left[J_{k'+b}\left(x'-R,\tau\right)-J_{k'}\left(x',\tau\right)\right]^{\frac{1}{2}}.$$

In the appendix (Lemma 2) we show that

$$J_k\left(x,\tau\right)=V_k\left(\tau\right)+e^{-r(\tau+\Delta)}x \tag{2}$$

satisfies (1), if and only if $V_k(\tau):\mathbb{K}\times[0,T]\to\mathbb{R}$ satisfies

$$\begin{aligned} V_k\left(\tau\right)=\mathbb{E}\Bigg\{&\int_0^{\min(\tau_\alpha,\tau)}e^{-rz}u_k dz+\mathbb{I}_{\{\tau_\alpha>\tau\}}e^{-r\tau}U_k \qquad (3)\\ &+\mathbb{I}_{\{\tau_\alpha\le\tau\}}e^{-r\tau_\alpha}\sum_{k'\in\mathbb{K}}n_{k'}\left(\tau-\tau_\alpha\right)\left[V_{k-b_{kk'}(\tau-\tau_\alpha)}\left(\tau-\tau_\alpha\right)+e^{-r(\tau+\Delta-\tau_\alpha)}R_{k'k}\left(\tau-\tau_\alpha\right)\right]\Bigg\}, \end{aligned}$$

[11] This axiomatic Nash solution can also be obtained from a strategic bargaining game in which, upon contact, Nature selects one of the banks with probability a half to make a take-it-or-leave-it offer which the other bank must either accept or reject on the spot. It is easy to verify that the expected equilibrium outcome of this game coincides with the solution to the Nash bargaining problem, subject to the obvious reinterpretation of $R_{\boldsymbol{s}'\boldsymbol{s}}(\tau)$ as an *expected* repayment, which is inconsequential. See Appendix C in Lagos and Rocheteau (2009).

for all $(k,\tau) \in \mathbb{K} \times [0,T]$, with

$$b_{kk'}(\tau) \in \arg\max_{b \in \Gamma(k,k')} \left[V_{k'+b}(\tau) + V_{k-b}(\tau) - V_{k'}(\tau) - V_k(\tau)\right] \tag{4}$$

$$e^{-r(\tau+\Delta)} R_{k'k}(\tau) = \frac{1}{2}\left[V_{k'+b_{kk'}(\tau)}(\tau) - V_{k'}(\tau)\right] + \frac{1}{2}\left[V_k(\tau) - V_{k-b_{kk'}(\tau)}(\tau)\right]. \tag{5}$$

In (4) and (5), we use $(b_{kk'}(\tau), R_{k'k}(\tau))$ (rather than $(b_{\boldsymbol{s}\boldsymbol{s}'}(\tau), R_{\boldsymbol{s}'\boldsymbol{s}}(\tau))$) to denote the bargaining outcome between a bank with individual state $\boldsymbol{s} \in \mathbb{K} \times \mathbb{R}$ and a bank with individual state $\boldsymbol{s}' \in \mathbb{K} \times \mathbb{R}$, in order to stress that this outcome is independent of the banks' net credit positions, x and x'. Hereafter, we use $\boldsymbol{V} \equiv [\boldsymbol{V}(\tau)]_{\tau \in [0,T]}$, with $\boldsymbol{V}(\tau) \equiv \{V_k(\tau)\}_{k \in \mathbb{K}}$, to denote the value function in (3).

When a pair of banks meet, they jointly decide on the size of the loan and the size of the repayment. The loan size determines the gain from trade, and the repayment implements a division of this gain between the borrower and the lender. For example, suppose that a bank with $i \in \mathbb{K}$ balances and a bank with $j \in \mathbb{K}$ balances meet with time τ until the end of the trading session, and negotiate a loan of size $b_{ij}(\tau) = i - k = s - j \in \Gamma(i,j)$. Then the implied joint gain from trade, the (*match*) *surplus*, corresponding to this transaction is

$$S_{ij}^{ks}(\tau) \equiv V_k(\tau) + V_s(\tau) - V_i(\tau) - V_j(\tau). \tag{6}$$

Thus, according to (4), the bargaining outcome always involves a loan size that maximizes the surplus. According to (5), the size of the repayment is chosen such that each bank's individual gain from trade equals a fraction of the joint gain from trade, with that fraction being equal to the bank's bargaining power. To see this more clearly, note that (2), (4) and (5) imply that the gain from trade to a bank with balance k who trades with a bank with balance k' when the time remaining is τ, namely $J_{k-b_{kk'}(\tau)}(x + R_{k'k}(\tau), \tau) - J_k(x,\tau)$, equals

$$\begin{aligned} & V_{k-b_{kk'}(\tau)}(\tau) + e^{-r(\tau+\Delta)} R_{k'k}(\tau) - V_k(\tau) \\ &= \frac{1}{2}\left[V_{k'+b_{kk'}(\tau)}(\tau) + V_{k-b_{kk'}(\tau)}(\tau) - V_{k'}(\tau) - V_k(\tau)\right]. \end{aligned} \tag{7}$$

Consider a bank with i balances that contacts a bank with j balances when the time until the end of the trading session is τ. Let $\phi_{ij}^{ks}(\tau)$ be the probability that the former and the latter hold k and s balances after the meeting, respectively, i.e., $\phi_{ij}^{ks}(\tau) \in [0,1]$, with $\sum_{k \in \mathbb{K}} \sum_{s \in \mathbb{K}} \phi_{ij}^{ks}(\tau) = 1$. Feasibility requires that $\phi_{ij}^{ks}(\tau) = 0$ if $(k,s) \notin \Pi(i,j)$. Given any feasible path for the distribution of trading probabilities, $\boldsymbol{\phi}(\tau) = \{\phi_{ij}^{ks}(\tau)\}_{i,j,k,s\in\mathbb{K}}$, the distribution of balances at time $T-\tau$, i.e., $\boldsymbol{n}(\tau) = \{n_k(\tau)\}_{k\in\mathbb{K}}$, evolves according to

$$\dot{n}_k(\tau) = f[\boldsymbol{n}(\tau), \boldsymbol{\phi}(\tau)] \quad \text{for all } k \in \mathbb{K}, \tag{8}$$

where

$$\begin{aligned} f[\boldsymbol{n}(\tau), \boldsymbol{\phi}(\tau)] \equiv\ & \alpha n_k(\tau) \sum_{i\in\mathbb{K}} \sum_{j\in\mathbb{K}} \sum_{s\in\mathbb{K}} n_i(\tau)\, \phi_{ki}^{sj}(\tau) \\ & - \alpha \sum_{i\in\mathbb{K}} \sum_{j\in\mathbb{K}} \sum_{s\in\mathbb{K}} n_i(\tau)\, n_j(\tau)\, \phi_{ij}^{ks}(\tau). \end{aligned} \tag{9}$$

The first term on the right side of (9) contains the total flow of banks that leave state k between time $t = T-\tau$ and time $t' = T-(\tau-\varepsilon)$ for a small $\varepsilon > 0$. The second term contains the total flow of banks into state k over the same interval of time.

The following proposition provides a sharper representation of the value function and the distribution of trading probabilities characterized in (3), (4) and (5).

Proposition 1 *The value function $\boldsymbol{V}$ satisfies (3), with (4) and (5), if and only if it satisfies*

$$\begin{aligned} V_i(\tau) = {} & v_i(\tau) + \alpha \int_0^\tau V_i(z)\, e^{-(r+\alpha)(\tau-z)} dz \\ & + \frac{\alpha}{2} \int_0^\tau \sum_{j\in\mathbb{K}} \sum_{k\in\mathbb{K}} \sum_{s\in\mathbb{K}} n_j(z)\, \phi_{ij}^{ks}(z) \left[V_k(z) + V_s(z) - V_i(z) - V_j(z)\right] e^{-(r+\alpha)(\tau-z)} dz, \end{aligned} \tag{10}$$

for all $(i,\tau) \in \mathbb{K} \times [0,T]$, with

$$v_i(\tau) = \left[1 - e^{-(r+\alpha)\tau}\right] \frac{u_i}{r+\alpha} + e^{-(r+\alpha)\tau} U_i, \tag{11}$$

for all $i \in \mathbb{K}$, and

$$\phi_{ij}^{ks}(\tau) = \begin{cases} \tilde{\phi}_{ij}^{ks}(\tau) & \text{if } (k,s) \in \Omega_{ij}[\boldsymbol{V}(\tau)] \\ 0 & \text{if } (k,s) \notin \Omega_{ij}[\boldsymbol{V}(\tau)], \end{cases} \tag{12}$$

for all $i,j,k,s \in \mathbb{K}$ and all $\tau \in [0,T]$, where $\tilde{\phi}_{ij}^{ks}(\tau) \geq 0$ and $\sum_{k\in\mathbb{K}} \sum_{s\in\mathbb{K}} \tilde{\phi}_{ij}^{ks}(\tau) = 1$, with

$$\Omega_{ij}[\boldsymbol{V}(\tau)] \equiv \arg \max_{(k',s')\in\Pi(i,j)} \left[V_{k'}(\tau) + V_{s'}(\tau) - V_i(\tau) - V_j(\tau)\right]. \tag{13}$$

The set $\Omega_{ij}[\boldsymbol{V}(\tau)]$ contains all the feasible pairs of post-trade balances that maximize the match surplus between a bank with i balances and a bank with j balances that is implied by the value function $\boldsymbol{V}(\tau)$ at time $T-\tau$. For any pair of banks with balances i and j, $\phi_{ij}^{ks}(\tau)$ defined in (12) is a probability distribution over the feasible pairs of post-trade portfolios that maximize the bilateral gain from trade

Definition 1 *An equilibrium is a value function,* $\boldsymbol{V}$*, a path for the distribution of reserve balances,* $\boldsymbol{n}(\tau)$*, and a path for the distribution of trading probabilities,* $\boldsymbol{\phi}(\tau)$*, such that: (a) given the value function and the distribution of trading probabilities, the distribution of balances evolves according to (8); and (b) given the path for the distribution of balances, the value function and the distribution of trading probabilities satisfy (10) and (12).*

Assumption A. For any $i, j \in \mathbb{K}$, and all $(k, s) \in \Pi(i, j)$, the payoff functions satisfy:

$$u_{\lceil \frac{i+j}{2} \rceil} + u_{\lfloor \frac{i+j}{2} \rfloor} \geq u_k + u_s \tag{DMC}$$

$$U_{\lceil \frac{i+j}{2} \rceil} + U_{\lfloor \frac{i+j}{2} \rfloor} \geq U_k + U_s, \text{ “} > \text{” unless } k \in \left\{ \left\lfloor \frac{i+j}{2} \right\rfloor, \left\lceil \frac{i+j}{2} \right\rceil \right\}, \tag{DMSC}$$

where $\lfloor x \rfloor \equiv \max\{k \in \mathbb{Z} : k \leq x\}$ and $\lceil x \rceil \equiv \min\{k \in \mathbb{Z} : x \leq k\}$ for any $x \in \mathbb{R}$.

In the appendix (Lemma 3) we show that conditions (DMC) and (DMSC) are equivalent to requiring that the payoff functions $\{u_k\}_{k \in \mathbb{K}}$ and $\{U_k\}_{k \in \mathbb{K}}$ satisfy *discrete midpoint concavity*, and *discrete midpoint strict concavity*, respectively. These are the natural discrete approximations to the notions of *midpoint concavity* and *midpoint strict concavity* of ordinary functions defined on convex sets.[12]

The following result provides a full characterization of equilibrium under Assumption A.

Proposition 2 *Let the payoff functions satisfy Assumption A. Then:*

(i) *An equilibrium exists, and the equilibrium paths for the maximum attainable payoffs,* $\boldsymbol{V}(\tau)$*, and the distribution of reserve balances,* $\boldsymbol{n}(\tau)$*, are uniquely determined.*

(ii) *The equilibrium path for the distribution of trading probabilities,* $\boldsymbol{\phi}(\tau) = \{\phi_{ij}^{ks}(\tau)\}_{i,j,k,s \in \mathbb{K}}$*, is given by*

$$\phi_{ij}^{ks}(\tau) = \begin{cases} \tilde{\phi}_{ij}^{ks}(\tau) & \text{if } (k,s) \in \Omega_{ij}^* \\ 0 & \text{if } (k,s) \notin \Omega_{ij}^* \end{cases} \tag{14}$$

for all $i, j, k, s \in \mathbb{K}$ *and all* $\tau \in [0, T]$*, where* $\tilde{\phi}_{ij}^{ks}(\tau) \geq 0$ *and* $\sum_{(k,s) \in \Omega_{ij}^*} \tilde{\phi}_{ij}^{ks}(\tau) = 1$*, where*

$$\Omega_{ij}^* = \begin{cases} \left\{ \left(\frac{i+j}{2}, \frac{i+j}{2} \right) \right\} & \text{if } i+j \text{ is even} \\ \left\{ \left(\left\lfloor \frac{i+j}{2} \right\rfloor, \left\lceil \frac{i+j}{2} \right\rceil \right), \left(\left\lceil \frac{i+j}{2} \right\rceil, \left\lfloor \frac{i+j}{2} \right\rfloor \right) \right\} & \text{if } i+j \text{ is odd.} \end{cases} \tag{15}$$

[12] Let X be a convex subset of $\mathbb{R}^n$, then a function $g : X \to \mathbb{R}$ is said to be *concave* if $g(\epsilon x + (1-\epsilon) y) \geq \epsilon g(x) + (1-\epsilon) g(y)$ for all $x, y \in X$, and all $\epsilon \in [0, 1]$. The function g is *midpoint concave* if $2g\left(\frac{x+y}{2}\right) \geq g(x) + g(y)$ for all $x, y \in X$. Clearly, if g is concave then it is midpoint concave. The converse is true provided g is continuous. The function $g : \mathbb{K} \to \mathbb{R}$ satisfies the *discrete midpoint concavity property* if $g\left(\left\lceil \frac{i+j}{2} \right\rceil\right) + g\left(\left\lfloor \frac{i+j}{2} \right\rfloor\right) \geq u_i + u_j$ for all $i, j \in \mathbb{K}$. See Murota (2003) for more on the midpoint concavity/convexity property and the role that it plays in the modern theory of discrete convex analysis.

(iii) $\boldsymbol{V}$ *is the unique bounded real-valued function that satisfies*

$$rV_i(\tau) + \dot{V}_i(\tau) = u_i + \frac{\alpha}{2}\sum_{j\in\mathbb{K}}\sum_{k\in\mathbb{K}}\sum_{s\in\mathbb{K}} n_j(\tau)\,\phi_{ij}^{ks}(\tau)\left[V_k(\tau) + V_s(\tau) - V_i(\tau) - V_j(\tau)\right] \quad (16)$$

for all $(i,\tau) \in \mathbb{K} \times [0,T]$*, with*

$$V_i(0) = U_i \quad \textit{for all } i \in \mathbb{K}, \quad (17)$$

and with the path for $\boldsymbol{\phi}(\tau)$ *given by (14), and the path for* $\boldsymbol{n}(\tau)$ *given by* $\dot{\boldsymbol{n}}(\tau) = f[\boldsymbol{n}(\tau), \boldsymbol{\phi}(\tau)]$.

(iv) *Suppose that at time* $T-\tau$*, a bank with balance* j *extends a loan of size* $j-s=k-i$ *to a bank with balance* i*. The present value of the equilibrium repayment from the latter to the former is*

$$e^{-r(\tau+\Delta)}R_{ij}^{ks}(\tau) = \frac{1}{2}\left[V_k(\tau) - V_i(\tau)\right] + \frac{1}{2}\left[V_j(\tau) - V_s(\tau)\right]. \quad (18)$$

The equilibrium distribution of trading probabilities (14) can be described intuitively as follows. At any point during the trading session, if a bank with balance i contacts a bank with balance j, then the post-transaction balance will necessarily be $\left\lfloor \frac{i+j}{2} \right\rfloor$ for one of the banks, and $\left\lceil \frac{i+j}{2} \right\rceil$ for the other. This property, and the uniqueness of the equilibrium paths for the distribution of reserve balances and maximum payoffs, hold under Assumption A. In the appendix (Corollary 1) we show that if we instead assume that u satisfies *discrete midpoint strict concavity* and U satisfies *discrete midpoint concavity*, then the existence and uniqueness results in Proposition 2 still hold.

5 Efficiency

In this section we use our theory to characterize the optimal process of reallocation of reserve balances in the fed funds market. The spirit of the exercise is to take as given the market structure, including the contact rate α and the regulatory variables $\{u_k, U_k\}_{k\in\mathbb{K}}$, and to ask whether decentralized trade in the over-the-counter market structure reallocates reserve balances efficiently, given these institutions. To this end, we study the problem of a social planner who solves

$$\begin{aligned}
&\max_{[\boldsymbol{\chi}(t)]_{t=0}^{T}} \left[\int_0^T \sum_{k\in\mathbb{K}} m_k(t)\, u_k e^{-rt} dt + e^{-rT} \sum_{k\in\mathbb{K}} m_k(T)\, U_k\right] \\
&\text{s.t. } \dot{m}_k(t) = -f\left[\boldsymbol{m}(t), \boldsymbol{\chi}(t)\right], \\
&\qquad \chi_{ij}^{ks}(t) \in [0,1], \text{ with } \chi_{ij}^{ks}(t) = 0 \text{ if } (k,s) \notin \Pi(i,j), \\
&\qquad \chi_{ij}^{ks}(t) = \chi_{ji}^{sk}(t), \text{ and } \sum_{k\in\mathbb{K}}\sum_{s\in\mathbb{K}} \chi_{ij}^{ks}(t) = 1,
\end{aligned} \tag{19}$$

for all $t \in [0,T]$, and all $i,j,k,s \in \mathbb{K}$. We have formulated the planner's problem in chronological time, so $m_k(t)$ denotes the measure of banks with balance k at time t. Since $\tau \equiv T - t$, we have $m_k(t) = m_k(T-\tau) \equiv n_k(\tau)$, and therefore $\dot{m}_k(t) = -\dot{n}_k(\tau)$. Hence the flow constraint is the real-time law of motion for the distribution of balances implied by the bilateral stochastic trading process. The control variable, $\boldsymbol{\chi}(t) = \{\chi_{ij}^{ks}(t)\}_{i,j,k,s\in\mathbb{K}}$, represents the planner's choice of reallocation of balances between any pair of banks that have contacted each other at time t. The first, second, and fourth constraints on $\boldsymbol{\chi}(t)$ ensure that $\{\chi_{ij}^{ks}(t)\}_{k,s\in\mathbb{K}}$ is a probability distribution for each $i,j \in \mathbb{K}$, and that the planner only chooses among feasible reallocations of balances between a pair of banks. We look for a solution that does not depend on the identities or "names" of banks, so the third constraint on $\boldsymbol{\chi}(t)$ recognizes the fact that $\chi_{ij}^{ks}(t)$ and $\chi_{ji}^{sk}(t)$ represent the same decision for the planner. That is, $\chi_{ij}^{ks}(t)$ and $\chi_{ji}^{sk}(t)$ both represent the probability that a pair of banks with balances i and j who contact each other at time t, exit the meeting with balances k and s, respectively.

Proposition 3 *A solution to the planner's problem is a path for the distribution of balances, $\boldsymbol{n}(\tau)$, a path for the vector of co-states associated with the law of motion for the distribution of balances, $\boldsymbol{\lambda}(\tau) = \{\lambda_k(\tau)\}_{k\in\mathbb{K}}$, and a path for the distribution of trading probabilities, $\boldsymbol{\psi}(\tau) = \{\psi_{ij}^{ks}(\tau)\}_{i,j,k,s\in\mathbb{K}}$. The necessary conditions for optimality are,*

$$r\lambda_i(\tau) + \dot{\lambda}_i(\tau) = u_i + \alpha \sum_{j\in\mathbb{K}}\sum_{k\in\mathbb{K}}\sum_{s\in\mathbb{K}} n_j(\tau)\, \psi_{ij}^{ks}(\tau) \left[\lambda_k(\tau) + \lambda_s(\tau) - \lambda_i(\tau) - \lambda_j(\tau)\right] \tag{20}$$

for all $(i,\tau) \in \mathbb{K} \times [0,T]$, with

$$\lambda_i(0) = U_i \quad \textit{for all } i \in \mathbb{K}, \tag{21}$$

with the path for $\boldsymbol{n}(\tau)$ given by $\dot{\boldsymbol{n}}(\tau) = f\left[\boldsymbol{n}(\tau), \boldsymbol{\psi}(\tau)\right]$, and with

$$\psi_{ij}^{ks}(\tau) = \begin{cases} \tilde{\psi}_{ij}^{ks}(\tau) & \textit{if } (k,s) \in \Omega_{ij}\left[\boldsymbol{\lambda}(\tau)\right] \\ 0 & \textit{if } (k,s) \notin \Omega_{ij}\left[\boldsymbol{\lambda}(\tau)\right], \end{cases} \tag{22}$$

for all $i, j, k, s \in \mathbb{K}$ *and all* $\tau \in [0, T]$*, where* $\tilde{\psi}_{ij}^{ks}(\tau) \geq 0$ *and* $\sum_{k \in \mathbb{K}} \sum_{s \in \mathbb{K}} \tilde{\psi}_{ij}^{ks}(\tau) = 1$.

The following result provides a full characterization of solution to the planner's problem under Assumption A.

Proposition 4 *Let the payoff functions satisfy Assumption A. Then:*

(i) *The optimal path for the distribution of trading probabilities,* $\boldsymbol{\psi}(\tau) = \{\psi_{ij}^{ks}(\tau)\}_{i,j,k,s \in \mathbb{K}}$*, is given by*

$$\psi_{ij}^{ks}(\tau) = \begin{cases} \tilde{\psi}_{ij}^{ks}(\tau) & \text{if } (k,s) \in \Omega_{ij}^{*} \\ 0 & \text{if } (k,s) \notin \Omega_{ij}^{*} \end{cases} \tag{23}$$

for all $i, j, k, s \in \mathbb{K}$ *and all* $\tau \in [0, T]$*, where* $\tilde{\psi}_{ij}^{ks}(\tau) \geq 0$ *and* $\sum_{(k,s) \in \Omega_{ij}^{*}} \tilde{\psi}_{ij}^{ks}(\tau) = 1$.

(ii) *Along the optimal path, the shadow value of a bank with* i *reserve balances is given by (20) and (21), with the path for* $\boldsymbol{\psi}(t)$ *given by (23), and the path for* $\boldsymbol{n}(\tau)$ *given by* $\dot{\boldsymbol{n}}(\tau) = f[\boldsymbol{n}(\tau), \boldsymbol{\psi}(\tau)]$.

Notice the similarity between the equilibrium conditions and planner's optimality conditions. First, from (12) and (22), we see that the equilibrium loan sizes are *privately efficient*. That is, given the value function $\boldsymbol{V}$, the equilibrium distribution of trading probabilities is the one that would be chosen by the planner. Second, the path for the equilibrium values, $\boldsymbol{V}(\tau)$, satisfies (16) and (17), while the path for the planner's shadow prices satisfies (20) and (21). These pairs of conditions would be identical were it not for the fact that the planner imputes to each agent gains from trade with frequency 2α, rather α, which is the frequency with which the agent generates gains from trade for himself in the equilibrium. This reflects a composition externality typical of random matching environments. The planner's calculation of the value of a marginal agent in state i includes not only the expected gain from trade to this agent, but also the expected gains from trade that having this marginal agent in state i generates for all other agents, by increasing their contact rates with agents in state i. In the equilibrium, the individual agent in state i internalizes the former, but not the latter.[13]

Under Assumption A, however, condition (14) is identical to (23), so the equilibrium paths for the distribution of balances and trading probabilities coincide with the optimal paths. This observation is summarized in the following proposition.

[13] In a labor market context, a similar composition externality arises in the competitive matching equilibrium of Kiyotaki and Lagos (2007).

Proposition 5 *Let the payoff functions satisfy Assumption A. Then, the equilibrium supports an efficient allocation of reserve balances.*

6 Positive implications

The performance of the fed funds market as a system that reallocates liquidity among banks, can be appraised by the behavior of empirical measures of the fed funds rate and of the effectiveness of the market to channel funds from banks with excess balances to those with shortages. In this section we derive the theoretical counterparts to these empirical measures, and argue that the theory is consistent with the most salient features of the actual fed funds market. We use the theory to identify the determinants of the fed funds rate, trade volume, and trading delays. We also describe the equilibrium dynamics of the fed fund balances of individual banks, and propose theory-based measures of the importance of bank-provided intermediation in the process of reallocation of fed funds among banks.

6.1 Trade volume and trading delays

The flow volume of trade at time $T-\tau$ is

$$\bar{\upsilon}(\tau)=\sum_{i\in\mathbb{K}}\sum_{j\in\mathbb{K}}\sum_{k\in\mathbb{K}}\sum_{s\in\mathbb{K}}\upsilon_{ij}^{ks}(\tau),$$

where

$$\upsilon_{ij}^{ks}(\tau)=\alpha n_i(\tau)\,n_j(\tau)\,\phi_{ij}^{ks}(\tau)\,|k-i|,$$

and the total volume traded during the whole trading session is

$$\bar{\upsilon}=\int_0^T\bar{\upsilon}(\tau)\,d\tau.$$

Notice that the arrival rate of specific trading opportunities is endogenous, as it depends on the equilibrium distribution of balances. For example, $\alpha n_j(\tau)\,\phi_{ij}^{ks}(\tau)$ is the rate at which agents with balance i trade a balance equal to $k-i$ with agents with balance j at time $T-\tau$. Therefore, even though the contact rate, α, is exogenous in our baseline formulation, trading delays—a key distinctive feature of over-the-counter markets—are determined by agents' trading strategies.

6.2 Fed funds rate

In our baseline formulation, banks negotiate loans and the present value of the loan repayment. It is possible to reformulate the negotiation in terms of a loan size and an interest rate. For example, consider a transaction between a bank with i balances and a bank with j balances in which the former borrows $k-i=j-s$ from the latter. We can think of the corresponding repayment, $R_{ij}^{ks}(\tau)$ in (18), as composed of the principal of the loan, augmented by continuously compounded interest, ρ. That is, we can write $R_{ij}^{ks}(\tau)=e^{\rho(\tau+\Delta)}(k-i)$, and solve for the transaction-specific interest rate,

$$\rho_{ij}^{ks}(\tau)=\frac{\ln\left[\frac{R_{ij}^{ks}(\tau)}{k-i}\right]}{\tau+\Delta}=r+\frac{\ln\left[\frac{V_j(\tau)-V_s(\tau)}{j-s}+\frac{\frac{1}{2}S_{ij}^{ks}(\tau)}{j-s}\right]}{\tau+\Delta}. \tag{24}$$

According to (24), the interest on a loan of size $j-s$ extended by a lender with balance j to a borrower with balance i at time $T-\tau$, is equal to the discount rate, r, plus a premium, which increases with the size of the joint gain from trade, $S_{ij}^{ks}(\tau)$, and with the lender's bargaining power (here equal to 1/2). Notice that according to the theory, there is no such thing as *the* fed funds rate, rather there is a time-varying *distribution* of rates. That is, empirically, in order to "explain" the rate determination in over-the-counter fed fund transactions, one would have to control for the opportunity cost of funds (r), the duration of the loan ($\tau+\Delta$), the size of the loan ($j-s$ in (24)), the bargaining power of the borrower and the lender (1/2 each in (24)), the present discounted value of the loss to the lender from giving up the funds ($V_j(\tau)-V_s(\tau)$), and the present discounted value of the gain to the borrower from obtaining the funds ($V_k(\tau)-V_i(\tau)$), both of which depend on the time until the end of the trading session (τ).

In the theory,

$$\bar{\rho}(\tau)=\sum_{i\in\mathbb{K}}\sum_{j\in\mathbb{K}}\sum_{k\in\mathbb{K}}\sum_{s\in\mathbb{K}}\omega_{ij}^{ks}(\tau)\,\rho_{ij}^{ks}(\tau)$$

is a weighted average of rates at each point in time, where $\omega_{ij}^{ks}(\tau)$ is a weighting function with $\omega_{ij}^{ks}(\tau)\geq 0$ and $\sum_{i,j,k,s\in\mathbb{K}}\omega_{ij}^{ks}(\tau)=1$. For example, if $\omega_{ij}^{ks}(\tau)=\upsilon_{ij}^{ks}(\tau)/\bar{\upsilon}(\tau)$, then $\bar{\rho}(\tau)$ is the value-weighted average fed funds rate at time $T-\tau$. Similarly,

$$\bar{\rho}=\int_0^T\sum_{i\in\mathbb{K}}\sum_{j\in\mathbb{K}}\sum_{k\in\mathbb{K}}\sum_{s\in\mathbb{K}}\omega_{ij}^{ks}(\tau)\,\rho_{ij}^{ks}(\tau)\,d\tau$$

for some weighting function $\omega_{ij}^{ks}(\tau)$ with $\omega_{ij}^{ks}(\tau) \geq 0$ and $\int_0^T \sum_{i,j,k,s \in \mathbb{K}} \omega_{ij}^{ks}(\tau)\, d\tau = 1$, is a weighted average of daily rates. For example, if $\omega_{ij}^{ks}(\tau) = \upsilon_{ij}^{ks}(\tau)/\bar{\upsilon}$, then $\bar{\rho}$ is a value-weighted daily average of rates akin to the *daily effective federal funds rate* published by the Federal Reserve.[14]

6.3 Equilibrium dynamics of fed fund balances

Consider a bank with balance $a(t_0) = i \in \mathbb{K}$ at time $t_0 \in [0, T)$, and let $t_1 \in (t_0, T)$ denote the time at which the bank receives its first trading opportunity on $[t_0, T]$. The probability distribution over post-trade balances at t_1, $a(t_1) \in \mathbb{K}$, is given by

$$\Pr[a(t_1) = j \mid a(t_0) = i] = \sum_{q \in \mathbb{K}} m_q(t_1)\, \phi_{iq}^{jq'}(t_1) \equiv \pi_{ij}(t_1),$$

where $q' \equiv q + i - j$ and $m_q(t_1)$ is the measure of banks with balance q at time t_1. Given a probability measure over $a(t_0) \in \mathbb{K}$, the $(K+1) \times (K+1)$ transition matrix $\mathbf{\Pi}(t_1) = [\pi_{ij}(t_1)]$ records the probabilities of making a transition from any balance $i \in \mathbb{K}$ to any balance $j \in \mathbb{K}$ at trading time t_1. More generally, consider a bank with balance $k_0 \in \mathbb{K}$ at t_0 that has N trading opportunities between time t_0 and time t, e.g., at times $\boldsymbol{t}^{(N)} = (t_1, t_2, \ldots, t_N)$, with $0 \leq t_0 < t_1 < t_2 < \cdots < t_N < t \leq T$. (We adopt the convention $\boldsymbol{t}^{(0)} = t_0$.) Then given the initial balance $k_0 \in \mathbb{K}$ and the realization of trading times $\boldsymbol{t}^{(N)} \in [t_0, t]^N$, the probability distribution over the sequence of post-trade balances at these trading times, i.e., $a(t_n) \in \mathbb{K}$ for all $n = 1, ..., N$, is given by

$$\Pr\left[a(t_1) = k_1, \ldots, a(t_N) = k_N \mid a(t_0) = k_0, \boldsymbol{t}^{(N)}\right] = \prod_{n=1}^{N} \pi_{k_{n-1}k_n}(t_n). \tag{25}$$

Given a probability measure over $a(t_0) \in \mathbb{K}$, the $(K+1) \times (K+1)$ transition matrix

$$\mathbf{\Pi}^{(N)}(\boldsymbol{t}^{(N)}) = \mathbf{\Pi}(t_1) \cdots \mathbf{\Pi}(t_N) \tag{26}$$

records the probabilities of making a transition from any balance $i \in \mathbb{K}$ to any other balance $j \in \mathbb{K}$ in N trades carried out at the realized trading times $\boldsymbol{t}^{(N)} = (t_1, ..., t_N)$. Notice that $\mathbf{\Pi}^{(1)}(\boldsymbol{t}^{(1)}) = \mathbf{\Pi}(t_1)$, and by convention, $\mathbf{\Pi}^{(0)}(\boldsymbol{t}^{(0)}) = \boldsymbol{I}$, where $\boldsymbol{I}$ denotes the $(K+1) \times (K+1)$ identity matrix. The following proposition provides a complete characterization of the stochastic process that rules the equilibrium dynamics of the balance held by an individual bank.

[14] The actual daily effective federal funds rate is a volume-weighted average of rates on trades arranged by major brokers. The Federal Reserve Bank of New York receives summary reports from the brokers, and every morning publishes the effective federal funds rate for the previous day.

Proposition 6 *For any $t_0 \in [0, T)$, and any $t \in [t_0, T]$, the transition function for the stochastic process that rules the equilibrium dynamics of individual balances is*

$$P(t|t_0) = \sum_{N=0}^{\infty} \alpha^N e^{-\alpha(t-t_0)} \int_{\mathbb{T}^{(N)}} \mathbf{\Pi}^{(N)}(\boldsymbol{t}^{(N)}) d\boldsymbol{t}^{(N)}, \tag{27}$$

where $\mathbb{T}^{(N)} = \left\{ \boldsymbol{t}^{(N)} \in [t_0, t]^N : t_0 < t_1 < \cdots < t_N < t \right\}$.

Let $p_{ij}(t|t_0)$ denote the (i, j) entry of the $(K+1) \times (K+1)$ matrix $P(t|t_0)$. Consider a bank with balance $i \in \mathbb{K}$ at time t_0, then $p_{ij}(t|t_0)$ is the probability the bank has balance $j \in \mathbb{K}$ at time t.

6.4 Intermediation and speculative trades

The equilibrium characterized in Proposition 2 (and by Proposition 5, the efficient allocation characterized in Proposition 4) exhibits endogenous intermediation in the sense that many banks act as dealers, buying and selling funds on their own account and channeling them from banks with larger balances to banks with smaller balances. To illustrate, consider a bank that starts the trading session with balance $a(0)$. Suppose, for example, that the bank in question only trades twice in the session, at times t_1 and t_2, with $0 < t_1 < t_2 < T$, first buying $a(t_1) - a(0)$, and then selling $a(t_1) - a(t_2)$, so that it ends the session with a balance $a(t_2)$, where $a(0) < a(t_2) < a(t_1)$. Throughout the trading session, this bank effectively intermediated a volume of funds equal to $a(t_1) - a(t_2)$, buying at time t_1 from a bank with some balance at least as large as $a(t_1)$, and then selling at a later time t_2 to a bank with some balance no larger than $a(t_2)$. This type of intermediation among participants is an important feature of the fed funds market. Next, we propose several theory-based empirical measures of the importance of intermediation in the process of reallocation of fed funds among banks.

Consider a bank with N trading opportunities between time t_0 and time t, e.g., at times $\boldsymbol{t}^{(N)} = (t_1, t_2, ..., t_N)$, with $0 \le t_0 < t_1 < t_2 < \cdots < t_N < t \le T$. Given the initial balance $k_0 \in \mathbb{K}$ and a realization $\boldsymbol{t}^{(N)} \in [t_0, t]^N$, the time-path of the bank's asset holdings during $[t_0, t]$ is described by a function $\boldsymbol{a}_{[t_0,t]} : [t_0, t] \to \mathbb{K}$ defined by

$$\boldsymbol{a}_{[t_0,t]}(x) = \begin{cases} k_0 & \text{for } t_0 \le x < t_1 \\ k_1 & \text{for } t_1 \le x < t_2 \\ \vdots & \qquad \vdots \\ k_N & \text{for } t_N \le x \le t, \end{cases}$$

where $k_n \in \mathbb{K}$ is the post-trade balance at time t_n for $n = 1, ..., N$. Given the initial balance k_0 at t_0, the realized path for a bank's balance during $[t_0, t]$ is completely described by the number of contacts, N, the vector of contact times, $\boldsymbol{t}^{(N)} \in [t_0, t]^N$, and the vector of post-trade balances at those contact times, $\boldsymbol{k}^{(N)} = (k_1, k_2, ..., k_N) \in \mathbb{K}^N$. Given k_0 and $\boldsymbol{k}^{(N)}$, define the bank's accumulated volume of purchases during $[t_0, t]$,

$$O^p(k_0, \boldsymbol{k}^{(N)}) = \sum_{n=1}^{N} \max \{k_n - k_{n-1}, 0\},$$

the accumulated volume of sales,

$$O^s(k_0, \boldsymbol{k}^{(N)}) = -\sum_{n=1}^{N} \min \{k_n - k_{n-1}, 0\},$$

and the *(signed) net trade*, $O^p(k_0, \boldsymbol{k}^{(N)}) - O^s(k_0, \boldsymbol{k}^{(N)}) = k_N - k_0$. Then

$$I(k_0, \boldsymbol{k}^{(N)}) = \min \left\{ O^p(k_0, \boldsymbol{k}^{(N)}), O^s(k_0, \boldsymbol{k}^{(N)}) \right\} \tag{28}$$

measures the volume of funds intermediated by the bank during the time interval $[t_0, t]$. Alternatively, $O^p(k_0, \boldsymbol{k}^{(N)}) + O^s(k_0, \boldsymbol{k}^{(N)})$ is the gross volume of funds traded by the bank, and $|O^p(k_0, \boldsymbol{k}^{(N)}) - O^s(k_0, \boldsymbol{k}^{(N)})|$ is the size of the bank's net trade over the period $[t_0, t]$, so

$$X(k_0, \boldsymbol{k}^{(N)}) = O^p(k_0, \boldsymbol{k}^{(N)}) + O^s(k_0, \boldsymbol{k}^{(N)}) - \left| O^p(k_0, \boldsymbol{k}^{(N)}) - O^s(k_0, \boldsymbol{k}^{(N)}) \right| \tag{29}$$

is a bank-level measure of *excess funds reallocation*, i.e., the volume of funds traded over and above what is required to accommodate the net trade. The measure $X(k_0, \boldsymbol{k}^{(N)})$ is an index of simultaneous buying and selling at the individual bank level during $[t_0, t]$. This leads to

$$\iota(k_0, \boldsymbol{k}^{(N)}) = \frac{X(k_0, \boldsymbol{k}^{(N)})}{O^p(k_0, \boldsymbol{k}^{(N)}) + O^s(k_0, \boldsymbol{k}^{(N)})}$$

as a natural measure of the proportion of the total volume of funds traded by a bank during $[0, t]$, that the bank intermediated during the same time period.

Having described the intermediation behavior of a single bank along a typical sample path, the next proposition shows how to calculate marketwide measures of intermediation.

Proposition 7 *Let $t_0 \in [0, T)$, and $t \in (t_0, T]$. During $[t_0, t]$:*

(i) *The aggregate cumulative volume of purchases (for $j=p$, sales, for $j=s$) is*

$$\bar{O}^{j}\left(t|t_0\right)=\sum_{k_0\in\mathbb{K}}m_{k_0}\left(t_0\right)\sum_{N=0}^{\infty}\alpha^{N}e^{-\alpha(t-t_0)}\int\limits_{\mathbb{T}^{(N)}}\tilde{O}^{j}(k_0,\boldsymbol{t}^{(N)})d\boldsymbol{t}^{(N)},\tag{30}$$

where

$$\tilde{O}^{j}(k_0,\boldsymbol{t}^{(N)})=\sum_{\boldsymbol{k}^{(N)}\in\mathbb{K}^{N}}\left(\prod_{n=1}^{N}\pi_{k_{n-1}k_n}\left(t_n\right)\right)O^{j}(k_0,\boldsymbol{k}^{(N)}).$$

(ii) *The aggregate cumulative volume of intermediated funds is*

$$\bar{I}\left(t|t_0\right)=\frac{1}{2}\bar{X}\left(t|t_0\right),\tag{31}$$

and the proportion of intermediated funds in the aggregate volume of traded funds is

$$\bar{\iota}\left(t|t_0\right)=\frac{\bar{X}\left(t|t_0\right)}{\bar{O}^{p}\left(t|t_0\right)+\bar{O}^{s}\left(t|t_0\right)},$$

where

$$\bar{X}\left(t|t_0\right)=\sum_{k_0\in\mathbb{K}}m_{k_0}\left(t_0\right)\sum_{N=0}^{\infty}\alpha^{N}e^{-\alpha(t-t_0)}\int\limits_{\mathbb{T}^{(N)}}\tilde{X}(k_0,\boldsymbol{t}^{(N)})d\boldsymbol{t}^{(N)}\tag{32}$$

is the aggregate excess reallocation of funds, with

$$\tilde{X}(k_0,\boldsymbol{t}^{(N)})=\sum_{\boldsymbol{k}^{(N)}\in\mathbb{K}^{N}}\left(\prod_{n=1}^{N}\pi_{k_{n-1}k_n}\left(t_n\right)\right)X(k_0,\boldsymbol{k}^{(N)}).$$

Notice that our measure of *excess funds reallocation*, $\bar{X}\left(t|t_0\right)$, is a real-time analogue to the notion of *excess job reallocation* used in empirical studies of job creation and destruction (e.g., Davis, Haltiwanger and Schuh, 1996).

7 Extensions

In this section we develop several extensions of the theory to allow for ex-ante heterogeneity in bank types. Each extension is motivated by a particular aspect of the fed funds market that our baseline model has abstracted from. First, according to practitioners, some banks (e.g., large banks) consistently exhibit a stronger bargaining position when trading against other (e.g., small) banks. Our first extension allows banks to differ in their bargaining power parameter. Second, empirical studies of the fed funds market have emphasized that a few banks trade with much higher intensity than others, and are consistently more likely to act as borrowers and as lenders during the same trading session.[15] Our second extension allows for banks to differ in the rate at which they contact and are contacted by potential trading partners. Third, in practice, in any given trading session institutions may value end-of-day reserve balances differently. For example, some banks may have balance sheets that call for larger balances to meet their reserve requirements. Policy considerations can also induce differences among fed funds participants, as the Federal Reserve remunerates the reserve balances of some participants, e.g., depository institutions, but not others, e.g., Government Sponsored Enterprises (GSEs). Our third extension allows for heterogeneity in the fed fund participants' payoffs from holding end-of-day balances.

For each extension, we describe the evolution of the distribution of balances and the value function, and the determination of the trading decisions, i.e., all the ingredients needed to define equilibrium. In each case, we give the relevant variables a superscript that identifies the bank's type. The set of types, $\mathbb{Y}$, is finite and η^y denotes the fraction of banks of type $y \in \mathbb{Y}$, i.e., $\eta^y \in [0,1]$ and $\sum_{y \in \mathbb{Y}} \eta^y = 1$. The measure of banks of type y with balance k at time $T-\tau$, is denoted $n_k^y(\tau)$, so $\sum_{k \in \mathbb{K}} n_k^y(\tau) = \eta^y$. In a meeting at time $T-\tau$ between a bank of type x with i balances and a bank of type y with j balances, $\phi_{ij,xy}^{ks}(\tau)$ is used to denote the probability that the former and the latter hold k and s balances after the meeting, respectively. In this section, $\boldsymbol{n}(\tau) = \left\{n_k^y(\tau)\right\}_{y \in \mathbb{Y}, k \in \mathbb{K}}$ and $\boldsymbol{V}(\tau) = \left\{V_k^y(\tau)\right\}_{y \in \mathbb{Y}, k \in \mathbb{K}}$ denote the distribution of balances and the value function, respectively, at time $T-\tau$. The distribution of trading probabilities at time $T-\tau$, $\boldsymbol{\phi}(\tau) = \{\{\phi_{ij,xy}^{ks}(\tau)\}_{x,y \in \mathbb{Y}}\}_{i,j,k,s \in \mathbb{K}}$, satisfies $\phi_{ij,xy}^{ks}(\tau) \in [0,1]$ with $\sum_{k \in \mathbb{K}} \sum_{s \in \mathbb{K}} \phi_{ij,xy}^{ks}(\tau) = 1$, and is feasible if $\phi_{ij,xy}^{ks}(\tau) = 0$ if $(k,s) \notin \Pi(i,j)$ for all $i,j,k,s \in \mathbb{K}$ and all $x, y \in \mathbb{Y}$.

7.1 Heterogeneous bargaining powers

Let $\theta_{xy} \in [0,1]$ be the bargaining power of a bank type $x \in \mathbb{Y}$ in negotiations with a bank of type $y \in \mathbb{Y}$, where $\theta_{xy} + \theta_{yx} = 1$.[16] Given any feasible path for the distribution of trading

[15] See Bech and Atalay (2008). The intensity of a bank's trading activity in the fed funds market is also correlated with the interest rates that the bank charges when it lends, and the rates that it pays when it borrows. Ashcraft and Duffie (2007) find that rates tend to be higher on loans that involve lenders who are more active in the federal funds market relative to the borrower. They also document that rates tend to be lower on loans that involve borrowers who are more active relative to the lender.

[16] For example, a natural specification would be $\mathbb{Y} = \{1, \ldots, N\}$ with $\theta_{xy} \leq \theta_{yx}$ if $x \leq y$. In this case, a higher type corresponds to a stronger bargaining power.

probabilities, $\boldsymbol{\phi}(\tau)$, the distribution of balances evolves according to

$$\dot{n}_k^x(\tau) = f^x[\boldsymbol{n}(\tau), \boldsymbol{\phi}(\tau)] \quad \text{for all } k \in \mathbb{K} \text{ and } x \in \mathbb{Y}, \tag{33}$$

where

$$\begin{aligned} f^x[\boldsymbol{n}(\tau), \boldsymbol{\phi}(\tau)] \equiv{}& \alpha n_k^x(\tau) \sum_{y\in\mathbb{Y}} \sum_{i\in\mathbb{K}} \sum_{j\in\mathbb{K}} \sum_{s\in\mathbb{K}} n_i^y(\tau)\, \phi_{ki,xy}^{sj}(\tau) \\ &- \alpha \sum_{y\in\mathbb{Y}} \sum_{i\in\mathbb{K}} \sum_{j\in\mathbb{K}} \sum_{s\in\mathbb{K}} n_i^x(\tau)\, n_j^y(\tau)\, \phi_{ij,xy}^{ks}(\tau). \end{aligned} \tag{34}$$

The value function satisfies

$$rV_i^x(\tau) + \dot{V}_i^x(\tau) = u_i + \alpha \sum_{y\in\mathbb{Y}} \sum_{j\in\mathbb{K}} \sum_{k\in\mathbb{K}} \sum_{s\in\mathbb{K}} n_j^y(\tau)\, \phi_{ij,xy}^{ks}(\tau)\, \theta_{xy} \left[V_k^x(\tau) + V_s^y(\tau) - V_i^x(\tau) - V_j^y(\tau)\right] \tag{35}$$

for all $(x, i, \tau) \in \mathbb{Y} \times \mathbb{K} \times [0, T]$, with

$$V_i^x(0) = U_i \quad \text{for all } x \in \mathbb{Y} \text{ and all } i \in \mathbb{K}. \tag{36}$$

The path for $\boldsymbol{\phi}(\tau)$ is given by

$$\phi_{ij,xy}^{ks}(\tau) = \begin{cases} \tilde{\phi}_{ij,xy}^{ks}(\tau) & \text{if } (k,s) \in \Omega_{ij,xy}[\boldsymbol{V}(\tau)] \\ 0 & \text{if } (k,s) \notin \Omega_{ij,xy}[\boldsymbol{V}(\tau)], \end{cases} \tag{37}$$

for all $x, y \in \mathbb{Y}$, all $i, j, k, s \in \mathbb{K}$, and all $\tau \in [0, T]$, where $\tilde{\phi}_{ij,xy}^{ks}(\tau) \geq 0$ and $\sum_{k\in\mathbb{K}} \sum_{s\in\mathbb{K}} \tilde{\phi}_{ij,xy}^{ks}(\tau) = 1$, with

$$\Omega_{ij,xy}[\boldsymbol{V}(\tau)] \equiv \arg \max_{(k',s')\in\Pi(i,j)} \left[V_{k'}^x(\tau) + V_{s'}^y(\tau) - V_i^x(\tau) - V_j^y(\tau)\right]. \tag{38}$$

If at time $T - \tau$, a bank of type y with balance j extends a loan of size $j - s = k - i$ to a bank of type x with balance i, the present value of the equilibrium repayment from the latter to the former is

$$e^{-r(\tau+\Delta)} R_{ij,x,y}^{ks}(\tau) = \frac{1}{2}[V_k^x(\tau) - V_i^x(\tau)] + \frac{1}{2}\left[V_j^y(\tau) - V_s^y(\tau)\right]. \tag{39}$$

7.2 Heterogeneous contact rates

Let α^x be the contact rate of a bank of type $x \in \mathbb{Y}$. Notice that from the perspective of any bank, the probability of finding a trading partner of type $y \in \mathbb{Y}$ with balance $j \in \mathbb{K}$ at time $T - \tau$, conditional on having contacted a random partner, is $\bar{\eta}^y n_j^y(\tau)$, where

$$\bar{\eta}^y \equiv \frac{\alpha^y \eta^y}{\sum_{x\in\mathbb{Y}} \alpha^x \eta^x}.$$

Hence the rate at which a bank of type x contacts a bank of type y who holds balance j at time $T-\tau$, is $\alpha^x \bar{\eta}^y n_j^y(\tau)$, and $\alpha^x \bar{\eta}^y n_j^y(\tau) n_i^x(\tau)$ is the measure of banks of type x who hold balance i, that meet a bank of type y who holds balance j. Therefore, given any feasible path for the distribution of trading probabilities, $\boldsymbol{\phi}(\tau)$, the distribution of balances evolves according to

$$\dot{n}_k^x(\tau) = f^x[\boldsymbol{n}(\tau), \boldsymbol{\phi}(\tau)] \quad \text{for all } k \in \mathbb{K} \text{ and } x \in \mathbb{Y}, \tag{40}$$

where

$$\begin{aligned} f^x[\boldsymbol{n}(\tau), \boldsymbol{\phi}(\tau)] \equiv\ & \alpha^x n_k^x(\tau) \sum_{y\in\mathbb{Y}} \sum_{i\in\mathbb{K}} \sum_{j\in\mathbb{K}} \sum_{s\in\mathbb{K}} \bar{\eta}^y n_i^y(\tau) \phi_{ki,xy}^{sj}(\tau) \\ & - \alpha^x \sum_{y\in\mathbb{Y}} \sum_{i\in\mathbb{K}} \sum_{j\in\mathbb{K}} \sum_{s\in\mathbb{K}} \bar{\eta}^y n_i^x(\tau) n_j^y(\tau) \phi_{ij,xy}^{ks}(\tau). \end{aligned} \tag{41}$$

The value function satisfies

$$rV_i^x(\tau) + \dot{V}_i^x(\tau) = u_i + \frac{\alpha^x}{2} \sum_{y\in\mathbb{Y}} \sum_{j\in\mathbb{K}} \sum_{k\in\mathbb{K}} \sum_{s\in\mathbb{K}} \bar{\eta}^y n_j^y(\tau) \phi_{ij,xy}^{ks}(\tau) \left[V_k^x(\tau) + V_s^y(\tau) - V_i^x(\tau) - V_j^y(\tau)\right]$$

for all $(x, i, \tau) \in \mathbb{Y} \times \mathbb{K} \times [0, T]$, subject to (36). Given $\boldsymbol{V}(\tau)$, the path for $\boldsymbol{\phi}(\tau)$ is as in (37), and the repayment as in (39).

7.3 Payoff heterogeneity

Let $U_k^y \in \mathbb{R}$ be the payoff to a bank of type $y \in \mathbb{Y}$ from holding a balance $k \in \mathbb{K}$ at the end of the trading session. Given any feasible path for the distribution of trading probabilities, $\boldsymbol{\phi}(\tau)$, the distribution of balances evolves according to (33) and (34). The value function satisfies (35), but with terminal condition

$$V_i^x(0) = U_i^x \quad \text{for all } x \in \mathbb{Y} \text{ and all } i \in \mathbb{K},$$

and $\theta_{xy} = 1/2$ for all $x, y \in \mathbb{Y}$. Given $\boldsymbol{V}(\tau)$, the path for $\boldsymbol{\phi}(\tau)$ is as in (37), and the repayment as in (39).

On October 9, 2008, the Federal Reserve began to pay interest on the required reserve balances and on the excess balances held by depository institutions, but not on the balances held by non-depository institutions.[17] This means that some large lenders in the federal funds

[17] The Financial Services Regulatory Relief Act of 2006 gives the Federal Reserve authority to pay interest on reserve balances only to depository institutions, including banks, savings associations, saving banks and credit unions, trust companies, and U.S. agencies and branches of foreign banks.

market which are non-depository institutions, such as the GSEs, do not receive interest on their reserve balances.[18] It has been argued (see Bech and Klee, 2009) that such institutions may have an incentive to lend at rates below the rate that banks receive on reserve balances, which might have contributed to an increase of their market share and to the effective federal funds rate (a daily volume-weighted average of brokered transactions rates) being lower than the rate of interest banks earn on reserve balances. In our extended model, this feature of GSEs, and its implication for the determination of the distribution of fed fund rates, can be handled by regarding GSEs as a particular type, $y_0 \in \mathbb{Y}$, with $U_k^{y_0} = 0$ for all $k \in \mathbb{K}$.

8 An analytical example

In this section we use the theory with $\mathbb{K} = \{0, 1, 2\}$ to study the effects that various institutional considerations and policies have on the performance of the market for federal funds. We interpret a bank with $k = 1$ as being "on target" (e.g., holding the level of required reserves), a bank with $k = 2$ as being "above target" (e.g., holding excess reserves), and a bank with $k = 0$ as being "below target" (e.g., unable to meet the level of required reserves). In this setting, the feasible sets of post-trade balances are: $\Pi(0,2) = \{(0,2), (1,1), (2,0)\}$, $\Pi(1,j) = \{(1,j), (j,1)\}$ for $j = 0, 2$, and $\Pi(i,i) = \{(i,i)\}$ for $i = 0, 1, 2$. Hence,

$$\max_{(k,s)\in\Pi(2,0)} S_{20}^{ks}(\tau) = \max\left\{S_{20}^{11}(\tau), 0\right\}, \text{ and}$$

$$\max_{(k,s)\in\Pi(i,i)} S_{ii}^{ks}(\tau) = \max_{(k,s)\in\Pi(1,j)} S_{1j}^{ks}(\tau) = 0 \quad \text{for all } i \in \mathbb{K}, \text{ and } j = 0, 2.$$

That is, in this special case there can only be profitable trade between a bank with $i = 2$ and a bank with $j = 0$ balances.[19] To simplify the notation, let $S(\tau) \equiv S_{20}^{11}(\tau)$, and refer to a bank with $i = 2$ and a bank with $j = 0$ as a *lender*, and *borrower*, respectively. Let $\theta \in [0, 1]$ be the bargaining power of the borrower. We conjecture that $S(\tau) > 0$ for all $\tau \in [0, T]$, and will later verify that this is indeed the case. In this case, the flows (8) and (9) lead to

$$\dot{n}_0(\tau) = \alpha n_2(\tau) n_0(\tau)$$

$$\dot{n}_2(\tau) = \alpha n_2(\tau) n_0(\tau),$$

[18] Fannie Mae and Freddie Mac are large lenders of fed funds because their business model involves using the fed funds market as a short-term investment for incoming mortgage payments, before passing the funds on to investors in the form of principal and/or interest payments. Similarly, the Federal Home Loan Banks use the fed funds market to keep their funds readily available to meet unexpected borrowing demands from members.

[19] Recall that from (6), we know that in general, $S_{ij}^{ks}(\tau) = S_{ji}^{ks}(\tau) = S_{ij}^{sk}(\tau) = S_{ji}^{sk}(\tau)$ for all $i, j, k, s \in \mathbb{K}$.

given the initial conditions $n_0(T)$ and $n_2(T)$. This implies

$$n_0(\tau) = \begin{cases} \frac{[n_2(T)-n_0(T)]n_0(T)}{n_2(T)e^{\alpha[n_2(T)-n_0(T)](T-\tau)}-n_0(T)} & \text{if } n_2(T) \neq n_0(T) \\ \frac{n_0(T)}{1+\alpha n_0(T)(T-\tau)} & \text{if } n_2(T) = n_0(T) \end{cases} \tag{42}$$

$$n_1(\tau) = 1 - n_0(\tau) - n_2(\tau) \tag{43}$$

$$n_2(\tau) = n_0(\tau) + n_2(T) - n_0(T). \tag{44}$$

The expression for the value function $\boldsymbol{V}$ in (16) and (17) (or (10)) leads to

$$rV_0(\tau) + \dot{V}_0(\tau) = u_0 + \alpha n_2(\tau)\theta S(\tau) \tag{45}$$

$$rV_1(\tau) + \dot{V}_1(\tau) = u_1 \tag{46}$$

$$rV_2(\tau) + \dot{V}_2(\tau) = u_2 + \alpha n_0(\tau)(1-\theta) S(\tau), \tag{47}$$

for all $\tau \in [0, T]$, given $V_i(0) = U_i$ for $i = 0, 1, 2$. Conditions (45), (46) and (47) imply

$$\dot{S}(\tau) + \delta(\tau) S(\tau) = \bar{u} \tag{48}$$

where $\bar{u} \equiv 2u_1 - u_2 - u_0$, and

$$\delta(\tau) \equiv \{r + \alpha[\theta n_2(\tau) + (1-\theta) n_0(\tau)]\}.$$

Given the boundary condition $S(0) = 2U_1 - U_2 - U_0$, the solution to (48) is

$$S(\tau) = \left(\int_0^\tau e^{-\left[\bar{\delta}(\tau)-\bar{\delta}(z)\right]} dz\right)\bar{u} + e^{-\bar{\delta}(\tau)} S(0), \tag{49}$$

where $\bar{\delta}(\tau) \equiv \int_0^\tau \delta(x)\,dx$.

Suppose that $\bar{u} \equiv 2u_1 - u_2 - u_0 \geq 0$ and $S(0) = 2U_1 - U_2 - U_0 > 0$, so Assumption A holds. Then it is clear from (49) that $S(\tau) > 0$ as conjectured. Then, with $S(\tau)$ given by (49), the unique equilibrium is simply the path for the distribution of reserve balances given by (42), (43) and (44), together with the distribution of trading probabilities given by $\phi_{ij}^{ks}(\tau) = 1$ if $(i, j, k, s) = (2, 0, 1, 1)$ or $(i, j, k, s) = (0, 2, 1, 1)$ and $\phi_{ij}^{ks}(\tau) = 0$ otherwise, and a value function $\boldsymbol{V}$ that satisfies the system of ordinary differential equations (45), (46), (47) with the boundary conditions $V_i(0) = U_i$ for $i = 0, 1, 2$. In equilibrium, the present value of the repayment is

$$e^{-r(\tau+\Delta)} R(\tau) = V_2(\tau) - V_1(\tau) + (1-\theta) S(\tau) = V_1(\tau) - V_0(\tau) - \theta S(\tau). \tag{50}$$

The interest rate implicit in the typical loan that promises to repay $R(\tau)$ at time $\tau+\Delta$ for one unit borrowed at time $T-\tau$ is

$$\begin{aligned}\rho(\tau) &= \frac{\ln R(\tau)}{\tau+\Delta} \\ &= r+\frac{\ln\left[V_2(\tau)-V_1(\tau)+(1-\theta)S(\tau)\right]}{\tau+\Delta}. \end{aligned} \tag{51}$$

The equilibrium in this example is a path for the distribution $\boldsymbol{n}(\tau)$, described explicitly by (42), (43) and (44); a path for the distribution of trading probabilities explicitly given by $\phi_{02}^{ks}(\tau)=\phi_{20}^{ks}(\tau)=\mathbb{I}_{\{(k,s)=(1,1)\}}$, $\phi_{ii}^{ks}(\tau)=0$ for all $(k,s)\in\Pi(i,i)$, for $i=0,1,2$, and $\phi_{1j}^{ks}(\tau)\in[0,1]$ for all $(k,s)\in\Pi(1,j)$, for $j=0,2$; and a path for the value function $V(\tau)$,

$$V_0(\tau)=\left(1-e^{-r\tau}\right)\frac{u_0}{r}+e^{-r\tau}U_0+\int_0^\tau e^{-r(\tau-z)}\alpha n_2(z)\,\theta S(z)\,dz \tag{52}$$

$$V_1(\tau)=\left(1-e^{-r\tau}\right)\frac{u_1}{r}+e^{-r\tau}U_1 \tag{53}$$

$$V_2(\tau)=\left(1-e^{-r\tau}\right)\frac{u_2}{r}+e^{-r\tau}U_2+\int_0^\tau e^{-r(\tau-z)}\alpha n_0(z)\,(1-\theta)\,S(z)\,dz, \tag{54}$$

which are given explicitly up to the path for the equilibrium surplus, $S(\tau)$. Some properties of the path for the equilibrium surplus are immediate from (49). For example, if $\bar{u}$ is small, then $\dot{S}(\tau)<0$ (the gain from trade is increasing chronological time, i.e., as t approaches T). Conversely, $\dot{S}(\tau)>0$ will be the case in parametrizations with $\bar{u}$ large, and small enough α and r. The following proposition reports the analytical expressions for the equilibrium surplus and interest rate.

Proposition 8 *The surplus of a match at time $T-\tau$ between a bank with balance $i=2$ and a bank with balance $j=0$ is*

$$S(\tau)=\begin{cases}\frac{e^{\alpha[n_2(T)-n_0(T)](T-\tau)}n_2(T)-n_0(T)}{e^{\{r-\alpha(1-\theta)[n_2(T)-n_0(T)]\}\tau}[n_2(T)-n_0(T)]}\left[\frac{\xi(\tau)\bar{u}}{n_0(T)}+\frac{[n_2(T)-n_0(T)]S(0)}{e^{\alpha[n_2(T)-n_0(T)]T}n_2(T)-n_0(T)}\right] & \text{if } n_2(T)\neq n_0(T)\\ \frac{1+\alpha(T-\tau)n_0(T)}{e^{r\tau}}\left[\frac{\xi(\tau)\bar{u}}{n_0(T)}+\frac{S(0)}{1+\alpha T n_0(T)}\right] & \text{if } n_2(T)=n_0(T),\end{cases}$$

where

$$\xi(\tau)\equiv\begin{cases}\sum_{k=1}^{\infty}\frac{\left[\frac{n_2(T)}{n_0(T)}\right]^{k-1}}{\frac{r}{n_0(T)-n_2(T)}+\alpha(k-\theta)}\frac{e^{\{r+\alpha[n_0(T)-n_2(T)](k-\theta)\}\tau}-1}{e^{\alpha[n_0(T)-n_2(T)](k-1)T}} & \text{if } n_2(T)<n_0(T)\\ \sum_{k=0}^{\infty}\frac{(-r)^k\left\{\left[\frac{1}{\alpha n_0(T)}+T\right]^k-\left[\frac{1}{\alpha n_0(T)}+T-\tau\right]^k\right\}}{\alpha k!k}e^{r\left[\frac{1}{\alpha n_0(T)}+T\right]} & \text{if } n_2(T)=n_0(T)\\ \sum_{k=0}^{\infty}\frac{\left[\frac{n_0(T)}{n_2(T)}\right]^{k+1}}{\frac{r}{n_2(T)-n_0(T)}+\alpha(k+\theta)}\frac{e^{\{r+\alpha[n_2(T)-n_0(T)](k+\theta)\}\tau}-1}{e^{\alpha[n_2(T)-n_0(T)](k+1)T}} & \text{if } n_0(T)<n_2(T).\end{cases}$$

Given $S(\tau)$, the equilibrium repayment is given by (50), with

$$V_1(\tau) - V_0(\tau) = e^{-r\tau}(U_1 - U_0) + \left(1 - e^{-r\tau}\right)\frac{u_1 - u_0}{r} - \theta \frac{e^{\alpha[n_2(T) - n_0(T)]T} n_2(T)}{n_0(T)} e^{-r\tau} \zeta[\tau, \bar{u}, S(0)],$$

where $\zeta[\tau, \bar{u}, S(0)]$ is a time-varying linear combination of $\bar{u}$ and $S(0)$.

8.1 Comparative dynamics

In this section we provide some analytical results on the effect that parameter changes have on the equilibrium paths for the trade surplus and the fed funds rate.

Proposition 9 describes the behavior of $S(\tau)$, namely the value of executing a trade (or the "value of a trade") between a borrower and a lender when the remaining time is τ. With $\bar{u} = 0$, (49) specializes to $S(\tau) = e^{-\bar{\delta}(\tau)}S(0)$, so $S(\tau)$ is a discounted version of $S(0)$, with effective discount rate given by $\bar{\delta}(\tau)$. More generally, for $\bar{u} \geq 0$, $S(\tau)$ is a linear combination of $\bar{u}$ and $S(0)$. There are two reasons why $S(0)$ appears discounted in the expression for $S(\tau)$. First, the actual gains from trade accrue at the end of the trading session, so $S(0)$ is discounted by the pure rate of time preference, r. Second, consider a meeting between a borrower and a lender when the remaining time is $\tau > 0$. The value $S(0)$ is discounted because both agents might meet alternative trading partners before the end of the session, and this increases their outside options. The borrower's outside option, $V_0(\tau)$, is increasing in the average rate at which he is able to contact a lender and reap gains from trade between time $T - \tau$ and T, i.e., $\alpha\theta \int_0^\tau n_2(s)\, ds$. Similarly, the lender's outside option, $V_2(\tau)$, is increasing in the average rate at which he is able to contact a borrower and reap gains from trade between time $T - \tau$ and T, i.e., $\alpha(1-\theta)\int_0^\tau n_0(s)\, ds$.

Proposition 9 *Assume $\bar{u} \geq 0$ and $S(0) > 0$. Then:*

(i) *The surplus at each point in time is decreasing in the discount rate, i.e., for all $\tau > 0$, $\frac{\partial S(\tau)}{\partial r} < 0$.*

(ii) *If the initial population of lenders is larger (smaller) than that of borrowers, then the surplus at each point in time during the trading session is decreasing (increasing) in the borrower's bargaining power. If the initial populations of lenders and borrowers are equal, then changes in the bargaining power have no effect on the surplus, i.e., for all $\tau > 0$, $\frac{\partial S(\tau)}{\partial \theta}$ is equal in sign to $n_0(T) - n_2(T)$.*

(iii) *The surplus at each point in time is increasing in the penalty for below-target end-of-day balances, increasing in the payoff for on-target end-of-day balances, and decreasing in the payoff for above-target end-of-day balances, i.e., for all* τ, $\frac{\partial S(\tau)}{\partial U_0} < 0$, $\frac{\partial S(\tau)}{\partial U_1} > 0$, *and* $\frac{\partial S(\tau)}{\partial U_2} < 0$.

Part *(i)* follows from the fact that a larger value of r increases the effective discount rate, $\bar{\bar{\delta}}(\tau)$, and also results in a deeper discount of the "dividend-flow gain from trade," $\bar{u}$. The effect of θ on $S(\tau) = 2V_1(\tau) - V_0(\tau) - V_2(\tau)$ is more subtle because a higher θ tends to increase $V_0(\tau)$ (benefits borrowers) and at the same time it tends to decrease $V_2(\tau)$ (hurts lenders). In part *(ii)* we show that the former effect dominates if and only if $n_2(\tau) > n_0(\tau)$, and in this case, the effective discount rate decreases with θ, which implies $S(\tau)$ decreases with θ for all $\tau > 0$. Finally, making the penalty for below-target end-of-day balances more severe (lowering U_0), making the payoff for holding above-target end-of-day balances less attractive, or increasing the payoff for holding on-target end-of-day balances, increases the terminal surplus $S(0)$, and consequently increases every surplus along the trading session, which explains part *(iii)*.

The following proposition considers the case with $\bar{u} = 0$. For example, this would be the case when banks are not remunerated for holding intraday balances and have access to intraday credit from the central bank at no cost.

Proposition 10 *Assume* $\bar{u} = 0$ *and* $S(0) > 0$. *Then:*

(i) *The fed funds rate at each point in time is increasing in the discount rate, i.e., for all* τ, $\frac{\partial \rho(\tau)}{\partial r} > 0$.

(ii) *The fed funds rate at each point in time is decreasing in the borrower's bargaining power, i.e., for all* $\tau > 0$, $\frac{\partial \rho(\tau)}{\partial \theta} < 0$.

(iii) *The fed funds rate at each point in time is increasing in the penalty for below-target end-of-day balances, i.e., for all* τ, $\frac{\partial \rho(\tau)}{\partial U_0} < 0$.

Proposition 10 describes the behavior of the fed funds rate at each point in time along the trading session. Parts *(i)–(iii)* follow from (51) and the fact that the size of the loan repayment $R(\tau)$ increases with r and U_0, and decreases with the borrower's bargaining power, θ.

8.2 Efficiency

Under Assumption A, the equilibrium paths for the distribution of balances and the distribution of trading probabilities coincide with the efficient paths. The planner's co-states satisfy

$$r\lambda_0(\tau) + \dot{\lambda}_0(\tau) = u_0 + \alpha n_2(\tau) S^*(\tau) \tag{55}$$

$$r\lambda_1(\tau) + \dot{\lambda}_1(\tau) = u_1 \tag{56}$$

$$r\lambda_2(\tau) + \dot{\lambda}_2(\tau) = u_2 + \alpha n_0(\tau) S^*(\tau), \tag{57}$$

for all $\tau \in [0, T]$, given $\lambda_i(0) = U_i$ for $i = 0, 1, 2$, where $S^*(\tau) \equiv 2\lambda_1(\tau) - \lambda_2(\tau) - \lambda_0(\tau)$ satisfies

$$\dot{S}^*(\tau) + \delta^*(\tau) S^*(\tau) = \bar{u} \tag{58}$$

with

$$\delta^*(\tau) \equiv \{r + \alpha[n_2(\tau) + n_0(\tau)]\}.$$

Given the boundary condition $S^*(0) = 2U_1 - U_2 - U_0$, the solution to (58) is

$$S^*(\tau) = \left(\int_0^\tau e^{-\left[\bar{\delta}^*(\tau) - \bar{\delta}^*(z)\right]} dz\right) \bar{u} + e^{-\bar{\delta}^*(\tau)} S(0),$$

where $\bar{\delta}^*(\tau) \equiv \int_0^\tau \delta^*(x)\, dx$.

The comparison between (45), (46) and (47), and (55), (56) and (57), illustrates the composition externality discussed in Section 5. For instance, since in this example meetings involving at least one agent who holds one unit of reserves never entail gains from trade, (46) and (56) confirm that the equilibrium value of a bank with one unit of balances coincides with the shadow price it is assigned by the planner. In contrast, comparing (45) to (55), and (47) to (57), reveals that the equilibrium gains from trade as perceived by an individual borrower and lender at time $T - \tau$ are $\theta S(\tau)$ and $(1-\theta) S(\tau)$, respectively, while according to the planner each of their marginal contributions equals $S^*(\tau)$.

Notice that $\delta^*(\tau) \geq \delta(\tau)$ for all $\tau \in [0, T]$, with "=" only for $\tau = 0$, so the planner effectively "discounts" more heavily than the equilibrium. It is easy to show that this implies $S(\tau) > S^*(\tau)$ for all $\tau \in (0, 1]$, with $S^*(0) = S(0) = 2U_1 - U_2 - U_0$. In words, due to the matching externality, the social value of a loan (loans are always of size 1 in this example) is smaller than the joint private value of a loan in the equilibrium. Intuitively, the planner internalizes the fact that borrowers and lenders who are searching make it easier for other lenders and borrowers to find trading partners, but these "liquidity provision services" to others receive no compensation in the equilibrium, so individual agents ignore them when calculating their equilibrium payoffs. Naturally, depending on the value of θ, the equilibrium payoff to lenders may be too high or too low relative to their shadow price in the planner's problem. It will be high if $(1-\theta)\,S(\tau) > S^*(\tau)$, as would be the case for example, if the borrower's bargaining power, θ, is small. As these considerations make clear, the efficiency proposition (Proposition 5) would typically become an *inefficiency* proposition in contexts where banks make some additional choices based on their private gains from trade (e.g., entry, search intensity decisions, etc.).

9 Quantitative analysis

In this section we parametrize the model and use it to illustrate and complement our analytical results. We also perform quantitative analyses of several policy-relevant issues.

9.1 Parametrization

The motives for trading, and the payoffs from holding fed funds positions are different for different types of fed funds market participants. Since commercial banks account for the bulk of the trade volume in the fed funds market, we will adopt their trading motives and payoffs as the baseline for our quantitative implementation.[20] The Federal Reserve imposes a minimum level of reserves on commercial banks and other depository institutions (all of whom we refer to as *banks*, for brevity). This reserve balance requirement applies to the average level of a bank's end-of-day balances during a two-week maintenance period.[21] End-of-day balances within a maintenance period may vary but remain in general positive as overnight overdrafts are considered unauthorized extensions of credit, and penalized.[22] In practice, banks typically target an average daily level of end-of-day balances and try to avoid overnight overdrafts. On October 9, 2008, the Federal Reserve began remunerating banks' positive end-of-day balances. Since December 18, 2008, the interest rate paid on both, required reserve balances, and excess balances, is 25 basis points (Federal Reserve, 2008). All these policy considerations are captured

[20] Ashcraft and Duffie (2007) report that commercial banks account for over 80 percent of the volume of federal funds traded in 2005, while 15 percent involves GSEs, and 5 percent corresponds to special situations involving nonbanks that hold reserve balances at the Federal Reserve. Their estimates are based on a sample of the top 100 institutions ranked by monthly volume of fed funds sent, including commercial banks, GSEs, and excluding transactions involving accounts held by central banks, federal or state governments, or other settlement systems.

[21] For an explanation of how these required operating balances are calculated, see Bennett and Hilton (1997) and Federal Reserve (2009, 2010b).

[22] The penalty fee charged on overnight overdrafts is generally 400 basis points over the effective fed funds rate, and it is increased by 100 basis points if there have been more than three overnight overdrafts in a year.

by the end-of-day payoffs $\{U_k\}_{k\in\mathbb{K}}$. Currently, the Fed does not pay interest on intraday balances, but it charges interest on daylight overdrafts. In the theory, the flow payoff to a bank from holding intraday balances during a trading session is captured by the vector $\{u_k\}_{k\in\mathbb{K}}$.

In the quantitative work we adopt the following formulation:

$$U_k = e^{-r\Delta_F}\left(k-\bar{k}_0\right)+F_k \tag{59}$$

with

$$F_k = \begin{cases} \bar{k}i^r + \left(k-\bar{k}_0-\bar{k}\right)i^e & \text{if } \bar{k}\le k-\bar{k}_0 \\ -P^r + \left(k-\bar{k}_0\right)i^r & \text{if } 0\le k-\bar{k}_0<\bar{k} \\ -\left[P^r+P^o+\left(\bar{k}_0-k\right)i^o\right] & \text{if } k-\bar{k}_0<0, \end{cases} \tag{60}$$

and

$$u_k = \begin{cases} \left(k-\bar{k}_0\right)^{1-\epsilon} i^d_+ & \text{if } 0\le k-\bar{k}_0 \\ \left(k-\bar{k}_0\right)^{1-\epsilon} i^d_- & \text{if } k-\bar{k}_0<0. \end{cases} \tag{61}$$

The parameter $\Delta_F\in\mathbb{R}_+$ represents the length of the period between the end of the trading session and the beginning of the following trading session, when the bank's reserves held overnight at the Federal Reserve become available (in practice, this period consists of the 2.5 hours between 6:30 pm and 9 pm). The parameter $\bar{k}\in\left\{0,...,K-\bar{k}_0\right\}$ represents the reserve requirement imposed on all banks. The parameter $\bar{k}_0$ is assumed to satisfy $0\le\bar{k}_0<K$, and allows us to consider translations of the set $\mathbb{K}$, which affords us a more a flexible interpretation of the elements of $\mathbb{K}$. Intuitively, $\bar{k}_0$ can be thought of as the overdraft threshold.[23] The overnight interest rate that a bank earns on required reserves is denoted $i^r\ge 0$, while $i^e\ge 0$ is the overnight interest rate on excess reserves ($i^r=i^e$ is the case currently in the United States), $i^o\ge 0$ is the overnight overdraft penalty rate, $P^r\ge 0$ is the pecuniary value of penalties for failing to meet reserve requirements, and $P^o\ge 0$ represents additional penalties resulting from the use of unauthorized credit. The interest rate that a bank earns on positive intraday balances is $i^d_+\ge 0$, and $i^d_-\ge 0$ is the interest rate it pays on daylight overdraft.[24] The parameter

[23] For example, in a parametrization with $\bar{k}_0=0$, $\mathbb{K}$ can be interpreted as the set of fed funds balances that can be held by an individual bank. More generally, we can instead regard $k\in\mathbb{K}$ as an abstract index, and interpret $k'\equiv k-\bar{k}_0$ as a bank's fed fund balance. Under this interpretation, fed fund balances (i.e., k') held by banks are in the set $\mathbb{K}'\equiv\left\{k':k'=k-\bar{k}_0 \text{ for some } k\in\mathbb{K}\right\}$. Then since $\mathbb{K}'=\left\{-\bar{k}_0,...,K-\bar{k}_0\right\}$, this formulation allows the payoff functions to accomodate the possibility of negative fed funds balances. In line with this more general interpretation, $\bar{k}$ represents the reserve requirement imposed on fed fund balances $k'\equiv k-\bar{k}_0$. (The reserve requirement stated in terms of the index k, would be $\bar{k}+\bar{k}_0$.)

[24] In practice, when an institution has insufficient funds in its Federal Reserve account to cover its settlement obligations during the operating day, it can incur in a daylight overdraft up to an individual maximum amount known as *net debit cap*. (This cap is equal to zero for some institutions.) On March 24, 2011, the Federal Reserve Board will implement major revisions to the Payment System Risk policy, which include a zero fee for

$\epsilon \in [0, 1)$ will be either set to zero or to a negligible value.[25]

We measure time in the model in days. Table 1 reports the parameter values for the baseline example. The model is meant to capture trade dynamics in the last 2.5 hours of the trading session, so we set $T = 2.5/24$. Since most transactions are settled through Fedwire, and Fedwire does not operate between 6.30pm and 9.00pm ET, $\Delta_F = 2.5/24$. By setting $\Delta = 22/24$, we ensure that all loans in the model have a maturity between 22 and 24.5 hours. The discount rate, r, is set to $0.04/365$, and the interest charge on daylight overdrafts, i^d_-, is set to $0.0036/360$. The interest rates on required and excess reserves, i^r and i^e are both set equal to $0.0025/360$, the rate currently paid by the Federal Reserve.[26] The interest penalty on overnight overdrafts is generally 400 basis points over the effective fed funds rate, so we set $i^o = 0.0425/360$. The pecuniary value of the additional penalties resulting from the use of unauthorized credit, $P^r + P^o$, is chosen to be about eight times the interest penalty on overnight overdrafts. For the contact rate, we choose $\alpha = 100$ which implies that on average agents have 10 meetings during the trading session, i.e., a trading opportunity every 15 minutes, on average. Borrowers and lenders have the same bargaining power in the baseline.

r	i^d_+	i^d_-	i^r	i^e	i^o	Δ	Δ_F	T	$P^r + P^o$	α	θ
$\frac{0.04}{365}$	$\frac{0.0005}{360}$	$\frac{0.0036}{360}$	$\frac{0.0025}{360}$	$\frac{0.0025}{360}$	$\frac{0.0425}{360}$	$\frac{22}{24}$	$\frac{2.5}{24}$	$\frac{2.5}{24}$	0.001	100	0.5

Table 1: Baseline parametrization for the model with $\mathbb{K} = \{0, 1, 2\}$

9.2 Trade dynamics

For the numerical exercises in this subsection, we set $\mathbb{K} = \{0, 1, 2\}$, $\bar{k}_0 = 0$, $\bar{k} = 1$, and $i^d_+ = 0$, $\epsilon = 0$, so $u_k = 0$ for $k = 0, 1, 2$, and the terminal payoff $\{U_k\}_{k=0}^{2}$ is

$$U_k = \begin{cases} e^{-r\Delta_F} k + i^r + (k-1)\, i^e & \text{if } 1 \leq k \\ -P^r & \text{if } k = 0. \end{cases}$$

All other parameter values are as described in Section 9.1.

collateralized daylight overdrafts and an increased fee for uncollateralized daylight overdrafts to 50 basis points, annual rate (from 36 basis points) (see Federal Reserve, 2010a).

[25] By setting ϵ to a negligible positive value we ensure that $\{u_k\}_{k\in\mathbb{K}}$ satisfies the discrete midpoint strict concavity property.

[26] The 360-day year is customary for interest rate calculation in money markets.

9.2.1 Bargaining power

Figure 1 (with actual time, $t = T - \tau$, on the horizontal axis) shows the time paths for the trade surplus, the opportunity cost to a lender from giving up the second unit of reserves, and the fed funds rate, for different values of the borrower's bargaining power, $\theta^L = 0.1$, $\theta = 0.5$ (the baseline), and $\theta^H = 0.9$. The top row of panels corresponds to the case in which the initial number of lenders is smaller than the initial number of borrowers, i.e., $n_2(T) = 0.3 < n_0(T) = 0.6$. First consider the left panel on the top row. Since $S(0) = 2U_1 - U_2 - U_0$, the trade surplus at the end of the session is the same for all values of θ. For all $t < T$, however, the time-path for the trade surplus is shifted upward as the borrower's bargaining power, θ, increases. The reason is that while for each τ, an increase in θ increases the borrower's outside option, $V_0(\tau)$, and decreases the lender's outside option, $V_2(\tau)$, the fact that $n_2(\tau) < n_0(\tau)$ for all τ, implies that the decrease in the lender's outside option is larger than the increase in the borrower's outside option, so the resulting trade surplus is larger at each point in time along the trading session. The middle panel shows that as θ increases, the path for the value of a lender is shifted down for all $\tau \in (0, T]$. (Agents with one unit of balances do not trade in this example, so the path for $V_1(\tau)$ is effectively exogenous.) The right panel confirms that the path for the fed funds rate is shifted down as the bargaining power of the borrower increases, as was to be expected from (51) and the effect of θ on $V_2(\tau) - V_1(\tau)$ illustrated in the middle panel. Intuitively, the borrower pays less for the loan when he has a stronger bargaining power in the negotiation of the loan rate.

The panels on the bottom row correspond to the case in which the initial number of borrowers is smaller than the initial number of lenders, i.e., $n_0(T) = 0.3 < n_2(T) = 0.6$. In this case an increase in θ still increases $V_0(\tau)$ and decreases $V_2(\tau)$ for each $\tau \in (0, T]$, but the fact that $n_0(\tau) < n_2(\tau)$ for all τ implies that the decrease in the lender's outside option is smaller than the increase in the borrower's outside option, so the resulting trade surplus is now smaller at each point during the trading session. As in the top panel, the path for the value of a lender is shifted down for all $\tau \in (0, T]$ as θ increases, but notice that the size of this effect is smaller for smaller $n_0(\tau)$ (because in this case the lender meets borrowers very infrequently, which makes his expected gain from trade small to begin with). Again, the right panel confirms that the path for the fed funds rate is shifted down as the bargaining power of the borrower increases. By comparing the right panel on the top row with the right panel on the bottom row, however, it is clear that time path of the fed funds rate is rather different in both cases. When there are more lenders than borrowers (the right panel on the bottom row) the fed funds rate tends to increase over time as the end of the trading session approaches. In contrast, the fed funds rate tends to decrease over time when there are more borrowers than lenders (the right panel on the top row), provided θ is not too small. In both cases $S(\tau)$ is increasing over time, which tends to make $\rho(\tau)$ increasing over time (see (51)). When the number of borrowers is large relative to the number of lenders, however, the difference $V_2(\tau) - V_1(\tau)$ is large and decreases over time, and this can (e.g., for θ large enough) dominate the dynamics of the fed funds rate, resulting in an equilibrium fed funds rate that decreases over time.

9.2.2 Deficiency charges

Figure 2 (with $t = T - \tau$, on the horizontal axis) shows the time paths for the trade surplus, the opportunity cost to a lender from giving up the second unit of reserves, and the fed funds rate, for different values of the penalty fee, $P \equiv P^r + P^o$, charged on banks for having balances below the end-of-day target. The different values considered are $P^L = 0$, $P = 0.001$ (the baseline), and $P^H = 0.005$. The panels on the top row correspond to the case in which the initial number of lenders is smaller than the initial number of borrowers, i.e., $n_2(T) = 0.3 < n_0(T) = 0.6$, while the bottom row corresponds to the case with $n_0(T) = 0.3 < n_2(T) = 0.6$. The left panels on the top and bottom rows show that making the penalty more severe shifts up the path of the surplus, an effect driven by the fact that the first-order effect of a larger penalty P is to reduce the borrower's outside option, $V_0(\tau)$, making it more valuable for borrowers to trade and avoid paying the end-of-period penalty. Naturally, this effect also causes the paths for the interest rate to shift up in response to the increase in the penalty. The middle panels show that an increase in P leads to an increase in the value of lenders for all $t \in [0, T)$.

9.2.3 Trading delays

Figure 3 (with $t = T - \tau$, on the horizontal axis) shows the time paths for the trade surplus, the opportunity cost to a lender from giving up the second unit of reserves, and the fed funds rate, for different values of the contact rate, $\alpha^L = 50$, $\alpha = 100$ (the baseline), and $\alpha^L = 200$. The panels on the top row correspond to the case in which the initial number of lenders is smaller than the initial number of borrowers, i.e., $n_2(T) = 0.3 < n_0(T) = 0.6$, while the bottom row corresponds to the case with $n_0(T) = 0.3 < n_2(T) = 0.6$. The middle panel on the top row shows that traders on the short side of the market benefit from increases in the contact rate.

In contrast, the middle panel on the bottom row shows that in this example, increases in α decrease the expected payoffs of the agents who are on the long side of the market. This is explained by the fact that, from the standpoint of the agents on the short side, a faster contact rate has the undesirable effect of taking scarce potential trading partners off the market, which can adversely affect the effective rate at which they are able to trade.[27]

For all $t < T$ the time-path for the trade surplus is shifted downward as α increases. In the parametrization illustrated in the top row, an increase in α increases $V_2(\tau)$ for all $\tau \in (0, T]$ and decreases $V_0(\tau)$ for all $\tau \in (0, T]$. However, the former outweights the latter since $n_2(\tau)$ is small relative to $n_0(\tau)$ for all τ. In the parametrization illustrated in the bottom row, an increase in α increases $V_0(\tau)$ for all $\tau \in (0, T]$ and decreases $V_2(\tau)$ for all $\tau \in (0, T]$ and the former effect outweights the latter since $n_0(\tau)$ is small relative to $n_2(\tau)$ for all τ.

Together, the dynamics of $V_2(\tau) - V_1(\tau)$ and $S(\tau)$ account for the pattern of interest rates displayed in the right panels of the top and bottom rows. In each case, the right panel shows that traders on the short side of the market benefit from increases in the contact rate. Specifically, when lenders are on the short side, increases in the contact rate take scarce lenders off the market which makes borrowers willing to pay higher rates for the loans. Similarly, when borrowers are on the short side, a faster contact rate takes scarce borrowers off the market making lenders more willing to accept lower rates for the loans.

[27] In general, the effect of changes in α on equilibrium payoffs can be subtle. For example, in some of our numerical simulations we have found that, if $n_2(T) < n_0(T)$, then $V_0(\tau)$ can be nonmonotonic in α: increasing in α for small values of α, but decreasing in α for large values. If $n_2(T) < n_0(T)$, however, $V_2(\tau)$ is typically increasing in α. We have found the converse to be the case for $n_2(T) > n_0(T)$, i.e., $V_0(\tau)$ is increasing in α, while increases in α from relatively small values tend to shift $V_2(\tau)$ up, while increases in α at large values tend to shift $V_2(\tau)$ down.

A Appendix

Lemma 1 *For any* $(k,k') \in \mathbb{K}\times\mathbb{K}$ *and any* $\tau \in [0,T]$*, consider the following problem:*

$$\max_{b\in\Gamma(k,k'),R\in\mathbb{R}} \left[V_{k-b}(\tau) - V_k(\tau) + e^{-r(\tau+\Delta)}R\right]^{\theta_{kk'}} \left[V_{k'+b}(\tau) - V_{k'}(\tau) - e^{-r(\tau+\Delta)}R\right]^{1-\theta_{kk'}} \quad (62)$$

where $\theta_{kk'} = 1 - \theta_{k'k} \in [0,1]$*, and* $V_k(\tau) : \mathbb{K}\times[0,T] \to \mathbb{R}$ *is bounded. The correspondence*

$$H^*\left(k,k',\tau;V\right) = \arg\max_{b\in\Gamma(k,k'),R\in\mathbb{R}} \left\{ \left[V_{k-b}(\tau) - V_k(\tau) + e^{-r(\tau+\Delta)}R\right]^{\theta_{kk'}} \right.$$
$$\left. \left[V_{k'+b}(\tau) - V_{k'}(\tau) - e^{-r(\tau+\Delta)}R\right]^{1-\theta_{kk'}} \right\}$$

is nonempty. Moreover, $(b_{kk'}(\tau), R_{k'k}(\tau)) \in H^*(k,k',\tau;V)$ *if and only if*

$$b_{kk'}(\tau) \in \arg\max_{b\in\Gamma(k,k')} \left[V_{k'+b}(\tau) + V_{k-b}(\tau) - V_{k'}(\tau) - V_k(\tau)\right], \text{ and} \quad (63)$$

$$e^{-r(\tau+\Delta)}R_{k'k}(\tau) = \theta_{kk'}\left[V_{k'+b_{kk'}(\tau)}(\tau) - V_{k'}(\tau)\right] + (1-\theta_{kk'})\left[V_k(\tau) - V_{k-b_{kk'}(\tau)}(\tau)\right]. \quad (64)$$

Proof of Lemma 1. Consider

$$\max_{(b,R)\in\tilde{\mathbf{\Gamma}}(k,k')} \left[V_{k-b}(\tau) - V_k(\tau) + e^{-r(\tau+\Delta)}R\right]^{\theta_{kk'}} \left[V_{k'+b}(\tau) - V_{k'}(\tau) - e^{-r(\tau+\Delta)}R\right]^{1-\theta_{kk'}} \quad (65)$$

where $\tilde{\mathbf{\Gamma}}(k,k') = \{(b,R) \in \Gamma(k,k') \times [-B,B]\}$ for some arbitrary real number $B > 0$. Clearly, this problem has at least one solution. Let (b^*,R^*) denote a solution to (65). If the constraints $-B \le R \le B$ are slack at (b^*,R^*), then (b^*,R^*) is also a solution to (62), and (b^*,R^*) must satisfy the following first-order condition

$$e^{-r(\tau+\Delta)}R^* = \theta_{kk'}\left[V_{k'+b^*}(\tau) - V_{k'}(\tau)\right] + (1-\theta_{kk'})\left[V_k(\tau) - V_{k-b^*}(\tau)\right]. \quad (66)$$

Suppose that (b^*,R^*) with R^* given by (66) is a solution to (65) with $-B \le R^* \le B$ (given (66), these inequalities can be guaranteed by choosing B large enough), but such that

$$b^* \notin \arg\max_{b\in\Gamma(k,k')} \left[V_{k'+b}(\tau) + V_{k-b}(\tau) - V_{k'}(\tau) - V_k(\tau)\right]. \quad (67)$$

Condition (66) implies

$$V_{k-b^*}(\tau) - V_k(\tau) + e^{-r(\tau+\Delta)}R^* = \theta_{kk'}\left[V_{k'+b^*}(\tau) + V_{k-b^*}(\tau) - V_{k'}(\tau) - V_k(\tau)\right]$$
$$V_{k'+b^*}(\tau) - V_{k'}(\tau) - e^{-r(\tau+\Delta)}R^* = (1-\theta_{kk'})\left[V_{k'+b^*}(\tau) + V_{k-b^*}(\tau) - V_{k'}(\tau) - V_k(\tau)\right],$$

so the value of (65) achieved by (b^*, R^*) is

$$\theta_{kk'}^{\theta_{kk'}} (1 - \theta_{kk'})^{1-\theta_{kk'}} \left[V_{k'+b^*}(\tau) + V_{k-b^*}(\tau) - V_{k'}(\tau) - V_k(\tau)\right] \equiv \xi^*.$$

But (67) implies that there exists $b' \in \Gamma(k, k')$ such that

$$\xi^* < \theta_{kk'}^{\theta_{kk'}} (1 - \theta_{kk'})^{1-\theta_{kk'}} \left[V_{k'+b'}(\tau) + V_{k-b'}(\tau) - V_{k'}(\tau) - V_k(\tau)\right].$$

Then since B can be chosen large enough so that

$$R' = e^{r(\tau+\Delta)} \left\{\theta_{kk'} \left[V_{k'+b'}(\tau) - V_{k'}(\tau)\right] + (1 - \theta_{kk'}) \left[V_k(\tau) - V_{k-b'}(\tau)\right]\right\} \in (-B, B),$$

it follows that (b', R') achieves a higher value than (b^*, R^*), so (b^*, R^*) is not a solution to (65); a contradiction. Hence, a solution (b^*, R^*) to (65) with $-B \leq R^* \leq B$ must satisfy (66) and

$$b^* \in \arg\max_{b \in \Gamma(k,k')} \left[V_{k'+b}(\tau) + V_{k-b}(\tau) - V_{k'}(\tau) - V_k(\tau)\right]. \tag{68}$$

Since the right side of (66) is bounded, R^* is finite and B can be chosen large enough such that $R^* \in (-B, B)$, so (62) has at least one solution, and any solution to (62) must satisfy (66) and (68). To conclude, we show that any (b^*, R^*) that satisfies (66) and (68) is a solution to (62). To see this, notice that for all $(b, R) \in \Gamma(k, k') \times \mathbb{R}$,

$$\begin{aligned}
&\left[V_{k-b}(\tau) - V_k(\tau) + e^{-r(\tau+\Delta)} R\right]^{\theta_{kk'}} \left[V_{k'+b}(\tau) - V_{k'}(\tau) - e^{-r(\tau+\Delta)} R\right]^{1-\theta_{kk'}} \\
&\leq \max_{R \in \mathbb{R}} \left[V_{k-b}(\tau) - V_k(\tau) + e^{-r(\tau+\Delta)} R\right]^{\theta_{kk'}} \left[V_{k'+b}(\tau) - V_{k'}(\tau) - e^{-r(\tau+\Delta)} R\right]^{1-\theta_{kk'}} \\
&= \theta_{kk'}^{\theta_{kk'}} (1 - \theta_{kk'})^{1-\theta_{kk'}} \left[V_{k'+b}(\tau) + V_{k-b}(\tau) - V_{k'}(\tau) - V_k(\tau)\right] \\
&\leq \theta_{kk'}^{\theta_{kk'}} (1 - \theta_{kk'})^{1-\theta_{kk'}} \max_{b \in \Gamma(k,k')} \left[V_{k'+b}(\tau) + V_{k-b}(\tau) - V_{k'}(\tau) - V_k(\tau)\right] = \xi^*. \blacksquare
\end{aligned}$$

Lemma 2 *The function $J_k(x, \tau)$ given in (2) satisfies (1) if and only if $V_k(\tau)$ satisfies (3), given (4) and (5).*

Proof of Lemma 2. Let $\boldsymbol{B}$ denote the space of bounded real-valued functions defined on $\mathbb{K} \times [0, T]$. Let $\boldsymbol{B'}$ denote the space of functions obtained by adding $e^{-r(\tau+\Delta)}x$ for some $x \in \mathbb{R}$, to each element of $\boldsymbol{B}$. That is,

$$\boldsymbol{B'} = \left\{g : \mathbb{S} \to \mathbb{R} \mid g(k, x, \tau) = w(k, \tau) + e^{-r(\tau+\Delta)} x \text{ for some } w \in \boldsymbol{B}\right\},$$

where $\mathbb{S} = \mathbb{K} \times \mathbb{R} \times [0, T]$. Let $\boldsymbol{s} = (k, x)$ and $\boldsymbol{s}' = (k', x')$ denote two elements of $\mathbb{K} \times \mathbb{R}$. For any $g \in \boldsymbol{B}'$ and any $(\boldsymbol{s}, \boldsymbol{s}', \tau) \in \mathbb{K} \times \mathbb{R} \times \mathbb{S}$, let

$$\tilde{H}\left(\boldsymbol{s}, \boldsymbol{s}', \tau; g\right) = \arg \max_{b \in \Gamma(k,k'), R \in \mathbb{R}} \Big\{ \left[g\left(k - b, x + R, \tau\right) - g\left(k, x, \tau\right)\right]^{\theta_{kk'}} \\ \left[g(k' + b, x' - R, \tau) - g(k', x', \tau)\right]^{1-\theta_{kk'}} \Big\},$$

where $\theta_{kk'} = 1 - \theta_{k'k} \in [0, 1]$ for any $k, k' \in \mathbb{K}$. Since $g \in \boldsymbol{B}'$, $\tilde{H}\left(\boldsymbol{s}, \boldsymbol{s}', \tau; g\right) = H^*\left(k, k', \tau; w\right)$, where

$$H^*\left(k, k', \tau; w\right) = \arg \max_{b \in \Gamma(k,k'), R \in \mathbb{R}} \left\{ \left[w\left(k - b, \tau\right) - w\left(k, \tau\right) + e^{-r(\tau+\Delta)} R\right]^{\theta_{kk'}} \right. \\ \left. \left[w(k' + b, \tau) - w(k', \tau) - e^{-r(\tau+\Delta)} R\right]^{1-\theta_{kk'}} \right\}$$

for some $w \in \boldsymbol{B}$, as defined in Lemma 1. By Lemma 1, $H^*\left(k, k', \tau; w\right)$ is nonempty, and $(b(k, k', \tau), R(k', k, \tau)) \in H^*\left(k, k', \tau; w\right)$ if and only if

$$b(k, k', \tau) \in \arg \max_{b \in \Gamma(k,k')} \left[w(k' + b, \tau) + w\left(k - b, \tau\right) - w(k', \tau) - w\left(k, \tau\right)\right] \tag{69}$$

and

$$e^{-r(\tau+\Delta)} R(k', k, \tau) = \theta_{kk'} \left\{w\left[k' + b(k, k', \tau), \tau\right] - w(k', \tau)\right\} \\ + (1 - \theta_{kk'}) \left\{w(k, \tau) - w\left[k - b(k, k', \tau), \tau\right]\right\}. \tag{70}$$

The right side of (1) defines a mapping $\mathcal{T}$ on $\boldsymbol{B}'$. That is, for any $g \in \boldsymbol{B}'$ and all $(k, x, \tau) \in \mathbb{S}$,

$$\left(\mathcal{T} g\right)(k, x, \tau) = \mathbb{E}\Bigg[\int_0^{\min(\tau_\alpha, \tau)} e^{-rz} u_k dz + \mathbb{I}_{\{\tau_\alpha > \tau\}} e^{-r\tau} \left(U_k + e^{-r\Delta} x\right) \\ + \mathbb{I}_{\{\tau_\alpha \le \tau\}} e^{-r\tau_\alpha} \int g\left[k - b(k, k', \tau - \tau_\alpha), x + R(k', k, \tau - \tau_\alpha), \tau - \tau_\alpha\right] \mu\left(d\boldsymbol{s}', \tau - \tau_\alpha\right)\Bigg]$$

where $b(k, k', \tau)$ satisfies (69) and $R(k', k, \tau)$ satisfies (70) (for the special case $\theta_{kk'} = 1/2$ for all $k, k' \in \mathbb{K}$), for $w \in \boldsymbol{B}$ defined by $w\left(k, \tau\right) = g\left(k, x, \tau\right) - e^{-r(\tau+\Delta)} x$ for all $(k, \tau) \in \mathbb{K} \times [0, T]$. Substitute $g\left(k, x, \tau\right) = w\left(k, \tau\right) + e^{-r(\tau+\Delta)} x$ on the right side of $\left(\mathcal{T} g\right)(k, x, \tau)$ to obtain

$$\left(\mathcal{T} g\right)(k, x, \tau) = \left(\mathcal{M} w\right)(k, \tau) + e^{-r(\tau+\Delta)} x, \tag{71}$$

where $\mathcal{M}$ is a mapping on $\boldsymbol{B}$ defined by

$$
\begin{aligned}
(\mathcal{M}w)(k,\tau) = \mathbb{E}\Bigg[&\int_0^{\min(\tau_\alpha,\tau)} e^{-rz}u_k dz + \mathbb{I}_{\{\tau_\alpha>\tau\}}e^{-r\tau}U_k \\
&+ \mathbb{I}_{\{\tau_\alpha\leq\tau\}}e^{-r\tau_\alpha}\int w\left[k-b(k,k',\tau-\tau_\alpha),\tau-\tau_\alpha\right]\mu\left(d\boldsymbol{s}',\tau-\tau_\alpha\right) \\
&+ \mathbb{I}_{\{\tau_\alpha\leq\tau\}}e^{-r\tau_\alpha}\int e^{-r(\tau+\Delta-\tau_\alpha)}R(k',k,\tau-\tau_\alpha)\mu\left(d\boldsymbol{s}',\tau-\tau_\alpha\right)\Bigg],
\end{aligned} \tag{72}
$$

for all $(k,\tau)\in\mathbb{K}\times[0,T]$. Since the right side of (72) is independent of the net credit position x, after recognizing that $\mu\left(\{(k',x)\in\mathbb{K}\times\mathbb{R}:k'=k\},\tau\right)=n_k(\tau)$, (72) can be written as

$$
\begin{aligned}
(\mathcal{M}w)(k,\tau) = &\,\mathbb{E}\left[\int_0^{\min(\tau_\alpha,\tau)} e^{-rz}u_k dz + \mathbb{I}_{\{\tau_\alpha>\tau\}}e^{-r\tau}U_k\right] \\
&+ \mathbb{E}\Bigg[\mathbb{I}_{\{\tau_\alpha\leq\tau\}}e^{-r\tau_\alpha}\sum_{k'\in\mathbb{K}} n_{k'}(\tau-\tau_\alpha)\,w\left[k-b(k,k',\tau-\tau_\alpha),\tau-\tau_\alpha\right] \\
&+ \mathbb{I}_{\{\tau_\alpha\leq\tau\}}e^{-r\tau_\alpha}\sum_{k'\in\mathbb{K}} n_{k'}(\tau-\tau_\alpha)\,e^{-r(\tau+\Delta-\tau_\alpha)}R(k',k,\tau-\tau_\alpha)\Bigg],
\end{aligned} \tag{73}
$$

for all $(k,\tau)\in\mathbb{K}\times[0,T]$. From (73), it is clear that $\mathcal{M}$ is the mapping defined by the right side of (3). Since $w\in\boldsymbol{B}$, and $(b(k,k',\tau),R(k',k,\tau))$ satisfy (69) and (70), it follows that $\mathcal{M}:\boldsymbol{B}\to\boldsymbol{B}$, and together with (71), this implies $\mathcal{T}:\boldsymbol{B}'\to\boldsymbol{B}'$. Notice that $g^*=w^*+e^{-r(\tau+\Delta)}x\in\boldsymbol{B}'$ is a fixed point of $\mathcal{T}$ if and only if $w^*\in\boldsymbol{B}$ is a fixed point of $\mathcal{M}$. In the statement of the lemma and in the body of the paper, the fixed points $g^*(k,x,\tau)$ and $w^*(k,\tau)$ are denoted $J_k(x,\tau)$ and $V_k(\tau)$, respectively. ■

Proof of Proposition 1. Start with the mapping (73), and notice that after writing out the expectation explicitly and performing a change of variable, it becomes

$$
(\mathcal{M}w)(k,\tau) = v_k(\tau)+\alpha\int_0^\tau\sum_{k'\in\mathbb{K}} n_{k'}(z)\left\{w\left[k-b(k,k',z),z\right]+e^{-r(z+\Delta)}R(k',k,z)\right\}e^{-(r+\alpha)(\tau-z)}dz,
$$

for all $(k,\tau)\in\mathbb{K}\times[0,T]$, where

$$
v_k(\tau)\equiv\mathbb{E}\left[\int_0^{\min(\tau_\alpha,\tau)} e^{-rz}u_k dz + \mathbb{I}_{\{\tau_\alpha>\tau\}}e^{-r\tau}U_k\right],
$$

which can be integrated to obtain the expression in (11). Since $b(k,k',\tau)$ and $R(k',k,\tau)$ satisfy

(69) and (70), the previous expression for the mapping $\mathcal{M}$ can be written as

$$(\mathcal{M}w)(k,\tau) = v_k(\tau) + \alpha \int_0^\tau w(k,z)\, e^{-(r+\alpha)(\tau-z)} dz$$
$$+ \alpha \int_0^\tau \Bigg[\sum_{k' \in \mathbb{K}} n_{k'}(z)\, \theta_{kk'} \{ w\left[k' + b(k,k',z), z\right] + w\left[k - b(k,k',z), z\right]$$
$$- w(k',z) - w(k,z) \} \Bigg] e^{-(r+\alpha)(\tau-z)} dz.$$

In turn, since

$$\begin{aligned} & w\left[k' + b(k,k',z), z\right] + w\left[k - b(k,k',z), z\right] - w(k',z) - w(k,z) \\ &= \max_{b \in \Gamma(k,k')} \left[w(k'+b,z) + w(k-b,z) - w(k',z) - w(k,z) \right] \\ &= \max_{(i,j) \in \Pi(k,k')} \left[w(j,z) + w(i,z) - w(k',z) - w(k,z) \right], \end{aligned}$$

we have

$$(\mathcal{M}w)(k,\tau) = v_k(\tau) + \alpha \int_0^\tau w(k,z)\, e^{-(r+\alpha)(\tau-z)} dz$$
$$+ \alpha \int_0^\tau \sum_{k' \in \mathbb{K}} n_{k'}(z)\, \theta_{kk'} \max_{(i,j) \in \Pi(k,k')} \left[w(i,z) + w(j,z) - w(k,z) - w(k',z) \right] e^{-(r+\alpha)(\tau-z)} dz,$$

for all $(k,\tau) \in \mathbb{K} \times [0,T]$. With a relabeling, this mapping can be rewritten as

$$(\mathcal{M}w)(i,\tau) = v_i(\tau) + \alpha \int_0^\tau w(i,z)\, e^{-(r+\alpha)(\tau-z)} dz \tag{74}$$
$$+ \alpha \int_0^\tau \sum_{j \in \mathbb{K}} \sum_{k \in \mathbb{K}} \sum_{s \in \mathbb{K}} n_j(z)\, \theta_{ij} \phi_{ij}^{ks}(z) \left[w(k,z) + w(s,z) - w(i,z) - w(j,z) \right] e^{-(r+\alpha)(\tau-z)} dz,$$

for all $(i,\tau) \in \mathbb{K} \times [0,T]$, with

$$\phi_{ij}^{ks}(z) = \begin{cases} \tilde{\phi}_{ij}^{ks}(z) & \text{if } (k,s) \in \Omega_{ij}\left[w(\cdot,z)\right] \\ 0 & \text{if } (k,s) \notin \Omega_{ij}\left[w(\cdot,z)\right], \end{cases}$$

for all $i,j,k,s \in \mathbb{K}$ and all $z \in [0,T]$, where $\tilde{\phi}_{ij}^{ks}(z) \geq 0$ and $\sum_{k \in \mathbb{K}} \sum_{s \in \mathbb{K}} \tilde{\phi}_{ij}^{ks}(z) = 1$, and

$$\Omega_{ij}\left[w(\cdot,z)\right] \equiv \arg \max_{(k',s') \in \Pi(i,j)} \left[w(k',z) + w(s',z) - w(i,z) - w(j,z) \right].$$

From (74) (with $\theta_{kk'} = 1/2$), it is clear that (10) is just $\boldsymbol{V} = \mathcal{M}\boldsymbol{V}$. ■

The following lemma establishes the equivalence between property (DMC) and discrete midpoint concavity.

Lemma 3 *Let g be a real-valued function on $\mathbb{K}$. Then g satisfies*

$$g\left(\left\lceil \tfrac{i+j}{2} \right\rceil\right) + g\left(\left\lfloor \tfrac{i+j}{2} \right\rfloor\right) \geq g(k) + g(s) \tag{75}$$

for any $i, j \in \mathbb{K}$ and all $(k,s) \in \Pi(i,j)$, if and only if it satisfies the discrete midpoint concavity property,

$$g\left(\left\lceil \tfrac{i+j}{2} \right\rceil\right) + g\left(\left\lfloor \tfrac{i+j}{2} \right\rfloor\right) \geq g(i) + g(j) \tag{76}$$

for all $i, j \in \mathbb{K}$.

Proof of Lemma 3. Suppose that g satisfies (75). Since the condition holds for all $(k,s) \in \Pi(i,j)$, and we know that $(i,j) \in \Pi(i,j)$, it holds for the special case $(k,s) = (i,j)$, so g satisfies (76). To show the converse, notice that since (76) holds for all $i, j \in \mathbb{K}$, it also holds for all $i, j \in \mathbb{K}$ such that $(i,j) \in \Pi(k,s)$ for any $k, s \in \mathbb{K}$. But for any such (i,j), we know that $i + j = k + s$, so (76) implies

$$g\left(\left\lceil \tfrac{k+s}{2} \right\rceil\right) + g\left(\left\lfloor \tfrac{k+s}{2} \right\rfloor\right) \geq g(i) + g(j)$$

for any $k, s \in \mathbb{K}$ and all $(i,j) \in \Pi(k,s)$, which is the same as (75) up to a relabeling. ■

The following two lemmas are used in the proof of Proposition 2.

Lemma 4 *For any given path $\boldsymbol{n}(\tau)$, there exists a unique $w^* \in \boldsymbol{B}$ that satisfies $w^* = \mathcal{M}w^*$, and a unique $g^* \in \boldsymbol{B}'$ that satisfies $g^* = \mathcal{T}g^*$, defined by $g^*(k,x,\tau) = w^*(k,\tau) + e^{-r(\tau+\Delta)}x$ for all $(k,x,\tau) \in \mathbb{S}$.*

Proof of Lemma 4. Write the mapping $\mathcal{M}$ defined in the proof of Proposition 1 (with $\theta_{kk'} = 1/2$), as

$$\begin{aligned}(\mathcal{M}w)(i,\tau) = {} & v_i(\tau) + \alpha \int_0^\tau w(i,z)\, e^{-(r+\alpha)(\tau-z)} dz \\ & + \frac{\alpha}{2} \int_0^\tau \sum_{j\in\mathbb{K}} n_j(z) \max_{(k,s)\in\Pi(i,j)} [w(k,z) + w(s,z) - w(i,z) - w(j,z)]\, e^{-(r+\alpha)(\tau-z)} dz,\end{aligned}$$

for all $(i,\tau) \in \mathbb{K} \times [0,T]$. For any $w, w' \in \boldsymbol{B}$, define the metric $D : \boldsymbol{B} \times \boldsymbol{B} \to \mathbb{R}$, by

$$D\left(w, w'\right) = \sup_{(i,\tau)\in\mathbb{K}\times[0,T]} \left[e^{-\beta\tau} \left| w(i,\tau) - w'(i,\tau) \right| \right],$$

where $\beta \in \mathbb{R}$ satisfies

$$\max\{0, 2\alpha - r\} < \beta < \infty. \tag{77}$$

For the case with $\beta = 0$, D reduces to the standard sup metric, d_∞. The metric space $(\boldsymbol{B}, d_\infty)$ is complete, and since $(\boldsymbol{B}, D)$ and $(\boldsymbol{B}, d_\infty)$ are strongly equivalent, it follows that $(\boldsymbol{B}, D)$ is also a complete metric space (see OK, 2007, p. 136 and 167). For any $w, w' \in \boldsymbol{B}$, and any $(i, \tau) \in \mathbb{K} \times [0, T]$,

$$\begin{aligned}
&e^{-\beta\tau} \left|(\mathcal{M}w)(i,\tau) - (\mathcal{M}w')(i,\tau)\right| = \\
&\quad = e^{-\beta\tau}\Bigg|\alpha \int_0^\tau w(i,z)\, e^{-(r+\alpha)(\tau-z)} dz - \alpha \int_0^\tau w'(i,z)\, e^{-(r+\alpha)(\tau-z)} dz \\
&\quad + \frac{\alpha}{2} \int_0^\tau \sum_{j\in\mathbb{K}} n_j(z) \max_{(k,s)\in\Pi(i,j)} \left[w(k,z) + w(s,z) - w(i,z) - w(j,z)\right] e^{-(r+\alpha)(\tau-z)} dz \\
&\quad - \frac{\alpha}{2} \int_0^\tau \sum_{j\in\mathbb{K}} n_j(z) \max_{(k,s)\in\Pi(i,j)} \left[w'(k,z) + w'(s,z) - w'(i,z) - w'(j,z)\right] e^{-(r+\alpha)(\tau-z)} dz\Bigg| \\
&\quad \le \alpha e^{-\beta\tau} \int_0^\tau \left|w(i,z) - w'(i,z)\right| e^{-(r+\alpha)(\tau-z)} dz \\
&\quad + \frac{\alpha}{2} e^{-\beta\tau} \int_0^\tau \sum_{j\in\mathbb{K}} n_j(z) \Bigg|\max_{(k,s)\in\Pi(i,j)} \left[w(k,z) + w(s,z) - w(i,z) - w(j,z)\right] \\
&\quad - \max_{(k,s)\in\Pi(i,j)} \left[w'(k,z) + w'(s,z) - w'(i,z) - w'(j,z)\right]\Bigg| e^{-(r+\alpha)(\tau-z)} dz.
\end{aligned}$$

Use $\left(k_{ij}^*(z), s_{ij}^*(z)\right)$ to denote a solution to the maximization on the right side of $\mathcal{M}w$, that is,

$$\left(k_{ij}^*(z), s_{ij}^*(z)\right) \in \max_{(k,s)\in\Pi(i,j)} \left[w(k,z) + w(s,z) - w(i,z) - w(j,z)\right].$$

A solution exists because $w \in \boldsymbol{B}$, and $\Pi(i,j)$ is a finite set for all $(i,j) \in \mathbb{K} \times \mathbb{K}$. Then

$$\begin{aligned}
&e^{-\beta\tau} \left|(\mathcal{M}w)(i,\tau) - (\mathcal{M}w')(i,\tau)\right| \le \\
&\le \alpha \int_0^\tau e^{-\beta z} \left|w(i,z) - w'(i,z)\right| e^{-(r+\alpha+\beta)(\tau-z)} dz \\
&+ \frac{\alpha}{2} \int_0^\tau \sum_{j\in\mathbb{K}} n_j(z) \Big\{ e^{-\beta z} \left|w\left(k_{ij}^*(z), z\right) - w'\left(k_{ij}^*(z), z\right)\right| + e^{-\beta z} \left|w\left(s_{ij}^*(z), z\right) - w'\left(s_{ij}^*(z), z\right)\right| \\
&+ e^{-\beta z} \left|w'(i,z) - w(i,z)\right| + e^{-\beta z} \left|w'(j,z) - w(j,z)\right| \Big\} e^{-(r+\alpha+\beta)(\tau-z)} dz \\
&\le \frac{3\alpha}{r+\alpha+\beta} \left[1 - e^{-(r+\alpha+\beta)\tau}\right] D(w, w') \\
&\le \frac{3\alpha}{r+\alpha+\beta} D(w, w').
\end{aligned}$$

Since this last inequality holds for all $(i,\tau) \in \mathbb{K} \times [0,T]$, and w and w' are arbitrary,

$$D\left(\mathcal{M}w, \mathcal{M}w'\right) \leq \frac{3\alpha}{r+\alpha+\beta} D\left(w, w'\right), \quad \text{for all } w, w' \in \boldsymbol{B}. \tag{78}$$

Notice that (77) implies $\frac{3\alpha}{r+\alpha+\beta} \in (0,1)$, so $\mathcal{M}$ is a contraction mapping on the complete metric space $(\boldsymbol{B}, D)$. By the Contraction Mapping Theorem (Theorem 3.2 in Stokey and Lucas, 1989), for any given path $\boldsymbol{n}(\tau)$, there exists a unique $w^* \in \boldsymbol{B}$ that satisfies $w^* = \mathcal{M}w^*$, and therefore, by (71), there exists a unique $g^* \in \boldsymbol{B}'$ that satisfies $g^* = \mathcal{T}g^*$, and it is defined by $g^*(k,x,\tau) = w^*(k,\tau) + e^{-r(\tau+\Delta)}x$ for all $(k,x,\tau) \in \mathbb{S}$. ■

Lemma 5 *Let $i,j,q \in \mathbb{K}$, and $(k,s) \in \Pi(i,j)$.*

(i) *If either $i+j$ or $s+q$ is even, then*

$$\left(\left\lceil \tfrac{k+q}{2} \right\rceil, \left\lfloor \tfrac{s+q}{2} \right\rfloor\right) \in \Pi\left(\left\lceil \tfrac{i+j}{2} \right\rceil, q\right) \quad \textit{and} \quad \left(\left\lfloor \tfrac{k+q}{2} \right\rfloor, \left\lceil \tfrac{s+q}{2} \right\rceil\right) \in \Pi\left(\left\lfloor \tfrac{i+j}{2} \right\rfloor, q\right).$$

(ii) *If $i+j$ and $s+q$ are odd, then*

$$\left(\left\lfloor \tfrac{k+q}{2} \right\rfloor, \left\lceil \tfrac{s+q}{2} \right\rceil\right) \in \Pi\left(\left\lceil \tfrac{i+j}{2} \right\rceil, q\right) \quad \textit{and} \quad \left(\left\lceil \tfrac{k+q}{2} \right\rceil, \left\lfloor \tfrac{s+q}{2} \right\rfloor\right) \in \Pi\left(\left\lfloor \tfrac{i+j}{2} \right\rfloor, q\right).$$

Proof of Lemma 5. Notice that for any $i,j,q \in \mathbb{K}$,

$$\Pi(i,j) = \{(i+j-y, y) \in \mathbb{K} \times \mathbb{K} : y \in \{0,1,\ldots,i+j\}\},$$

so

$$\Pi\left(\left\lceil \tfrac{i+j}{2} \right\rceil, q\right) = \left\{\left(\left\lceil \tfrac{i+j}{2} \right\rceil + q - y, y\right) \in \mathbb{K} \times \mathbb{K} : y \in \left\{0,1,\ldots,\left\lceil \tfrac{i+j}{2} \right\rceil + q\right\}\right\} \tag{79}$$

$$\Pi\left(\left\lfloor \tfrac{i+j}{2} \right\rfloor, q\right) = \left\{\left(\left\lfloor \tfrac{i+j}{2} \right\rfloor + q - y, y\right) \in \mathbb{K} \times \mathbb{K} : y \in \left\{0,1,\ldots,\left\lfloor \tfrac{i+j}{2} \right\rfloor + q\right\}\right\}. \tag{80}$$

For any $i,j,q \in \mathbb{K}$, define

$$\tilde{\Pi}(i,j,q) = \left\{\left(\left\lceil \tfrac{k+q}{2} \right\rceil, \left\lfloor \tfrac{s+q}{2} \right\rfloor\right) \in \mathbb{K} \times \mathbb{K} : (k,s) \in \Pi(i,j)\right\}$$
$$\hat{\Pi}(i,j,q) = \left\{\left(\left\lfloor \tfrac{k+q}{2} \right\rfloor, \left\lceil \tfrac{s+q}{2} \right\rceil\right) \in \mathbb{K} \times \mathbb{K} : (k,s) \in \Pi(i,j)\right\},$$

and recall that $(k,s) \in \Pi(i,j)$ implies $k+s = i+j$.

(i) Assume that either $i+j$ or $s+q$ is even. We first show that given any $i,j,q \in \mathbb{K}$, $(k,s) \in \Pi(i,j)$ implies $\left(\left\lceil \frac{k+q}{2} \right\rceil, \left\lfloor \frac{s+q}{2} \right\rfloor\right) \in \Pi\left(\left\lceil \frac{i+j}{2} \right\rceil, q\right)$. Notice that if either $i+j$ or $s+q$ is even, then

$$\left\lceil \frac{k+q}{2} \right\rceil + \left\lfloor \frac{s+q}{2} \right\rfloor = \left\lceil \frac{i+j}{2} \right\rceil + q. \tag{81}$$

With (81),

$$\begin{aligned}\tilde{\Pi}(i,j,q) &= \left\{\left(\left\lceil \tfrac{k+q}{2} \right\rceil, \left\lceil \tfrac{i+j}{2} \right\rceil + q - \left\lceil \tfrac{k+q}{2} \right\rceil\right) \in \mathbb{K}\times\mathbb{K} : (k, i+j-k) \in \Pi(i,j)\right\} \\ &= \left\{\left(y, \left\lceil \tfrac{i+j}{2} \right\rceil + q - y\right) \in \mathbb{K}\times\mathbb{K} : y \in \left\{\left\lceil \tfrac{q}{2} \right\rceil, \left\lceil \tfrac{q+1}{2} \right\rceil, \dots, \left\lceil \tfrac{q+i+j}{2} \right\rceil\right\}\right\} \\ &\equiv \tilde{\Pi}^e\left(\left\lceil \tfrac{i+j}{2} \right\rceil, q\right).\end{aligned} \tag{82}$$

By construction, given any $i,j,q \in \mathbb{K}$, $\left(\left\lceil \frac{k+q}{2} \right\rceil, \left\lfloor \frac{s+q}{2} \right\rfloor\right) \in \tilde{\Pi}^e\left(\left\lceil \frac{i+j}{2} \right\rceil, q\right)$ for all $(k,s) \in \Pi(i,j)$. Since $0 \le \left\lceil \frac{q}{2} \right\rceil$, and $\left\lceil \frac{q+i+j}{2} \right\rceil \le \left\lceil \frac{i+j}{2} \right\rceil + q$, it follows from (79) and (82) that $\tilde{\Pi}^e\left(\left\lceil \frac{i+j}{2} \right\rceil, q\right) \subseteq \Pi\left(\left\lceil \frac{i+j}{2} \right\rceil, q\right)$ for all $i,j,q \in \mathbb{K}$, which implies $\left(\left\lceil \frac{k+q}{2} \right\rceil, \left\lfloor \frac{s+q}{2} \right\rfloor\right) \in \Pi\left(\left\lceil \frac{i+j}{2} \right\rceil, q\right)$ for all $(k,s) \in \Pi(i,j)$, and any $i,j,q \in \mathbb{K}$.

Next, we show that given any $i,j,q \in \mathbb{K}$, $(k,s) \in \Pi(i,j)$ implies $\left(\left\lfloor \frac{k+q}{2} \right\rfloor, \left\lceil \frac{s+q}{2} \right\rceil\right) \in \Pi\left(\left\lfloor \frac{i+j}{2} \right\rfloor, q\right)$. Notice that if either $i+j$ or $s+q$ is even, then

$$\left\lfloor \frac{k+q}{2} \right\rfloor + \left\lceil \frac{s+q}{2} \right\rceil = \left\lfloor \frac{i+j}{2} \right\rfloor + q. \tag{83}$$

With (83),

$$\begin{aligned}\hat{\Pi}(i,j,q) &= \left\{\left(\left\lfloor \tfrac{k+q}{2} \right\rfloor, \left\lfloor \tfrac{i+j}{2} \right\rfloor + q - \left\lfloor \tfrac{k+q}{2} \right\rfloor\right) \in \mathbb{K}\times\mathbb{K} : (k, i+j-k) \in \Pi(i,j)\right\} \\ &= \left\{\left(y, \left\lfloor \tfrac{i+j}{2} \right\rfloor + q - y\right) \in \mathbb{K}\times\mathbb{K} : y \in \left\{\left\lfloor \tfrac{q}{2} \right\rfloor, \left\lfloor \tfrac{q+1}{2} \right\rfloor, \dots, \left\lfloor \tfrac{q+i+j}{2} \right\rfloor\right\}\right\} \\ &\equiv \hat{\Pi}^e\left(\left\lfloor \tfrac{i+j}{2} \right\rfloor, q\right).\end{aligned} \tag{84}$$

By construction, given any $i,j,q \in \mathbb{K}$, $\left(\left\lfloor \frac{k+q}{2} \right\rfloor, \left\lceil \frac{s+q}{2} \right\rceil\right) \in \hat{\Pi}^e\left(\left\lfloor \frac{i+j}{2} \right\rfloor, q\right)$ for all $(k,s) \in \Pi(i,j)$. Since $0 \le \left\lfloor \frac{q}{2} \right\rfloor$, and $\left\lfloor \frac{q+i+j}{2} \right\rfloor \le \left\lfloor \frac{i+j}{2} \right\rfloor + q$, it follows from (80) and (84) that $\hat{\Pi}^e\left(\left\lfloor \frac{i+j}{2} \right\rfloor, q\right) \subseteq \Pi\left(\left\lfloor \frac{i+j}{2} \right\rfloor, q\right)$ for all $i,j,q \in \mathbb{K}$, which implies $\left(\left\lfloor \frac{k+q}{2} \right\rfloor, \left\lceil \frac{s+q}{2} \right\rceil\right) \in \Pi\left(\left\lfloor \frac{i+j}{2} \right\rfloor, q\right)$ for all $(k,s) \in \Pi(i,j)$, and any $i,j,q \in \mathbb{K}$.

(ii) Suppose that $i+j$ and $s+q$ are odd. We first show that given any $i,j,q \in \mathbb{K}$, $(k,s) \in \Pi(i,j)$ implies $\left(\left\lfloor \frac{k+q}{2} \right\rfloor, \left\lceil \frac{s+q}{2} \right\rceil\right) \in \Pi\left(\left\lceil \frac{i+j}{2} \right\rceil, q\right)$. Notice that if $i+j$ and $s+q$ are odd, then

$$\left\lfloor \frac{k+q}{2} \right\rfloor + \left\lceil \frac{s+q}{2} \right\rceil = \left\lceil \frac{i+j}{2} \right\rceil + q. \tag{85}$$

With (85),

$$\begin{aligned}\hat{\Pi}(i,j,q) &= \left\{\left(\left\lceil \tfrac{i+j}{2}\right\rceil + q - \lceil \tfrac{s+q}{2}\rceil, \lceil \tfrac{s+q}{2}\rceil\right) \in \mathbb{K}\times\mathbb{K} : (k,s)\in\Pi(i,j)\right\}\\ &= \left\{\left(\left\lceil \tfrac{i+j}{2}\right\rceil + q - y, y\right) \in \mathbb{K}\times\mathbb{K} : y \in \left\{\lceil \tfrac{q}{2}\rceil, \left\lceil \tfrac{q+1}{2}\right\rceil, \ldots, \left\lceil \tfrac{q+i+j}{2}\right\rceil\right\}\right\}\\ &\equiv \hat{\Pi}^o\left(\left\lceil \tfrac{i+j}{2}\right\rceil, q\right).\end{aligned} \tag{86}$$

By construction, given any $i,j,q\in\mathbb{K}$, $\left(\left\lfloor \frac{k+q}{2}\right\rfloor, \lceil \frac{s+q}{2}\rceil\right) \in \hat{\Pi}^o\left(\left\lceil \frac{i+j}{2}\right\rceil, q\right)$ for all $(k,s)\in\Pi(i,j)$. Since $0 \le \lceil \frac{q}{2}\rceil$, and $\left\lceil \frac{q+i+j}{2}\right\rceil \le \left\lceil \frac{i+j}{2}\right\rceil + q$, it follows from (79) and (86) that $\hat{\Pi}^o\left(\left\lceil \frac{i+j}{2}\right\rceil, q\right) \subseteq \Pi\left(\left\lceil \frac{i+j}{2}\right\rceil, q\right)$ for all $i,j,q\in\mathbb{K}$, which implies $\left(\left\lfloor \frac{k+q}{2}\right\rfloor, \lceil \frac{s+q}{2}\rceil\right) \in \Pi\left(\left\lceil \frac{i+j}{2}\right\rceil, q\right)$ for all $(k,s)\in\Pi(i,j)$, and any $i,j,q\in\mathbb{K}$.

Finally, we show that given any $i,j,q\in\mathbb{K}$, $(k,s)\in\Pi(i,j)$ implies $\left(\left\lceil \frac{k+q}{2}\right\rceil, \lfloor \frac{s+q}{2}\rfloor\right) \in \Pi\left(\left\lfloor \frac{i+j}{2}\right\rfloor, q\right)$. Notice that if $i+j$ and $s+q$ are odd, then

$$\left\lceil \frac{k+q}{2}\right\rceil + \left\lfloor \frac{s+q}{2}\right\rfloor = \left\lfloor \frac{i+j}{2}\right\rfloor + q. \tag{87}$$

With (87),

$$\begin{aligned}\tilde{\Pi}(i,j,q) &= \left\{\left(\left\lfloor \tfrac{i+j}{2}\right\rfloor + q - \lfloor \tfrac{s+q}{2}\rfloor, \lfloor \tfrac{s+q}{2}\rfloor\right) \in \mathbb{K}\times\mathbb{K} : (k,s)\in\Pi(i,j)\right\}\\ &= \left\{\left(\left\lfloor \tfrac{i+j}{2}\right\rfloor + q - y, y\right) \in \mathbb{K}\times\mathbb{K} : y \in \left\{\lfloor \tfrac{q}{2}\rfloor, \left\lfloor \tfrac{q+1}{2}\right\rfloor, \ldots, \left\lfloor \tfrac{q+i+j}{2}\right\rfloor\right\}\right\}\\ &\equiv \tilde{\Pi}^o\left(\left\lfloor \tfrac{i+j}{2}\right\rfloor, q\right).\end{aligned} \tag{88}$$

By construction, given any $i,j,q\in\mathbb{K}$, $\left(\left\lceil \frac{k+q}{2}\right\rceil, \lfloor \frac{s+q}{2}\rfloor\right) \in \tilde{\Pi}^o\left(\left\lfloor \frac{i+j}{2}\right\rfloor, q\right)$ for all $(k,s)\in\Pi(i,j)$. Since $0 \le \lfloor \frac{q}{2}\rfloor$, and $\left\lfloor \frac{q+i+j}{2}\right\rfloor \le \left\lfloor \frac{i+j}{2}\right\rfloor + q$, it follows from (80) and (88) that $\tilde{\Pi}^o\left(\left\lfloor \frac{i+j}{2}\right\rfloor, q\right) \subseteq \left(\left\lfloor \frac{i+j}{2}\right\rfloor, q\right)$ for all $i,j,q\in\mathbb{K}$, which implies $\left(\left\lceil \frac{k+q}{2}\right\rceil, \lfloor \frac{s+q}{2}\rfloor\right) \in \Pi\left(\left\lfloor \frac{i+j}{2}\right\rfloor, q\right)$ for all $(k,s)\in\Pi(i,j)$, and any $i,j,q\in\mathbb{K}$. ■

Proof of Proposition 2. Consider the metric space $(\boldsymbol{B}, D)$ used in the proof of Lemma 4. A function $w\in\boldsymbol{B}$ satisfies the *bilateral-trade asset-holding Equalization Property* (EP) if for all $(i,j,\tau)\in\mathbb{K}\times\mathbb{K}\times[0,T]$,

$$\begin{aligned}&\max_{(k,s)\in\Pi(i,j)} \left[w(k,\tau) + w(s,\tau) - w(i,\tau) - w(j,\tau)\right]\\ &\qquad = w\left(\left\lceil \tfrac{i+j}{2}\right\rceil, \tau\right) + w\left(\left\lfloor \tfrac{i+j}{2}\right\rfloor, \tau\right) - w(i,\tau) - w(j,\tau).\end{aligned} \tag{EP}$$

A function $w \in \boldsymbol{B}$ satisfies the *bilateral-trade asset-holding Strict Equalization Property* (SEP) if for all $(i,j,\tau) \in \mathbb{K} \times \mathbb{K} \times [0,T]$,

$$\arg \max_{(k,s)\in\Pi(i,j)} \left[w\left(k,\tau\right)+w\left(s,\tau\right)-w\left(i,\tau\right)-w\left(j,\tau\right)\right]=\Omega_{ij}^{*}, \tag{SEP}$$

where Ω_{ij}^{*} is defined in (15). Let

$$\boldsymbol{B}'' = \{w \in \boldsymbol{B} : w \text{ satisfies (EP)}\}$$
$$\boldsymbol{B}''' = \{w \in \boldsymbol{B} : w \text{ satisfies (SEP)}\}.$$

Clearly, $\boldsymbol{B}''' \subseteq \boldsymbol{B}'' \subseteq \boldsymbol{B}$.

We first establish that $\boldsymbol{B}''$ is a closed subset of $\boldsymbol{B}$. Let $\{w_n\}_{n=0}^{\infty}$ be a sequence of functions in $\boldsymbol{B}''$, with $\lim_{n\to\infty} w_n = \bar{w}$. If $\bar{w} \notin \boldsymbol{B}''$, then there exists some $(k,s) \in \Pi(i,j)$ and $\varsigma \in \mathbb{R}$ such that

$$0<\varsigma=\bar{w}\left(k,\tau\right)+\bar{w}\left(s,\tau\right)-\left[\bar{w}\left(\left\lceil \tfrac{i+j}{2}\right\rceil,\tau\right)+\bar{w}\left(\left\lfloor \tfrac{i+j}{2}\right\rfloor,\tau\right)\right],$$

for some $(i,j,\tau) \in \mathbb{K} \times \mathbb{K} \times [0,T]$. This implies

$$\begin{aligned} w_n\left(k,\tau\right)+w_n\left(s,\tau\right)=\; & w_n\left(\left\lceil \tfrac{i+j}{2}\right\rceil,\tau\right)+w_n\left(\left\lfloor \tfrac{i+j}{2}\right\rfloor,\tau\right)+\varsigma\\ & -\left\{\bar{w}\left(k,\tau\right)+\bar{w}\left(s,\tau\right)-\left[w_n\left(k,\tau\right)+w_n\left(s,\tau\right)\right]\right\}\\ & +\bar{w}\left(\left\lceil \tfrac{i+j}{2}\right\rceil,\tau\right)+\bar{w}\left(\left\lfloor \tfrac{i+j}{2}\right\rfloor,\tau\right)-\left[w_n\left(\left\lceil \tfrac{i+j}{2}\right\rceil,\tau\right)+w_n\left(\left\lfloor \tfrac{i+j}{2}\right\rfloor,\tau\right)\right]. \end{aligned}$$

For this particular $(i,j,\tau) \in \mathbb{K} \times \mathbb{K} \times [0,T]$, for all n large enough we can ensure that

$$\left|\bar{w}\left(k,\tau\right)+\bar{w}\left(s,\tau\right)-\left[w_n\left(k,\tau\right)+w_n\left(s,\tau\right)\right]\right|<\frac{\varsigma}{4}$$

and

$$\left|\bar{w}\left(\left\lceil \tfrac{i+j}{2}\right\rceil,\tau\right)+\bar{w}\left(\left\lfloor \tfrac{i+j}{2}\right\rfloor,\tau\right)-\left[w_n\left(\left\lceil \tfrac{i+j}{2}\right\rceil,\tau\right)+w_n\left(\left\lfloor \tfrac{i+j}{2}\right\rfloor,\tau\right)\right]\right|<\frac{\varsigma}{4},$$

but then

$$0<\varsigma/2<w_n\left(k,\tau\right)+w_n\left(s,\tau\right)-\left[w_n\left(\left\lceil \tfrac{i+j}{2}\right\rceil,\tau\right)+w_n\left(\left\lfloor \tfrac{i+j}{2}\right\rfloor,\tau\right)\right],$$

which contradicts the fact that $w_n \in \boldsymbol{B}''$. Thus, we conclude that $\bar{w} \in \boldsymbol{B}''$, so $\boldsymbol{B}''$ is closed.

The second step is to show that the mapping $\mathcal{M}$ defined in (72) preserves property (EP), i.e., that $\mathcal{M}(\boldsymbol{B}'') \subseteq \boldsymbol{B}''$. That is, we wish to show that for any $w \in \boldsymbol{B}''$, $w' = \mathcal{M}w \in \boldsymbol{B}''$, or equivalently, that

$$w\left(\left\lceil \tfrac{i+j}{2}\right\rceil,\tau\right)+w\left(\left\lfloor \tfrac{i+j}{2}\right\rfloor,\tau\right)\geq w\left(k,\tau\right)+w\left(s,\tau\right) \text{ for all } (k,s)\in\Pi\left(i,j\right),$$

for any $(i,j,\tau) \in \mathbb{K} \times \mathbb{K} \times [0,T]$, implies that

$$w'\left(\left\lceil \tfrac{i+j}{2} \right\rceil, \tau\right) + w'\left(\left\lfloor \tfrac{i+j}{2} \right\rfloor, \tau\right) - w'(k,\tau) - w'(s,\tau) \geq 0 \text{ for all } (k,s) \in \Pi(i,j), \tag{89}$$

for any $(i,j,\tau) \in \mathbb{K} \times \mathbb{K} \times [0,T]$. Since $w \in \boldsymbol{B}''$, using (74) (with $\theta_{kk'} = 1/2$ for all $k,k' \in \mathbb{K}$),

$$\begin{aligned}
(\mathcal{M}w)(i,\tau) &= v_i(\tau) + \alpha \int_0^\tau w(i,z)\, e^{-(r+\alpha)(\tau-z)} dz \\
&\quad + \frac{\alpha}{2} \int_0^\tau \sum_{q\in\mathbb{K}} n_q(z) \left[w\left(\left\lceil \tfrac{i+q}{2} \right\rceil, z\right) + w\left(\left\lfloor \tfrac{i+q}{2} \right\rfloor, z\right) - w(i,z) - w(q,z)\right] e^{-(r+\alpha)(\tau-z)} dz,
\end{aligned}$$

for all $(i,\tau) \in \mathbb{K} \times [0,T]$. For any $(i,j,\tau) \in \mathbb{K} \times \mathbb{K} \times [0,T]$ and $(k,s) \in \Pi(i,j)$, let $G(i,j,k,s,\tau)$ denote the left side of inequality (89). Then,

$$\begin{aligned}
G(i,j,k,s,\tau) &= v_{\lceil \frac{i+j}{2} \rceil}(\tau) + v_{\lfloor \frac{i+j}{2} \rfloor}(\tau) - v_k(\tau) - v_s(\tau) \\
&\quad + \alpha \int_0^\tau \left[w\left(\left\lceil \tfrac{i+j}{2} \right\rceil, z\right) + w\left(\left\lfloor \tfrac{i+j}{2} \right\rfloor, z\right) - w(k,z) - w(s,z)\right] e^{-(r+\alpha)(\tau-z)} dz \\
&\quad + \frac{\alpha}{2} \int_0^\tau \sum_{q\in\mathbb{K}} n_q(z) \Bigg[w\left(\left\lceil \frac{\left\lceil \frac{i+j}{2} \right\rceil + q}{2} \right\rceil, z\right) + w\left(\left\lfloor \frac{\left\lceil \frac{i+j}{2} \right\rceil + q}{2} \right\rfloor, z\right) \\
&\qquad\qquad - w\left(\left\lceil \tfrac{i+j}{2} \right\rceil, z\right) - w(q,z)\Bigg] e^{-(r+\alpha)(\tau-z)} dz \\
&\quad + \frac{\alpha}{2} \int_0^\tau \sum_{q\in\mathbb{K}} n_q(z) \Bigg[w\left(\left\lceil \frac{\left\lfloor \frac{i+j}{2} \right\rfloor + q}{2} \right\rceil, z\right) + w\left(\left\lfloor \frac{\left\lfloor \frac{i+j}{2} \right\rfloor + q}{2} \right\rfloor, z\right) \\
&\qquad\qquad - w\left(\left\lfloor \tfrac{i+j}{2} \right\rfloor, z\right) - w(q,z)\Bigg] e^{-(r+\alpha)(\tau-z)} dz \\
&\quad - \frac{\alpha}{2} \int_0^\tau \sum_{q\in\mathbb{K}} n_q(z) \left[w\left(\left\lceil \tfrac{k+q}{2} \right\rceil, z\right) + w\left(\left\lfloor \tfrac{k+q}{2} \right\rfloor, z\right) - w(k,z) - w(q,z)\right] e^{-(r+\alpha)(\tau-z)} dz \\
&\quad - \frac{\alpha}{2} \int_0^\tau \sum_{q\in\mathbb{K}} n_q(z) \left[w\left(\left\lceil \tfrac{s+q}{2} \right\rceil, z\right) + w\left(\left\lfloor \tfrac{s+q}{2} \right\rfloor, z\right) - w(s,z) - w(q,z)\right] e^{-(r+\alpha)(\tau-z)} dz.
\end{aligned}$$

With (11) and after deleting redundant terms, this expression can be rearranged to yield

$$
\begin{aligned}
G(i,j,k,s,\tau) = {} & \frac{1-e^{-(r+\alpha)\tau}}{r+\alpha}\left(u_{\lceil\frac{i+j}{2}\rceil}+u_{\lfloor\frac{i+j}{2}\rfloor}-u_k-u_s\right) \\
& + e^{-(r+\alpha)\tau}\left[U_{\lceil\frac{i+j}{2}\rceil}+U_{\lfloor\frac{i+j}{2}\rfloor}-U_k-U_s\right] \\
& + \frac{\alpha}{2}\int_0^\tau \left[w\left(\left\lceil\tfrac{i+j}{2}\right\rceil,z\right)+w\left(\left\lfloor\tfrac{i+j}{2}\right\rfloor,z\right)-w(k,z)-w(s,z)\right]e^{-(r+\alpha)(\tau-z)}dz \\
& + \frac{\alpha}{2}\int_0^\tau \sum_{q\in\mathbb{K}} n_q(z)\left\{w\left(\left\lceil\frac{\left\lceil\frac{i+j}{2}\right\rceil+q}{2}\right\rceil,z\right)+w\left(\left\lfloor\frac{\left\lceil\frac{i+j}{2}\right\rceil+q}{2}\right\rfloor,z\right)\right. \\
& \qquad -w\left(\left\lceil\tfrac{k+q}{2}\right\rceil,z\right)-w\left(\left\lfloor\tfrac{s+q}{2}\right\rfloor,z\right) \\
& \qquad -w\left(\left\lfloor\tfrac{k+q}{2}\right\rfloor,z\right)-w\left(\left\lceil\tfrac{s+q}{2}\right\rceil,z\right) \\
& \qquad \left.+w\left(\left\lceil\frac{\left\lfloor\frac{i+j}{2}\right\rfloor+q}{2}\right\rceil,z\right)+w\left(\left\lfloor\frac{\left\lfloor\frac{i+j}{2}\right\rfloor+q}{2}\right\rfloor,z\right)\right\}e^{-(r+\alpha)(\tau-z)}dz.
\end{aligned}
$$

What needs to be shown is that $w \in \boldsymbol{B}''$ implies that for any $(i,j,\tau) \in \mathbb{K}\times\mathbb{K}\times[0,T]$, $G(i,j,k,s,\tau) \geq 0$ for all $(k,s) \in \Pi(i,j)$. The fact that $w \in \boldsymbol{B}''$ immediately implies that the first integral in the last expression is nonnegative. By Lemma 5, $w \in \boldsymbol{B}''$ also implies that the second integral in the last expression is nonnegative. Together with Assumption A, these observations imply

$$
\begin{aligned}
0 < {} & \frac{1-e^{-(r+\alpha)\tau}}{r+\alpha}\left(u_{\lceil\frac{i+j}{2}\rceil}+u_{\lfloor\frac{i+j}{2}\rfloor}-u_k-u_s\right)+e^{-(r+\alpha)\tau}\left[U_{\lceil\frac{i+j}{2}\rceil}+U_{\lfloor\frac{i+j}{2}\rfloor}-U_k-U_s\right] \\
\leq {} & G(i,j,k,s,\tau),
\end{aligned} \tag{90}
$$

so we conclude that $\mathcal{M}(\boldsymbol{B}'') \subseteq \boldsymbol{B}''' \subseteq \boldsymbol{B}''$.

The third step is to show that (14) is the equilibrium distribution of trading probabilities. From Lemma 4, we know that $\mathcal{M}$ is a contraction mapping on the complete metric space $(\boldsymbol{B}, D)$, so it has a unique fixed point $w^*(k,\tau) \equiv V_k(\tau) \in \boldsymbol{B}$. In addition, we have now established that $\boldsymbol{B}''$ is a closed subset of $\boldsymbol{B}$, and that $\mathcal{M}(\boldsymbol{B}'') \subseteq \boldsymbol{B}''' \subseteq \boldsymbol{B}''$. Therefore, by Corollary 1 in Stokey and Lucas (1989, p. 52) we conclude that $V_k(\tau) \in \boldsymbol{B}'''$. This implies that the set $\Omega_{ij}[\boldsymbol{V}(\tau)]$ defined in (13) reduces to Ω^*_{ij} for all $(i,j,\tau) \in \mathbb{K}\times\mathbb{K}\times[0,T]$, and consequently, that (12) reduces to (14) for all $(i,j,\tau) \in \mathbb{K}\times\mathbb{K}\times[0,T]$. This establishes part (ii) in the statement of the proposition.

We can now show that the paths $\boldsymbol{n}(\tau)$ and $\boldsymbol{V}(\tau)$ are uniquely determined. Since (by Lemma 4) the fixed point $V_k(\tau) \in \boldsymbol{B}'''$ is unique given any path for the distribution of reserve balances, $\boldsymbol{n}(\tau)$, all that has to be shown is that given the initial condition $\{n_k(T)\}_{k\in\mathbb{K}}$, and given that the path $\boldsymbol{\phi}(\tau)$ satisfies (14), the system of first-order ordinary differential equations, $\dot{\boldsymbol{n}}(\tau) = f[\boldsymbol{n}(\tau), \boldsymbol{\phi}(\tau)]$, has a unique solution. But since f is continuously differentiable, this follows from Propositions 6.3 and 7.6 in Amann (1990). This establishes part (i) in the statement of the proposition.

By Proposition 1, the equilibrium value function, $\boldsymbol{V}$, satisfies (10). Notice that (10) implies (17). Differentiate both sides of (10) with respect to τ, and rearrange terms to obtain

$$\dot{V}_i(\tau)+rV_i(\tau) = \dot{v}_i(\tau)+(r+\alpha)v_i(\tau)+\frac{\alpha}{2}\sum_{j\in\mathbb{K}}\sum_{k\in\mathbb{K}}\sum_{s\in\mathbb{K}} n_j(\tau)\phi_{ij}^{ks}(\tau)\left[V_k(\tau)+V_s(\tau)-V_i(\tau)-V_j(\tau)\right],$$

which together with the fact that $\dot{v}_i(\tau) = u_i - (r+\alpha)v_i(\tau)$ implies (16). This establishes part (iii) in the statement of the proposition.

Suppose that at time $T-\tau$, a bank with balance j extends a loan of size b to a bank with balance i. Then (5) implies that the present discounted value of the repayment from the latter to the former is

$$\frac{1}{2}\left[V_{i+b}(\tau)-V_i(\tau)\right]+\frac{1}{2}\left[V_j(\tau)-V_{j-b}(\tau)\right],$$

which reduces to the right side of (18) if the loan size is $b = j-s = k-i$, as specified by part (iv) in the statement of the proposition. ■

Corollary 1 *Assume that $\{U_k\}_{k\in\mathbb{K}}$ satisfies the discrete midpoint concavity property and $\{u_k\}_{k\in\mathbb{K}}$ satisfies the discrete midpoint strict concavity property. An equilibrium exists, and the equilibrium paths for the distribution of reserve balances, $\boldsymbol{n}(\tau)$, and maximum attainable payoffs, $\boldsymbol{V}(\tau)$, are uniquely determined, and identical to those in Proposition 2. The equilibrium distribution of trading probabilities is*

$$\phi_{ij}^{ks}(\tau) = \begin{cases} \tilde{\phi}_{ij}^{ks}(\tau) & \text{if } (k,s)\in\Omega_{ij}^{*}(\tau) \\ 0 & \text{if } (k,s)\notin\Omega_{ij}^{*}(\tau) \end{cases} \tag{91}$$

for all $i,j,k,s\in\mathbb{K}$ and $\tau\in[0,T]$, with $\tilde{\phi}_{ij}^{ks}(\tau)\geq 0$ and $\sum_{(k,s)\in\Omega_{ij}^{*}(\tau)}\tilde{\phi}_{ij}^{ks}(\tau) = 1$, and where $\Omega_{ij}^{*}(\tau) = \Omega_{ij}^{*}$, with Ω_{ij}^{*} given by (15) for all $\tau\in(0,T]$, and $\Omega_{ij}^{*}(0) = \Omega_{ij}^{*}\cup\Omega_{ij}^{0}$, where

$$\Omega_{ij}^{0} = \left\{(k,s)\in\Pi(i,j) : U_k+U_s = U_{\lceil\frac{i+j}{2}\rceil}+U_{\lfloor\frac{i+j}{2}\rfloor}\right\}.$$

Proof of Corollary 1. The proof proceeds exactly as the proof of Proposition 2 up to (90). Notice that under Assumption A, (90) holds for all $\tau \in [0, T]$. Instead, under the assumption that $\{U_k\}_{k\in\mathbb{K}}$ satisfies *discrete midpoint concavity* and $\{u_k\}_{k\in\mathbb{K}}$ satisfies *discrete midpoint strict concavity*, the inequality in (90) holds as a strict inequality for all $\tau \in (0, T]$, but only as a weak inequality for $\tau = 0$. As before, the unique fixed point $V_k(\tau) \in \boldsymbol{B}''$, but now $V_k(\tau) \notin \boldsymbol{B}'''$, since $V_k(\tau)$ satisfies (SEP) for all $(i, j, \tau) \in \mathbb{K} \times \mathbb{K} \times (0, T]$, rather than for all $(i, j, \tau) \in \mathbb{K} \times \mathbb{K} \times [0, T]$. However, it is clear from (90) that in this case $\mathcal{M}V_k(\tau) = V_k(\tau) \in \boldsymbol{B}_0'''$, where $\boldsymbol{B}_0'''$ is the subset of elements of $\boldsymbol{B}$ that satisfy (SEP) for all $(i, j, \tau) \in \mathbb{K} \times \mathbb{K} \times (0, T]$. This implies that the set $\Omega_{ij}[\boldsymbol{V}(\tau)]$ defined in (13) now reduces to the set $\Omega^*_{ij}(\tau)$ defined in the statement of the corollary for all $\tau \in [0, T]$, and consequently, that (12) reduces to (91) for all $\tau \in [0, T]$. Notice that despite the potential multiplicity of optimal post-trade portfolios in bilateral meetings at $\tau = 0$ (which is the only difference between this case and the one treated in Proposition 2), as can be seen from (74) and (91), the mapping $\mathcal{M}$ is unaffected by this multiplicity, and hence so is its fixed point. Therefore, (by Lemma 2) the fixed point $V_k(\tau) \in \boldsymbol{B}_0'''$ is unique given any path for $\boldsymbol{n}(\tau)$. Finally, if we cast (8) in integral equation form,

$$n_k(\tau) = n_k(T) - \alpha \int_\tau^T \sum_{i\in\mathbb{K}} \sum_{j\in\mathbb{K}} \sum_{s\in\mathbb{K}} n_i(z) \left[n_k(z)\, \phi^{sj}_{ki}(z) - n_j(z)\, \phi^{ks}_{ij}(z) \right] dz \tag{92}$$

for all $k \in \mathbb{K}$, then it becomes clear that for all $k \in \mathbb{K}$ and all $\tau \in [0, T]$, $n_k(\tau)$ is independent of $\phi^{ks}_{ij}(0)$ (changing the integral at a single point leaves the right side of (92) unaffected). Therefore, by the same arguments used in the final step of the proof of Proposition 2, there exists a unique $\boldsymbol{n}(\tau)$ that solves the system (92), and it is the same solution that obtains under Assumption A. ■

Proof of Proposition 3. The planner's current-value Hamiltonian can be written as

$$L = \sum_{k\in\mathbb{K}} m_k(t)\, u_k + \alpha \sum_{i\in\mathbb{K}} \sum_{j\in\mathbb{K}} \sum_{k\in\mathbb{K}} \sum_{s\in\mathbb{K}} m_i(t)\, m_j(t)\, \chi^{ks}_{ij}(t) \left[\mu_k(t) - \mu_i(t) \right],$$

where $\boldsymbol{\mu}(t) = \{\mu_k(t)\}_{k\in\mathbb{K}}$ is the vector of co-states associated with the law of motion for the distribution of banks across reserve balances. In an optimum, the co-states and the controls must satisfy $\frac{\partial L}{\partial m_i(t)} = r\mu_i(t) - \dot{\mu}_i(t)$, and

$$\chi_{ij}^{ks}(t) \begin{cases} = 1 & \text{if } \left.\frac{\partial L}{\partial \chi_{ij}^{ks}(t)}\right|_{\chi_{ji}^{sk}(t)=\chi_{ij}^{ks}(t)} > 0 \\ \in [0,1] & \text{if } \left.\frac{\partial L}{\partial \chi_{ij}^{ks}(t)}\right|_{\chi_{ji}^{sk}(t)=\chi_{ij}^{ks}(t)} = 0 \\ 0 & \text{if } \left.\frac{\partial L}{\partial \chi_{ij}^{ks}(t)}\right|_{\chi_{ji}^{sk}(t)=\chi_{ij}^{ks}(t)} < 0. \end{cases}$$

Notice that

$$\left.\frac{\partial L}{\partial \chi_{ij}^{ks}(t)}\right|_{\chi_{ji}^{sk}(t)=\chi_{ij}^{ks}(t)} = \alpha m_i(t)\, m_j(t)\left[\mu_k(t) + \mu_s(t) - \mu_i(t) - \mu_j(t)\right],$$

and that given $\chi_{ji}^{sk}(t) = \chi_{ij}^{ks}(t)$,

$$\frac{\partial L}{\partial m_i} = u_i + \alpha \sum_{j\in\mathbb{K}}\sum_{k\in\mathbb{K}}\sum_{s\in\mathbb{K}} m_j(t)\,\chi_{ij}^{ks}(t)\left[\mu_k(t) + \mu_s(t) - \mu_i(t) - \mu_j(t)\right].$$

Thus the necessary conditions for optimality are:

$$\chi_{ij}^{ks}(t) = \begin{cases} \tilde{\chi}_{ij}^{ks}(t) & \text{if } (k,s) \in \Omega_{ij}[\boldsymbol{\mu}(t)] \\ 0 & \text{if } (k,s) \notin \Omega_{ij}[\boldsymbol{\mu}(t)], \end{cases} \tag{93}$$

for all $i,j,k,s \in \mathbb{K}$ and all $t \in [0,T]$, where $\tilde{\chi}_{ij}^{ks}(t) \geq 0$ and $\sum_{k\in\mathbb{K}}\sum_{s\in\mathbb{K}} \tilde{\chi}_{ij}^{ks}(t) = 1$, the Euler equations,

$$r\mu_i(t) - \dot{\mu}_i(t) = u_i + \alpha \sum_{j\in\mathbb{K}}\sum_{k\in\mathbb{K}}\sum_{s\in\mathbb{K}} m_j(t)\,\chi_{ij}^{ks}(t)\left[\mu_k(t) + \mu_s(t) - \mu_i(t) - \mu_j(t)\right] \tag{94}$$

for all $i \in \mathbb{K}$, with the path for $\boldsymbol{m}(t)$ given by (19), and

$$\mu_i(T) = U_i \quad \text{for all } i \in \mathbb{K}. \tag{95}$$

In summary, the necessary conditions are (19), (93), (94), and (95). Next, we use the fact that $\tau \equiv T - t$ to define $m_k(t) = m_k(T-\tau) \equiv n_k(\tau)$, $\chi_{ij}^{ks}(t) = \chi_{ij}^{ks}(T-\tau) \equiv \psi_{ij}^{ks}(\tau)$, and $\mu_i(t) = \mu_i(T-\tau) \equiv \lambda_i(\tau)$. With these new variables, (94) leads to (20), (19) leads to $\dot{\boldsymbol{n}}(\tau) = f[\boldsymbol{n}(\tau), \boldsymbol{\psi}(\tau)]$, (95) leads to (21), and (93) leads to (22). ■

Proof of Proposition 4. The function $\boldsymbol{\lambda} \equiv [\lambda(\tau)]_{\tau\in[0,T]}$ satisfies (20) and (21) if and only if it satisfies

$$\lambda_i(\tau) = v_i(\tau) + \alpha\int_0^\tau \lambda_i(z)\, e^{-(r+\alpha)(\tau-z)}dz$$
$$+\alpha\int_0^\tau \sum_{j\in\mathbb{K}}\sum_{k\in\mathbb{K}}\sum_{s\in\mathbb{K}} n_j(z)\,\psi_{ij}^{ks}(z)\,[\lambda_k(z)+\lambda_s(z)-\lambda_i(z)-\lambda_j(z)]\,e^{-(r+\alpha)(\tau-z)}dz.$$

The right side of this functional equation defines a mapping $\mathcal{P}:\boldsymbol{B}\to\boldsymbol{B}$, that is for any $w\in\boldsymbol{B}$,

$$(\mathcal{P}w)(i,\tau) = v_i(\tau) + \alpha\int_0^\tau w(i,z)\, e^{-(r+\alpha)(\tau-z)}dz$$
$$+\alpha\int_0^\tau \sum_{j\in\mathbb{K}}\sum_{k\in\mathbb{K}}\sum_{s\in\mathbb{K}} n_j(z)\,\psi_{ij}^{ks}(z)\,[w(k,z)+w(s,z)-w(i,z)-w(j,z)]\,e^{-(r+\alpha)(\tau-z)}dz,$$

for all $(i,\tau)\in\mathbb{K}\times[0,T]$. Hence a function $\boldsymbol{\lambda}$ satisfies (20) and (21) if and only if it satisfies $\boldsymbol{\lambda}=\mathcal{P}\boldsymbol{\lambda}$. Rewrite the mapping $\mathcal{P}$ as

$$(\mathcal{P}w)(i,\tau) = v_i(\tau) + \alpha\int_0^\tau w(i,z)\, e^{-(r+\alpha)(\tau-z)}dz \tag{96}$$
$$+\alpha\int_0^\tau \sum_{j\in\mathbb{K}} n_j(z)\max_{(k,s)\in\Pi(i,j)}[w(k,z)+w(s,z)-w(i,z)-w(j,z)]\,e^{-(r+\alpha)(\tau-z)}dz,$$

and for any $w,w'\in\boldsymbol{B}$, define the metric $D^*:\boldsymbol{B}\times\boldsymbol{B}\to\mathbb{R}$ by

$$D^*\left(w,w'\right) = \sup_{(i,\tau)\in\mathbb{K}\times[0,T]}\left[e^{-\kappa\tau}\left|w(i,\tau)-w'(i,\tau)\right|\right],$$

where $\kappa\in\mathbb{R}$ satisfies

$$\max\{0,5\alpha-r\}<\kappa<\infty. \tag{97}$$

The metric space $(\boldsymbol{B},D^*)$ is complete (by the same argument used to argue that $(\boldsymbol{B},D)$ is complete, in the proof of Lemma 4). For any $w,w'\in\boldsymbol{B}$, and any $(i,\tau)\in\mathbb{K}\times[0,T]$, the same steps that led to (78), now lead to

$$D^*\left(\mathcal{P}w,\mathcal{P}w'\right)\le\frac{5\alpha}{r+\alpha+\kappa}D^*\left(w,w'\right),\qquad\text{for all } w,w'\in\boldsymbol{B}.$$

Notice that (97) implies $\frac{5\alpha}{r+\alpha+\kappa}\in(0,1)$, so $\mathcal{P}$ is a contraction mapping on the complete metric space $(\boldsymbol{B},D^*)$. By the Contraction Mapping Theorem (Theorem 3.2 in Stokey and Lucas, 1989), for any given path $\boldsymbol{n}(\tau)$, there exists a unique $\boldsymbol{\lambda}\in\boldsymbol{B}$ that satisfies $\boldsymbol{\lambda}=\mathcal{P}\boldsymbol{\lambda}$.

Consider the sets $\boldsymbol{B}''$ and $\boldsymbol{B}'''$ defined in the proof of Proposition 2. By following the same steps as in the first part of that proof, it can be shown that $\boldsymbol{B}''$ is closed under D^*. Next we show that the mapping $\mathcal{P}$ defined in (96) preserves property (EP), i.e., that $\mathcal{P}(\boldsymbol{B}'') \subseteq \boldsymbol{B}''$. That is, we wish to show that for any $w \in \boldsymbol{B}''$, $w' = \mathcal{P}w \in \boldsymbol{B}''$, or equivalently, that

$$w\left(\left\lceil \tfrac{i+j}{2} \right\rceil, \tau\right) + w\left(\left\lfloor \tfrac{i+j}{2} \right\rfloor, \tau\right) \geq w(k,\tau) + w(s,\tau) \text{ for all } (k,s) \in \Pi(i,j),$$

for any $(i,j,\tau) \in \mathbb{K} \times \mathbb{K} \times [0,T]$, implies that

$$w'\left(\left\lceil \tfrac{i+j}{2} \right\rceil, \tau\right) + w'\left(\left\lfloor \tfrac{i+j}{2} \right\rfloor, \tau\right) - w'(k,\tau) - w'(s,\tau) \geq 0 \text{ for all } (k,s) \in \Pi(i,j), \tag{98}$$

for any $(i,j,\tau) \in \mathbb{K} \times \mathbb{K} \times [0,T]$. Since $w \in \boldsymbol{B}''$,

$$\begin{aligned}(\mathcal{P}w)(i,\tau) &= v_i(\tau) + \alpha \int_0^\tau w(i,z)\, e^{-(r+\alpha)(\tau-z)} dz \\ &\quad + \alpha \int_0^\tau \sum_{q\in\mathbb{K}} n_q(z) \left[w\left(\left\lceil \tfrac{i+q}{2} \right\rceil, z\right) + w\left(\left\lfloor \tfrac{i+q}{2} \right\rfloor, z\right) - w(i,z) - w(q,z) \right] e^{-(r+\alpha)(\tau-z)} dz,\end{aligned}$$

for any $(i,\tau) \in \mathbb{K} \times [0,T]$. For any $(i,j,\tau) \in \mathbb{K} \times \mathbb{K} \times [0,T]$ and $(k,s) \in \Pi(i,j)$, let $G'(i,j,k,s,\tau)$ denote the left side of inequality (98). Then,

$$\begin{aligned}G'(i,j,k,s,\tau) &= \frac{1-e^{-(r+\alpha)\tau}}{r+\alpha}\left(u_{\lceil \frac{i+j}{2} \rceil} + u_{\lfloor \frac{i+j}{2} \rfloor} - u_k - u_s\right) \\ &\quad + e^{-(r+\alpha)\tau}\left[U_{\lceil \frac{i+j}{2} \rceil} + U_{\lfloor \frac{i+j}{2} \rfloor} - U_k - U_s\right] \\ &\quad + \alpha \int_0^\tau \sum_{q\in\mathbb{K}} n_q(z) \Bigg\{ w\left(\left\lceil \frac{\left\lceil \frac{i+j}{2} \right\rceil + q}{2} \right\rceil, z\right) + w\left(\left\lfloor \frac{\left\lceil \frac{i+j}{2} \right\rceil + q}{2} \right\rfloor, z\right) \\ &\qquad - w\left(\left\lceil \tfrac{k+q}{2} \right\rceil, z\right) - w\left(\left\lfloor \tfrac{s+q}{2} \right\rfloor, z\right) \\ &\qquad - w\left(\left\lfloor \tfrac{k+q}{2} \right\rfloor, z\right) - w\left(\left\lceil \tfrac{s+q}{2} \right\rceil, z\right) \\ &\qquad + w\left(\left\lceil \frac{\left\lfloor \frac{i+j}{2} \right\rfloor + q}{2} \right\rceil, z\right) + w\left(\left\lfloor \frac{\left\lfloor \frac{i+j}{2} \right\rfloor + q}{2} \right\rfloor, z\right) \Bigg\} e^{-(r+\alpha)(\tau-z)} dz.\end{aligned}$$

What needs to be shown is that $w \in \boldsymbol{B}''$ implies that for any $(i,j,\tau) \in \mathbb{K} \times \mathbb{K} \times [0,T]$, $G'(i,j,k,s,\tau) \geq 0$ for all $(k,s) \in \Pi(i,j)$. By Lemma 5, $w \in \boldsymbol{B}''$ implies that the integral in the last expression is nonnegative. Together with Assumption A, this implies

$$\begin{aligned}0 &< \frac{1-e^{-(r+\alpha)\tau}}{r+\alpha}\left(u_{\lceil \frac{i+j}{2} \rceil} + u_{\lfloor \frac{i+j}{2} \rfloor} - u_k - u_s\right) + e^{-(r+\alpha)\tau}\left[U_{\lceil \frac{i+j}{2} \rceil} + U_{\lfloor \frac{i+j}{2} \rfloor} - U_k - U_s\right] \\ &\leq G'(i,j,k,s,\tau),\end{aligned}$$

so $\mathcal{M}(\boldsymbol{B}'') \subseteq \boldsymbol{B}''' \subseteq \boldsymbol{B}''$.

At this point, we have shown that $\mathcal{P}$ is a contraction mapping on the complete metric space $(\boldsymbol{B}, D^*)$, so it has a unique fixed point $\boldsymbol{\lambda} \in \boldsymbol{B}$. We have also established that $\boldsymbol{B}''$ is a closed subset of $\boldsymbol{B}$, and that $\mathcal{M}(\boldsymbol{B}'') \subseteq \boldsymbol{B}''' \subseteq \boldsymbol{B}''$. Therefore, by Corollary 1 in Stokey and Lucas (1989, p. 52), $\boldsymbol{\lambda} = \mathcal{P}\boldsymbol{\lambda} \in \boldsymbol{B}'''$, that is, the unique fixed point $\boldsymbol{\lambda}$ satisfies (SEP). This implies that the set $\Omega_{ij}[\boldsymbol{\lambda}(\tau)]$ in (22) reduces to Ω^*_{ij} (as defined in (15)) for all $(i, j, \tau) \in \mathbb{K} \times \mathbb{K} \times [0, T]$, and consequently, that (22) reduces to (23). This establishes part (i) in the statement of the proposition.

Given the initial condition $\{n_k(T)\}_{k \in \mathbb{K}}$, and given that the path $\boldsymbol{\psi}(\tau)$ satisfies (23), the system of first-order ordinary differential equations, $\dot{\boldsymbol{n}}(\tau) = f[\boldsymbol{n}(\tau), \boldsymbol{\psi}(\tau)]$ is identical to the one in part (iii) of Proposition 2, and therefore also has a unique solution. Given the resulting path $\boldsymbol{n}(\tau)$, according to Proposition 3, the path for the vector of co-states must satisfy the necessary condition $\boldsymbol{\lambda} = \mathcal{P}\boldsymbol{\lambda}$, or equivalently, (20) and (21), which establishes part (ii) in the statement of the proposition. ■

Proof of Proposition 6. The (i, j) element of the transition matrix $\boldsymbol{\Pi}^{(N)}(\boldsymbol{t}^{(N)})$ as defined in (26), denoted $\pi^{(N)}_{ij}(\boldsymbol{t}^{(N)})$, is the probability that a bank with balance i at time t_0 has balance j at time t, conditional on a realization of the number of trading opportunities, $N \in \{0, 1, 2, \ldots\}$, and a realization of the corresponding trading times, $\boldsymbol{t}^{(N)} \in \mathbb{T}^{(N)}$. Let $n_{[t_0,t]}$ denote the random number of trading opportunities that a bank encounters during the time interval $[t_0, t]$. Since trading opportunities follow a Poisson process with intensity α,

$$\Pr\left(n_{[t_0,t]} = N\right) = \frac{[\alpha(t - t_0)]^N e^{-\alpha(t-t_0)}}{N!}. \tag{99}$$

Let $h(\boldsymbol{t}^{(N)}|n_{[t_0,t]} = N)$ denote the probability density of $\boldsymbol{t}^{(N)} \in \mathbb{T}^{(N)}$ conditional on N trading opportunities in $[t_0, t]$, and notice that $h(\boldsymbol{t}^{(N)}|n_{[t_0,t]} = N) = h'((T_1, \ldots, T_N)\,|n_{[t_0,t]} = N)$, where $h'((T_1, \ldots, T_N)\,|n_{[t_0,t]} = N)$ is the conditional probability density for the N interarrival times, $T_n \equiv t_n - t_{n-1}$, for $n = 1, \ldots, N$. Then, by the definition of conditional density,

$$\begin{aligned} h'(T_1, \ldots, T_N|n_{[t_0,t]} = N) &= \frac{\Pr\left(n_{[t_0,t]} = N \mid T_1, \ldots, T_N\right) \prod_{n=1}^{N} \left(\alpha e^{-\alpha(T_n - T_{n-1})}\right)}{\Pr\left(n_{[t_0,t]} = N\right)} \\ &= \frac{\Pr\left(T_{N+1} > t - t_N\right) \alpha^N e^{-\alpha(t_N - t_0)}}{\Pr\left(n_{[t_0,t]} = N\right)} \\ &= \frac{N!}{(t - t_0)^N}. \end{aligned} \tag{100}$$

Notice that the volume of $[t_0, t]^N$ is $(t - t_0)^N$, but the volume of $\mathbb{T}^{(N)}$ is $(t - t_0)^N / N!$, since for all possible draws of N-vectors from $[t_0, t]^N$, the ascending ordering $\boldsymbol{t}^{(N)} = (t_1, t_2, ..., t_N)$ is only one of $N!$ possible orderings. Thus by (100), the conditional probability distribution for the trading times $\boldsymbol{t}^{(N)}$ given $n_{[t_0,t]} = N$, is uniform on $\mathbb{T}^{(N)}$. For a bank holding any balance in $\mathbb{K}$ at time t_0, we can now use (99) and (100) to write the unconditional transition probabilities to any balance at time t, as

$$P(t|t_0) = \sum_{N=0}^{\infty} \frac{[\alpha(t - t_0)]^N e^{-\alpha(t-t_0)}}{N!} \int_{\mathbb{T}^{(N)}} \mathbf{\Pi}^{(N)}(\boldsymbol{t}^{(N)}) \frac{N!}{(t - t_0)^N} d\boldsymbol{t}^{(N)},$$

which simplifies to (27). ■

Proof of Proposition 7. Given an initial balance $a(t_0) = k_0 \in \mathbb{K}$, and given the realization of trading times $\boldsymbol{t}^{(N)} \in [t_0, t]^N$, the probability distribution over the post-trade balances at these trading times, i.e., over vectors $(a(t_1), \ldots, a(t_N)) = \boldsymbol{k}^{(N)} \in \mathbb{K}^N$, is given by (25). Hence,

$$\mathbb{E}\left[O^j(k_0, \boldsymbol{k}^{(N)}) \mid k_0, \boldsymbol{t}^{(N)}\right] = \sum_{\boldsymbol{k}^{(N)} \in \mathbb{K}^N} \left(\prod_{n=1}^{N} \pi_{k_{n-1}k_n}(t_n) \right) O^j(k_0, \boldsymbol{k}^{(N)}) \equiv \tilde{O}^j(k_0, \boldsymbol{t}^{(N)})$$

is the expected cumulative volume of funds purchased (for $j = p$, or sold, for $j = s$) during $[t_0, t]$ by banks that hold balance k_0 at t_0 and have N trading opportunities, at times $\boldsymbol{t}^{(N)} = (t_1, \ldots, t_N)$. By (99) and (100), the expected cumulative volume of funds purchased (for $j = p$, or sold, for $j = s$) during $[t_0, t]$ by banks that hold balance k_0 at t_0 is

$$\mathbb{E}\left[\tilde{O}^j(k_0, \boldsymbol{t}^{(N)}) \mid k_0\right] = \sum_{N=0}^{\infty} \frac{[\alpha(t - t_0)]^N e^{-\alpha(t-t_0)}}{N!} \int_{\mathbb{T}^{(N)}} \tilde{O}^j(k_0, \boldsymbol{t}^{(N)}) \frac{N!}{(t - t_0)^N} d\boldsymbol{t}^{(N)}.$$

Since the density of banks with balance k_0 at time t_0 is $m_{k_0}(t_0)$,

$$\mathbb{E}\left[\mathbb{E}\left[\tilde{O}^j(k_0, \boldsymbol{t}^{(N)}) \mid k_0\right]\right] = \sum_{k_0 \in \mathbb{K}} m_{k_0}(t_0) \sum_{N=0}^{\infty} \frac{[\alpha(t - t_0)]^N e^{-\alpha(t-t_0)}}{N!} \int_{\mathbb{T}^{(N)}} \tilde{O}^j(k_0, \boldsymbol{t}^{(N)}) \frac{N!}{(t - t_0)^N} d\boldsymbol{t}^{(N)}$$

is the expected cumulative volume of funds purchased (for $j = p$, or sold, for $j = s$) by all banks during $[t_0, t]$, which after simplification reduces to $\bar{O}^j(t|t_0)$ in (30). An identical calculation but replacing $\bar{O}^j(k_0, \boldsymbol{k}^{(N)})$ with $X(k_0, \boldsymbol{k}^{(N)})$ leads to (32). Finally, from (28) and (29) it is easy to check that $I(k_0, \boldsymbol{k}^{(N)}) = \frac{1}{2} X(k_0, \boldsymbol{k}^{(N)})$ for all $(k_0, \boldsymbol{k}^{(N)}) \in \mathbb{K}^{N+1}$, which implies (31). ■

Proof of Proposition 8. The right side of (49) can be integrated to obtain the closed-form expression for $S(\tau)$, where

$$\xi(\tau) = [n_2(T) - n_0(T)] \int_0^\tau \frac{e^{\{r+\theta\alpha[n_2(T)-n_0(T)]\}z} n_0(T)}{n_2(T) e^{\alpha[n_2(T)-n_0(T)]T} - e^{\alpha[n_2(T)-n_0(T)]z} n_0(T)} dz$$

can be integrated to yield the expression reported in the statement of the proposition. Conditions (45), (46) and (47) imply

$$\dot{V}_1(\tau) - \dot{V}_0(\tau) + r[V_1(\tau) - V_0(\tau)] = u_1 - u_0 - \theta\alpha n_2(\tau) S(\tau),$$

a differential equation in $V_1(\tau) - V_0(\tau)$, with boundary condition $V_1(0) - V_0(0) = U_1 - U_0$. The solution to this differential equation is

$$V_1(\tau) - V_0(\tau) = e^{-r\tau}(U_1 - U_0) + \int_0^\tau [u_1 - u_0 - \theta\alpha n_2(z) S(z)] e^{-r(\tau-z)} dz. \tag{101}$$

With (44) and the closed-form expression for $S(\tau)$, the integral on the right side of (101) can be calculated explicitly to yield

$$\left(1 - e^{-r\tau}\right)\frac{u_1 - u_0}{r} - e^{-r\tau}\frac{e^{\alpha[n_2(T)-n_0(T)]T} n_2(T)}{n_0(T)}\theta\zeta[\tau, \bar{u}, S(0)],$$

with

$$\zeta[\tau, \bar{u}, S(0)] = \sum_{k=1}^{\infty} \frac{\left[\frac{n_2(T)}{n_0(T)}\right]^{k-1}}{\frac{r}{\alpha[n_0(T)-n_2(T)]} + k - \theta} \frac{\frac{e^{\{r+\alpha k[n_0(T)-n_2(T)]\}\tau}-1}{r+\alpha k[n_0(T)-n_2(T)]} - \frac{e^{\alpha\theta[n_0(T)-n_2(T)]\tau}-1}{\alpha\theta[n_0(T)-n_2(T)]}}{e^{\alpha(k-1)[n_0(T)-n_2(T)]T}}\bar{u}$$
$$+ \frac{\left[e^{\alpha\theta[n_0(T)-n_2(T)]\tau}-1\right]n_0(T)}{\theta\left[n_0(T) - e^{-\alpha[n_0(T)-n_2(T)]T} n_2(T)\right]} S(0)$$

if $n_2(T) < n_0(T)$,

$$\zeta[\tau, \bar{u}, S(0)] = e^{r\left[\frac{1}{\alpha n_0(T)}+T\right]} \sum_{k=0}^{\infty} \frac{(-r)^k}{k!k} \left\{\left[\frac{1}{\alpha n_0(T)} + T\right]^k \tau - \frac{\left\{\left[\frac{1}{\alpha n_0(T)}+T\right]^{k+1} - \left[\frac{1}{\alpha n_0(T)}+T-\tau\right]^{k+1}\right\}}{k+1}\right\}\bar{u}$$
$$+ \frac{\tau}{\frac{1}{\alpha n_0(T)}+T} S(0)$$

if $n_2(T) = n_0(T)$, and

$$\zeta[\tau, \bar{u}, S(0)] = \sum_{k=0}^{\infty} \frac{\left[\frac{n_0(T)}{n_2(T)}\right]^{k+1}}{\frac{r}{\alpha[n_2(T)-n_0(T)]} + \theta + k} \frac{\frac{e^{\{r+\alpha k[n_2(T)-n_0(T)]\}\tau}-1}{r+\alpha k[n_2(T)-n_0(T)]} - \frac{1-e^{-\alpha\theta[n_2(T)-n_0(T)]\tau}}{\alpha\theta[n_2(T)-n_0(T)]}}{e^{\alpha(k+1)[n_2(T)-n_0(T)]T}}\bar{u}$$
$$+ \frac{\left[1-e^{-\alpha\theta[n_2(T)-n_0(T)]\tau}\right]n_0(T)}{\theta\left[e^{\alpha[n_2(T)-n_0(T)]T} n_2(T) - n_0(T)\right]} S(0)$$

if $n_0(T) < n_2(T)$. ■

Proof of Proposition 9. From (49), since $\bar{u} \geq 0$ and $S(0) > 0$, we have $S(\tau) > 0$ for all $\tau \in [0, T]$.

(i) Differentiate (49) to obtain

$$\frac{\partial S(\tau)}{\partial r} = -\left[\left(\int_0^\tau (\tau - z)\, e^{-\left[\bar{\delta}(\tau)-\bar{\delta}(z)\right]} dz\right) \bar{u} + \tau e^{-\bar{\delta}(\tau)} S(0)\right],$$

which is clearly negative for $\tau > 0$.

(ii) Differentiate (49) to obtain

$$\frac{\partial S(\tau)}{\partial \theta} = -\alpha \left\{\bar{u} \int_0^\tau (\tau - z)\, e^{-\left[\bar{\delta}(\tau)-\bar{\delta}(z)\right]} dz + \tau e^{-\bar{\delta}(\tau)} S(0)\right\} [n_2(T) - n_0(T)],$$

which has the sign of $n_0(T) - n_2(T)$.

(iii) Differentiate (49) to obtain $\frac{\partial S(\tau)}{\partial U_0} = \frac{\partial S(\tau)}{\partial U_2} = -\frac{1}{2}\frac{\partial S(\tau)}{\partial U_1} = -\frac{S(\tau)}{S(0)} < 0$. ■

Proof of Proposition 10. For $\bar{u} = 0$, $R(\tau)$ is given by (50), but with $S(\tau)$ given by

$$S(\tau) = e^{-\int_0^\tau \{r+\alpha[\theta n_2(s)+(1-\theta)n_0(s)]\}ds} S(0),$$

and with $V_1(\tau) - V_0(\tau)$ given by

$$V_1(\tau) - V_0(\tau) = e^{-r\tau}(U_1 - U_0) + \left(1 - e^{-r\tau}\right)\frac{u_1 - u_0}{r} - \frac{\left[e^{-r\tau} - e^{-\{r+\alpha\theta[n_2(T)-n_0(T)]\}\tau}\right] n_2(T)}{n_2(T) - e^{-\alpha[n_2(T)-n_0(T)]T} n_0(T)} S(0)$$

for the case $n_2(T) \neq n_0(T)$, and

$$V_1(\tau) - V_0(\tau) = e^{-r\tau}(U_1 - U_0) + \left(1 - e^{-r\tau}\right)\frac{u_1 - u_0}{r} - \frac{\tau e^{-r\tau}}{\frac{1}{\alpha n_0(T)} + T}\theta S(0)$$

for the case $n_2(T) = n_0(T)$. From (51),

$$\frac{\partial \rho(\tau)}{\partial x} = \frac{1}{\tau + \Delta}\frac{1}{R(\tau)}\frac{\partial R(\tau)}{\partial x},$$

for $x = \theta$, r, U_0.

(i) Differentiate (50) to obtain

$$\frac{\partial R(\tau)}{\partial r} = R(\tau)\Delta - e^{r(\tau+\Delta)}\frac{u_1 - u_0}{r^2}\left(1 - r\tau - e^{-r\tau}\right) > 0,$$

since $1 - r\tau - e^{-r\tau} \leq 0$. Thus, $\frac{\partial \rho(\tau)}{\partial r} > 0$.

(ii) For any $\tau > 0$, differentiate (50) to obtain

$$\frac{\partial R(\tau)}{\partial \theta} = -e^{r(\tau+\Delta)} \left[1 + \frac{\left\{\theta e^{\alpha[n_2(T)-n_0(T)]\tau} n_0(T) + (1-\theta) e^{\alpha[n_2(T)-n_0(T)]T} n_2(T)\right\}[n_2(T)-n_0(T)]\alpha\tau}{e^{\alpha[n_2(T)-n_0(T)]T} n_2(T) - e^{\alpha[n_2(T)-n_0(T)]\tau} n_0(T)}\right] S(\tau) < 0$$

for the case $n_2(T) \neq n_0(T)$, and

$$\frac{\partial R(\tau)}{\partial \theta} = -e^{r(\tau+\Delta)} \left[1 + \frac{\alpha\tau n_0(T)}{1 + \alpha(T-\tau) n_0(T)}\right] S(\tau) < 0$$

for the case $n_2(T) = n_0(T)$. Hence $\frac{\partial \rho(\tau)}{\partial \theta} < 0$.

(iii) Differentiate (50) to obtain

$$\frac{\partial R(\tau)}{\partial U_0} = -e^{r(\tau+\Delta)} \frac{(1-\theta) e^{\alpha[n_2(T)-n_0(T)]T} n_2(T) - \left[1 - \theta e^{\alpha(1-\theta)[n_2(T)-n_0(T)]\tau}\right] e^{\alpha\theta[n_2(T)-n_0(T)]\tau} n_0(T)}{e^{\alpha[n_2(T)-n_0(T)]T} n_2(T) - e^{\alpha[n_2(T)-n_0(T)]\tau} n_0(T)} \frac{S(\tau)}{S(0)}$$

for the case $n_2(T) \neq n_0(T)$, and

$$\frac{\partial R(\tau)}{\partial U_0} = -e^{r(\tau+\Delta)} \frac{(1-\theta)[1 + \alpha T n_0(T)]}{1 + \alpha(T-\tau) n_0(T)} \frac{S(\tau)}{S(0)}$$

for the case $n_2(T) = n_0(T)$. It can be verified that $\frac{\partial R(\tau)}{\partial U_0} < 0$ in both cases, so we conclude that $\frac{\partial \rho(\tau)}{\partial U_0} < 0$. ■

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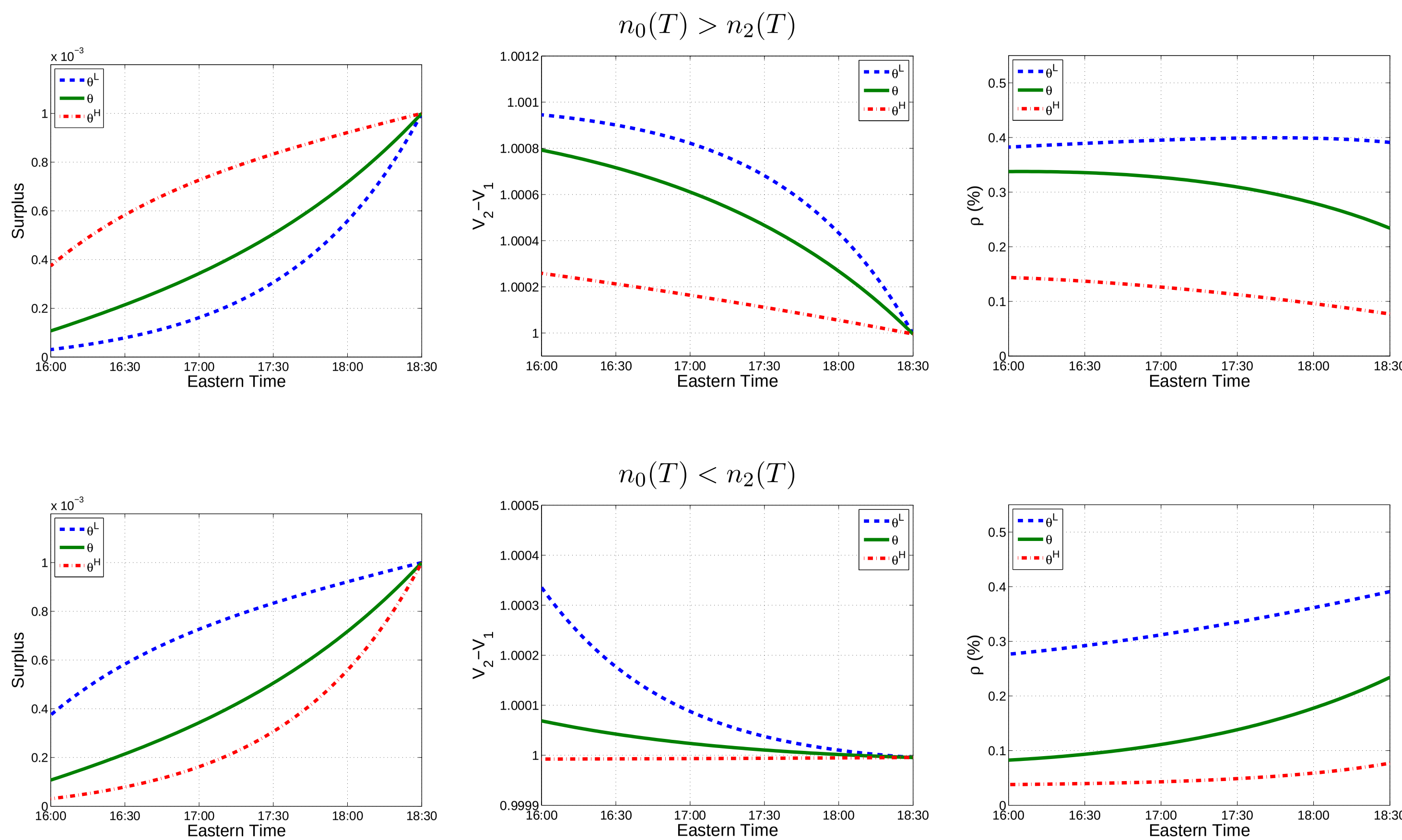

Figure 1: Surplus (left), $V_2(t) - V_1(t)$ (center) and fed funds rate (right) for different values of the bargaining power θ when $n_0(T) > n_2(T)$ (top row) and when $n_0(T) < n_2(T)$ (bottom row). Parameter values: $\theta^L = 0.1$, $\theta = 0.5$, $\theta^H = 0.9$, $\alpha = 100$, $T = 2.5/24$, $\Delta = 22/24$, $n_2(T) = 0.6$ (0.3), $n_0(T) = 0.3$ (0.6), $r = 0.04/365$, $i^d = 0.0036/360$, $i^o = 0.0425/360$, $i^r = i^e = 0.0025/360$ and $P^r + P^o = 0.001$.

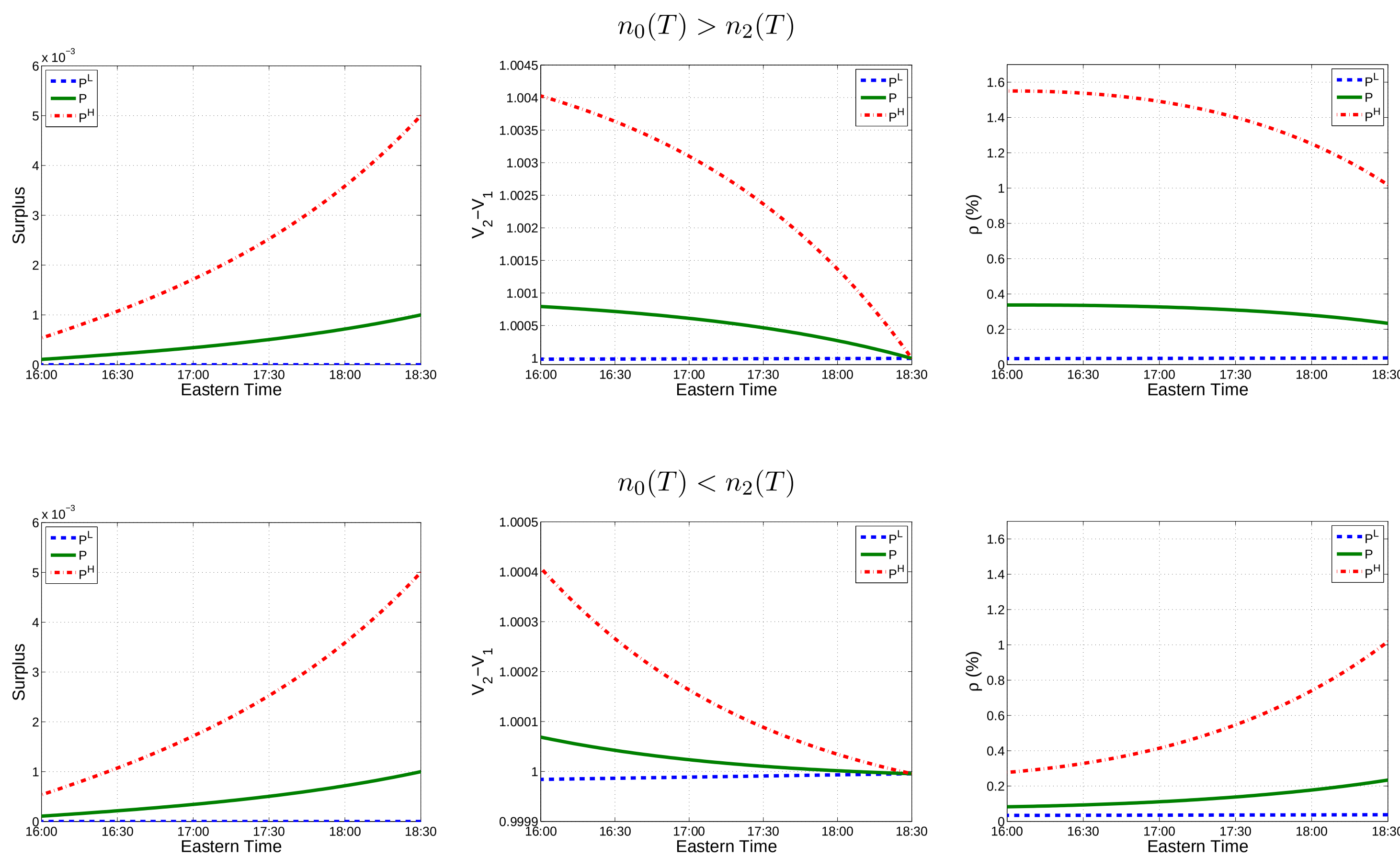

Figure 2: Surplus (left), $V_2(t) - V_1(t)$ (center) and fed funds rate (right) for different values of the penalty fee for holding an end-of-day balances below target when $n_0(T) > n_2(T)$ (top row) and when $n_0(T) < n_2(T)$ (bottom row). Parameter values: $P^L = 0$, $P = 0.001$, $P^H = 0.005$, $\alpha = 100$, $T = 2.5/24$, $\Delta = 22/24$, $n_2(T) = 0.6$ (0.3), $n_0(T) = 0.3$ (0.6), $r = 0.04/365$, $i^d = 0.0036/360$, $i^o = 0.0425/360$, $i^r = i^e = 0.0025/360$ and $\theta = 1/2$.

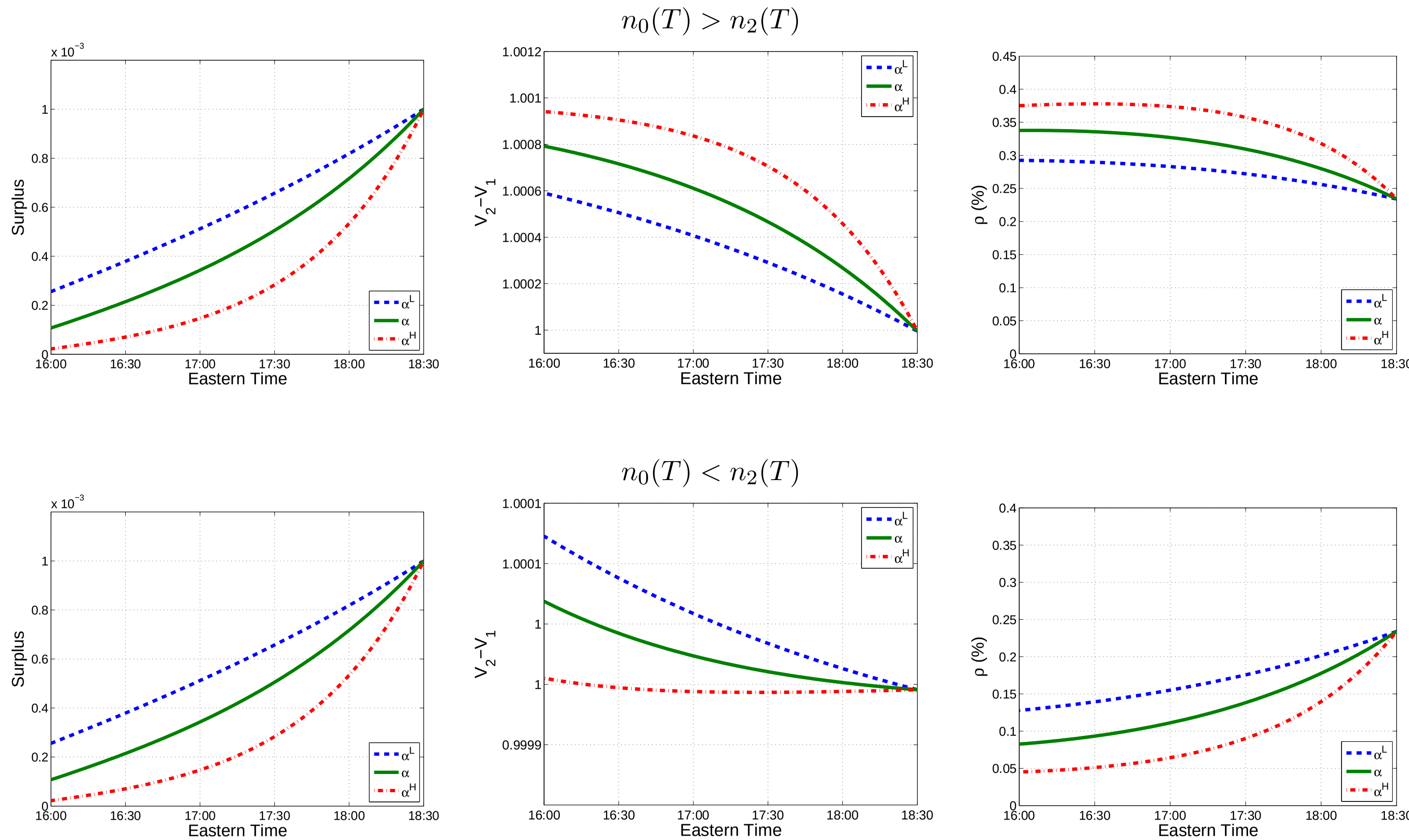

Figure 3: Surplus (left), $V_2(t) - V_1(t)$ (center) and fed funds rate (right) for different values of the frequency of meetings α when $n_0(T) > n_2(T)$ (top row) and when $n_0(T) < n_2(T)$ (bottom row). Parameter values: $\alpha^L = 50$, $\alpha = 100$, $\alpha^H = 200$, $T = 2.5/24$, $\Delta = 22/24$, $n_2(T) = 0.6$ (0.3), $n_0(T) = 0.3$ (0.6), $r = 0.04/365$, $i^d = 0.0036/360$, $i^o = 0.0425/360$, $i^r = i^e = 0.0025/360$, $\theta = 1/2$ and $P^r + P^o = 0.001$.