Multidimensional Asymmetric Information, Adverse Selection, and Efficiency*

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Abstract

This paper develops a framework for the analysis of how asymmetric information impacts on adverse selection and market efficiency. We adopt Akerlof's (1970) unit-demand model extended to a setting with multidimensional public and private information. Adverse selection and efficiency are defined quantitatively as real valued random variables. We characterize how public information disclosure and private information acquisition affect the relationship between adverse selection and efficiency. Gaussian information structures can be summarized by three parameters with natural interpretations as the quality of public information about match quality, a measure of the difference between buyers and the seller in the extent of information about the seller's value, and a measure of the relevance of the seller's valuation to buyers' valuation. In particular, the characterization allows us to describe how a quantity of adverse selection can be imposed efficiently. We conclude by discussing implications and an application to the labour market.

The Akerlof (1970) unit-demand model has proved fundamental in developing our understanding in a wide variety of applications ranging from resale to corporate securities,

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and insurance to labour markets. In these and other applications asymmetric information affects both the terms of trade and the efficiency of markets and, of course, may lead markets to break down entirely.

Akerlof's model is deliberately rather stylized to highlight these effects clearly. Specifically, it supposes that buyers and seller valuations are perfectly correlated so that information can be captured by a scalar parameter and it supposes that buyers have no information at all. Both assumptions are unrealistic. For example, in the second-hand car market, there may be reasons other than low quality, such as moving to another continent, which would lead the seller to have a low value for keeping the car. In addition, while of course one can condition on the information to consider a car with a particular mileage, age, observable condition and so on, the classical approach does not allow for exploration of how changes in information affect market outcomes—the focus of our analysis. These limitations have, of course, been noted and inspired further theoretical development. Finkelstein and McGarry (2006), for example, provide convincing evidence of the inadequacy of the scalar type assumption to account for observed behavior in insurance markets, and Einav, Finkelstein and Cullen (2010) present a general framework that does not impose the scalar assumption, though focused primarily on price rather than information effects. Levin (2001) explores when more information can increase or decrease adverse selection for the scalar model (see also Kessler (2001) and Creane (2008)).

The model we analyze is an extension of the Akerlof (1970) unit-demand model that allows for multivariate private and public information, where identical buyers compete by making offers for the good, after which the seller accepts one of the offers or keeps the good. It is useful to distinguish between the joint distribution of seller and buyer valuation, assumed to be common knowledge and the joint distribution of these valuations with random variables representing the seller's private information and the public information available to all. Bergemann and Morris (2014) term the former the basic game and the latter the information structure (see also Gossner (2000) who highlights this distinction). A burgeoning literature explores how information structures map into economic outcomes in contexts including information disclosure (Kamenica and Gentzkow, 2011), price discrimination Bergemann, Brooks and Morris (forthcoming) and independent private value auctions Bergemann, Brooks and Morris (2013). We conduct a related exercise for the unit-demand trade model with a more-informed seller. In our analysis information structure is the main parameter of the model and adverse selection and efficiency are the equilibrium outcomes.

In order to quantify the effects of asymmetric information, we provide a definition of the quantity of adverse selection conditional on public information as the difference in the average quality of sellers (as perceived by buyers) and the average quality of those who select to trade. Efficiency is also defined as a quantity: specifically, it is defined relative to no trade at all and measures the difference between the average value to the buyer and the seller for traded goods, weighted by the probability of trade. Since adverse selection and efficiency are defined conditionally on public information, they are functions of public information and therefore from an ex ante perspective they are random variables. Understanding how asymmetric information impacts on adverse selection and market efficiency therefore amounts to understanding how the joint distribution of adverse selection and efficiency depends on the structure of asymmetric information.

An important aspect of the information structure is the public estimate of the difference between the value of the good to the seller and to buyers. We term this random variable (which obviously depends on public information) the apparent match quality.

We consider all Gaussian information structures, as do Bergemann, Heumann and Morris (2014) for example. We begin by showing that even though information may be of high dimension, the distributions of adverse selection and efficiency depend on the information only through a three dimensional parameter space. These parameters have natural economic interpretations as the quality of public information about the apparent match value; a measure of the difference between buyers and the seller in the amount of information about the seller's value (which we term the "information gap"); and a measure of the relevance of the seller's valuation to buyers' valuation.

Our first characterization results compare market outcomes for fixed realizations of public information. We find that adverse selection is increasing and efficiency is decreasing in the apparent match quality. These results are intuitive: Buyers who believe that the private value to the seller is likely to be much higher than the value to buyers naturally offer lower prices for fear that if they obtain the good it is a lemon. Instead, buyers who believe that the seller's value is much lower than the value to buyers are likely to offer prices close to the expected valuation based only on public information and ignoring the seller's private information. The latter case naturally leads to more efficient trade, whereas in the former case (since the prices offered by buyers are far from their estimate of the value of the good) trade is less efficient.

While for any realization of a given information structure, there is less efficiency when there is more adverse selection, this relationship need not arise for aggregate outcomes.[1,2] Intuitively, information structures can impose adverse selection more efficiently by concentrating it at public realizations where the good is more efficiently retained by the seller. Our next set of results is about aggregate market outcomes, and how they depend on key features of the information structure.

Expected adverse selection is shown to be increasing in each of the three parameters described above: aggregate adverse selection is higher when there is better public information about the apparent match value, the difference between buyers and the seller in the amount of information about the seller's value is higher, and seller's valuation are more closely related to buyers' valuation. The relationship with expected efficiency is more complex. Comparing two information structures that are otherwise similar, the information structure for which the seller's value is more relevant for buyers entails lower expected efficiency. On the other hand, holding the information gap and informed party relevance fixed (with the latter not too large), better public information about match quality leads to greater expected efficiency.

Moreover, adverse selection and efficiency satisfy a Spence-Mirrlees condition. Comparing two information structures with the same information gap and which generate the same quantity of expected adverse selection, the information structure with better public information about match quality leads to more expected efficiency.

We use these result to characterize how aggregate efficiency and aggregate adverse selection may be optimally achieved; that is, we characterize the upper frontier of the set of outcomes that may be achieved by any information structure. As we discuss at greater length below and in related work (Bar-Isaac, Jewitt and Leaver (2015)), this is an important exercise in the context of labour market competition where firms compete to hire workers by offering both current wages and future opportunities (that depend on the information available to potential rival employers). We first consider information structure with the same fixed inside information; that is we consider information disclosure. We establish that these efficient public information disclosures take a rather simple form whereby public estimates of the buyer and seller value are conditionally collinear given public information. Second, we explore the effect of improving private information. We show that, providing

[1] That a relationship may hold at individual realizations but might be reversed in the aggregate is a familiar result in the statistics literature associated with Yule (1903) and Simpson (1951).

[2] Note that this distinction between results at the level of an individual realization of public information and between aggregate outcomes does not arise in the context of classic results where there is only a single possible realization of public information (namely null information).

there is sufficient freedom to transmit information publicly, improving private information facilitates greater efficiency in imposing adverse selection.

Information structures that impose a quantity of adverse selection efficiently have the interesting implication that the average price of goods that are traded is higher than the average price offered for goods that are not traded—in the labor market this corresponds to higher wages for workers that an employer does not seek to retain earning more than workers whom the firm chooses to keep.[3] Relatedly, a test for adverse selection based on aggregate data that was unable to condition on the public information available to market participants would under-estimate the extent of adverse selection in the market.[4] More broadly, we provide a condition under which prices for traded goods are higher or lower than non-traded goods and whether aggregate tests overestimate or underestimate adverse selection. We discuss our findings in the context of a labour market application.

1 Model

We present a unit-demand model of trade. Numerous identical buyers compete in offering prices to a seller. There is asymmetric information whereby the seller has additional information not available to buyers. The value to the seller is denoted by the random variable V_S. The value to a buyer is denoted V_B. The joint distribution of (V_S, V_B) is exogenously given and unchanging through our analysis. Instead, our focus of interest is the information structure which specifies a $2 + n_Q + n_T$ dimensional vector (Q, T) of real valued random variables jointly distributed with (V_S, V_B). Q is to be read as information private to the seller, T represents public information available to all market participants. Note that depending on the realization of (V_S, V_B), trade may be efficient (if $V_B > V_S$) or inefficient, in the context of the insurance literature the possibility that trade may be inefficient typically relies on positive loading factors (see for example, Fang and Wu, 2016).

Buyers compete with each other through price offers that naturally depend on a realization of the public information T. The seller, who observes a realization Q can accept

[3] The empirical finding in Bidwell (2011) that external hires into a large investment bank are on average paid 18 percent more than internal promotions into identical positions is consistent with this result. See also Lang and Weinstein (2016) who argue that empirical findings on wage growth for employees who move to new firms rather than stay at existing ones are hard to square one-dimensional models of asymmetric information and competitive labour markets.

[4] Chiappori and Salanie (forthcoming) provide a useful overview of testing for adverse selection in the context of insurance markets. In the context of labour markets Gibbons and Katz (1991) is a seminal paper.

or reject any of the offers. That T represents public information can be understood as suggesting that it is contained with Q and implying that V_B and V_S are conditionally independent of T given Q. We suppose that given the information structure, expectations of the values to the seller and buyers exist.

In equilibrium, the seller selects one of the highest price offers, as long as it is higher than, her own estimate of her value. Buyers make price offers that maximize expected profits given beliefs about the price offers of other firms and the seller's strategy.

We focus on Gaussian information structures.[5] A Gaussian structure can be represented by its characteristic function $\phi(u) = \exp\left(u'\mu\boldsymbol{i} - \frac{1}{2}u'\Sigma u\right)$, where Σ is a symmetric positive semidefinite matrix. Without loss of generality, we may normalize means of Q and T to be zero. Hence, if information structures are Gaussian, they are identified with a set of symmetric positive semidefinite covariance matrices.

It is convenient to introduce the following notation.

Notation 1 *We can uniquely define:*

$$B \underset{def}{=} \mathbb{E}[V_B|Q], \;\; S \underset{def}{=} \mathbb{E}[V_S|Q], \;\; M \underset{def}{=} S - B.$$

The supports of Q and T are denoted respectively by $\mathbf{Q}$ and $\mathbf{T}$. The supports of B and S and supports conditional on $T = t$, but for the degenerate cases of null information will be $\mathbb{R}$.

The random variable B is the seller's estimate (the best estimate given all available information) of the buyers' value of the good. Similarly, the random variable S is the best estimate of the seller's value of the worker in employment at the current, training firm. A feature of this paper which distinguishes it from much of the existing literature (e.g. Akerlof 1970, Levin 2001) is that we do not assume that B and S are increasing functions of some underlying scalar type (making them co-monotone random variables). Neither do we assume that B and S are co-monotone. Finally, the random variable M is the best estimate of the difference between the seller's value and the buyers' value. In what follows, we refer to the expectation of M given public information, $\mathbb{E}[M|T]$, as the *apparent match quality.*[6]

[5] As we discuss below, several results apply beyond the Gaussian case. An earlier working paper, Bar-Isaac, Jewitt and Leaver (2014) contains further discussion.

[6] Alternatively, one could view $E[-M|T]$ as the *apparent grounds for trade.*

It will become convenient, both for interpretation and reduction of dimensionality, to refer to Gaussian distributions in terms of regression coefficients.

Notation 2 *Following standard notation, we denote the linear regression coefficient of B on S adjusting for T by $\beta_{BS.T}$ and the total regression coefficient obtained by marginalizing over T by β_{BS}. Cochran's (1938) identity is*

$$\beta_{BS} = \beta_{BS.T} + \beta_{BT.S}\beta_{ST}.$$

We write σ_{BS} for the covariance of B and S, σ^2_B for the variance of B, $\sigma^2_{S|T}$ for the conditional variance of S given T and $\sigma_{BS|T}$ for the conditional covariance of B and S given T. If S is contained in T, then we will set $\beta_{BS.T} = 0$.

We maintain the following assumption throughout the paper whose import is discussed in the following section.

Assn $0 < \beta_{BS.T} < 1$.

1.1 Willingness to Pay

In estimating their value, buyers need only do so second hand, via estimating B. In particular, the law of iterated expectations implies

$$\mathbb{E}[V_B|T, S] = \mathbb{E}[B|T, S], \tag{1}$$

and consequently for each $t \in \boldsymbol{T}$, $p \in \boldsymbol{S}(t)$,

$$\mathbb{E}[V_B|T = t, S < p] = \mathbb{E}[B|T = t, S < p]. \tag{2}$$

The quantity in equation (2) represents the expected value to buyers in case the seller is willing to sell at a price p. It therefore represents buyers' willingness to pay for a good traded at price p.

Definition 1 (The willingness to pay map) *Let C be the set of real valued functions $\phi : \mathbf{T} \to \mathrm{Conv}(\mathbf{B})$. We define the* willingness to pay map $\Psi : C \to C$, *by*

$$\begin{aligned}\Psi(\phi)(t) &= \mathbb{E}[B|T = t, S < \phi(t)], \text{ for } \phi(t) \geq \inf \boldsymbol{S}(t)\\ &= \mathbb{E}[B|T = t, S = \inf \boldsymbol{S}(t)], \text{ for } \phi(t) < \inf \boldsymbol{S}(t).\end{aligned}$$

We will make use of the fact (established as Lemma 1) that under the maintained assumption that $\beta_{BS.T} < 1$, Ψ is *contractive* on an appropriately defined metric space. This fact not only allows us to settle existence and uniqueness questions since it implies that the willingness to pay map has a unique fixed point, but it also supplies a means of verifying properties of adverse selection. Evidently, without the assumption $\beta_{BS.T} < 1$ the equilibrium price is not defined.

2 Equilibrium

Our objective in this section is to establish the analog of the familiar (Akerlof 1970) market outcomes as equilibria. We begin by establishing the existence and uniqueness of a fixed point of the willingness to pay map Ψ, we term this solution the *price schedule*.

Definition 2 *We call the unique fixed point of the willingness to pay map* Ψ the price schedule $p : \mathbf{T} \to \mathbb{R}$. *The random variable* $P = p(T)$ *is called the price.*

Lemma 1 Ψ *is a contraction mapping on an appropriately defined metric space and has a unique fixed point.*

Proof. Specifically, define the distance $d : \mathcal{C} \times \mathcal{C} \to [-\infty, \infty]$, $d(\phi, \varphi) = \sup_{t \in \boldsymbol{T}} |\phi(t) - \varphi(t)|$. It is classical that the set of bounded functions mapping from an arbitrary set into the reals is a complete metric space when endowed with the sup norm (see e.g. Dunford and Schwartz p.258). If $\boldsymbol{B}$ is compact, therefore, $M = (\mathcal{C}, d)$ is a complete metric space. However, apart from the trivial case where Q is orthogonal to V_B, $\boldsymbol{B}$ is not compact. Evidently, given some $\phi^* \in \mathcal{C}$ if we denote by $\mathcal{C}^* \subset \mathcal{C}$ the set of $\phi \in \mathcal{C}$ such that $\phi - \phi^*$ is bounded, $(\mathcal{C}^*, d)$ is a complete metric space. Specifically, take $\phi^*(t) = \mathbb{E}[B|T = t] - \frac{\beta_{BS.T}}{1-\beta_{BS.T}} \max\{\mathbb{E}[M|T = t], 0\}$.

The map Ψ is defined by

$$\Psi(\phi)(t) = \mathbb{E}[B|T = t] - \beta_{BS.T} \sigma_{S|T} h\left(\frac{\mathbb{E}[S|T = t] - \phi(t)}{\sigma_{S|T}}\right),$$

where h is the Gaussian $\mathcal{N}(0, 1)$ hazard function. Confirmation of this fact appears in Proposition 4. The hazard is continuously differentiable and satisfies $0 < h' < 1$, therefore it follows immediately from the mean value theorem that $|\Psi(\phi)(t) - \Psi(\varphi)(t)| < \beta_{BS.T} |\varphi(t) - \phi(t)|$, hence $d(\Psi(\phi), \Psi(\varphi)) \leq \beta_{BS.T} d(\varphi, \phi)$. Finally, it is routine to establish that $\phi \in \mathcal{C}^* \Rightarrow \Psi(\phi) \in \mathcal{C}^*$. Hence, Ψ is a contraction on the complete metric space $(\mathcal{C}^*, d)$. Application of the Banach fixed point theorem completes the proof. ■

With the a well-defined and unique price, the following equilibrium of the trading game can readily be established.

Proposition 1 *There exists a PBE in which each buyer offers a price $P = p(T)$. The seller sells if and only if $S < P$.*

Proof. We start by confirming that there is no incentive to deviate from these strategies. Clearly, the sellers sells if and only if $S < P$. It follows that it cannot be optimal for buyers to deviate from posting an offer that is a fixed point of the willingness to pay map in Definition 1. To see this, suppose $\mathbb{E}[V_B|T = t, S < p(t)] \neq p(t)$ for some $t \in \mathbf{T}$. Here, either the successful buyer will make an expected loss, or there is a price offer that is a profitable deviation by some buyer. ■

In this PBE, equilibrium market outcomes are determined by a price schedule that is the unique fixed point of the willingness to pay map. It is an immediate consequence of Definition 1 that in equilibria identified by Proposition 1, if there is a positive probability of trade (i.e. if $\Pr[S < p(t)|T = t] > 0$), then the following price equation holds

Price Equation

$$p(t) = \mathbb{E}[B|T = t, S < p(t)]. \tag{3}$$

For the analysis a key outcome of interest is the *expected total surplus* corresponding to the information structure. This is defined as $\mathbb{E}[TS] = \mathbb{E}[S] - \mathbb{E}[M1_{\{S<P\}}] = \mathbb{E}[B] + \mathbb{E}[M1_{\{S>P\}}]$.

2.1 Adverse Selection and Efficiency: Definitions

Given all the information in the economy (and specifically, including the private information of the buyer), we define the quantity of adverse selection as the difference between the average quality of the good as perceived by buyers and the average quality contingent on trade. That is, it measures how the fact that a good is selected to trade changes buyers' estimates of the good's value. This is consistent with the early usage in the insurance literature discussed in Akerlof (1970), and is similar to measuring the extent of the winner's curse in a common value auction.

Definition 3 (Quantity of adverse selection) *The* quantity of adverse selection *is defined as the random variable*

$$AS = \mathbb{E}[B|T] - \mathbb{E}[B|T, S < P]. \tag{4}$$

The adverse selection schedule *is defined as the map* $as : \mathbf{T} \to R$ *with* $as(t) = E[B|T = t] - p(t)$.

Since the adverse selection schedule differs from (minus) the price schedule only by the function $t \mapsto \mathbb{E}[B|T = t]$, it follows from that the adverse selection schedule may also be defined in terms of the fixed point of a contraction mapping. This has the evident consequence that any property preserved under the contraction mapping is a property which will be displayed by adverse selection.

Although adverse selection is defined as the difference in average (i.e. expected) value between two distributions, the competitive nature of equilibria and the fact that valuations and prices are measured in the same units means that the quantity of adverse selection can also be viewed as a difference in prices. We may view adverse selection as the difference between the equilibrium price which would obtain, through competition, in a counterfactual game without private information (i.e. $\mathbb{E}[B|T]$) and the equilibrium price in the game itself (P). Viewed in this way, adverse selection is a price effect.

A familiar thought, certainly since Akerlof (1970), is that adverse selection depresses prices and discourages trade, even when trade is warranted on efficiency grounds. To explore the equilibrium relationship between adverse selection, as quantified in Definition 3, the likelihood of trade, and market efficiency, we introduce two further definitions and associated notation.

Definition 4 (Probability of trade) *The* probability of trade *is defined as the random variable*

$$R = \mathbb{E}[1_{\{S<P\}}|T] = \Pr[S < P|T]. \tag{5}$$

The probability of trade schedule *is defined as the map* $r : \mathbf{T} \to \mathbb{R}_+$ *with* $r(t) = \Pr[S < w(t)|T = t]$.

In contrast to Akerlof (1970) trade is not always warranted on efficiency grounds, so increased probability of trade is *not* a synonym for increased efficiency.

Definition 5 (Efficiency contribution) *The* efficiency contribution *(relative to the no trade status quo) is defined as the random variable*

$$EC = -\mathbb{E}[M1_{\{S<P\}}|T]. \tag{6}$$

The efficiency contribution schedule *is defined as the map* $ec : \mathbf{T} \to \mathbb{R}$ *with* $ec(t) = -E[M1_{\{S<p(t)\}}|T = t]$.

Expected total surplus is given by $\mathbb{E}[S] - \mathbb{E}[M1_{\{S<P\}}]$. Since $\mathbb{E}[S]$ is independent of the information structure, it is immediate that the expected surplus depends on the information structure only through its impact on EC. Note that EC is always positive, this is established in the following remark, which is an easy consequence of the price equation.

Remark 1

$$EC = \mathbb{E}[(P - S)^+ |T] \geq 0. \tag{7}$$

We characterize the random variables AS, R, EC and the relationship between them. A key simplification follows from identifying a natural orthogonality condition to which we now turn.

3 Characterization

3.1 Fixed Information Structure

The difference between the estimate of the (buyers') value based on public information and the estimate based on the superior private information of the seller is $\mathbb{E}[V_B|T] - \mathbb{E}[V_B|Q] = \mathbb{E}[B|T] - B$. Since inside information Q contains public information T, $\mathbb{E}[B|T] - B$ represents the error in the buyers' estimate of the good's value. Similarly, $\mathbb{E}[S|T] - S$ is the corresponding error in the public estimate of the seller's value.

Notation 3 *Errors in public estimates, compared to private estimates.*

$$\not{B} \underset{def}{=} \mathbb{E}[B|T] - B, \;\; \not{S} \underset{def}{=} \mathbb{E}[S|T] - S.$$

The first thing to note is that these errors are by construction uncorrelated with T. Since AS is a function of T, it follows that these differences are also uncorrelated with AS. For Gaussian information structures, this lack of correlation implies statistical independence.

Using the definition of AS in (4), we can write price as $P = \mathbb{E}[S|T] - \mathbb{E}[M|T] - AS$. Hence, the trading event $S < P$ can be re-expressed as $AS + \mathbb{E}[M|T] < \not{S}$. That is, trade takes place when the error in the public estimate of the buyers' value exceeds the sum of adverse selection and the apparent match quality. The trading event, therefore, is an inequality between two independent random variables. A number of important consequences follow.

Proposition 2 *In the equilibrium identified in Proposition 1, adverse selection, AS, and the apparent match quality, $E[M|T]$, are comonotone random variables. Specifically, there exists an increasing function $\widehat{as} : \mathbb{R} \to \mathbb{R}^+$ such that $AS = as(T) = \widehat{as}(E[M|T])$.*

Proof. By Lemma 1 and Definition 3, a unique adverse selection schedule exists, allowing us to write (4) recursively as $AS = \mathbb{E}\left[\not{B}|T, AS + \mathbb{E}[M|T] < \not{S}\right]$. Since $(\not{B}, \not{S}) \perp T$, we may therefore write $as(t) = \mathbb{E}[\not{B}|as(t) + \mathbb{E}[M|T = t] < \not{S}]$. Hence, $as(t)$ depends on t only through $\mathbb{E}[M|T = t]$. This establishes the existence of $\widehat{as} : \mathbb{R} \to \mathbb{R}$ such that $AS = as(T) = \widehat{as}(\mathbb{E}[M|T])$.

Monotonicity follows from the positive dependence of $\not{B}$ and $\not{S}$. Details are in Appendix A. ■

Proposition 2 establishes an important property of adverse selection: AS depends on information (Q, T) only through the *scalar* $\mathbb{E}[M|T]$ and the relationship is a non-decreasing one. This is also intuitive: There is a genuine reason for trade if $\mathbb{E}[M|T]$ is small (negative) but not if it is large (positive).

It is worth noting that in general adverse selection can be either positive or negative. Negative adverse selection reflects the possibility of asymmetric information leading to *propitious or advantageous selection,* as has been noted by Hemenway (1990) and de Meza and Webb (2001).

Using the properties of AS established in Proposition 2, it becomes straightforward to provide a representation of the probability of trade R and the efficiency contribution EC.

Proposition 3 *Under the conditions of Proposition 2, the probability of trade R and the efficiency contribution EC depend on information (Q, T) only through the apparent match* *quality* $E[M|T]$. *Specifically, there exist* $\widehat{r} : \mathbb{R} \to \mathbb{R}_+$ *and* $\widehat{ec} : \mathbb{R} \to \mathbb{R}_+$ *such that*

$$\begin{aligned} R &= r(T) = \widehat{p}(\mathbb{E}[M|T]), \\ EC &= ec(T) = \widehat{ec}(\mathbb{E}[M|T]). \end{aligned}$$

Moreover, $a \mapsto \widehat{r}(a)$ *and* $a \mapsto \widehat{ec}(a)$ *are decreasing. Hence,* $R, EC, -AS$ *and* $-E[M|T]$ *are all comonotone.*

Proof. We can write (5) as $r(t) = \Pr\left[AS + \mathbb{E}[M|T] < \not{S} | T = t\right]$. The fact that $\not{S} \perp T$, implies $r(t) = \Pr\left[as(t) + \mathbb{E}[M|T = t] < \not{S}\right]$. Noting the obvious fact that this probability is decreasing in $as(t) + \mathbb{E}[M|T = t]$, Proposition 2 implies $R = r(T) = \widehat{r}(\mathbb{E}[M|T])$ for some nonnegative decreasing function $\widehat{r} : \mathbb{R} \to \mathbb{R}_+$ as required. Using Remark 1, the efficiency contribution at $T = t$ is $ec(t) = \mathbb{E}[(P - S)^+ | T = t] = \mathbb{E}\left[(\not{S} - AS - \mathbb{E}[M|T])^+ | T = t\right]$. Since $\not{S} \perp T$ it follows that $ec(t) = \varphi\left(\widehat{as}(\mathbb{E}[M|T = t]) + \mathbb{E}[M|T = t]\right)$ for the decreasing convex function $\varphi : \mathbb{R} \to \mathbb{R}_+$, $\varphi(z) = \mathbb{E}\left[(\not{S} - z)^+\right]$. Hence, $ec(t)$ is a decreasing transformation of $\mathbb{E}[M|T = t]$. ■

Propositions 2 and 3 highlight the important role played by the apparent match quality $\mathbb{E}[M|T]$ in determining adverse selection and efficiency. As might be expected, higher adverse selection, as defined here, occurs with reduced trade probability and lower efficiency contribution. These results do not rely on the Gaussian structure beyond the implied orthogonality of the errors in public estimates with the public information; that is, $(\not{B}, \not{S}) \perp T$.[7]

Gaussian information, of course, imposes more restrictions. The following result establishes that the marginal distribution of adverse selection depends on the information structure only through a three dimensional parameter space.

Proposition 4 $\widehat{as} : R \to R_+$ *is implicitly defined by*

$$\widehat{as} = \beta_{BS.T}\sigma_{S|T} h\left(\frac{\widehat{as} + \mathbb{E}[M|T = t]}{\sigma_{S|T}}\right), \tag{8}$$

where $h : R \to R_+$ *is the inverse Mills ratio (i.e. the normal hazard function).*

[7] Further detail on the non-Gaussian case including conditions that ensure the existence of a well-defined equilibrium and adverse selection schedule appear in Bar-Isaac, Jewitt and Leaver (2014).

Proof. As in the proof of Proposition 2, we can write $as(t) = \mathbb{E}[\not{B}|as(t) + \mathbb{E}[M|T = t] < \not{S}]$. Noting that, by the Law of Iterated Expectations $\mathbb{E}[\not{B}|z \leq \not{S}] = \mathbb{E}[\mathbb{E}[\not{B}|\not{S}]|z \leq \not{S}]$, $\mathbb{E}[\not{B}|\not{S}] = \beta_{BS.T}\not{S}$, $\not{S} \sim N\left(0, \sigma_{S|T}\right)$, and $h\left(z\right) = \mathbb{E}[\not{S}|z \leq \not{S}]$ is the inverse Mills ratio, establishes the implicit representation in (8). ■

This result implies that even though the information structure may be of high dimension, three parameters of the information structure characterize the adverse selection schedule, according to (8) and $\mathbb{E}[AS]$ as discussed below. Moreover, these have a natural economic interpretation. Specifically:

- $\sigma_{S|T}$ is the *information gap* regarding the seller's value of the good. It is a measure of the difference in the amount of information about V_S held by the informed and uninformed parties. For Gaussian distributions, the variance of the conditional expectation $Var(\mathbb{E}[V_S|Q])$ is a bona fide measure of how informative Q is about V_S, similarly $Var(\mathbb{E}[V_S|T])$ measures how much information is in T. By the law of total variance, we have $\sigma^2_{S|T} = Var(\mathbb{E}[V_S|Q]) - Var(\mathbb{E}[V_S|T])$.

- $\beta_{BS.T}$ is *relevance of the seller's value to the buyers' value.* It is a measure of the relevance of the seller's estimate of its own value, S, to the buyers' value B given public information T. Note that we may also write $\beta^2_{BS.T}\sigma^2_{S|T} = Var(\mathbb{E}[V_B|S,T]) - Var(\mathbb{E}[V_B|T])$. Therefore given $\sigma_{S|T}$, $\beta_{BS.T}$ is a measure of how much extra information (S,T) contains about B than just the public information T.

- $\sigma_{\mathbb{E}[M|T]}$ is the *quality of public information about match quality.* It is a measure of how much information is available publicly about $V_S - V_B$.[8]

3.2 Comparing Information Structures

The results in Section 3.1 characterize how, for a given information structure, adverse selection and efficiency depend on the information structure only through the apparent match quality and that the information structure can be summarised by three parameters relating to the information gap regarding the seller's value, the relevance of the seller's value to buyer's value, and the quality of public information about match quality.

[8] Note $Var(\mathbb{E}[M|T]) = Var(\mathbb{E}[V_S - V_B|T])$ determines the distribution of $\mathbb{E}[M|T]$. By construction, $\mathbb{E}[\mathbb{E}[M|T]] = \mathbb{E}[V_S - V_B]$ independently of the information structure so, in the Gaussian case, its distribution is determined solely by its variance, $Var(\mathbb{E}[M|T])$.

In particular, for a given realization of public information of a fixed information structure, adverse selection increases and efficiency decreases in the apparent quality of the current match (that is the estimate of the difference between the seller's and buyers' valuation). These relationships need not hold at the aggregate level when comparing different information structures—that is, it may be that aggregate adverse selection and efficiency are both higher in one information structure than another.

The characterization of the adverse selection schedule, established in Proposition 4, allows us to address the question of how overall efficiency *aggregated across submarkets* depends on the information structure, and how this relates to aggregate adverse selection.

Proposition 5 *$E[AS]$ and $E[EC]$ depend on the information structure only through the three dimensional parameter $(\beta_{BS.T}, \sigma_{S|T}, \sigma_{\mathbb{E}[M|T]}) \in [0,1) \times R_+^2$. Furthermore,*

1. *$E[AS]$ is increasing in $(\beta_{BS.T}, \sigma_{S|T}, \sigma_{\mathbb{E}[M|T]})$.*

2. *For each $(\sigma_{S|T}, \sigma_{\mathbb{E}[M|T]}) \in R_+^2$, $E[EC]$ is decreasing in $\beta_{BS.T}$.*

3. *For each $(\beta_{BS.T}, \sigma_{S|T}) \in [0, \frac{\pi}{4}] \times R_+, E[EC]$ is increasing in $\sigma_{\mathbb{E}[M|T]}$.*

Proof. *Part 1.* It is immediate from (8) that for each $t \in T$, $\widehat{as}(\mathbb{E}[M|T=t])$ is increasing in $\beta_{BS.T}$. Similarly, $\widehat{as}(\mathbb{E}[M|T=t])$ is seen to be increasing in $\sigma_{S|T}$ by the star-shaped property of the normal hazard function (Lemma D2). The convexity property of the normal hazard (Lemma B) is inherited by $\widehat{as}$, hence it follows from Jensen's inequality that $\mathbb{E}[AS] = \mathbb{E}[\widehat{as}(\mathbb{E}[M|T])]$ is increasing in $\sigma_{\mathbb{E}[M|T]}$.

Part 2. Straightforward manipulations show that EC has the representation

$$EC = \sigma_{S|T}\Psi\left(\frac{\mathbb{E}[M|T] + \widehat{as}\left(\mathbb{E}[M|T]\right)}{\sigma_{S|T}}\right)$$

where, $\Psi(x) = \mathrm{E}[(\varepsilon - x)^+]$ with $\varepsilon \sim \mathcal{N}(0,1)$. Note that Ψ is a decreasing function. Since $\widehat{as}$ is increasing in $\beta_{BS.T}$, EC is made smaller in first-order stochastic dominance order and therefore has smaller expectation.

Part 3. This is more delicate. Using the above representation, we can write $\mathrm{E}[EC]$ as

$$\mathbb{E}[EC] = \sigma_{S|T}\mathbb{E}\left[\Psi\left(\frac{\sigma_{\mathbb{E}[M|T]}\varepsilon + \widehat{as}\left(\sigma_{\mathbb{E}[M|T]}\varepsilon\right)}{\sigma_{S|T}}\right)\right]. \tag{9}$$

Differentiation gives

$$\frac{\partial \mathbb{E}[EC]}{\partial \sigma_{\mathbb{E}[M|T]}} = \mathbb{E}\left[\Psi'.\left(\widehat{as}'\varepsilon + \varepsilon\right)\right].$$

Differentiating the representation for $\widehat{as}$ in (8) gives

$$\widehat{as}' = \widehat{as}'\left(\sigma_{\mathbb{E}[M|T]}\varepsilon\right) = \frac{\beta_{BS.T}h'}{1-\beta_{BS.T}h'}.$$

Using this together with the fact that $\Psi' = -(1-F)$ (F is the standard normal cdf) gives

$$\frac{\partial \mathbb{E}[EC]}{\partial \sigma_{\mathbb{E}[M|T]}} = -\mathbb{E}\left[\left(\frac{1-F}{1-\beta_{BS.T}h'}\varepsilon\right)\right].$$

Setting $k(x,\beta) = \frac{1-F(x)}{1-\beta h'(x)}$ and $a(\varepsilon) = \frac{\sigma_{\mathbb{E}[M|T]}t+\widehat{as}\left(\sigma_{\mathbb{E}[M|T]}\varepsilon\right)}{\sigma_{S|T}}$, this becomes

$$\frac{\partial \mathbb{E}[EC]}{\partial \sigma_{\mathbb{E}[M|T]}} = -\int_{-\infty}^{\infty} k(a(\eta), \beta_{BS.T})\eta f(\eta) d\eta.$$

Using symmetry of the normal density around zero,

$$\frac{\partial \mathbb{E}[EC]}{\partial \sigma_{\mathbb{E}[M|T]}} = \int_{0}^{\infty} \left(k(a(-\eta), \beta_{BS.T}) - k(a(\eta), \beta_{BS.T})\right)\eta f(\eta) d\eta.$$

The desired result now follows from the following facts. For each $0 \leq \beta < 1$, $x \mapsto k(x,\beta)$ is quasiconcave and for each $x \in \mathbb{R}$, $\beta \mapsto k(x,\beta)$ is increasing on $[0,1)$ and satisfies $\lim_{x\to-\infty} k(x,\beta) = 1$ and $k(0,\beta) = 0.5\left(1-\frac{2\beta}{\pi}\right)^{-1}$. Hence, $k(0,\frac{\pi}{4}) = 1$. It follows that for $\beta \leq \frac{\pi}{4}$, (1) $\min\limits_{x\leq 0} k(x,\beta) \geq \max\limits_{x\geq 0} k(x,\beta)$, (2) $x \mapsto k(x,\beta)$ is decreasing on $\mathbb{R}_+$. Hence, $x > 0$, $x' < x$ imply $k(x',\beta) > k(x,\beta)$. Since a is an increasing function and it is positive on $\mathbb{R}_{++}$, it follows that $k(a(-t),\beta) > k(a(t),\beta)$ for $t \in \mathbb{R}_{++}$, therefore $\frac{\partial \mathbb{E}[EC]}{\partial \sigma_{\mathbb{E}[M|T]}} > 0$ as required. ■

Proposition 5, Part 1, establishes that expected adverse selection is increasing in all three parameters. It is intuitive that $\mathbb{E}[AS]$ is increasing in the information gap and relevance of the seller's value parameters. The reason why $\mathbb{E}[AS]$ is increasing in the quality of public information about match quality is slightly less obvious but it arises via Jensen's inequality from a natural convexity—when apparent match quality is high adverse selection will be positive and large, but when apparent match quality is low adverse selection remains bounded from below by zero.

Parts 2 and 3 of the proposition establish the effect of the parameters on expected efficiency. Comparing two information structures with the same $\sigma_{S|T}$ and $\sigma_{\mathbb{E}[M|T]}$[9], the information structure for which $\beta_{BS.T}$ is larger leads to less expected efficiency. Hence, there is a trade off between $\mathbb{E}[AS]$ and $\mathbb{E}[EC]$ for fixed $\sigma_{S|T}$ and $\sigma_{\mathbb{E}[M|T]}$. This is consistent with the familiar intuition from models with scalar types and no public information. On the other hand, holding $\sigma_{S|T}$ and $\beta_{BS.T}$ fixed (with the latter not too large), better public information about the match quality increases both expected adverse selection *and* expected efficiency.

Proposition 6 *For each fixed $\sigma_{S|T}$, $E\left[EC\right]$ and $E\left[AS\right]$ satisfy a strict Spence-Mirrlees condition in each of $\beta_{BS.T}$ and $\sigma_{\mathbb{E}[M|T]}$. Specifically,*

$$\frac{\partial \mathbb{E}\left[AS\right]}{\partial \sigma_{\mathbb{E}[M|T]}} / \frac{\partial \mathbb{E}\left[AS\right]}{\partial \beta_{BS.T}} > \frac{\partial \mathbb{E}\left[EC\right]}{\partial \sigma_{\mathbb{E}[M|T]}} / \frac{\partial \mathbb{E}\left[EC\right]}{\partial \beta_{BS.T}}.$$

Proof. Implicit differentiation of (9) gives

$$\begin{aligned} \frac{\partial \mathbb{E}[EC]}{\partial \sigma_{\mathbb{E}[M|T]}} &= \mathbb{E}\left[\Psi'.\left(\frac{\partial AS}{\partial \sigma_{\mathbb{E}[M|T]}} + \varepsilon\right)\right], \\ \frac{\partial \mathbb{E}[EC]}{\partial \beta_{BS.T}} &= \mathbb{E}\left[\Psi'.\frac{\partial AS}{\partial \beta_{BS.T}}\right] < 0, \end{aligned}$$

the inequality following from $\Psi' < 0$. Similarly, implicit differentiation of the representation for $\widehat{as}$ in (8) gives

$$\begin{aligned} \frac{\partial \mathbb{E}[AS]}{\partial \sigma_{\mathbb{E}[M|T]}} &= \mathbb{E}\left[\frac{\partial AS}{\partial \sigma_{\mathbb{E}[M|T]}}\right] = \mathbb{E}\left[\frac{\beta_{BS.T} h'}{1 - \beta_{BS.T} h'} \varepsilon\right] > 0, \\ \frac{\partial \mathbb{E}[AS]}{\partial \beta_{BS.T}} &= \mathbb{E}\left[\frac{\partial AS}{\partial \beta_{BS.T}}\right] = \mathbb{E}\left[\frac{\sigma_{S|T} h}{1 - \beta_{BS.T} h'}\right] > 0. \end{aligned}$$

The marginal rate of substitution between $\sigma_{\mathbb{E}[M|T]}$ and β for the function $\mathbb{E}[AS]$ is given

[9] For instance, this would arise under two information structures with the same public information and with private information equally good for estimating V_S. To see this, note first that private information Q does not affect $\sigma_{\mathbb{E}[M|T]}$ (recall $Var(\mathbb{E}[M|T]) = Var(\mathbb{E}[V_B - V_S|T])$). Second, $\sigma_{S|T}$ is a measure of the quality of information about V_S.

by

$$\frac{\frac{\partial \mathbb{E}[AS]}{\partial \sigma_{\mathbb{E}[M|T]}}}{\frac{\partial \mathbb{E}[AS]}{\partial \beta}} = \frac{\beta}{\sigma_{S|T}} \frac{\mathbb{E}\left[\frac{h'}{1-\beta h'}T\right]}{\mathbb{E}\left[\frac{h}{1-\beta h'}\right]} = k > 0.$$

Thus, with k so defined, the operator D_k defined as $D_k = \left(\frac{\partial}{\partial \sigma_{\mathbb{E}[M|T]}} - k\frac{\partial}{\partial \beta}\right)$ satisfies

$$D_k\left(\mathbb{E}[AS]\right) = \mathbb{E}[D_k\left(AS\right)] = 0.$$

We will show

$$D_k\left(\mathbb{E}[EC]\right) = \mathbb{E}[D_k\left(EC\right)] = \mathbb{E}\left[\Psi'.\left(D_k\left(AS\right) + \varepsilon\right)\right] > 0.$$

First, noting that Ψ' is increasing in ε, hence, Ψ' and ε have a positive covariance $\mathbb{E}\left[\Psi'.\varepsilon\right] > 0$. It suffices therefore to show that

$$\mathbb{E}\left[\Psi'.D_k\left(AS\right)\right] > 0.$$

Note that

$$D_k\left(AS\right) = \frac{\beta_{BS.T}h'}{1-\beta_{BS.T}h'}\left(\varepsilon - \frac{k\sigma_{S|T}}{\beta_{BS.T}}\frac{h}{h'}\right).$$

The function $x \mapsto h(x)/h'(x)$ is increasing and starshaped (Lemma B1 in Appendix B). It is a consequence of Lemma B1 that $\frac{\widehat{as}(\sigma_{\mathbb{E}[M|T]}\varepsilon)+\sigma_{\mathbb{E}[M|T]}\varepsilon}{\sigma_{S|T}}$ is starshaped in ε (which is inherited from h). Hence, since starshaped functions are closed under composition, $D_k\left(AS\right)$ has a single sign change from negative to positive as ε traverses from $-\infty$ to ∞. The rest of the proof is standard. Suppose $D_k\left(AS\right)$ is zero at some realisation $\varepsilon = \varepsilon'$, then setting c equal to the value taken by Ψ' at ε' and using the fact that Ψ' is increasing in ε, it is established that $\left(\Psi' - c\right).D_k\left(AS\right) \geq 0$ on $\mathbb{R}$, the inequality being strict except at ε'. Therefore,

$$0 < \mathbb{E}\left[\left(\Psi' - c\right).D_k\left(AS\right)\right] = \mathbb{E}\left[\Psi'.D_k\left(AS\right)\right] - c\mathbb{E}\left[D_k\left(AS\right)\right] = \mathbb{E}\left[\Psi'.D_k\left(AS\right)\right].$$

This establishes $D_k\left(\mathbb{E}[EC]\right) > 0$. Since $\frac{\partial \mathbb{E}[EC]}{\partial \beta}_{BS.T} < 0$, this implies the result. ■

Comparing two information structures with $\sigma_{S|T}$ fixed and which generate the same amount of expected adverse selection, the information structure with better public information about match quality leads to higher expected efficiency.

Propositions 5 and 6 have a useful implication which is illustrated in Figure 1. The figure illustrates the level sets of $\mathbb{E}[AS]$ and $\mathbb{E}[EC]$, their general shape are as established in the Propositions. It is assumed that the vertex of the cone which is at $(0, \beta)$ satisfies $\beta \leq \frac{\pi}{4}$. On the positively oriented extreme ray of the cone, Proposition 6 applies to ensure that there is a direction of increase of both $\mathbb{E}[AS]$ and $\mathbb{E}[EC]$ pointing into the interior of the cone. Similarly, on the negatively oriented extreme ray of the cone, Proposition 5 applies to ensure that there is a direction of increase of both $\mathbb{E}[AS]$ and $\mathbb{E}[EC]$ pointing into the interior of the cone. Hence, there is always a direction pointing into the interior of the cone in which both $\mathbb{E}[AS]$ and $\mathbb{E}[EC]$ are increasing. Moreover, as illustrated, this direction may be chosen such that such that $\sigma_{\mathbb{E}[M|T]}$ is increasing and $|\beta - \beta_{BS.T}|$ is decreasing. This is stated formally in Corollary 1 below.

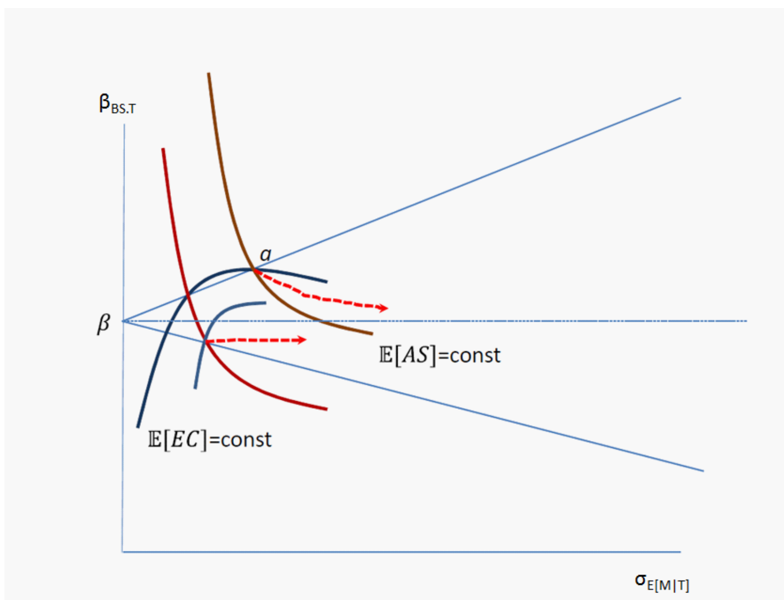

Fig 1: Implication of Propositions 5 and 6: Paths into the body of the cone.

Definition 6 *Let C be a cone in $R_+ \times R$ with vertex $(0, \beta)$ and non-empty interior which contains the horizontal line segment through its vertex, i.e. $\{(a_1, \beta) \in R_+^2 | a_1 \geq 0\}$. Let $\widehat{a} = (\widehat{a}_1, \widehat{a}_2)$ lie on an extreme ray of C. We say the path $p : R_+ \to C$, $p(a_1) = (a_1, \varphi(a_1))$ with $\varphi : R_+ \to R$ is from $\widehat{a}$ into the body of the cone if $p(\widehat{a}_1) = (\widehat{a}_1, \widehat{a}_2)$ and $a_1 \mapsto |\varphi(a_1) - \beta|$ is decreasing on $[a_1, \infty)$.*

Corollary 1 *Let C be a closed convex cone satisfying the conditions of Definition 6 for some $0 \le \beta \le \frac{\pi}{4}$. Suppose $\widehat{a} = (\sigma_{\mathbb{E}[M|T]}, \beta_{BS.T}) \in R_+ \times [0,1)$ lies on an extreme ray of C, there is a path from $\widehat{a}$ into the body of the cone along which both $E[AS]$ and $E[EC]$ are increasing.*

Proof. On the positively oriented extreme ray of the cone, Proposition 6 together with the fact that $\mathbb{E}[AS]$ is increasing in both $\sigma_{\mathbb{E}[M|T]}$ and $\beta_{BS.T}$ applies to ensure that there is a direction of increase of $\sigma_{\mathbb{E}[M|T]}$ and decrease of $\beta_{BS.T}$ which increases both $\mathbb{E}[AS]$ and $\mathbb{E}[EC]$. This establishes to conclusion if $\widehat{a}$ is on the positively oriented extreme ray. Similarly, on the negatively oriented extreme ray of the cone, Proposition 5 applies—increasing $\sigma_{\mathbb{E}[M|T]}$ with $\beta_{BS.T}$ constant suffices. ■

3.3 A Sampling Example

The following example illustrates different possible relationships between aggregate adverse selection and efficiency that may arise through different information structures.

Suppose seller and buyers obtain noisy samples of the valuations:

$$\begin{aligned} Q &= \left(V_S + \epsilon_1, ..., V_S + \epsilon_{N_S},\ V_B + \nu_1, ..., V_B + \nu_{N_B}\right), \\ T &= \left(V_B + \nu_1, ..., V_B + \nu_{N_T}\right),\ N_T \le N_B. \end{aligned}$$

Private information Q consists of N_S samples of the seller's valuation $V_S = V_B + V_M$ and N_B samples of buyers' valuation V_B. Some of the samples of buyers' are publicly available in T. Hence, elements of Γ are identified with three numbers: N_S, N_B, and N_T, $\Gamma \subset \{(N_S, N_B, N_T) \in \mathbb{N}^3 \mid 0 \le N_T \le N_B, 0 \le N_S\}$.

All random variables are independently distributed, and assumed to be $\mathcal{N}(0,1)$ except for $V_B \sim \mathcal{N}\left(\mu, \sigma^2_{V_B}\right)$ and $V_M \sim \mathcal{N}\left(0, \sigma^2_{V_M}\right)$. Below, we describe how changes in the information structure, that increase the asymmetry in information between the seller and buyers, affect adverse selection and efficiency.

a. *Fewer public samples of buyer value.*
Suppose that the information structure (Q', T') involves public information T' that contains fewer samples of buyer value V_B than the public information T associated with the information structure (Q, T) (lower N_T with no change in N_B and N_S). Both the information gap and informed party relevance are bigger under (Q', T') than

(Q, T). Hence, $AS' > AS$. To see that $EC' < EC$, note that $\beta_{BS.T} = 1 - \beta_{MS.T}$. Using the orthogonality of M and T, it follows that $1 - \beta_{BS.T} = \frac{\sigma_{MS}}{\sigma^2_{S|T}}$, therefore, since σ_{MS} is unaffected by information disclosure, $\sigma_{S|T}(1 - \beta_{BS.T}) = \frac{\sigma_{MS}}{\sigma_{S|T}}$ is decreasing in the information gap. This gives the result.

b. *More private samples of buyer value.*
Suppose (Q', T') contains more samples of V_B than (Q, T) (higher N_B with no change in N_S and N_T). It can be shown that both the information gap and informed party relevance are bigger under (V_B, V_S, Q', T') than (Q', T'). Hence, it is immediate that $AS' > AS$. We can also show that $EC' < EC$. As in part a., this is established by showing that collecting more samples of V_B *decreases* $\beta_{MS.T}\sigma_{S|T}$. The intuition is that collecting more samples of V_B makes S a noisier measure of M given public information. Note that $\beta_{MS.T}\sigma_{S|T}$ is a measure of how much information S contains about M.

c. *More private samples of seller value.*
In a. and b. above, adverse selection and efficiency go in opposite directions. Here is a case where they go together. Suppose $Q = T = V_B + \nu_1$, then $AS = EC = 0$. However, with $Q' = (V_B + V_M + \epsilon_1, V_B + \nu_1)$, $T' = V_B + \nu_1$ (higher N_S with no change in N_B and N_T) there is both positive adverse selection and efficiency contribution.

4 Efficient Adverse Selection

As argued above, in aggregate comparisons of information structures, no clear relationship need arise between aggregate adverse selection and efficiency. To gain intuition, and to speak to applications were information structures might arise endogenously,[10] we explore how keeping fixed the private information of the seller, different public information disclosure can impose a fixed quantity of adverse selection most efficiently.

4.0.1 Preliminaries

In the Gaussian case, an information structure $I = (V_S, V_B, Q, T)$ from a set Γ of feasible information structures can be identified one-to-one with a set of symmetric positive

[10] We speak to one such application in the labour market explicitly in Section 5.2, and at much greater length in Bar-Isaac, Jewitt and Leaver (2015).

semidefinite covariance matrices partitioned as

$$\Sigma_I = \begin{bmatrix} \Sigma_V & \Sigma_{VQ} & \Sigma_{VT} \\ \Sigma_{QV} & \Sigma_Q & \Sigma_{QT} \\ \Sigma_{TV} & \Sigma_{TQ} & \Sigma_T \end{bmatrix}, \tag{10}$$

each containing Σ_V as the first 2×2 elements and with Σ_Q and Σ_T positive definite (if necessary delete redundant elements). V denotes the vector of employment values (V_S, V_B). Note that we write Σ_V rather than $\Sigma_{V_S V_B}$ for later convenience. Using this notation, that the private information Q contains public information T can be written as $\Sigma_{VT} = \Sigma_{VQ}\Sigma_Q^{-1}\Sigma_{QT}$.

Definition 7 *The seller has* fixed private information Q *if for each* $I \in \Gamma$, (V, Q_I, T_I) *is equal in distribution to* (V, Q, T_I). *Hence,* $S_I = E[V_S|Q] = S$ *and* $B_I = E[V_B|Q] = B$, *for each* $I \in \Gamma$.

Notation 4 *Let* S_n *denote the set of* $n \times n$ *symmetric matrices,* $S_n^+ \subset S_n$ *positive semi-definite,* $S_n^{++} \subset S_n^+$ *positive definite. For* $n = 2 + n_Q + n_{T_I}$, $S^+\left(\Sigma_{(V,Q)}\right)$ *denotes the set of all finite symmetric positive semidefinite matrices partitioned as in (10) with*

$$\Sigma_{(V,Q)} = \begin{bmatrix} \Sigma_V & \Sigma_{VQ} \\ \Sigma_{QV} & \Sigma_Q \end{bmatrix} \tag{11}$$

as the first $(2 + n_Q) \times (2 + n_Q)$ *elements. For* $A, B \in S_n$, *we write* $A \preceq B$ *if* $B - A \in S_n^+$ *and* $A \prec B$ *if* $B - A \in S_n^{++}$. $0 \in S_n$ *denotes the n-dimensional null matrix, hence* $A \succ 0$ *means* A *is positive definite.* Σ_{SB} *and* $\Sigma_{SB|T}$ *denote the* 2×2 *unconditional and conditional covariance matrices:*

$$\Sigma_{SB} = \begin{pmatrix} \sigma_S^2 & \sigma_{SB} \\ \sigma_{SB} & \sigma_B^2 \end{pmatrix}, \; \Sigma_{SB|T} = \begin{pmatrix} \sigma_{S|T}^2 & \sigma_{SB|T} \\ \sigma_{SB|T} & \sigma_{B|T}^2 \end{pmatrix}.$$

4.0.2 Disclosure with Fixed Private Information

With fixed private information, different elements of Γ correspond to different public information disclosures. We are especially interested in the case of Gaussian information structures where there are no further constraints on information disclosure apart from the fact that one cannot disclose what is not known. We write $\Gamma = \Gamma_Q$ to denote the set of such Gaussian information structures with fixed private information Q. Γ_Q can be identified one-to-one with the subset of $S^+\left(\Sigma_{(V,Q)}\right)$ satisfying $\Sigma_{VT}=\Sigma_{VQ}\Sigma_Q^{-1}\Sigma_{QT_I}$. We can therefore define the set of possible expected adverse selection-efficiency pairs that such an information structure could generate as

$$\Omega(\Gamma_Q) \underset{def}{=} \left\{(\mathbb{E}[AS],\mathbb{E}[EC]) \mid \Sigma_I \in S^+\left(\Sigma_{(V,Q)}\right), \Sigma_{VT_I}=\Sigma_{VQ}\Sigma_Q^{-1}\Sigma_{QT_I}\right\}.$$

The set $S^+\left(\Sigma_{(V,Q)}\right)$ of positive semidefinite matrices with (11) as the initial block of elements is a rather large one from which to choose, especially since the overall dimensionality is not specified *a priori*. However, Proposition 5 established that adverse selection $\mathbb{E}[AS]$ and efficiency $\mathbb{E}[EC]$ depend on a much lower dimensional parameter space. The following lemma achieves a similar reduction in dimensionality to three (the elements of a symmetric 2×2 matrix) bounded by semidefinite constraints.

Lemma 2 *The set $\Omega(\Gamma_{\Sigma_{VQ}})$ can be parameterized equivalently as*

$$\Omega(\Gamma_Q)=\left\{(\mathbb{E}[AS_I],\mathbb{E}[EC_I]) \mid \Sigma_{SB|T_I}\in\mathcal{S}_2, 0\preceq\Sigma_{SB|T_I}\preceq\Sigma_{SB}\right\}.$$

Proof. Proposition 5 applies wherever AS is defined. Hence, the marginal distribution of AS is determined by the three quantities $\sigma_{S|T},\beta_{BS.T},\sigma_{\mathbb{E}[M|T]}$. Similarly for EC. By the law of total variance $\sigma^2_{\mathbb{E}[M|T]}=\sigma^2_M-\sigma^2_{M|T}=\sigma^2_M-(\sigma_{S|T}-\sigma_{B|T})^2$. σ^2_M is determined only by the private information of training firm, Q, so remains constant. Hence $\sigma^2_{\mathbb{E}[M|T]}$ is determined by the conditional covariance matrix. It follows immediately from the standard formula for univariate regression that $\beta_{BS.T}=\sigma_{BS|T}/\sigma^2_{S|T}$. Therefore, $\beta_{BS.T}$ is also determined by the conditional covariance matrix. Hence, $\mathbb{E}[EC]$ and $\mathbb{E}[AS]$ depend on $\Sigma\in\mathcal{S}^+\left(\Sigma_{(Y,Q_I)}\right)$ only through $\Sigma_{(S,B)|T}\in\mathcal{S}_2$. This establishes the first part of the parameterization.

It remains to establish the semidefiniteness conditions. The constraint $\mathbf{0}\preceq\Sigma_{(S,B)|T}$ arises because $\Sigma_{(S,B)|T}$ is a covariance matrix and so must be positive semidefinite. The second semidefinite constraint states that $\Sigma_{(S,B)|T}$ must also be smaller in the semidefiniteness order than the *unconditional* covariance matrix for (S,B). To see this, note by the law of total variance $Var\left((S,B)\right)=\mathbb{E}[Var\left((S,B)|T\right)]+Var((\mathbb{E}[S|T],\mathbb{E}[B|T]))$. ■

Given inside information Q, what type of public information T_I generates the maximum expected efficiency for a given expected quantity of adverse selection? Lemma 2 suggests that the problem may be formulated as the following nonlinear semidefinite program:

$$V(\Sigma_{SB}, \overline{AS}) = \max_{\Sigma_{SB|T_I} \in \mathcal{S}_2} \mathbb{E}[EC_I] \qquad \textbf{(P1)}$$
$$\text{subject to } \mathbb{E}[AS_I] \geq \overline{AS}, \text{ and } 0 \preceq \Sigma_{SB|T_I} \preceq \Sigma_{SB}.$$

Thus the problem is reduced to choosing a 2×2 *conditional* covariance matrix $\Sigma_{SB|T_I} \in \mathcal{S}_2$ for (S, B). In what follows, we will say $I \in \Gamma$ solves the program **P1** if $\Sigma_{SB|T_I} \in \mathcal{S}_2$ solves **P1**. Our next result establishes conditions for this optimum. Since the statistic T_I is not uniquely defined (because any full rank linear transformation of T_I carries the same information), the T_I identified in the statement of this proposition should be understood as unique up to equivalence classes.

Proposition 7 *Suppose information structures are Gaussian with fixed inside information Q, and suppose $\beta_{SB} \leq \frac{\pi}{4}$. If $I \in \Gamma$ solves program* **P1** *for some $\overline{AS} \geq 0$, then $\Sigma_{SB|T_I}$ is singular. Furthermore,*

1. *If B and S are colinear ($B = \beta_0 + \beta_{BS}S$), then for some $\varepsilon \perp B, \varepsilon \sim N(0, \sigma_\varepsilon)$, $T = B + \varepsilon$.*

2. *If B and S are not colinear, T_I is two dimensional and for some $\alpha \in R^2$, $B = \beta_{BS.T}S + \alpha \cdot T$.*

Proof. Proposition 6 implies that at least one of the constraints $0 \preceq \Sigma_{SB|T}$ or $\Sigma_{SB|T} \preceq \Sigma_{SB}$ must bind, therefore it suffices to show that $\Sigma_{SB|T} \preceq \Sigma_{SB}$ does not bind alone. Equivalently, that $\Sigma_{SM|T} \preceq \Sigma_{SM}$ does not bind alone. The constraint $\Sigma_{SM|T} \preceq \Sigma_{SM}$ expressed in terms of $\beta_{BS.T}, \sigma_{\mathbb{E}[M|T]}, \sigma^2_{S|T}$ becomes

$$0 \preceq \begin{pmatrix} \sigma_S^2 - \sigma^2_{S|T} & (1 - \beta_{BS.T})\sigma^2_{S|T} \\ (1 - \beta_{BS.T})\sigma^2_{S|T} & \sigma^2_M - \sigma^2_{\mathbb{E}[M|T]} \end{pmatrix}.$$

That is, $(\sigma_{\mathbb{E}[M|T]}, \beta_{BS.T})$ is contained in a cone with vertex at $(0, \overline{\beta})$ with $(1-\overline{\beta})\sigma^2_{S|T} = \sigma_{SM}$, i.e. $\overline{\beta} = 1 - \frac{\sigma_{SM}}{\sigma^2_{S|T}} \leq 1 - \frac{\sigma_{SM}}{\sigma^2_{S_\gamma}} = \frac{\sigma^2_S - \sigma_{SM}}{\sigma^2_S} = \frac{\sigma_{SB}}{\sigma^2_S} = \beta_{BS}$. If $\beta_{BS} \leq \frac{\pi}{4}$, this cone satisfies the conditions of Corollary 1 and applying the corollary, there is a direction of increase for $\mathbb{E}[AS]$, $\mathbb{E}[EC]$ pointing into the interior of the cone. Hence the constraint $\Sigma_{SM|T} \preceq \Sigma_{SM}$

does not bind alone. Therefore $\Sigma_{SB|T}$ is singular as required. This argument establishes the result for the nondegenerate cases $\sigma^2_{S|T}$, $0 < \sigma^2_{S|T} < \sigma^2_S$. If $\sigma^2_{S|T} = \sigma^2_S$, T is orthogonal to S and the cone has no interior. However, in this case the cone degenerates into the line segment $\beta = \beta_{BS}$. Proposition 5 implies that traversing this line segment in a direction of increasing $\sigma_{\mathbb{E}[M|T]}$ increases $\mathbb{E}[EC]$ and $\mathbb{E}[AS]$, so the result still holds. If $\sigma^2_{S|T} = 0$, then $\Sigma_{SB|T}$ is singular so there is nothing to prove. ■

The result identifies (up to equivalence classes) the public information disclosures associated with efficient imposition of adverse selection. The constraints in **P1** bind in a particular way—such that the matrix $\Sigma_{SB|T_I}$ is singular. This condition implies that, given T_I, S and B have a singular covariance matrix and therefore that there is an affine relationship between them.

The basic idea of the proof follows easily from Corollary 1. Either the constraint $\Sigma_{SB|T_I} \preceq \Sigma_{SB}$ binds, or $0 \preceq \Sigma_{SB|T}$ binds. In the latter case $\Sigma_{SB|T}$ is evidently singular as claimed. The constraint $\Sigma_{SB|T} \preceq \Sigma_{SB}$ can be expressed as a conical one to which Corollary 1 applies, implying there is a path of increasing $\mathbb{E}[AS]$ and $\mathbb{E}[EC]$ into the body of the cone—in other words, the constraint $\Sigma_{SB|T} \preceq \Sigma_{SB}$ does not bind (in isolation). Hence, the result.

4.1 Information Acquisition

4.1.1 Information Acquisition

In this section, we explore how the quality of *private* information affects the trade-off between adverse selection and efficiency. Hence, we are interested in the incentives for firms' to acquire information. We begin by establishing that it is not always possible to 'neutralize' the effect of acquiring more private information via a public information disclosure. Notwithstanding this result, we are still able to show that information acquisition expands the expected adverse selection-efficiency frontier.

The Difficulty of Neutralizing Private Information It should be obvious that the effect of acquiring more inside information on expected adverse selection and efficiency will depend on disclosure, i.e. whether this information is passed into the public domain. A more subtle point is that acquiring more inside information *and* disclosing it will typically result in a different combination of expected adverse selection and efficiency relative to not acquiring the information in the first place. To see this, it is helpful to return to our concrete example in Section 3.3.

Sampling example (continued) Suppose that under information structure γ, $Q = V_S$ and $T_\gamma = \varnothing$ (i.e. large M, $N = n = 0$), and under information structure γ', $Q_{\gamma'} = (V_S, V_B)$, and $T_{\gamma_I} = V_B$ (i.e. large N_S, large $N_B = N_T$). The change from information structure γ_I to γ_I' therefore corresponds to the situation where the seller learns, and discloses, the buyer's value; i.e. it passes *all* of its extra private information into the public domain. Under information structure γ', there is full efficiency and no adverse selection. In contrast under information structure γ, there is positive adverse selection but not full efficiency (even relative to the worse information).

In light of this example, it is natural to ask whether there is some (other) disclosure policy that will neutralize the acquisition of more inside information.

Remark 2 *In the Gaussian case, suppose Q' is more informative than Q. The following statements are equivalent.*

1. *For any $\gamma \in \Gamma_Q$, there is a $\gamma' \in \Gamma_{Q'}$ such that $(AS_{\gamma'}, EC_{\gamma'})$ is equal in distribution to (AS_γ, EC_γ).*
2. *The improvement in information leaves unchanged the correlation between B and S, that is $\sigma_{SM} = \sigma_{S'M'}$.*

Proof. It is required that for each disclosure T_γ under private information Q, there exists a choice of T' under private information Q' such that (1) $\mathbb{E}[M'|T']$ has the same distribution as $\mathbb{E}[M|T]$, i.e. $Var(\mathbb{E}[M'|T']) = Var(\mathbb{E}[M|T])$. (2) the functional forms of $\widehat{aM}$ and $\widehat{ec}$ are the same in the two cases. Hence, we require it be possible to set $\sigma^2_{S'|T'} = \sigma^2_{S|T}$ and $\beta_{B'S'.T'} = \beta_{BS.T}$ and $Var(\mathbb{E}[M'|T']) = Var(\mathbb{E}[M|T])$. Equivalently, $\sigma_{M'S'|T'} = \sigma_{MS|T}$, $\sigma^2_{S'|T'} = \sigma^2_{S|T}$ and $Var(\mathbb{E}[M'|T']) = Var(\mathbb{E}[M|T])$. By the law of total variance $Var(\mathbb{E}[M|T]) = \sigma^2_M - \sigma^2_{M|T}$, similarly for $Var(\mathbb{E}[M'|T'])$. Hence, we require $\sigma^2_{M'|T'} = \sigma^2_{M'} - \sigma^2_M + \sigma^2_{M|T}$. It is also required that the semidefinite constraints $0 \preceq \Sigma_{S'M'|T'} \preceq \Sigma_{S'M'}$ are satisfied. In sum, we require that for any 2×2 symmetric matrix $\Sigma_{SM|T}$ satisfying $0 \preceq \Sigma_{SM|T} \preceq \Sigma_{SM}$, it is also true that $0 \preceq \Sigma_{SM|T} + \begin{pmatrix} 0 & 0 \\ 0 & \sigma^2_{M'} - \sigma^2_M \end{pmatrix} \preceq \Sigma_{S'M'}$, where $\sigma^2_{M'} - \sigma^2_M \geq 0$ by the fact that Q' is more informative than Q. Since

$\begin{pmatrix} 0 & 0 \\ 0 & \sigma^2_{M'} - \sigma^2_M \end{pmatrix}$ is positive semidefinite, $0 \preceq \Sigma_{(S,M)|T} + \begin{pmatrix} 0 & 0 \\ 0 & \sigma^2_{M'} - \sigma^2_M \end{pmatrix}$ is evidently satisfied so the condition reduces to $\Sigma_{(S,M)|T} + \begin{pmatrix} 0 & 0 \\ 0 & \sigma^2_{M'} - \sigma^2_M \end{pmatrix} \preceq \Sigma_{(S',M')}$. Given $\Sigma_{(S,M)|T} \preceq \Sigma_{(S,M)}$, it suffices that $\Sigma_{(S,M)} + \begin{pmatrix} 0 & 0 \\ 0 & \sigma^2_{M'} - \sigma^2_M \end{pmatrix} \preceq \Sigma_{(S',M')}$. Equivalently, $0 \preceq \begin{pmatrix} \sigma^2_{S'} - \sigma^2_S & \sigma^2_{S'M'} - \sigma^2_{SM} \\ \sigma^2_{S'M'} - \sigma^2_{SM} & 0 \end{pmatrix}$ which holds if and only if $\sigma^2_{S'} \geq \sigma^2_S$ (which is ensured by Q' is more informative than Q) and $\sigma^2_{S'M'} = \sigma^2_{SM}$, which is the desired result.

■

The condition in Remark 2 is strong. Typically, acquisition will improve information about match quality, and it will not be possible to neutralize the effects of more private information simply by passing it on. This difficulty arises because of the parameter $\sigma^2_{\mathbb{E}[M|T]} = \sigma^2_M - \sigma^2_{M|T}$. To retain this at its previous level when $\Sigma_{(S,B)}$ increases to $\Sigma_{(S',B')}$, $\sigma^2_{B|T}$ must also increase by the same amount.

Information Acquisition Increases Efficiency If it were possible to neutralize private information via disclosure, then establishing that information acquisition (weakly) increases efficiency would be trivial. Notwithstanding Remark 2, we can use the results of Section 4 via Corollary 1 to establish that private information acquisition can expand the adverse selection-efficiency frontier.

Proposition 8 *With free disclosure of information, information acquisition increases efficiency. Specifically, $V\left(\Sigma_{(S,B)}, \overline{AS}\right)$ is increasing in $\Sigma_{(S,B)}$ in positive semidefinite order in the region $\beta_{BS} \leq \frac{\pi}{4}$.*

Proof. As in the proof of Proposition 7, for fixed $\sigma_{S|T}$, the constraint $\Sigma_{(S,B)|T} \preceq \Sigma_{(S,B)}$ represents, in $(\sigma_{\mathbb{E}[S|T]}, \beta_{BS.T})$ space as a cone satisfying the conditions of Corollary 1. The corresponding region for the constraint $0 \preceq \Sigma_{(S,B)|T}$ is a convex elliptical region containing the vertex of the cone. Improving the information from $\Sigma_{(S,B)}$ to $\Sigma_{(S',B')} \succeq \Sigma_{(S,B)}$ enlarges the elliptical region by set inclusion, but shifts the cone so that neither of the two constraint sets generally contains the other. The general situation is as illustrated in Figure 2 in which the cone with vertex at $(0, \beta)$ represents the constraint set with less private information and the one with vertex at $(0, \beta')$ represents the constraint set with better private information. The two constraint sets have been drawn to satisfy the conclusion of Proposition 8 which asserts that the set of $\beta_{BS.T}$ achievable with less private information is also achievable with more private information. By Corollary 1 from any point in the constraint with vertex at $(0, \beta)$, there must be a path along which both $\mathbb{E}[EC]$ and $\mathbb{E}[AS]$ are increasing which eventually passes into the constraint set with vertex at $(0, \beta')$. (Note, one can always construct a new cone which is a subset of the one with vertex at $(0, \beta)$—the construction is shown for the path initiating on the elliptical part of the constraint.) This establishes the conclusion. ■

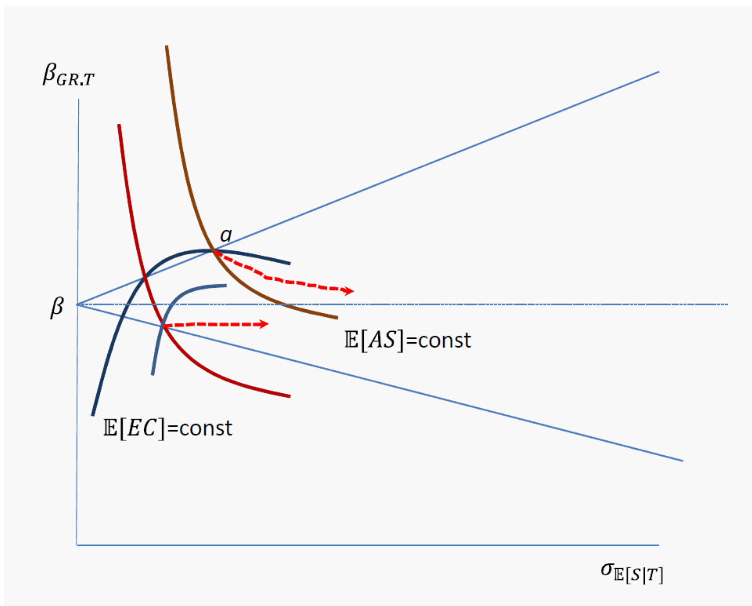

Illustration for Proof of Proposition 8.

Proposition 8 establishes that, if β_{BS} is not too large, private information acquisition expands the adverse selection-efficiency frontier. As with Proposition 7 the gist of the proof follows easily given Corollary 1. The constraints $\Sigma_{(S,B)|T} \preceq \Sigma_{(S,B)}$ and $0 \preceq \Sigma_{(S,B)|T}$ boil down respectively to a cone satisfying the conditions of Corollary 1 and a convex elliptical set containing the vertex of the cone. Improving private information for fixed T_{γ_I}, expands the elliptical set in the sense of set inclusion, but generally shifts the vertex of the conical set. So the intersection of the two constraint sets are not generally ordered by set inclusion as private information improves (Remark 2 already implies this). Notwithstanding this, it is very easy to show that the set of attainable $\beta_{BS.T}$ under better private information contains the set of attainable $\beta_{BS.T}$ under worse private information. Given this fact, Corollary 1 is easily seen to imply that from any point in the intersection of constraint sets with worse private information, there is a path along which which $\mathbb{E}[AS]$ and $\mathbb{E}[EC]$ are both increasing and which passes into the intersection of the constraint sets for better private information. Hence, the conclusion of the proposition is established.

5 Discussion and Applications

5.1 Empirical Consequences

Comparing prices of traded and non-traded goods In the classic Akerlof (1970) with only a single (null) realization of public information, the price of traded and non-traded goods is identical. However, in aggregating across realizations of public information, since both the propensity for different public realizations and the probability of trade associated with each realization vary, this is a meaningful comparison. The following result provides a characterization. We discuss a labor market implication in Section 5.2 below.

Proposition 9 *Suppose the conditions of Proposition 2 hold.*

1. *If $E[B|T]$ and $E[M|T]$ are negatively quadrant dependent (NQD),*[11] *then the average price of trade goods is higher than the average price of non-traded goods; that is, $E[P|S < P] > E[P|S \geq P]$.*

2. *If (T, S, B) is affiliated and $t \mapsto E[M|T = t]$ is increasing, then $E[B|T]$ and $E[M|T]$ are positively quadrant dependent (PQD),*[12] *moreover, the average price of traded goods is lower than the average price of non-traded goods.*

Proof. *1.* From the price equation (3), $\mathbb{E}[RP] = \mathbb{E}[\mathbb{E}[B|T]\, R] - \mathbb{E}[(AS)R]$. If $\mathbb{E}[B|T]$ and $\mathbb{E}[M|T]$ are NQD, then since R is a decreasing function of $\mathbb{E}[M|T]$, $\mathbb{E}[B|T]$ and R are PQD and, consequently, have a positive covariance. Therefore, $\mathbb{E}[\mathbb{E}[B|T]\, R] \geq \mathbb{E}[B|T]\, \mathbb{E}[R]$.

[11] A sufficient condition for NQD is $\sigma_{BM} > \sigma_{BM|T}$; i.e. if disclosure of public information decreases the correlation between B and M. Appendix C briefly reviews statistical dependence concepts.

[12] A sufficient condition is that $\beta_{BT.S} \geq 0$, $\beta_{BT} \geq 0$, $\beta_{MT} \geq 0$ and T is affiliated.

Since Proposition 11 below asserts that $\mathbb{E}[(AS)R] < \mathbb{E}[AS]\mathbb{E}[R]$, it follows that $\mathbb{E}[RP] > \mathbb{E}[P]\mathbb{E}[R]$, i.e. the average wage of released workers is higher than the average wage, as required.

2. Suppose (T, S, B) is affiliated. Theorem 5 of Milgrom and Weber (1981) implies that the willingness to pay map Ψ of Definition 3 maps increasing functions into increasing functions. Hence, the wage schedule, which is the unique fixed point of this map, is an increasing function. If $t \mapsto \mathbb{E}[M|T = t]$ is increasing, then by Proposition 3, $t \mapsto r(t)$ is decreasing. Hence, since T is affiliated, the covariance inequality $\mathbb{E}[p(T)r(T)] \leq \mathbb{E}[p(T)]\mathbb{E}[r(T)]$ holds. Since $\mathbb{E}[r(T)] = \Pr[S < P]$, Bayes law gives $\mathbb{E}[P] \geq \mathbb{E}[P|S < P]$ as required. ■

The proposition uses Lehmann's (1966) notions of positive and negative quadrant dependence for pairs of random variables (PQD and NQD respectively). If a pair of random variables are PQD, they are also positively correlated. For the readers convenience, Appendix C gives a brief summary. While our framework allows for multidimensional public and private information, this result makes clear the importance of the correlation or otherwise of just two scalar statistics, $\mathbb{E}[B|T]$ and $\mathbb{E}[M|T]$. The basic intuition follows from our result that the probability of trade, R, is high when $\mathbb{E}[M|T]$ is low. If $\mathbb{E}[B|T]$ and $\mathbb{E}[M|T]$ are 'negatively correlated', then it is intuitive that the workers more likely to leave will be those more highly valued by outside firms. The proposition makes this intuition precise.

Finding adverse selection when public information is not observed by the econometrician In our unit-demand model, the positive correlation test for adverse selection (Chiappori and Salanie 2000) becomes a simple test of whether adverse selection is positive. As was stressed by Chiappori and Salanie, since theory implies different observable types should be offered different contracts, one should control for public information and hence test whether $AS = \mathbb{E}[B|T] - \mathbb{E}[B|T, S < P] > 0$. However, if there is unobserved heterogeneity, i.e. T is poorly observed by the econometrician, this control is difficult. A natural surrogate is to estimate the quantity of adverse selection in the aggregated market constructed by ignoring T, specifically $\mathbb{E}[B] - \mathbb{E}[B|S < P]$. Our next result speaks to what may be learned from this quantity.

Since $\mathbb{E}[AS] = \mathbb{E}[B] - \mathbb{E}[P]$, the difference between the quantity of adverse selection in the aggregated market and the average quantity of adverse selection can be written as

$$\mathbb{E}[B] - \mathbb{E}[B|S < P] - \mathbb{E}[AS] = \mathbb{E}[P] - \mathbb{E}[P|S < P]. \tag{12}$$

It follows that the quantity of adverse selection in the aggregated market will overestimate (underestimate) the average quantity of adverse selection in cases where the average price of traded goods is higher (lower) than of non-traded goods. Proposition 9 therefore has the immediate corollary that the aggregated market will overestimate (underestimate) the average quantity of adverse selection according to whether Condition 1. or 2. of Proposition 9 obtain. The following proposition strengthens the bound obtained in the first part of this corollary and provides conditions for a weaker bound in the second part.

Proposition 10 *Suppose the conditions of Proposition 2 hold. If $E[B|T]$ and $E[M|T]$ are NQD then*

$$\mathbb{E}[B] - \mathbb{E}[B|S < P] \leq \mathbb{E}[AS|S < P].$$

If $E[B|T]$ and $E[M|T]$ are PQD the direction of inequality is reversed.

Proof. By the definition of adverse selection and the wage equation $\mathbb{E}[AS|S < P] = \mathbb{E}[\mathbb{E}[B|T] - B|S < P]$. Hence, by straightforward manipulations $\mathbb{E}[B] - \mathbb{E}[B|S < P] = \mathbb{E}[AS|S < P] + \mathbb{E}[\mathbb{E}[B] - \mathbb{E}[B|T]|S < P]$. The result now follows from noting that $\mathbb{E}[\mathbb{E}[B] - \mathbb{E}[B|T]|S < P] = \mathbb{E}[(\mathbb{E}[B] - \mathbb{E}[B|T])\, R]/\mathbb{E}[R]$. Hence, $\mathbb{E}[B] - \mathbb{E}[B|S < P] = \mathbb{E}[AS|S < P] - \mathrm{Cov}\left(\mathbb{E}[B|T,], \frac{R}{\mathbb{E}[R]}\right)$. Since $\mathrm{Cov}\left(\mathbb{E}[B|T,], \frac{R}{\mathbb{E}[R]}\right)$ is positive if $\mathbb{E}[B|T]$ and $E[M|T]$ are NQD and negative if PQD, the result is established. ■

This result raises the possibility of an effect reversal (Yule 1903; Simpson 1953) in which $\mathbb{E}[B] - \mathbb{E}[B|S < P] \leq 0$ notwithstanding the fact that $AS > 0$. The intuition for this effect reversal is clearly similar to that of Proposition 9. The probability of trade R is high when $\mathbb{E}[M|T]$ is low. So if $\mathbb{E}[B|T]$ is high when $\mathbb{E}[M|T]$ is low, the value of traded goods might be higher than the value of the population not trading.

Adverse selection falls mainly on trades that do not take place As noted above, in the empirical investigation of markets (potentially) impacted by adverse selection data may well be incomplete. For example, prices of used cars are generally only available for those used cars which are actually traded.

Proposition 11 *Suppose the conditions of Proposition 2 hold. The average quantity of adverse selection conditional on trade taking place is an underestimate of the average quantity of adverse selection. Hence,*

$$0 \leq \mathbb{E}[AS|S < P] \leq \mathbb{E}[AS].$$

Proof. Since AS and R are respectively increasing and decreasing functions of $\mathbb{E}[M|T]$, they are negatively correlated, it follows that $\mathbb{E}[AS|S < P] = \frac{\mathbb{E}[(AS)R]}{\mathbb{E}[R]} \leq \mathbb{E}[AS]$. ∎

This proposition establishes that adverse selection falls mainly on trades that do not take place. Hendren (2013) makes a similar point. He argues that severity of asymmetric information explains insurance rejections in non-group health insurance markets. Hence, failure of the positive correlation test for those who obtain (some) insurance does not necessarily mean that there is no adverse selection in the market overall. Hendren's model retains the scalar type assumption and adverse selection is not always 'interior' (in the language of the current paper, for some realization of T the probability of trade is zero, hence insurance rejections). Although the models are different, both point to the obvious importance of taking into account trades which do not occur when estimating the amount of adverse selection in the economy and its impact on market efficiency.

Proposition 11 means that adverse selection concentrates disproportionately where trades do not take place, and we remarked that depending on data availability this might impede its detection.

5.2 Labour Market Application

In the labor market interpretation pursued at length in Bar-Isaac, Jewitt and Leaver (2015), the seller can be seen as a current employer, and buyers as rival employers who might try to raid an employee, along the lines of Waldman (1984) and Greenwald (1986). Thus p would be understood as a wage, and the quantity of adverse selection and its effect on wages is of direct interest in speaking to the distribution of surplus between employer and worker. In this interpretation, B can be understood as general human capital, S the productivity of the worker at the current employer, and $M = S - B$, the match-specific component of worker productivity at the current employer.

Our results highlight that the information structure associated with a particular employer determines a worker's outside opportunities and their wages.[13] Indeed, to the extent that firms can commit to information structures through rules, review procedures and organizational processes, firms would compete to attract workers through offering information structures in addition to wages. Creating adverse selection (and thereby depressing wages) is a means of transferring rents from worker to firm; however, it is clear that if attracting

[13] The empirical importance of such effects has been noted in the recent work of Pallais (2014), for example, who contrasts wage prospects for workers under two different information structures.

workers is a concern for a firm then it would seek to claw back rents in the most efficient way possible. In particular, for a firm with such concerns the analysis in Section 4.0.2 is pertinent and the firm would adopt a disclosure policy characterized through Proposition 7.

In the interesting case where productivity at the current and rival firms is not colinear, then by Proposition 7, B and M have a positive conditional correlation given T.[14] It follows from the law of total probability (law of total covariance) that $\mathrm{Cov}\left(\mathbb{E}\left[B|T\right], \mathbb{E}\left[M|T\right]\right) < \mathrm{Cov}(B, M)$. Hence, if $\mathrm{Cov}(B, M) \leq 0$, as will be reasonable to assume in certain applications—e.g. it is true in the specification of the sampling example when private information is very good (in the limit of large N_B, N_S)—then $\mathrm{Cov}(\mathbb{E}\left[B|T\right], \mathbb{E}\left[S|T\right]) < 0$. Since negatively correlated Gaussian random variables are NQD (Lehmann 1996, p.1139), we may apply Proposition 9.

Corollary 2 *Suppose the conditions of Proposition 7 hold and* $\mathrm{Cov}(B, M) \leq 0$. *If adverse selection is induced efficiently, then on average wages of those that stay with the employer are lower than average wages of those who leave.*

Corollary 2 is intuitive. The efficiency contribution $\mathbb{E}[EC]$ determines the amount of surplus to be distributed between the employer and the worker. Since the quantity of adverse selection $\mathbb{E}[AS]$ determines the expected employment wage of the worker, it also determines the distribution of this surplus between the training firm and worker. Firm profits are maximized by inducing adverse selection efficiently. Applying Proposition 7, the way to do this is to impose adverse selection on those workers that it is efficient to keep, and allow higher wages for those workers who are more productively employed elsewhere.

In the sampling example, if the employer has maximal private information, then $\mathrm{Cov}(B, M) = 0$. Thus, applying Corollary 2, if the employer has the market power to depress wages (for example, through providing good training opportunities), it *will* be a low wage firm. Moreover, this implies following Proposition 10 that adverse selection in the aggregated market will underestimate expected adverse selection. Effect reversals are therefore a possibility, implying that the standard test (Gibbons and Katz 1991) is no longer a necessary condition for adverse selection.

[14]To see this note that, using $S = B + M$, we can rewrite the expression in the Proposition as $G = \frac{\beta_{BS.T}}{1-\beta_{BS.T}} S + \alpha \cdot T$, where to satisfy regression to the mean, $\beta_{BS.T} \in (0, 1)$.

Specifically, Gibbons and Katz (1991) test for adverse selection by comparing wages for workers who are selectively 'laid off' and those who become unemployed due to exogenous plant closures. With plant closures, there is no stigma so wages should on average be $\mathbb{E}[B]$; however, for workers who are laid off, expected wages are $\mathbb{E}[P|S < P] = \mathbb{E}[B|S < P]$. Hence, (absent the econometrician observing T) the Gibbons and Katz test reduces to testing whether there is adverse selection in the aggregated market: $\mathbb{E}[B]-\mathbb{E}[B|S < P] > 0$. Our analysis suggests that this test may be problematic if the current employer can freely commit to information from a broad class and is sufficiently attractive to workers that it has scope to claw back worker rents through imposing adverse selection efficiently. A necessary condition for adverse selection for the Gibbons and Katz (1991) test to find adverse selection is if $\mathbb{E}[B|T]$ and $\mathbb{E}[M|T]$ are PQD—in the Gaussian case, this would correspond to $\text{Cov}(\mathbb{E}[B|T], \mathbb{E}[M|T]) \geq 0$. Evidently, an over-sufficient condition for this is if M and T are orthogonal; that is, if public information does not speak to the match between worker and current employer.

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A Material Omitted from Section 2

Lemma A1 Suppose $z \leq z'$, then $E[\not{B}|z < \not{S}] \leq E[\not{B}|z' < \not{S}]$.

Proof. By the law of iterated expectations

$$\mathbb{E}[\not{B}|z \leq \not{S}] = \mathbb{E}\left[\mathbb{E}\left[B|T\right] - \mathbb{E}[B|T,S] \mid z \leq \not{S}\right].$$

Hence, by the law of iterated expectations again,

$$\mathbb{E}[\not{B}|z \leq \not{S}] = \int \mathbb{E}[\mathbb{E}\left[B|T=t\right] - \mathbb{E}[B|T=t,S] \mid T=t, z \leq \not{S}]dF_{T|z\leq\not{S}}(t).$$

Where, $F_{T|z\leq\not{S}}$ is the conditional cdf of T given the event $z \leq \not{S}$. Since $\not{S} \perp T$, $F_{T|z\leq\not{S}} = F_T$, the unconditional cdf. Hence,

$$\mathbb{E}[\not{B}|z \leq \not{S}] = \int \mathbb{E}[\mathbb{E}\left[B|T=t\right] - \mathbb{E}[B|T=t,S] \mid T=t, z \leq \mathbb{E}\left[S|T=t\right] - S]dF_T(t).$$

The result now follows from the fact that, given $0 < \beta_{BS.T} < 1$, for each fixed $t \in \boldsymbol{T}$ the integrand is increasing in z. ■

Proposition 2 In the equilibrium identified in Proposition 1, adverse selection, AS, and the apparent match quality, $\mathbb{E}[M|T]$, are comonotone random variables. Specifically, there exists an increasing function $\widehat{as} : \mathbb{R} \to \mathbb{R}_+$ such that $AS = as(T) = \widehat{as}(\mathbb{E}[M|T])$.

Proof. (continued) It remains to confirm positivity and monotonicity. Lemma 1 implies that adverse selection at $T = t$, $as(t)$, is uniquely defined as the limit of the convergent sequence $a_0, a_1, a_2, \ldots$ with $a_{i+1} = \mathbb{E}[\not{B}|a_i + \mathbb{E}[M|T=t] < \not{S}]$. $as(t')$ is defined by a similarly defined convergent sequence $a_0, a_1', a_2' \ldots$. Suppose that $\mathbb{E}[M|T = t'] > \mathbb{E}[M|T = t]$. It follows immediately from the positive dependence of $(\not{B}, \not{S})$ (Lemma A1) that $a_1' = \mathbb{E}[\not{B}|a_0 + \mathbb{E}[M|T = t'] < \not{S}] \geq a_1$, hence $a_2' \geq a_2'$. Similarly $a_i' \geq a_i$ for all i, therefore $as(t') \geq as(t)$. Positivity follows directly from Lemma B1. ■

B Material Omitted from Section 3.2

Properties of the normal hazard function

It is convenient to collect here some mainly familiar properties of the normal hazard function (inverse Mills ratio).

Lemma B1 The normal hazard function $h : \mathbb{R} \to \mathbb{R}_+$, satisfies:

1. Level bounds. $[x]^+ < h(x) < \frac{\sqrt{4+x^2}-x}{2} < \left|x + \frac{1}{x}\right|$ on R. This implies $lim_{x\to\infty}\frac{h(x)}{x} = lim_{x\to\infty}h'(x) = 1$.

2. Gradient bounds. $x \mapsto h(x)$ and $x \mapsto x - h(x)$ are both strictly increasing on $\mathbb{R}$. $0 < h' < 1$ on $\mathbb{R}$.

3. Convexity bounds. h is strictly convex and log concave on $\mathbb{R}$. $0 < \frac{d}{dx}\frac{h(x)}{h'(x)} < 1$.

4. Starshaped. $x \mapsto h(x)$ and $x \mapsto \frac{h(x)}{h'(x)}$ are both strictly starshaped on $\mathbb{R}$. We call a function $g : \mathbb{R} \to \mathbb{R}$ with $g(0) > 0$ strictly starshaped if for all $x \in \mathbb{R}$, $0 < \alpha < 1$, $\alpha g(x) < g(\alpha x)$.

Proof. *Property 1.* The lower bound and larger upper bound are in Gordon (1941). The tighter upper bound was established by Birnbaum (1942).

Property 2. One easily verifies $h'(x) = h(x)(h(x) - x)$ which, given Property 1, is positive. Since convexity implies the derivative is increasing, the final statement of 1. implies $h' < 1$.

Property 3. Convexity was conjectured by Birnbaum (1950), and proved by Sampford (1953). Logconcavity is established in Marshall and Olkin (2007) Lemma B.4 p.437. Marshall and Olkin establish the result using an inequality of Birnbaum. A simpler proof follows from using $h'(x) = h(x)(h(x)-x)$ and using Property 2. That h/h' is increasing follows immediately from the logconcavity of h. That $\frac{d}{dx}\frac{h(x)}{h'(x)} < 1$, follows from the convexity of h follows immediately from observing $\frac{d}{dx}\frac{h(x)}{h'(x)} = 1 - \frac{h(x)h''(x)}{(h'(x))^2}$.

Property 4. A continuous positive function is starshaped on $\mathbb{R}$ if it is starshaped on $(0, \infty)$. h is starshaped if $\frac{h'(x)}{h(x)} \leq \frac{1}{x}$ on $(0, \infty)$. One easily verifies $h'(x) = h(x)(h(x) - x)$. Hence, it suffices to show that $(h(x) - x)x \leq 1$, the result therefore follows from 1. To establish $x \mapsto \frac{h(x)}{h'(x)}$ is starshaped on $(0, \infty)$, we require that $h/h'x$ is decreasing on $x > 0$, this follows from h being starshaped (h/x is decreasing) and by the convexity of h, $\frac{1}{h'}$ is decreasing. Hence, the result follows from the fact that the product of two decreasing positive functions is decreasing. ■

C Material Omitted from Section 5: Brief Notes on Statistical Dependence Concepts

The pair of scalar random variables (X_1, X_2) are said to be *positively quadrant dependent* (PQD) Lehmann (1966) if for all x_1, x_2 in the support of (X_1, X_2)

$$\Pr[X_1 \leq x_1, X_2 \leq x_2] \geq \Pr[X_1 \leq x_1] \Pr[X_2 \leq x_2].$$

If the inequality is reversed (X_1, X_2) is said to be *negatively quadrant dependant* (NQD). If (X, Y) are PQD, then for any non-decreasing functions φ_1, φ_2 $(\varphi_1(X_1), \varphi_2(X_2))$ is PQD. (Lehmann 1966, Lemma 1). Lehmann Lemma 3 establishes if (X_1, X_2) are PQD, for all non-decreasing functions φ_1, φ_2 such that the expectations exist

$$\mathbb{E}[\varphi_1(X_1)\varphi_2(X_2)] \geq \mathbb{E}[\varphi_1(X_1)]\mathbb{E}[\varphi_2(X_2)],$$

$\varphi_1(X_1)$ and $\varphi_2(X_2)$ have a positive covariance. This condition is an equivalence so may serve as an alternative definition of PQD.

Applying Bayes rule, PQD evidently implies $\Pr[X_1 \leq x_1 | X_2 \leq x_2] \geq \Pr[X_1 \leq x_1]$ which in turn implies, if the expectations exist, $\mathbb{E}[X_1 | X_2 \leq x_2] \leq \mathbb{E}[X_1]$. Kowalczyk and Pleszcynska (1977) say (X_1, X_2) is EDQ^+ if for all x_2 in the support of X_2,

$$\mathbb{E}[X_1 | X_2 \leq x_2] \leq \mathbb{E}[X_1].$$

If the inequality is reversed, then Kowalczyk and Pleszcynska (1977) say (X_1, X_2) are EDQ^-.

Let the vector of random variables $X = (X_1, ..., X_n)$ have a density with respect to Lebesgue measure f, X is said to be affiliated if $f(x \vee x')f(x \wedge x') \geq f(x)f(x')$ a.e. on $\mathbb{R}^{2n}$. Equivalently, for each pair of nondecreasing functions and sublattice Z $\mathbb{E}[\varphi_1(X)\varphi_2(X)|Z] \geq \mathbb{E}[\varphi_1(X)|S]\mathbb{E}[\varphi_2(X)|Z]$.

It is known that: (X_1, X_2) affiliated $\Rightarrow$ (X_1, X_2) PQD $\Rightarrow$ (X_1, X_2) EDQ^+.