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Downside Risk Efficiency Under Market Distress

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Abstract

In moments of financial distress downside risk measures like lower partial moments are more appropriate than the standard variance to characterize risk. The goal of this paper is to study how to choose optimal portfolios in these periods. In order to do this we extend the definition of lower partial moments to this environment, derive the corresponding mean-risk dominance set and define the concept of stochastic dominance under distress. The paper shows the close connection between the mean-risk dominance set and the stochastic dominance frontier in these situations. The advantage of using stochastic dominance is that we can readily compare investors' preferences over investment portfolios in a meaningful way regardless their degree of risk aversion. We do this by proposing a hypothesis test. Our novel family of test statistics for testing stochastic dominance under distress makes allowance for testing orders of dominance higher than one, for general forms of dependence between portfolios and can be extended to residuals of regression models. These results are illustrated in an empirical application for data from US stocks. We show that mean- variance strategies are stochastically dominated by mean-risk efficient portfolios in episodes of financial distress.

JEL classification: C1, C2, G1.

Keywords: Downside risk, Lower partial moments, Market distress, Mean-risk models, Mean-variance models, Stochastic dominance

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1 Introduction

It was Markowitz (1952) who formalized the concept of portfolio diversification by showing that investors should choose assets as if they care only about the mean and variance of the returns on an investment portfolio and therefore should penalize equally departures from expected wealth in both sides. Alternatively, Roy (1952) developed the concept of safety-first portfolios where investors' aim consisted on minimizing the likelihood of a dread event, this identified with an outcome in the tail of the distribution of portfolio returns. Roy, as Markowitz, also confined himself to distributions defined by the first two statistical moments. Following this alternative interpretation of risk Markowitz (1959) proposed the semivariance, risk measure that only focused on deviations of the return on the portfolio below a target return determined by the expected return on the investment or the return on the risk-free asset.

Hogan and Warren (1974), Bawa (1975), Arzac and Bawa (1977), and Bawa and Lindenberg (1977) continued on the idea of risk based on dread events and proposed different risk measures based on penalizing the chance of these events. Thus, building on Roy's (1952) formulation of risk and extending the semivariance of Markowitz (1959) these authors introduced lower partial moments (LPM) of the distribution of returns to describe risk. The corresponding counterpart of the mean-variance dominance set in this downside risk framework is the mean-risk dominance set, see Stone (1973), Porter (1974) and Fishburn (1977). The optimal portfolio choice for these investors is the solution of the following optimization problem,

$$\min_{\{w\}} LPM_q^{R^P}(w,\tau), \tag{1}$$

with $LPM_q^{R^P}(w,\tau) = \int_{-\infty}^{\tau}(\tau - R^P)^q dF(R^P)$, being the risk measure of a portfolio P with distribution function F and its returns defined by $R^P = \sum_{j=1}^{m} w_j R_j$, in which m is the number of assets, $w = (w_1, \ldots, w_m)$ is the vector of weights of each asset and $\sum_{j=1}^{m} w_j = 1$. For sake of notation we will use hereafter $LPM_q^P(\tau)$ to denote $LPM_q^{R^P}(w,\tau)$. This minimization problem is subject to the constraint $\sum_{j=1}^{m} w_j \mu(j) \leq \mu^*(P)$, with $\mu(j) = E[R_j]$ and $\mu^*(P)$ some target return level.

Bawa ((1975), (1976), (1978)) provided a microeconomic foundation for these risk measures by introducing a family of utility functions consistent with them that described the preferences of downside risk averse investors. These functions take the form

$$U(R^P; q, \tau) = R^P - k(\tau - R^P)^q I(R^P \leq \tau), \tag{2}$$

where R^P is the return on a portfolio P; τ is the threshold denoting the target return; k a scale parameter, $I(\cdot)$ an indicator function that takes the value one if $R^P \leq \tau$ and zero otherwise, and q the degree of risk aversion of the investor. Further, Bawa and Lindenberg (1977) and Harlow and Rao (1989) showed that the optimal portfolio choice was obtained from maximizing the expected value of (2) subject to the constraints discussed above. A review and recent development of downside risk management can be found in Sortino and Satchell (2001).

The difference between the mean-risk dominance approach and the expected utility framework is not without importance, thus, whereas the former method relies on the minimization by the investor of an objective risk function the latter is based on strong assumptions on individuals' preferences given by the axioms of the von Neumann-Morgenstern class of utility functions.

Fishburn (1977) in his seminal paper studies the optimal portfolio choice under each paradigm and shows the close relationship between the different concepts. Thus, he discusses separately the relation between mean-risk dominance models and the concept of stochastic dominance, and between stochastic dominance and the maximization of expected utility by rational individuals. This author finds that the efficient sets obtained from minimizing LPM_q measures are a subset of the different efficient first, second and third stochastic dominance sets. An excellent monograph on the theory of stochastic dominance is Levy (2006). Results involving hypothesis tests for stochastic dominance between investment portfolios are found in Post (2003), Post and Versijp (2004), Linton, Maasoumi, and Whang (2005) or more recently in Kopa and Post (2008), Kopa and Chovanec (2008) and Scaillet and Topaloglou (2009). Related tests for the hypothesis in different contexts are found in McFadden (1989), Kaur, Rao and Singh (1994), Anderson (1996), Davidson and Duclos (2000) or Barrett and Donald (2003). Most of these studies, however, see for example Linton, Maasoumi, and Whang (2005), use stochastic dominance to compare portfolios because of its relation with the optimal portfolio choice problem under uncertainty given by the maximization of the von Neumann-Morgenstern class of utility functions. It is worth noting that this powerful statistical methodology to compare portfolios is only meaningful under restrictive assumptions implying, for example, that risk averse investors' choices can be described as the result of maximizing the expected value of some function that is nondecreasing and concave over its domain. Otherwise, the results from the tests for stochastic dominance are difficult to interpret. Alternatively, mean-risk dominance models solve this problem by defining a clear-cut objective function to minimize. Investors in this framework construct their preferences over risky portfolios by comparing the value of this function representing the amount of risk in the portfolio. In the specific case of downside risk averse investors this objective function is a risk measure penalizing weighted dispersions below a target return.

Furthermore, it is not clear that the preferences of investors remain constant under periods of financial distress, in which the likelihood of adverse events is bound to increase. In these cases one can construct a new utility function within the von Neumann-Morgenstern class to describe individuals' preferences or, alternatively, it can construct a different risk measure and derive optimal portfolios under the new environment. This paper takes the view that the latter approach is less restrictive and proposes methods based on lower partial moments to choose optimal portfolios in periods of financial distress. It also introduces the definition of stochastic dominance under distress and shows the close connection between the mean-risk dominance set and the stochastic dominance frontier under distress. The advantage of using stochastic dominance is that we can readily compare investors' preferences over investment portfolios in a meaningful way regardless their degree of risk aversion. In this way this paper complements the influential papers of Porter (1974) and Fishburn (1977).

The second contribution is to propose hypothesis tests for stochastic dominance of different orders between portfolios, that can be naturally extended to test for stochastic dominance under distress. We further derive a decomposition of the LPM risk measures that makes possible to obtain a simple and estimable form of the asymptotic distribution of the different test statistics for each family of hypothesis tests. Also, by a simple transformation of the test statistic our method allows to test the reverse stochastic dominance hypotheses using the same asymptotic critical values and therefore without any extra computational effort. Finally, as in Linton, Maasoumi and Whang (2005) and unlike Barrett and Donald (2003), we make allowance for dependence between portfolios when testing for the different hypotheses, and discuss briefly the extension to testing stochastic dominance for residuals of time series regression models. In this way our study on stochastic dominance tests complements and extends the pioneering works of Barrett and Donald (2003) and Linton, Maasoumi and Whang (2005) in three directions. First, the asymptotic distribution function of our test statistics for testing the relevant hypotheses have a parametric closed form that allows to approximate the critical value of the tests in finite samples without the need of bootstrap as in Barrett and Donald (2003) or subsampling methods as in Linton, Maasoumi and Whang (2005). Second, we use the concept of stochastic dominance in portfolio theory to test for efficiency among investment portfolios; and finally we extend stochastic dominance tests to stochastic dominance under distress episodes of the market.

The paper is structured as follows. Section 2 introduces the definition of stochastic dominance under distress and its relation with mean-risk efficiency. Section 3 introduces different estimators of the LPM risk measures, derives the relevant hypothesis tests for testing the different forms of stochastic dominance and the corresponding asymptotic theory. In Section 4 we carry out a Monte Carlo simulation experiment to study the finite sample performance of the proposed tests. Section 5 compares the mean-variance and mean-risk efficient portfolios via stochastic dominance for real data from the US equity market. Finally Section 6 concludes with the main findings of the paper. Proofs and tables are gathered in the appendix.

2 Mean-Risk and Stochastic Dominance Under Market Distress

This section studies the relation between the mean-risk dominance set determined by LPM risk measures and stochastic dominance between portfolios in situations of market distress in which all the assets available in the marketplace have returns below a threshold u. The section also introduces a decomposition of the LPM measures that permits to disentangle the downside risk exposure of a portfolio due to the likelihood of market distress from the risk exposure due to the investor's asset allocation. In periods of financial distress optimal portfolio allocations are not necessarily the optimal allocations obtained from calm periods. Before showing these results we need to impose the following assumptions.

Assumption A.1: The vector of weights characterizing portfolio P satisfies that $0 \leq w_j \leq 1$, for all j, and $\sum\limits_{j=1}^{m} w_j = 1$.

Assumption A.2: $\tau, u \in \Omega$, with Ω a compact set in $\mathbb{R}$.

Assumption A.3: The distribution functions $F(\tau) = P\left(R^P \leq \tau\right)$, $F_u(\tau) = P\left(R^P \leq \tau | R_1 \leq u, \ldots, R_m \leq u\right)$ and $\lambda(u) = P\left(R_1 \leq u, \ldots, R_m \leq u\right)$ are continuous and differentiable in the $\mathbb{R}$ and $\mathbb{R}^m$ domain, respectively.

Assumption A.4: $E[(R^P)^{2q}] < \infty$ for R^P the return on portfolio P, and $q \geq 0$.

Assumption A.1 ensures that investors can only take long positions in the assets comprising the portfolio and implies that $F_u(u) = 1$. This assumption is standard in the literature, see for instance Post (2003) and Scaillet and Topaloglou (2009). Assumption A.2-A.4 guarantee the existence of the different LPM measures determined by q and of their variance.

The efficient portfolio frontier in models in which risk is measured by probability weighted dispersions below a target τ is defined by those portfolios achieving certain objective return $\mu^*(P)$, and minimizing a risk measure of the type

$$\rho(F) = \int_{-\infty}^{\tau} \varphi(\tau - x) dF(x), \tag{3}$$

in which $\varphi(y)$ is a nonnegative nondecreasing function in y with $\varphi(0) = 0$. More specifically, for investors equipped with (q, τ) risk measures defined by $\varphi(\tau - x) = (\tau - x)^q$ those portfolios are the result of minimizing LPM_q measures under the constraints introduced in (1). Further, the preference relation between any two risky portfolios follows from this result:

Portfolio A dominates Portfolio B in the mean-risk model defined by the pair (q, τ) *if and only if* $\mu(A) \geq \mu(B)$ *and* $LPM_q^A(\tau) \leq LPM_q^B(\tau)$, *with at least one strict inequality.*

The following statement taken from Fishburn (1977) shows the close connection between the preference relations between two portfolios in the different mean-risk models determined by τ and stochastic dominance efficiency.

- *A first stochastic dominates (FSD) B if and only if* $F^A \neq F^B$ *and* $LPM_0^A(\tau) \leq LPM_0^B(\tau)$ *for all* τ.
- *A second stochastic dominates (SSD) B if and only if* $F^A \neq F^B$ *and* $LPM_1^A(\tau) \leq LPM_1^B(\tau)$ *for all* τ.
- *A third stochastic dominates (TSD) B if and only if* $F^A \neq F^B$, $\mu(A) \geq \mu(B)$, *and* $LPM_2^A(\tau) \leq LPM_2^B(\tau)$ *for all* τ,

with F^A and F^B the distribution functions of two portfolios A and B.

The proof of these statements follows as an immediate consequence of the definitions of stochastic dominance for different orders, see for example Fishburn (1977, p 122). This can be shown by integrating by parts the relevant LPM_q measures. For SSD for example, $LPM_1^A(\tau) = \int_{-\infty}^{\tau}(\tau - R^A)dF(R^A) = \int_{-\infty}^{\tau} F(R^A)dx$. Further, A SSD B implies by definition $\int_{-\infty}^{\tau} F(R^A)dx \leq \int_{-\infty}^{\tau} F(R^B)dx$ for all τ, that in turn is equivalent to $LPM_1^A(\tau) \leq LPM_1^B(\tau)$.

In the framework of von Neumann-Morgenstern axioms for the preferences of individuals it is worthwhile to mention the well known connection between stochastic dominance and utility functions. For example, if A FSD B then $\mu(A) > \mu(B)$ and $E[v^A] \geq E[v^B]$ for v a nondecreasing utility function, implying that A is preferred to B by risk neutral and risk loving individuals. Equally, if A SSD B then $\mu(A) \geq \mu(B)$ and $E[v^A] \geq E[v^B]$ for v nondecreasing and concave, and therefore portfolio A is preferred to portfolio B for risk averse investors, and so on for higher orders of stochastic dominance and risk aversion.

Unlike for the mean-variance efficient portfolio frontier each of the mean-risk sets is characterized by the risk aversion parameters q and τ, nevertheless, the concept of stochastic dominance permits to uncover mean-risk dominant portfolios independently of these parameters; the converse result is not true however. Further, our concept of portfolio (mean-risk and stochastic dominance) efficiency is taken from Porter (1974) and Fishburn (1977), and implies portfolios not dominated by other portfolios; in contrast to the definition of efficiency discussed by De Giorgi and Post (2008) that implies a portfolio that dominates the rest of possible portfolios. The following theorem formalizes these results.

Theorem 1 (Fishburn (1977, p. 123)):

- *If A FSD B then A dominates B in the mean-risk model defined by the pair (q, τ) for all $q \geq 0$.*
- *If A SSD B then A dominates B in the mean-risk model defined by the pair (q, τ) for all $q \geq 1$, except when $\mu(A) = \mu(B)$ and $LPM_q^A(\tau) = LPM_q^B(\tau)$.*
- *If A TSD B then A dominates B in the mean-risk model defined by the pair (q, τ) for all $q \geq 2$, except when $\mu(A) = \mu(B)$ and $LPM_q^A(\tau) = LPM_q^B(\tau)$.*

These results show that efficient portfolio sets corresponding to investors minimizing LPM_q measures defined by the pair (q, τ) are a subset of the FSD efficient set for $q \geq 0$; of the SSD efficient set for $q \geq 1$ and of the TSD efficient set for $q \geq 2$; except in the noted cases.

As discussed in the introduction the distinction between the mean-risk approach and the expected utility framework is not vacuous. As an illustrative example consider the case of an investor whose preferences can be described by a $LPM_q(\tau)$ measure, with $q \geq 1$, but such that once the return on the portfolio exceeds this value he prefers the riskier alternative against the corresponding certainty equivalent. This behavior can be described by a utility function that is concave up to τ (risk aversion) and becomes convex afterwards (risk loving). Note that whereas the connection between stochastic dominance of orders higher than one and the mean-risk model can produce a ranking of preferences, the ranking in terms of expected utility does not since the utility function is not concave in the whole domain of the distribution and therefore there is no equivalence between, for example, SSD and the ranking produced by the expected value of the utility function.

In what follows we extend the results on mean-risk dominance and stochastic dominance to periods of market distress characterized in this paper by values of the return on every asset in the market below a threshold u, see assumption A.3 for notation. The relevant mean-risk model is obtained from minimizing the following risk measure

$$LPM_{q,u}^{P}(\tau) = \int_{-\infty}^{\tau} (\tau - x)^{q} dF_{u}(x), \tag{4}$$

with $F_u(x)$ denoting the conditional distribution function of P on market distress, and subject to the constraints $\sum_{j=1}^{m} w_j \mu_u(j) \leq \mu^{*}(P)$, with $\mu_u(j) = E[R^j | R_1 \leq u, \ldots, R_m \leq u]$ the corresponding expected values conditional on market distress, and $\sum_{j=1}^{m} w_j = 1$. The corresponding mean-risk dominance set is now defined by the parameters (q, τ, u), with $\tau \leq u$.

The concept of stochastic dominance under market distress is the counterpart of stochastic dominance applied now to the conditioning set $\{R_1 \leq u, R_2 \leq u, \ldots, R_m \leq u\}$. In particular, we define conditional stochastic dominance (CSD) of first order by the condition $F_u^A(\tau) \leq F_u^B(\tau)$ for all $\tau \leq u$. CSD of second order by $\int_{-\infty}^{\tau} F_u^A(\tau) \leq \int_{-\infty}^{\tau} F_u^B(\tau)$ for all $\tau \leq u$, and so on. From these definitions we can characterize (CSD) of different orders from the corresponding conditional $LPM_{q,u}$ measures.

Proposition 1:

- *A first conditional stochastic dominates (FCSD) B if and only if $F_u^A \neq F_u^B$ and $LPM_{0,u}^A(\tau) \leq LPM_{0,u}^B(\tau)$ for all $\tau \leq u$.*
- *A second conditional stochastic dominates (SCSD) B if and only if $F_u^A \neq F_u^B$ and $LPM_{1,u}^A(\tau) \leq LPM_{1,u}^B(\tau)$ for all $\tau \leq u$.*
- *A third conditional stochastic dominates (TCSD) B if and only if $F_u^A \neq F_u^B$, $\mu_u(A) \geq \mu_u(B)$, and $LPM_{2,u}^A(\tau) \leq LPM_{2,u}^B(\tau)$ for all $\tau \leq u$,*

with F_u^A and F_u^B the relevant conditional distribution functions introduced before.

The proof of this result follows, as before, from integrating by parts the relevant $LPM_{q,u}(\tau)$ measure. Using the same arguments as in the unconditional case and imposing appropriate assumptions it can be shown that a similar result holds when the preferences of individuals can be expressed in the expected utility framework. The characterization introduced in this proposition extends naturally the relationship between mean-risk dominance sets and stochastic dominance to a conditional environment characterized by market distress.

Theorem 2:

- *If A FCSD B then A dominates B in the mean-risk model defined by the parameters* (q, τ, u) *for all* $q \geq 0$ *and given* $\tau \leq u$.
- *If A SCSD B then A dominates B in the mean-risk model defined by the parameters* (q, τ, u) *for all* $q \geq 1$ *and given* $\tau \leq u$*, except when* $\mu_u(A) = \mu_u(B)$ *and* $LPM^A_{q,u}(\tau) = LPM^B_{q,u}(\tau)$.
- *If A TCSD B then A dominates B in the mean-risk model defined by the parameters* (q, τ, u) *for all* $q \geq 2$ *and given* $\tau \leq u$*, except when* $\mu_u(A) = \mu_u(B)$ *and* $LPM^A_{q,u}(\tau) = LPM^B_{q,u}(\tau)$.

The proof of this result is straightforward and follows the same steps of theorem 1. For the first case, for example, consider a mean-risk model defined by the parameters (q, τ, u), with $\tau \leq u$. Proposition 1 implies that if $A\ FCSD\ B$ then $LPM^A_{0,u}(\tau) \leq LPM^B_{0,u}(\tau)$. The definition of $FCSD$ also implies that $\mu_u(A) \geq \mu_u(B)$, hence the mean-risk dominance of A over B. The extension to models with $q > 0$ follows from noting that $A\ FCSD\ B$ implies $A\ SCSD\ B$, $A\ SCSD\ B$ implies $A\ TCSD\ B$ and so on.

This result entails different optimal portfolio choices contingent on the state of the market, that is, this theorem provides the tools for disentangling individuals' state dependent preferences without the need to define state dependent utility functions and assuming axioms, *e.g.* extended independence axiom, that extend the expected utility representation.

In order to make the conditions for stochastic dominance in theorems 1 and 2 statistically testable we develop in the next section hypothesis tests for stochastic dominance and stochastic dominance under distress of different orders. Before this we introduce a very convenient decomposition of the risk measures in (1) and (4) that will enable us to derive the asymptotic distribution of the relevant test statistics and that, in contrast to existing literature, can be easily estimated for any order of q. Specifically, our decomposition improves Anderson (1996) that uses a trapezoidal approximation of the LPM-integrals, and Davidson and Duclos (2000) and Barrett and Donald (2003) that integrate directly the empirical processes.

Proposition 2: *Assume A.1-A.4 hold, and let* $LPM^P_q(\cdot)$ *and* $LPM^P_{q,u}(\cdot)$ *for* $q \geq 0$ *be the downside risk measures defined in* (1) *and* (4)*, respectively. Then*

$$LPM^P_q(\tau) = E[(\tau - R^P)^q | R^P \leq \tau] LPM^P_0(\tau), \tag{5}$$

and

$$LPM^P_{q,u}(\tau) = E[(\tau - R^P)^q | R^P \leq \tau, R_1 \leq u, \ldots, R_m \leq u] LPM^P_{0,u}(\tau). \tag{6}$$

This proposition can be used to derive another decomposition of the unconditional downside risk measure in terms of the downside risk components corresponding to distress and no distress periods, for any order q.

Corollary 1: *Let* LPM_q^P *for* $q \geq 0$ *be the downside risk measure defined in* (1). *Then*

$$LPM_q^P(\tau) = \lambda(u)\theta_{q,u}(\tau)LPM_{q,u}^P(\tau) + (1-\lambda(u))\theta_{q,u}^c(\tau)LPM_{q,u}^{c\,P}(\tau), \tag{7}$$

with $\theta_{q,u}(\tau) = \frac{E[(\tau-R^P)^q|R^P\leq\tau]}{E[(\tau-R^P)^q|R^P\leq\tau,R_1\leq u,R_2\leq u,\ldots,R_m\leq u]}$, $\theta_{q,u}^c(\tau) = \frac{E[(\tau-R^P)^q|R^P\leq\tau]}{E[(\tau-R^P)^q|R^P\leq\tau,R_1>u \text{ or } R_2>u \text{ or}\ldots\text{or } R_m>u]}$ *and* $LPM_{q,u}^{c\,P}(\tau) = E[(\tau - R^P)^q | R^P \leq \tau, R_1 > u \text{ or } R_2 > u \text{ or}\ldots\text{or } R_m > u]\, P(R^P \leq \tau | R_1 > u \text{ or } R_2 > u \text{ or}\ldots\text{or } R_m > u)$.

This decomposition allow us to disentangle the risk exposure of the portfolio due to the probability $\lambda(u)$ of market distress from the risk exposure produced by the allocation of weights in each market regime. This corollary also shows that optimal portfolio choices are not necessarily optimal choices under distress and the other way around. These choices depend indirectly on the threshold u and the values taken by the assets comprising the investment portfolio.

3 Estimation and Inference

Suppose we have n independent and identically distributed ($i.i.d$) vectors of observations obtained from m different random variables $R_1,\ldots,R_m$. Then, natural estimators of $LPM_0(\tau)$ and $LPM_{0,u}(\tau)$, for τ nonstochastic are

$$\widehat{LPM}_0(\tau) = \frac{1}{n}\sum_{i=1}^{n} I(R_i^P \leq \tau), \tag{8}$$

and

$$\widehat{LPM}_{0,u}(\tau) = \frac{1}{n_u}\sum_{i=1}^{n} I(R_i^P \leq \tau, R_{1,i} \leq u, R_{2,i} \leq u, \ldots, R_{m,i} \leq u), \tag{9}$$

with n_u the number of vectors satisfying $R_1 \leq u, R_2 \leq u, \ldots, R_m \leq u$. The empirical counterpart of $\lambda(u)$ is

$$\widehat{\lambda}(u) = \frac{1}{n}\sum_{i=1}^{n} I(R_{1,i} \leq u, R_{2,i} \leq u, \ldots, R_{m,i} \leq u). \tag{10}$$

For τ sufficiently close to the left end point of the distribution one can use parametric distributions, as the Generalized Pareto or Pareto distributions obtained from extreme value theory to estimate the relevant lower partial moments, see Pickands (1975) or Embrechts, Klüppelberg, and Mikosch (1997) for a thorough review of extreme value theory. These parametric distributions, however, are good approximations to the correct underlying distribution only for very large sample sizes, the specific nature of these depending on the form of the distribution $F(\cdot)$ and the corresponding normalizing sequences. Therefore, and given our interest also in more moderate τs, we will not use this semiparametric approach further in the paper.

The different expected values necessary to compute LPM_q measures of higher orders are estimated by their corresponding empirical counterparts

$$\widehat{E}[(\tau - R^P)^q | R^P \leq \tau] = \frac{1}{n_p} \sum_{i=1}^{n} (\tau - R_i^P)^q I(R_i^P \leq \tau), \tag{11}$$

and

$$\begin{aligned} &\widehat{E}[(\tau - R^P)^q | R^P \leq \tau, R_1 \leq u, R_2 \leq u, \ldots, R_m \leq u] = \\ &= \frac{1}{n'_p} \sum_{i=1}^{n} (\tau - R_i^P)^q I(R_i^P \leq \tau, R_{1,i} \leq u, \ldots, R_{m,i} \leq u), \end{aligned} \tag{12}$$

with n_p the number of observations in the sample satisfying $R^P \leq \tau$ and n'_p the number of observations satisfying $R^P \leq \tau$ and $R_1 \leq u, R_2 \leq u, \ldots, R_m \leq u$. Therefore, the relevant downside risk estimators are

$$\widehat{LPM}_q(\tau) = \frac{1}{n} \sum_{i=1}^{n} (\tau - R_i^P)^q I(R_i^P \leq \tau), \tag{13}$$

and

$$\widehat{LPM}_{q,u}(\tau) = \frac{1}{n_u} \sum_{i=1}^{n} (\tau - R_i^P)^q I(R_i^P \leq \tau, R_{1,i} \leq u, R_{2,i} \leq u, \ldots, R_{m,i} \leq u). \tag{14}$$

By a strong law of large numbers in the univariate and multivariate setting these estimators are consistent estimators of the population parameters for $n'_p \to \infty$. Note that the latter implies $n_p, n_u \to \infty$ since $n'_p \leq n_p$, $n'_p \leq n_u$ for all n.

3.1 A Hypothesis Test for Stochastic Dominance

The development of hypothesis tests for stochastic dominance is an open problem widely investigated in economics and finance in general; and in particular in the income distribution literature and in portfolio theory, see McFadden (1989), Larsen and Resnick (1993), Kaur, Rao and Singh (1994), Anderson (1996), Davidson and Duclos (2000), Barrett and Donald (2003) or recently Linton, Maasoumi and Whang (2005), Davidson and Duclos (2006), Linton, Song and Whang (2008) and Scaillet and Topaloglou (2009).

Our approach for testing stochastic dominance differs from these influential papers in three aspects: first, due to the decompositions of the LPM measures in proposition 2 we can test for any order of stochastic dominance by using simple modifications of the test statistics. Further, all these different statistics follow asymptotically the supremum of different centered Gaussian processes with distribution that can be tabulated if the joint distribution function of the two portfolios is known. In this case the critical values of the asymptotic distribution of the tests can be approximated by simulation methods as in Koul and Ling (2006, p. 7). Second, as in Linton, Maasoumi and Whang (2005) and unlike Barrett and Donald (2003), the different tests for stochastic dominance make allowance for dependence between portfolios; and third, we extend these tests to scenarios of market distress, characterized by values of the vector of random variables comprising the portfolio below a given threshold u.

Our test statistic is of Kolmogorov-Smirnov type and shares the spirit of those proposed in most of related literature. We focus on the hypothesis test

$$\begin{cases} H_{0,\gamma}: LPM_{\gamma}^{A}(\tau) \leq LPM_{\gamma}^{B}(\tau), & \text{for all } \tau \in \Omega, \\ H_{1,\gamma}: LPM_{\gamma}^{A}(\tau) > LPM_{\gamma}^{B}(\tau), & \text{for some } \tau \in \Omega, \end{cases} \tag{15}$$

rather than on the strict inequality for testing for stochastic dominance[1] between two portfolios A and B. Alternatively, and following the notation in Linton, Maasoumi and Whang (2005) we define $D_{\gamma}(\tau) = LPM_{\gamma}^{A}(\tau) - LPM_{\gamma}^{B}(\tau)$ and write the hypothesis test above as

$$\begin{cases} H_{0,\gamma}: D_{\gamma}(\tau) \leq 0, & \text{for all } \tau \in \Omega, \\ H_{1,\gamma}: D_{\gamma}(\tau) > 0, & \text{for some } \tau \in \Omega. \end{cases} \tag{16}$$

Using the relation between stochastic dominance, mean-risk dominance and investor's expected utility maximization problem discussed in the previous section we can say that under $H_{0,0}$ A dominates B in the mean-risk sense for risk-neutral and risk-averse investors, under $H_{0,1}$ A dominates B for risk-averse investors except when $\mu(A) = \mu(B)$, and under $H_{0,2}$ and $\mu(A) \geq \mu(B)$ A dominates B for risk-averse investors with increasing absolute risk aversion levels. Other testing methods for this hypothesis reverse the roles of the hypotheses and have the alternative hypothesis as corresponding to strong stochastic dominance. These methods are formulated using a slightly different definition of stochastic dominance that involves strict inequality (strong stochastic dominance) in (15), and are usually based on the minimum distance rather than on the maximum, see for example Kaur, Rao and Singh (1994).

The asymptotic theory for LPM risk measures determined by τ fixed is given in the following proposition. In contrast to most of the existing literature this result is possible for general orders of q due to the decompositions discussed in proposition 2.

Proposition 3: *Suppose we have n i.i.d observations from a random variable R, let $\widehat{LPM}_{\gamma}(\tau)$ be the estimator introduced in* (13) *and assume A.1-A.4 hold. Then*

$$\sqrt{n}\left(\widehat{LPM}_{\gamma}(\tau) - LPM_{\gamma}(\tau)\right) \xrightarrow{d} N\left(0,\ E[(\tau - R)^{2\gamma} | R \leq \tau] F(\tau) - E[(\tau - R)^{\gamma} | R \leq \tau]^{2} F^{2}(\tau)\right), \tag{17}$$

for all fixed $\tau \in \Omega$, and $\gamma \geq 0$.

Before introducing the asymptotic theory relevant to the composite hypothesis test we need the following notation and one further assumption. Let A and B denote two portfolios with returns characterized by the random variables R^A and R^B respectively. Denote $F^{A,B}(\tau_s, \tau_t) = P(R^A \leq \tau_s, R^B \leq \tau_t)$, $k_{\gamma}^{i}(\tau) = E[(\tau - R^i)^{\gamma} | R^i \leq \tau]$ with $i = A, B$, and $k_{\gamma}^{A,B}(\tau_s, \tau_t) = E[(\tau_s - R^A)^{\gamma}(\tau_t - R^B)^{\gamma} | R^A \leq \tau_s, R^B \leq \tau_t]$, and $\Sigma(\tau_s, \tau_t)$ is

[1]Hereafter q denotes the order of investor's risk aversion and γ the order of stochastic dominance.

the asymptotic covariance matrix of the vector $(\widehat{LPM}_{\gamma}^{A}(\tau_s) - LPM_{\gamma}^{A}(\tau_s),\, \widehat{LPM}_{\gamma}^{B}(\tau_t) - LPM_{\gamma}^{B}(\tau_t))$.

Assumption A.5: $\inf_{\tau\in\Omega} det(\Sigma(\tau,\tau)) > 0$.

Assumption A.5 ensures that (17) can be extended to describe the asymptotic bivariate distribution of $\widehat{LPM}_{\gamma}^{A}(\tau)$ and $\widehat{LPM}_{\gamma}^{B}(\tau)$ for all fixed $\tau \in \Omega$. Applying the Cramer-Wold device, the limiting distribution of the difference between these random variables also converges to a normal distribution

$$\sqrt{n}\left(\widehat{D}_{\gamma}(\tau) - D_{\gamma}(\tau)\right) \xrightarrow{d} N\left(0, V_{\gamma}(\tau)\right) \tag{18}$$

with

$$\begin{gathered}V_{\gamma}(\tau) = \left(k_{2\gamma}^{A}(\tau)F^{A}(\tau) - (k_{\gamma}^{A}(\tau)F^{A}(\tau))^2\right) + \left(k_{2\gamma}^{B}(\tau)F^{B}(\tau) - (k_{\gamma}^{B}(\tau)F^{B}(\tau))^2\right) - \\ 2\left(k_{\gamma}^{A,B}(\tau,\tau)F^{A,B}(\tau,\tau) - k_{\gamma}^{A}(\tau)F^{A}(\tau)k_{\gamma}^{B}(\tau)F^{B}(\tau)\right).\end{gathered}$$

Theorem 3: *Under assumptions A.1-A.5,*

$$\sqrt{n}\sup_{\tau\in\Omega}(\widehat{D}_{\gamma}(\tau) - D_{\gamma}(\tau)) \xrightarrow{d} \sup_{\tau\in\Omega} G_{\gamma}(\tau), \tag{19}$$

with $G_{\gamma}(\tau)$ a Gaussian process with zero mean and covariance function given by

$$\begin{gathered}E[G_{\gamma}(\tau_s)G_{\gamma}(\tau_t)] = \\ \left(k_{2\gamma}^{A}(\tau_s\wedge\tau_t)F^{A}(\tau_s\wedge\tau_t) - k_{\gamma}^{A}(\tau_s)F^{A}(\tau_s)k_{\gamma}^{A}(\tau_t)F^{A}(\tau_t)\right) + \\ \left(k_{2\gamma}^{B}(\tau_s\wedge\tau_t)F^{B}(\tau_s\wedge\tau_t) - k_{\gamma}^{B}(\tau_s)F^{B}(\tau_s)k_{\gamma}^{B}(\tau_t)F^{B}(\tau_t)\right) - \\ \left(k_{\gamma}^{A,B}(\tau_s,\tau_t)F^{A,B}(\tau_s,\tau_t) - k_{\gamma}^{A}(\tau_s)F^{A}(\tau_s)k_{\gamma}^{B}(\tau_t)F^{B}(\tau_t)\right) - \\ \left(k_{\gamma}^{A,B}(\tau_t,\tau_s)F^{A,B}(\tau_t,\tau_s) - k_{\gamma}^{A}(\tau_t)F^{A}(\tau_t)k_{\gamma}^{B}(\tau_s)F^{B}(\tau_s)\right),\end{gathered} \tag{20}$$

for all $\tau_s, \tau_t \in \Omega$, and $\gamma \geq 0$.

Our family of test statistics is defined by $T_{n,\gamma} = \sqrt{n}\sup_{\tau\in\Omega}\widehat{D}_{\gamma}(\tau)$, and as in Barrett and Donald (2003) focuses on the least favorable case given by the equality of functions $LMP_{\gamma}^{A}(\tau) = LPM_{\gamma}^{B}(\tau)$, for every $\tau \in \Omega$. These authors and particularly Linton, Maasoumi and Whang (2005) discuss the problem of assuming equality of functions and argue that the convergence of test statistics of Kolmogorov-Smirnov and Cramér-von Mises type is not uniform over the probabilities under the null hypothesis. This problem is partially overcome by using subsampling methods to approximate the relevant null and alternative hypotheses. This resampling method has the particular advantage of exhibiting more power for the boundary of the null hypothesis for some forms of alternative hypotheses. On the other hand, and as discussed by these authors as well, subsampling does not make use of the full sample, and as such it may lose power for alternatives that are far from the boundary.

More recently, Linton, Song and Whang (2008) investigate in more detail this problem and show that discontinuity of convergence arises precisely between the interior points of the null hypothesis and the boundary points of the null hypothesis. In order to solve this issue these authors propose bootstrap procedures to obtain stochastic dominance tests with asymptotic coverage exactly equal to the nominal level of the test over the boundary of points and therefore valid over the whole null hypothesis. We acknowledge this technical issue with the test, nevertheless in order to keep the testing procedure as simple as possible to be applied in practice, we concentrate in this paper as most of the literature, on the least favorable case and derive asymptotic critical values with correct coverage under equality of the LPM_q risk functions across $\tau \in \Omega$. Thus, under A.1-A.5 and $H_{0,\gamma}$,

$$T_{n,\gamma} \xrightarrow{d} \sup_{\tau \in \Omega} G_\gamma(\tau). \tag{21}$$

Further, the asymptotic critical values of these tests indexed by γ are given by

$$c_\gamma(1-\alpha) = \inf_{x \in \Re} \{x \,|\, P\left(\sup_{\tau \in \Omega} G_\gamma(\tau) \leq x\right) \geq 1-\alpha\}, \tag{22}$$

with α denoting the significance level.

Proposition 4: *Given Assumptions A.1-A.5 and the test statistic $T_{n,\gamma}$, then:*

(i) *Under $H_{0,\gamma}$,*

$$\lim_{n \to \infty} P(\textit{reject } H_{0,\gamma}) = \lim_{n \to \infty} P(T_{n,\gamma} > c_\gamma(1-\alpha)) \leq \alpha, \tag{23}$$

with equality when $F^A(\tau) = F^B(\tau)$ for every $\tau \in \Omega$.

(ii) *If $H_{0,\gamma}$ is false,*

$$\lim_{n \to \infty} P(\text{reject } H_{0,\gamma}) = \lim_{n \to \infty} P(T_{n,\gamma} > c_\gamma(1-\alpha)) = 1. \tag{24}$$

Next we determine the power of the test against a sequence of contiguous alternatives converging to the boundary $D_\gamma(\tau) = 0$ for all τ, at a rate $n^{-1/2}$. We define the sequence of local alternatives $F^A(\tau) = F^B(\tau) + \frac{\delta(\tau)}{\sqrt{n}}$, that implies $D_\gamma(\tau) = \frac{\delta(\tau)}{\sqrt{n}}$ for each $\tau \in \Omega$, and with $\delta(\tau)$ such that $\sup\limits_{\tau \in \Omega} \delta(\tau) > 0$.

Proposition 5: *Under $H_{1,\gamma} : D_\gamma(\tau) = \frac{\delta(\tau)}{\sqrt{n}}$ with $\sup\limits_{\tau \in \Omega} \delta(\tau) > 0$, we have*

$$\lim_{n \to \infty} P\left(\text{reject } H_{0,\gamma}\right) = \lim_{n \to \infty} P\left(T_{n,\gamma} > c_\gamma(1-\alpha)\right) \geq \lim_{n \to \infty} P\left(\sup_{\tau \in \Omega} G_\gamma(\tau) > c_\gamma(1-\alpha) - \sup_{\tau \in \Omega} \delta(\tau)\right). \tag{25}$$

Then, the power of the test against local alternatives is nontrivial since

$$\lim_{n \to \infty} P\left(\sup_{\tau \in \Omega} G_\gamma(\tau) > c_\gamma(1-\alpha) - \sup_{\tau \in \Omega} \delta(\tau)\right) > \alpha.$$

The asymptotic critical values of the different tests depend on the joint distribution function $F^{A,B}$ generating the data that is usually unknown. This, as acknowledged by other authors as well, implies that $c_\gamma(1-\alpha)$ is not distribution-free and cannot be universally tabulated. Nevertheless, if $F^{A,B}(\tau,\tau)$ is assumed to be known the supremum of the different Gaussian processes indexed by γ, G_γ, can be tabulated via Monte-Carlo simulation methods, as Koul and Ling (2006, p. 7) does in a different context. The choice of the number of Monte-Carlo iterations and the partition of the grid is up to the econometrician, making the accuracy of this approximation as fine as the econometrician desires. The algorithm is as follows.

Algorithm:

1. Determine an equally spaced partition of $\Omega \subseteq \mathbb{R}$. In practice we choose l points such that $-a < \tau_1 < \tau_2 < \ldots < \tau_l < a$, with $a > 0$, $a \in \Omega$ and $\tau_i - \tau_{i-1} = 2a/l$.
2. Compute the asymptotic covariance matrix with elements defined in (20) and such that $\tau_s, \tau_t \in [-a, a]$.
3. Generate $\{G_{l,\gamma}(\tau_i) : i = 1, \ldots, l\}$ with $G_{l,\gamma}$ the corresponding multivariate finite-dimensional version (l-multivariate centered normal distribution) of the Gaussian process $G_\gamma(\tau)$ with mean zero and covariance matrix $E[G_\gamma(\tau_i)G_\gamma(\tau_j)]$, and calculate $\sup\limits_{1\leq i\leq l} G_{l,\gamma}(\tau_i)$.
4. Repeat step 3 $\mathbb{B}$ times to generate $\mathbb{B}$ independent replications of $G_{l,\gamma}(\tau)$, $\{G^b_{l,\gamma}(\tau), b = 1, ..., \mathbb{B}\}$, and compute $\sup\limits_{1\leq i\leq l} G^b_{l,\gamma}(\tau_i)$ in each replication to obtain the critical values of the test (21).

The interest of these tests is, however, when the nuisance parameters of the asymptotic distribution are not known. In this case there are two alternatives explored in the literature, namely, simulation methods and resampling methods. Simulation methods as in Barrett and Donald (2003), based on the multiplier method of Hansen (1996) and van der Vaart and Wellner (1996); and resampling methods as bootstrap in Barrett and Donald (2003), block bootstrap in Scaillet and Topaloglou (2009), and subsampling in Linton, Maasoumi and Whang (2005). Our method can be considered a particular example of simulation method. In order to to be consistent with the case of known distribution functions and to keep the test as simple as possible, we propose here to exploit the parametric form of the asymptotic distribution of the functional of G_γ, and approximate the critical value of the asymptotic distribution of the test with the critical value of the supremum of the corresponding centered Gaussian process with covariance function estimated by the empirical versions of the distribution functions and conditional statistical moments.

This method only approximates the critical values of the finite-dimensional version of the asymptotic Gaussian process but, as shown in Carrasco and Florens (2000), it is not able to approximate consistently the critical values of the asymptotic process even for very large values of n. As discussed by Davidson and Duclos (2000) this problem can be partially mitigated by choosing a random grid and/or a very dense partition of the compact set Ω for estimating the covariance kernel. In practice, our Monte-Carlo study in next section shows a good performance of the finite-dimensional approximation in terms of size and power and lends support to this claim. Furthermore, our approximation of the critical values is valid when there is dependence between the prospects A and B and can be extended naturally to a conditional setting. In contrast to subsampling techniques there is the advantage of not having to rely on the choice of a subsample size that can influence heavily the results in small samples. Finally, it is worth mentioning that the symmetry of the asymptotic distribution of the different test statistics $T_{n,\gamma}$ under $H_{0,\gamma}$ allows us to carry out the reverse hypothesis test $H_{0,\gamma}^{*} : LPM_{\gamma}^{B} \leq LPM_{\gamma}^{A}$ without the need of extra calculations. The asymptotic critical value of this test is also $c_{\gamma}(1-\alpha)$, and the relevant test statistic $T_{n,\gamma}^{*}$ can be computed from $T_{n,\gamma}$ by exploiting that $T_{n,\gamma}^{*} = -\sqrt{n} \inf\limits_{\tau \in \Omega} \widehat{D}_{\gamma}$. In practice then we only need to compute this value along with $T_{n,\gamma}$ to extract meaningful conclusions about the reverse test in case $H_{0,\gamma}$ is rejected.

3.2 Stochastic Dominance Hypothesis Tests Under Distress

The results above can be easily extended to testing stochastic dominance under distress and with it the mean-risk dominance set in episodes of market turmoil. For ease of exposition we will assume that the universe of eligible assets for constructing both portfolios is the same.

Define now the parameter space $\Omega_u = \Omega \cap (-\infty, u]$. The relevant hypothesis test in this environment is

$$\begin{cases} H_{0,\gamma,u} : D_{\gamma,u}(\tau) \leq 0, & \text{for all } \tau \in \Omega_u, \\ H_{1,\gamma,u} : D_{\gamma,u}(\tau) > 0, & \text{for some } \tau \in \Omega_u, \end{cases} \tag{26}$$

with $D_{\gamma,u}(\tau) = LPM_{\gamma,u}^{A}(\tau) - LPM_{\gamma,u}^{B}(\tau)$.

The asymptotic theory follows from the previous results for the unconditional stochastic dominance tests. Assumptions A.2-A.5 are sufficient to guarantee that the covariance function of the corresponding conditional functional process is well defined. Before introducing the main result of this section we need further notation. Let $F_u^{A,B}(\tau_s, \tau_t) = P(R^A \leq \tau_s, R^B \leq \tau_t | R_1 \leq u, \ldots, R_m \leq u)$, $k_{\gamma,u}^{i}(\tau) = E[(\tau - R^i)^{\gamma} | R^i \leq \tau, R_1 \leq u, \ldots, R_m \leq u]$ with $i = A, B$, $k_{\gamma,u}^{A,B}(\tau_s, \tau_t) = E[(\tau_s - R^A)^{\gamma}(\tau_t - R^B)^{\gamma} | R^A \leq \tau_s, R^B \leq \tau_t, R_1 \leq u, \ldots, R_m \leq u]$, and $\lambda(u)$ defined in A.3.

Theorem 4: *Under A.1-A.5,*

$$\sqrt{n_u} \sup_{\tau \in \Omega_u} (\widehat{D}_{\gamma,u}(\tau) - D_{\gamma,u}(\tau)) \xrightarrow{d} \sup_{\tau \in \Omega_u} G_{\gamma,u}(\tau), \tag{27}$$

with $G_{\gamma,u}(\tau)$ a Gaussian process with zero mean and covariance function given by

$$\begin{gathered} E[G_{\gamma,u}(\tau_s) G_{\gamma,u}(\tau_t)] = \\ \left(k_{2\gamma,u}^{A}(\tau_s \wedge \tau_t) F_u^{A}(\tau_s \wedge \tau_t) - k_{\gamma,u}^{A}(\tau_s) F_u^{A}(\tau_s) k_{\gamma,u}^{A}(\tau_t) F_u^{A}(\tau_t) \lambda(u)\right) + \\ \left(k_{2\gamma,u}^{B}(\tau_s \wedge \tau_t) F_u^{B}(\tau_s \wedge \tau_t) - k_{\gamma,u}^{B}(\tau_s) F_u^{B}(\tau_s) k_{\gamma,u}^{B}(\tau_t) F_u^{B}(\tau_t) \lambda(u)\right) - \\ \left(k_{\gamma,u}^{A,B}(\tau_s, \tau_t) F_u^{A,B}(\tau_s, \tau_t) - k_{\gamma,u}^{A}(\tau_s) F_u^{A}(\tau_s) k_{\gamma,u}^{B}(\tau_t) F_u^{B}(\tau_t) \lambda(u)\right) - \end{gathered} \tag{28}$$

$$\left(k_{\gamma,u}^{A,B}(\tau_t,\tau_s)F_u^{A,B}(\tau_t,\tau_s) - k_{\gamma,u}^{A}(\tau_t)F_u^{A}(\tau_t)k_{\gamma,u}^{B}(\tau_s)F_u^{B}(\tau_s)\lambda(u)\right),$$

for all $\tau_s, \tau_t \in \Omega_u$, *and* $\gamma \geq 0$.

The family of test statistics for testing stochastic dominance under distress are $T_{n_u,\gamma} = \sqrt{n_u} \sup_{\tau \in \Omega_u} \widehat{D}_{\gamma,u}(\tau)$, that under A.1-A.5, and $H_{0,\gamma,u}$, with $u \in \Omega$, satisfy

$$T_{n_u,\gamma} \xrightarrow{d} \sup_{\tau \in \Omega_u} G_{\gamma,u}(\tau), \tag{29}$$

with $G_{\gamma,u}(\tau)$ a Gaussian process with zero mean and covariance function given in expression (28). Further, the asymptotic critical values of these tests are given by

$$c_{\gamma,u}(1-\alpha) = \inf_{x \in \Re}\{x \,|\, P\left(\sup_{\tau \in \Omega_u} G_{\gamma,u}(\tau) \leq x\right) \geq 1-\alpha\}, \tag{30}$$

with α denoting the significance level.

Simulation procedures as the multiplier method or bootstrap can be proposed here as well to approximate the critical value of the test. Instead, and as before, we propose to approximate the relevant critical values from the restricted supremum of the estimated version of the Gaussian process $G_{\gamma,u}(\tau)$.

The next subsection discusses the extensions of these tests to residuals of time series regression models.

3.3 Stochastic Dominance Hypothesis Tests for Residual Processes

In many situations of practical interest the realizations of the random variables under study are serially dependent. In these cases researchers can proceed in two ways. One possibility is to develop hypothesis tests for stochastic dominance accounting for the possibility of serial dependence, see for example Linton, Maasoumi and Whang (2005) and Scaillet and Topaloglou (2009). In our framework the asymptotic distributions in theorems 3 and 4 would need to incorporate the presence of serial correlation and heteroscedasticity in the data, implying more convoluted covariance structures of the respective asymptotic distributions. Appropriate heteroscedastic and autocorrelation consistent (HAC) estimators of the conditional expected values and distribution functions would need to be used instead. The sequences under study also need to satisfy some mixing conditions.

Alternatively, it can be assumed that the serial dependence in the data can be removed by regressing the returns on an appropriate conditional information set. In this paper we follow this approach and apply our test to the corresponding residuals. More concretely, let $Z_t^T = \{(1, R_{t-j}^A, R_{t-j}^B, X_{t+1-j}),\ j = 1, \ldots\}$ be a vector of regressors, X_t denotes a vector of random variables different from R_t^A and R_t^B and T the transpose vector operation. The relevant regression equation is

$$R_t^i = Z_t^T \beta^i + a_t^i, \tag{31}$$

with β^i the parameter vector and $a_t^i = h_t^i \varepsilon_t^i$ the innovation variables corresponding to each regression equation. These sequences consist of a volatility process h_t^i and an $i.i.d$ error sequence ε_t^i for $i = A, B$. Let $\widehat{T}_{n,\gamma}$ be the family of test statistics computed from the residual sequences $\widehat{\varepsilon}_t^i = \frac{R_t^i - Z_t^T \widehat{\beta}^i}{\widehat{h}_t^i}$ for $i = A, B$, with $\widehat{\beta}^i$ the vector of parameter estimates and $\widehat{h}_t^i$ the estimated volatility process. In what follows we show that theorems 3 and 4 still hold for these alternative tests based on the residual sequences.

Assumption A.6: (i) $\left\{(R_t^i, Z_t) : t = 1, \ldots, n\right\}$ is a strictly stationary and ergodic sequence for $i = A, B$. (ii) The conditional distribution of ε_t^i given the vector Z_t has bounded density with respect to Lebesgue measure almost sure $(a.s.)$ for $i = A, B$, and $t \geq 1$. (iii) $\sqrt{n}(\widehat{\beta}^i - \beta^i) = O_p(1)$ and $\sqrt{n}(\widehat{h}_t^i - h_t^i) = O_p(1)$.

Corollary 2: *Suppose assumptions A.1-A.6 are satisfied. Then, under $H_{0,\gamma}$,*

$$\widehat{T}_{n,\gamma} \xrightarrow{d} \sup_{\tau \in \Omega} G_\gamma(\tau), \tag{32}$$

with $\widehat{T}_{n,\gamma} = \sqrt{n} \sup_{\tau \in \Omega} \left(\widehat{LPM}_\gamma^{A}(\tau) - \widehat{LPM}_\gamma^{B}(\tau) \right)$, $\widehat{LPM}_\gamma^{i}$ the downside risk measure computed from the estimated residuals from (31)*, $i = A, B$, and $G_\gamma(\tau)$ the Gaussian process introduced in* (20).

Without the need of imposing more assumptions this corollary can also be formulated for stochastic dominance under distress using residual processes. This result is omitted for sake of space.

Summarizing, the tests proposed in this section complement and extend in three ways the existing methods for testing stochastic dominance. First, by deriving a testing framework for general degrees of stochastic dominance that makes allowance for different forms of dependence between portfolios without relying on bootstrap and subsampling techniques; second, by introducing alternative tests for the hypothesis of stochastic dominance under distress episodes of the market, and third by showing the applicability of these techniques to residuals from time series regression models. The implications of these techniques in optimal portfolio theory are of much interest. A simple application for financial data is described in Section 5. Next section illustrates via simulation experiments the findings of this section.

4 Mote-Carlo Simulation Experiments

In this section we consider a small Monte Carlo experiment to gauge the extent to which the preceding asymptotic arguments hold in finite samples. We are interested, in particular, in comparing the approximation of the critical values given by our asymptotic theory and the approximation offered by the multiplier method discussed in Barrett and Donald (2003). We study these approximations for stochastic dominance tests of first and second order; and also, for the corresponding tests of stochastic dominance under distress. In the second block of simulations we carry out a small study of the power of the tests against local alternatives. In this case we only focus on our method.

In all of these simulations the critical values used for computing size and power of the different tests correspond to the finite-dimensional approximation obtained from estimating the covariance kernel of the asymptotic Gaussian processes. Since in this experiment $F^{A,B}$ is known the asymptotic true critical values can also be approximated via Monte-Carlo simulation. We have carried out both experiments finding similar results. Since the former technique corresponds to what one would have to do in practice we choose to report size and power derived from the finite-dimensional approach. The results of the other experiment are available upon request.

Tables 1 and 2 report empirical sizes under both methods for $H_{0,\gamma}$ for $\gamma = 0, 1$ and when the correlation parameter between the random variables is $\rho = 0$, 0.4 and 0.8. The significance levels studied are 10%, 5% and 1% and the data generating processes are bivariate Student-t distributions with $\nu = 5$ and $\nu = 10$ degrees of freedom. We choose these distributions as plausible candidates to describe the unconditional generating process for pairs of financial returns, or more usually, to describe the sequence of innovations of the standard processes encountered in the modeling of financial time series, see Bollerslev (1987). These distributions belong to the elliptical family of distribution functions and are therefore completely characterized by the first two statistical moments and the correlation function. Nevertheless, unlike the Gaussian distribution these processes are capable of generating asymptotic tail dependence as ρ increases. The impact of this phenomenon in the size and power of the tests can be observed in the different simulations reported.

[INSERT TABLES 1 AND 2 ABOUT HERE]

The results for the empirical size for stochastic dominance under distress are reported in tables 3 and 4. Note that in order to have a simulation exercise comparable to the unconditional case we need to have conditional samples of $n_u = 50, 100$ and 500 observations. For the independent case this can be achieved for a threshold $u = 0$ by generating random samples of $n = 200, 400$ and 2000 observations. For values of ρ greater than zero the asymptotic tail dependence present in the data generates subsamples in the conditioning region with more than $n_u = 50, 100$ and 500 observations and yields in turn, due to a higher sample size, better approximations of size and power of the tests.

[INSERT TABLES 3 and 4 ABOUT HERE]

The study of the power of the tests against local alternatives is designed as follows. The family of alternative hypotheses is defined by a random variable $R^A = X - \frac{c}{\sqrt{n}}$, where X, as R^B, follows an standardized mean-zero Student-t distribution with ν degrees of freedom, and such that $Cov(X, R^B) = \rho$. The distribution function of R^A is given by $F^A(\tau) = F^B(\tau + \frac{c}{\sqrt{n}})$, that by a Taylor expansion satisfies

$F^A(\tau) = F^B(\tau) + \frac{cf^B(\tau)}{\sqrt{n}} + o\left(\frac{1}{\sqrt{n}}\right)$ with $f^B(\tau)$ the density function of the centered Student-t distribution.[2] The distribution of R^A can be written as $F^A(\tau) = F^B(\tau) + \frac{\delta(\tau)}{\sqrt{n}}$ with $\delta(\tau) = cf^B(\tau)$ and such that $\sup_{\tau\in\Omega} \delta(\tau) > 0$. For our examples we consider $c = 0.5, 1$ and 5.

We study the power for the three dependence structures considered before. Tables 5 and 6 report the results for the unconditional tests and tables 7 and 8 the results corresponding to the conditional tests representing financial distress. The data generating processes are Student-t distributions with $\nu = 5$ and 10.

[INSERT TABLES 5 - 8 ABOUT HERE]

Some remarks on the simulations:

1. The different proposed test statistics show a good finite sample performance in terms of size and power. This performance improves as n increases.
2. The approximations of the different critical values by those obtained from the estimated asymptotic distribution and by the multiplier method are of similar accuracy. A by-product of our simulations is that the multiplier method used by Barret and Donald (2003) can be extended to dependent portfolios.
3. The power of the tests increases as the correlation between the random variables gets higher.
4. The conclusions from the simulations for stochastic dominance under distress are very similar and are omitted for sake of space. It is remarkable the substantial increase in power in these cases compared to their unconditional counterparts with same sample sizes. This is due to a larger conditional sample size produced by the positive dependence.

In the next section we implement these tests for evaluating efficient investment portfolios and compare them in normal and crises episodes of the market.

5 An Empirical Application to US Equity Markets

We study a portfolio of risky and heavily traded stocks in the US economy that cover very different and important sectors: Microsoft (MSFT), General Electric (GE), Bank of America Corporation (BAC) and Verizon Communications (VZ). The data set spans the period 02/01/2000 - 30/12/2007 with daily frequency and is obtained from Yahoo Finance website. In contrast to studies using financial indexes each asset in this case is not a diversified instrument per se and is, therefore, bound to be affected by negative and positive idiosyncratic shocks. The marginal unconditional distribution functions exhibit rather heavy tails and can

[2]The supremum of $f^B(\tau)$ is achieved at $\tau = 0$ and takes the value 0.380 for $\nu = 5$ and 0.389 for $\nu = 10$.

invalidate, in turn, approximations of the distribution of the portfolio given by normal distributions, and that thereby support mean-variance efficient sets consisting of aggregation of uncorrelated assets.

With these data we choose two portfolios determined by the vectors of weights $w_o = [0.05 \quad 0.85 \quad 0.05 \quad 0.05]$ and $w_{mv} = [0.20 \quad 0.15 \quad 0.30 \quad 0.35]$. The portfolio defined by w_o is obtained from minimizing the LPM_0 measure for $\tau = 0$ and that by w_{mv} from minimizing the unconditional variance. The upper panel in figure 1 shows the unconditional distribution function of returns from each strategy.

As a by-product of theorem 1 we know that the mean-LPM_0 efficient portfolio, w_0, is not stochastically dominated in first order by any other feasible portfolio. This does not imply, however, that this portfolio first stochastically dominates the mean-variance counterpart w_{mv}. In order to see this we carry out the hypothesis test $H_{0,0} : LPM_0^{w_o} \leq LPM_0^{w_{mv}}$ with simulated critical values at 10%, 5% and 1% given by 1.029, 1.158 and 1.364, respectively. The relevant test statistic given by 2.710 leads us to conclude that no portfolio dominates the other in terms of first stochastic dominance. This is also apparent from the upper panel in figure 1 that shows the two relevant empirical distribution functions crossing each other over the real domain.

The above findings do not imply any obvious result concerning higher orders of stochastic dominance. Thus, we need to carry out the hypothesis tests corresponding to second stochastic dominance. In this case the critical values are 1.594, 1.982 and 2.813, respectively, with test statistics 7.305 for $H_{0,1}$ and -2.480 for the reverse $H_{0,1}^*$ hypothesis; and yielding sufficient statistical evidence to conclude that the mean-variance strategy w_{mv} dominates w_0 stochastically in second order, and hence in higher orders.

The efficiency analysis between portfolios is repeated now under market distress defined by different threshold values, in particular $u = 0.5$ and $u = 1$. The mean-risk portfolio for both thresholds is $w_{o,u} = [0.05 \quad 0.05 \quad 0.05 \quad 0.85]$ and the mean-variance portfolio is $w_{mv,0.5} = [0.10 \quad 0.10 \quad 0.40 \quad 0.40]$ and $w_{mv,1} = [0.15 \quad 0.10 \quad 0.35 \quad 0.40]$. As before, the efficiency of the LPM-type portfolio implies by construction, see theorem 2, that it is not dominated by any other portfolio. To compare this portfolio against the mean-variance alternatives under distress it only remains to test the hypothesis $H_{0,0,u} : LPM_{0,u}^{w_{o,u}} \leq LPM_{0,u}^{w_{mv,u}}$, for $u = 0.5, 1$. The simulated critical values of this test are 0.932, 1.038 and 1.233 for $u = 0.5$, and the relevant test statistic is 0.298; showing no statistical evidence to reject the null hypothesis of stochastic dominance of $w_{o,0.5}$ over $w_{mv,0.5}$ under distress. This result shows that in periods of market distress the corresponding mean-risk efficient portfolio dominates the efficient mean-variance counterpart for all $\tau \leq 0.5$. To check the robustness of this result to the choice of u the same test is carried out now for $u = 1$. The critical values of the test are now 0.947, 1.048 and 1.279, and the test statistic 0.762, yielding the same conclusions as for $u = 0.5$. The middle panel (for $u = 0.5$) of figure 1 supports these findings; the lower panel (for $u = 1$), on the other hand, exhibits some crossing of the empirical conditional distributions, hence the importance of carrying out the formal statistical test.

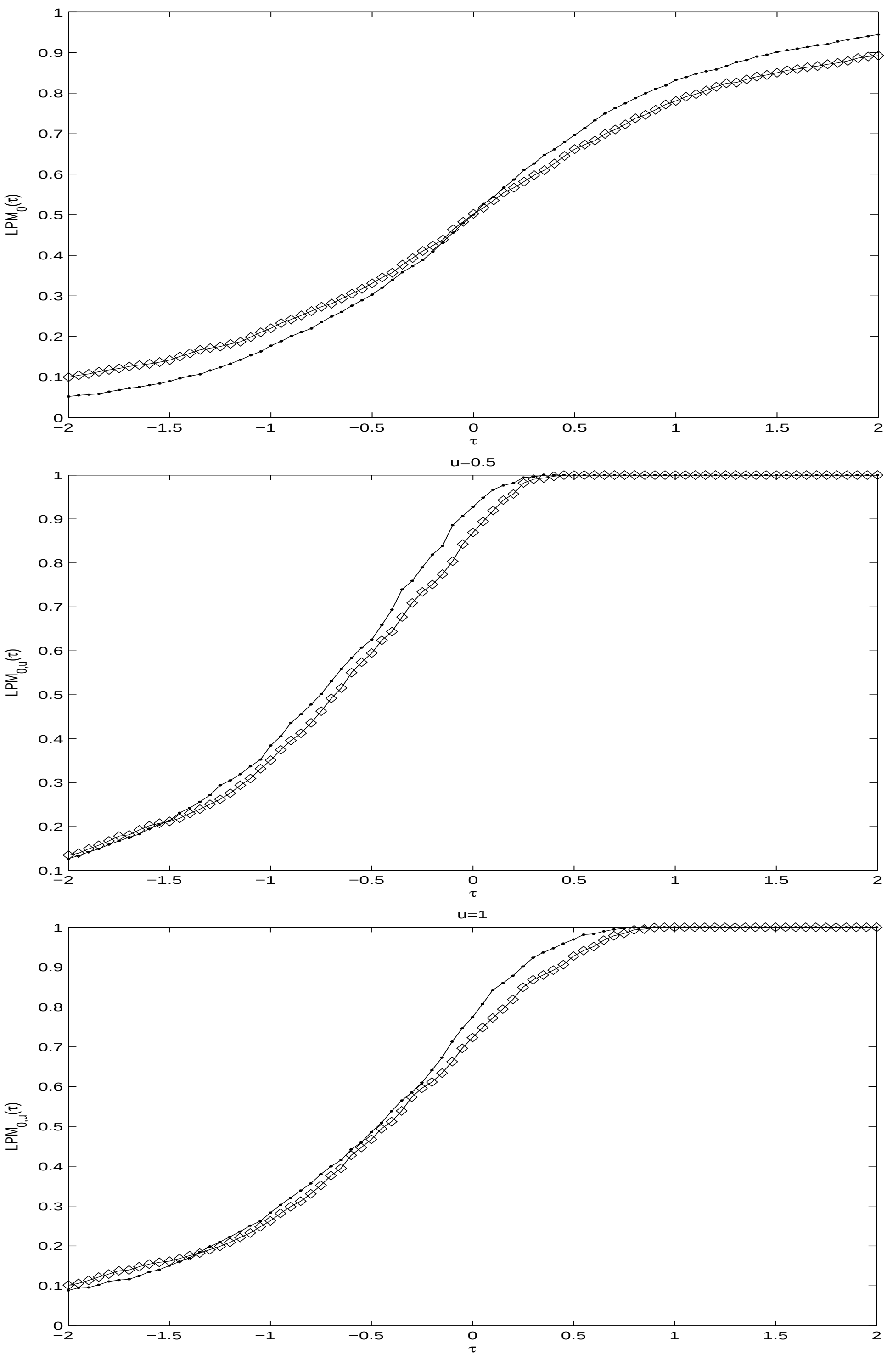

FIGURE 1. Nonparametric estimates of the conditional risk measures LPM_0 (upper panel) and $LPM_{0,u}$ (middle and lower panels). ($\Diamond-$) describes Portfolio w_o and $w_{o,u}$, and ($\cdot-$) Portfolio w_{mv} and $w_{mv,u}$ for $u = 0.5$, 1. The assets comprising the portfolio are $(GE, MSFT, VZ, BAC)$ covering the period 02/01/2000 - 30/12/2007 with daily frequency.

Clearly, the return data is not $i.i.d$ and therefore, the reader should take the previous testing results with caution. In order to remove the serial dependence from the data we carry out this experiment using two more methodologies. For the first one we entertain the abnormal returns of each portfolio obtained from removing the dependence from the market portfolio, proxied in this example by the Dow-Jones Industrial Average Stock Index over the same period. We find, however, no statistical significance at 5% of the systematic risk β parameter. Therefore, the results on stochastic dominance obtained before do not vary now. The second experiment contemplates the residual sequence of each time series after filtering for the presence of serial dependence in the data. In particular, we have estimated each optimal portfolio independently using an ARMA(1,1)-GARCH(1,1) process and a pure GARCH(1,1) process. Whereas the ARMA components are not statistically significant at 5%, the parameters of the volatility model are highly significant. The process for the downside risk portfolio is

$$R_t^{w_o} = h_{o,t}\varepsilon_t^{w_0}, \;\; with \;\; h_{o,t}^2 = \underset{(0.006)}{0.033} + \underset{(0.010)}{0.123} R_{t-1}^2 + \underset{(0.010)}{0.876} h_{o,t-1}^2,$$

with $\varepsilon_t^{w_0}$ the corresponding error term, and where standard errors of the estimates are in brackets. For the mean-variance efficient portfolio,

$$R_t^{w_{mv}} = h_{mv,t}\varepsilon_t^{w_{mv}}, \;\; with \;\; h_{mv,t}^2 = \underset{(0.002)}{0.006} + \underset{(0.007)}{0.060} R_{mv,t-1}^2 + \underset{(0.007)}{0.938} h_{mv,t-1}^2,$$

with $\varepsilon_t^{w_{mv}}$ the error term.

The results in this case are more supportive of the stochastic dominance of downside risk strategies for the complete domain of the random variables. The test statistics for the unconditional case are $\widehat{T}_{n,0} = 0.866$ for $H_{0,0}$, and $\widehat{T}_{n,0}^* = 1.577$ for the reverse hypothesis $H_{0,0}^*$, and the critical value of both tests at 5% is 1.163. Hence we do not reject the hypothesis of stochastic dominance of the downside risk portfolio. Finally, the results from the hypothesis test under market distress confirm these findings. In this case the relevant test statistics are $\widehat{T}_{n_{0.5},0} = 0.042$ and $\widehat{T}_{n_{0.5},0}^* = 5.051$, and the critical value at 5% is 1.271. Similar results are obtained for $u = 1$.

6 Conclusions

The mean-risk efficient frontier is comprised by all the assets that are not dominated in terms of downside risk by any other asset available in the market. Downside risk is given by a lower partial moment measure characterized by two parameters: τ that determines the target return and q that describes the intensity of risk aversion. Under periods of financial distress portfolios in this frontier are not necessarily optimal. The mean-risk efficient set in this environment is the result of minimizing, given an expected return constraint, a lower partial moment measure $LPM_{q,u}$ adapted to the distress event. These portfolios are also efficient in stochastic dominance defined now under distress, and therefore, are also in the optimal set of investors maximizing expected utility. More specifically, if portfolios are first stochastic dominant under distress then they are in the optimal frontier of risk neutral and risk averse investors; if they are second stochastic dominant under distress then these portfolios maximize expected utility under distress of risk averse investors, and so on.

In order to compare portfolios across the risk aversion parameters τ and q we carry out hypothesis tests for stochastic dominance. The asymptotic distribution of these tests is simplified by using the relationship between LPM and stochastic dominance under distress derived in this paper. The methods discussed in this paper can be of much interest for investors and practitioners whose level of risk aversion cannot be modeled by a simple threshold level given, for example, by the return on the risk-free asset or by a zero return, but is within an interval of possible threshold values. In particular, in an empirical application with data on US equity markets we find that mean-risk efficient portfolios dominate stochastically mean-variance portfolios under market distress.

Finally note that the hypothesis tests developed in this paper concentrate on pairwise comparisons. Nevertheless, following the techniques in Barret and Donald (2003), Linton, Maasoumi and Whang (2005) or more recently in Scaillet and Topaloglou (2009) we can extend our method naturally to testing for more than two alternatives.

Future research goes in the direction of constructing realized versions of these LPM measures, following the approach introduced for the semivariance by Barndorff-Nielsen, Kinnebrock and Shephard (2008), in order to analyze the dynamic evolution of these risk measures.

Mathematical appendix

Proof of Proposition 2: The proof of the first result in this proposition is trivial. For the second equality we have that

$$LPM^P_{0,u}(\tau) = \int_{-\infty}^{\tau} dF_u(x) = F_u(\tau),$$

with $F_u(\tau) = P(R^P \leq \tau \big| R_1 \leq u, \ldots, R_m \leq u)$. Now, by definition,

$$LPM^P_{q,u}(\tau) = \int_{-\infty}^{\tau} (\tau - x)^q \frac{dF_u(x)}{F_u(\tau)} LPM^P_{0,u}(\tau).$$

Notice that $\frac{dF_u(x)}{F_u(\tau)}$ is the density function of the distribution function $\frac{F_u(x)}{F_u(\tau)} = P\{R^P \leq x | R^P \leq \tau, R_1 \leq u, \ldots, R_m \leq u\}$, for $x \leq \tau$. Therefore

$$LPM^P_{q,u}(\tau) = E\left[(\tau - R^P)^q | R^P \leq \tau, R_1 \leq u, \ldots, R_m \leq u\right] LPM^P_{0,u}(\tau).$$

Proof of Proposition 3: The proof of this result follows from assumption A.4 and central limit theorem arguments for $i.i.d$ random variables. In particular, we have

$$\frac{\widehat{LPM}_\gamma(\tau) - E[\widehat{LPM}_\gamma(\tau)]}{\sqrt{V(\widehat{LPM}_\gamma(\tau))}} \xrightarrow{d} N(0,1).$$

The next steps derive the expression of the first two statistical moments of the LPM estimators. Suppose we have n $i.i.d$ vectors of observations from a random variable R, and let $\widehat{LPM}_\gamma(\tau)$ be the estimator of $LPM_\gamma(\tau)$ introduced in (13),

$$\widehat{LPM}_\gamma(\tau) = \tfrac{1}{n}\sum_{i=1}^{n} (\tau - x_i)^\gamma I(x_i \leq \tau).$$

By the law of iterated expectations,

$$E[\widehat{LPM}_\gamma(\tau)] = E[(\tau - X)^\gamma | X \leq \tau] F(\tau).$$

The unbiasedness result is obtained from the definition of LPM_0 and proposition 2.

To obtain the expression for the variance term we use similar but more tedious calculus. By definition, we know that

$$V(\widehat{LPM}_\gamma(\tau)) = E[\widehat{LPM}^2_\gamma(\tau)] - E^2[(\tau - X)^\gamma | X \leq \tau] F^2(\tau).$$

Using the above arguments and exploiting the serial independence between the observations, the first term on the right can be expressed as

$$E[\widehat{LPM}_{\gamma}^{2}(\tau)] = E\left[\frac{1}{n^2}\sum_{i=1}^{n}(\tau - x_i)^{2\gamma}\big|x_i \le \tau\right]F(\tau) + E\left[\frac{1}{n^2}\sum_{i=1}^{n}\sum_{\substack{j=1\\ j\neq i}}^{n}(\tau - x_i)^{\gamma}(\tau - x_j)^{\gamma}\big|x_i \le \tau, x_j \le \tau\right]F^2(\tau).$$

After some algebra we obtain

$$E[\widehat{LPM}_{\gamma}^{2}(\tau)] = \frac{1}{n}\Big(E\left[(\tau - X)^{2\gamma}|X \le \tau\right]F(\tau) - E\left[(\tau - X)^{\gamma}|X \le \tau\right]^2 F^2(\tau)\Big) + E\left[(\tau - X)^{\gamma}|X \le \tau\right]^2 F^2(\tau).$$

It follows then that

$$V[\widehat{LPM}_{\gamma}(\tau)] = \frac{1}{n}\Big(E\left[(\tau - X)^{2\gamma}|X \le \tau\right]F(\tau) - E\left[(\tau - X)^{\gamma}|X \le \tau\right]^2 F^2(\tau)\Big).$$

Therefore,

$$\sqrt{n}\frac{\widehat{LPM}_{\gamma}(\tau) - LPM_{\gamma}(\tau)}{\sqrt{\Big(E\left[(\tau - X)^{2\gamma}|X \le \tau\right]F(\tau) - E\left[(\tau - X)^{\gamma}|X \le \tau\right]^2 F^2(\tau)\Big)}} \xrightarrow{d} N(0,1). \tag{33}$$

Proof of Theorem 3: From proposition 3 and assumptions A.1-A.5, it is straightforward to show that the finite-dimensional version of the Gaussian process G_{γ} is a multivariate normal distribution. Thus, for a finite vector determined by the partition $-\infty < \tau_1 < \tau_2 < \ldots < \tau_l < \infty$ we have that

$$\sqrt{n}\left(\widehat{D}_{\gamma}(\tau_1) - D_{\gamma}(\tau_1), \ldots, \widehat{D}_{\gamma}(\tau_l) - D_{\gamma}(\tau_l)\right) \xrightarrow{d} \left(G_{\gamma}(\tau_1), \ldots, G_{\gamma}(\tau_l)\right), \tag{34}$$

with the vector on the right following a mean-zero multivariate normal distribution. The novelty of this result resides on the structure of the covariance kernel that is given by

$$\begin{aligned} E[G_{\gamma}(\tau_s)G_{\gamma}(\tau_t)] = {} & \big(k_{2\gamma}^{A}(\tau_s \wedge \tau_t)F^{A}(\tau_s \wedge \tau_t) - k_{\gamma}^{A}(\tau_s)F^{A}(\tau_s)k_{\gamma}^{A}(\tau_t)F^{A}(\tau_t)\big) + \\ & \big(k_{2\gamma}^{B}(\tau_s \wedge \tau_t)F^{B}(\tau_s \wedge \tau_t) - k_{\gamma}^{B}(\tau_s)F^{B}(\tau_s)k_{\gamma}^{B}(\tau_t)F^{B}(\tau_t)\big) - \\ & \big(k_{\gamma}^{A,B}(\tau_s, \tau_t)F^{A,B}(\tau_s, \tau_t) - k_{\gamma}^{A}(\tau_s)F^{A}(\tau_s)k_{\gamma}^{B}(\tau_t)F^{B}(\tau_t)\big) - \\ & \big(k_{\gamma}^{A,B}(\tau_t, \tau_s)F^{A,B}(\tau_t, \tau_s) - k_{\gamma}^{A}(\tau_t)F^{A}(\tau_t)k_{\gamma}^{B}(\tau_s)F^{B}(\tau_s)\big), \end{aligned}$$

for all $\tau_s, \tau_t \in \Omega$. Notice that assumption A.4 ensures the existence of this covariance.

We follow the same procedure as in the previous proof to derive this structure; in particular, we only show the contribution of $Cov\left(\widehat{LPM}_{\gamma}^{A}(\tau_s), \widehat{LPM}_{\gamma}^{A}(\tau_t)\right)$ and $Cov\left(\widehat{LPM}_{\gamma}^{A}(\tau_s), \widehat{LPM}_{\gamma}^{B}(\tau_t)\right)$ to the asymptotic covariance. The other two terms follow the same algebra. Thus

$$\begin{aligned} E[\widehat{LPM}_{\gamma}^{A}(\tau_s)\widehat{LPM}_{\gamma}^{A}(\tau_t)] = {} & E\left[\frac{1}{n^2}\sum_{i=1}^{n}(\tau_s - x_i)^{\gamma}(\tau_t - x_i)^{\gamma}\big|x_i \le \tau_s \wedge \tau_t\right]F^{A}(\tau_s \wedge \tau_t) + \\ & E\left[\frac{1}{n^2}\sum_{i=1}^{n}\sum_{\substack{j=1\\ j\neq i}}^{n}(\tau_s - x_i)^{\gamma}(\tau_t - x_j)^{\gamma}\big|x_i \le \tau_s, x_j \le \tau_t\right]F^{A}(\tau_s)F^{A}(\tau_t). \end{aligned}$$

By the serial independence between the observations this expression becomes

$$E[\widehat{LPM}_{\gamma}^{A}(\tau_s)\widehat{LPM}_{\gamma}^{A}(\tau_t)] =$$

$$\frac{1}{n}\left(E\left[(\tau_s - X)^{\gamma}(\tau_t - X)^{\gamma}|X \leq \tau_s \wedge \tau_t\right] F^{A}(\tau_s \wedge \tau_t) - E\left[(\tau_s - X)^{\gamma}|X \leq \tau_s\right] E\left[(\tau_t - X)^{\gamma}|X \leq \tau_t\right] F^{A}(\tau_s)F^{A}(\tau_t)\right) +$$

$$E\left[(\tau_s - X)^{\gamma}|X \leq \tau_s\right] E\left[(\tau_t - X)^{\gamma}|X \leq \tau_t\right] F^{A}(\tau_s)F^{A}(\tau_t).$$

Then, it follows that

$$\lim_{n\to\infty} n\, Cov\left(\widehat{LPM}_{\gamma}^{A}(\tau_s), \widehat{LPM}_{\gamma}^{A}(\tau_t)\right) =$$

$$E\left[(\tau_s - X)^{\gamma}(\tau_t - X)^{\gamma}|X \leq \tau_s \wedge \tau_t\right] F^{A}(\tau_s \wedge \tau_t) - E\left[(\tau_s - X)^{\gamma}|X \leq \tau_s\right] E\left[(\tau_t - X)^{\gamma}|X \leq \tau_t\right] F^{A}(\tau_s)F^{A}(\tau_t).$$

For the covariance term denoting cross dependence the procedure is similar. Let $\{y_j\}_{j=1}^{n}$ denote the sequence of observations from B. Now,

$$E[\widehat{LPM}_{\gamma}^{A}(\tau_s)\widehat{LPM}_{\gamma}^{B}(\tau_t)] = E\left[\frac{1}{n^2}\sum_{i=1}^{n}(\tau_s - x_i)^{\gamma}(\tau_t - y_i)^{\gamma}\middle|x_i \leq \tau_s, y_i \leq \tau_t\right] F^{A,B}(\tau_s, \tau_t) +$$

$$E\left[\frac{1}{n^2}\sum_{i=1}^{n}\sum_{\substack{j=1 \\ j\neq i}}^{n}(\tau_s - x_i)^{\gamma}(\tau_t - y_j)^{\gamma}\middle|x_i \leq \tau_s, y_j \leq \tau_t\right] F^{A}(\tau_s)F^{B}(\tau_t).$$

By the serial independence between the observations and the cross independence between x_i and y_j for $i \neq j$ the previous expression becomes

$$E[\widehat{LPM}_{\gamma}^{A}(\tau_s)\widehat{LPM}_{\gamma}^{B}(\tau_t)] = \frac{1}{n}E\left[(\tau_s - X)^{\gamma}(\tau_t - X)^{\gamma}|X \leq \tau_s, Y \leq \tau_t\right] F^{A,B}(\tau_s, \tau_t) -$$

$$\frac{1}{n}E\left[(\tau_s - X)^{\gamma}|X \leq \tau_s\right] E\left[(\tau_t - Y)^{\gamma}|Y \leq \tau_t\right] F^{A}(\tau_s)F^{B}(\tau_t) +$$

$$E\left[(\tau_s - X)^{\gamma}|X \leq \tau_s\right] E\left[(\tau_t - Y)^{\gamma}|Y \leq \tau_t\right] F^{A}(\tau_s)F^{B}(\tau_t).$$

Then, it follows that

$$\lim_{n\to\infty} n\, Cov\left(\widehat{LPM}_{\gamma}^{A}(\tau_s), \widehat{LPM}_{\gamma}^{B}(\tau_t)\right) = E\left[(\tau_s - X)^{\gamma}(\tau_t - Y)^{\gamma}|X \leq \tau_s, Y \leq \tau_t\right] F^{A,B}(\tau_s, \tau_t) -$$

$$E\left[(\tau_s - X)^{\gamma}|X \leq \tau_s\right] E\left[(\tau_t - Y)^{\gamma}|Y \leq \tau_t\right] F^{A}(\tau_s)F^{B}(\tau_t).$$

The tightness of the corresponding process follows from observing that the process characterized by $\widehat{LPM}_{\gamma}(\cdot)$ for $\gamma > 0$ is a linear functional of the empirical process defined for $\gamma = 0$, which satisfies the weak convergence result and therefore it is tight. The proof finishes by applying the continuous mapping theorem to the supremum functional.

Proof of Proposition 4: It is similar to the proof of proposition 1 in Barrett and Donald (2003). The proof of (i) involves characterizing the distribution of the test statistic and then using the covariance structure in theorem 3 to prove an inequality between suprema of Gaussian random variables.

Proof of Proposition 5: The power of the asymptotic test in theorem 3 is defined under $H_{1,\gamma}$ by

$P(T_{n,\gamma} > c_\gamma(1-\alpha))$. Substracting $\sup_{\tau\in\Omega}\delta(\tau)$ in both sides of the probability expression we obtain

$$P(T_{n,\gamma} > c_\gamma(1-\alpha)) = P\left(\sqrt{n}\sup_{\tau\in\Omega}\widehat{D}_\gamma(\tau) - \sup_{\tau\in\Omega}\delta(\tau) > c_\gamma(1-\alpha) - \sup_{\tau\in\Omega}\delta(\tau)\right)$$

$$\geq P\left(\sqrt{n}\sup_{\tau\in\Omega}\left(\widehat{D}_\gamma(\tau) - \frac{\delta(\tau)}{\sqrt{n}}\right) > c_\gamma(1-\alpha) - \sup_{\tau\in\Omega}\delta(\tau)\right),$$

and

$$\lim_{n\to\infty} P\left(\sqrt{n}\sup_{\tau\in\Omega}\left(\widehat{D}_\gamma(\tau) - \frac{\delta(\tau)}{\sqrt{n}}\right) > c_\gamma(1-\alpha) - \sup_{\tau\in\Omega}\delta(\tau)\right) > \alpha, \tag{35}$$

since $\sqrt{n}\sup_{\tau\in\Omega}\left(\widehat{D}_\gamma(\tau) - \frac{\delta(\tau)}{\sqrt{n}}\right)$ converges to $\sup_{\tau\in\Omega}G_\gamma(\tau)$, as does $T_{n,\gamma}$ under H_0. Now, by definition of the process $\sup_{\tau\in\Omega}\delta(\tau)$, the quantile of the asymptotic distribution in (35) is to the left of the asymptotic critical value $c_\gamma(1-\alpha)$ and implies therefore a rejection probability greater than α.

Proof of Theorem 4: The proof of this result follows directly from the proof of theorem 3. We need first to see that the result in Proposition 3 holds for the conditional lower partial moments measure. In particular we will show for Portfolio A that

$$\sqrt{n_u}\left(\widehat{LPM}^A_{\gamma,u}(\tau) - LPM^A_{\gamma,u}(\tau)\right) \xrightarrow{d} N\left(0, k^A_{2\gamma,u}(\tau)F^A_u(\tau) - (k^A_{\gamma,u}(\tau)F^A_u(\tau))^2\lambda(u)\right), \tag{36}$$

for all fixed $\tau \in \Omega$, and $\gamma \geq 0$.

Note that in this case, in contrast to the unconditional case, the standardizing sequence n_u is a random sequence. In fact, by Chebychev's inequality we have that

$$\widehat{\lambda}(u) = \lambda(u) + o_P(1), \tag{37}$$

with $\lambda(u)$ defined in assumption A.3 and $\widehat{\lambda}(u)$ in expression (10). Now, we prove (36).

Proof: Suppose we have n $i.i.d$ vectors of observations from the random variable R^A, and let $\widehat{LPM}^A_{\gamma,u}(\tau)$ be the estimator of $LPM^A_{\gamma,u}(\tau)$ introduced in (14),

$$\widehat{LPM}^A_{\gamma,u}(\tau) = \tfrac{1}{n_u}\sum_{i=1}^n(\tau - R^A_i)^\gamma I(R^A_i \leq \tau, R_1 \leq u, \ldots, R_m \leq u).$$

Now, by the law of iterated expectations

$\frac{1}{n}E[\sum_{i=1}^n(\tau - R^A_i)^\gamma I(R^A_i \leq \tau, R_{1,i} \leq u, \ldots, R_{m,i} \leq u)] =$

$E[(\tau - R^A)^\gamma | R^A \leq \tau, R_1 \leq u, \ldots, R_m \leq u]P(R^A \leq \tau, R_1 \leq u, \ldots, R_m \leq u).$

Further, by the properties of the variance and the serial independence of the observations it is immediate that

$V\left(\frac{1}{n}\sum_{i=1}^{n}(\tau - R_i^A)^\gamma I(R_i^A \leq \tau, R_{1,i} \leq u, \ldots, R_{m,i} \leq u)\right) =$

$\frac{1}{n}E[(\tau - R^A)^{2\gamma}|X \leq \tau, R_1 \leq u, \ldots, R_m \leq u)]P(R^A \leq \tau, R_1 \leq u, \ldots, R_m \leq u)-$

$-\frac{1}{n}E^2[(\tau - R^A)^\gamma|R^A \leq \tau, R_1 \leq u, \ldots, R_m \leq u](P(R^A \leq \tau, R_1 \leq u, \ldots, R_m \leq u))^2,$

that converges to zero as $n \to \infty$. Therefore, using the above notation we have

$$\frac{1}{n}\sum_{i=1}^{n}(\tau - R_i^A)^\gamma I(R_i^A \leq \tau, R_{1,i} \leq u, \ldots, R_{m,i} \leq u) \xrightarrow{p} k_{\gamma,u}^A(\tau)P(R^A \leq \tau, R_1 \leq u, \ldots, R_m \leq u), \tag{38}$$

and

$$\sqrt{n}\frac{\frac{1}{n}\sum_{i=1}^{n}(\tau - R_i^A)^\gamma I(R_i^A \leq \tau, R_{1,i} \leq u, \ldots, R_{m,i} \leq u) - k_{\gamma,u}^A(\tau)P(R^A \leq \tau, R_1 \leq u, \ldots, R_m \leq u)}{\sqrt{k_{2\gamma,u}^A(\tau)P(R^A \leq \tau, R_1 \leq u, \ldots, R_m \leq u) - (k_{\gamma,u}^A(\tau)P(R^A \leq \tau, R_1 \leq u, \ldots, R_m \leq u))^2}} \xrightarrow{d} N(0,1). \tag{39}$$

The consistency of the conditional LPM measure follows from (38), (37) and Slutsky theorem. Thus

$$\widehat{LPM}_{\gamma,u}^A(\tau) \xrightarrow{p} k_{\gamma,u}^A(\tau)F_u^A(\tau) = LPM_{\gamma,u}^A(\tau), \tag{40}$$

since $F_u^A(\tau) = \frac{P(R^A \leq \tau, R_1 \leq u, \ldots, R_m \leq u)}{\lambda(u)}$.

More importantly, dividing numerator and denominator in (39) by $\widehat{\lambda}(u)$, and using Slutsky theorem we obtain (36)

$$\sqrt{n_u}\left(\widehat{LPM}_{\gamma,u}^A(\tau) - LPM_{\gamma,u}^A(\tau)\right) \xrightarrow{d} N\left(0, k_{2\gamma,u}^A(\tau)F_u^A(\tau) - (k_{\gamma,u}^A(\tau)F_u^A(\tau))^2\lambda(u)\right).$$

With this result and assumption A.5 it is not difficult to show that

$$\sqrt{n_u}\left(\widehat{D}_{\gamma,u}(\tau) - D_{\gamma,u}(\tau)\right) \xrightarrow{d} N\left(0, V_{\gamma,u}(\tau)\right), \tag{41}$$

with

$$V_{\gamma,u}(\tau) = \left(k_{2\gamma,u}^A(\tau)F_u^A(\tau) - (k_{\gamma,u}^A(\tau)F_u^A(\tau))^2\lambda(u)\right) + \left(k_{2\gamma,u}^B(\tau)F_u^B(\tau) - (k_{\gamma,u}^B(\tau)F_u^B(\tau))^2\lambda(u)\right) -$$
$$2\left(k_{\gamma,u}^{A,B}(\tau,\tau)F_u^{A,B}(\tau,\tau) - k_{\gamma,u}^A(\tau)F_u^A(\tau)k_{\gamma,u}^B(\tau)F_u^B(\tau)\lambda(u)\right).$$

With these results in place, the proof of theorem 4 follows from the proof of theorem 3 after applying Slutsky theorem and replacing n by n_u appropriately.

Proof of Corollary 2: Let ε_t^i be the error sequence of a possibly heteroscedastic time series defined in (31), and $\widehat{\varepsilon}_t^i$ be the corresponding residual sequence. The relevant test statistics are $T_{n,\gamma}$ and $\widehat{T}_{n,\gamma}$ respectively.

The latter test statistic can be expressed as

$$\widehat{T}_{n,\gamma} = \sqrt{n} \sup_{\tau \in \Omega} \left(\left(\widehat{LPM}_{\gamma}^{\widehat{A}}(\tau) - \widehat{LPM}_{\gamma}^{A}(\tau) \right) - \left(\widehat{LPM}_{\gamma}^{\widehat{B}}(\tau) - \widehat{LPM}_{\gamma}^{B}(\tau) \right) + \widehat{D}_{\gamma}(\tau) \right), \tag{42}$$

with $\widehat{LPM}_{\gamma}^{\widehat{i}}$ denoting the downside risk measure computed from the estimated residuals of the time series regression models for $i = A, B$.

It is sufficient to show that $\sup_{\tau \in \Omega} \left| \sqrt{n} \left(\widehat{LPM}_{\gamma}^{\widehat{i}}(\tau) - \widehat{LPM}_{\gamma}^{i}(\tau) \right) \right| \xrightarrow{p} 0$ for $i = A, B$, to obtain the desired result. Without loss of generality and to ease notation we will denote the error and residual variables without using the index i. Then, the difference above can be written as

$$\sup_{\tau \in \Omega} \left| \frac{1}{\sqrt{n}} \sum_{t=1}^{n} (\tau - \widehat{\varepsilon}_t)^{\gamma} I(\widehat{\varepsilon}_t \leq \tau) - \frac{1}{\sqrt{n}} \sum_{t=1}^{n} (\tau - \varepsilon_t)^{\gamma} I(\varepsilon_t \leq \tau) \right|, \tag{43}$$

for both portfolios A and B.

In order to show the convergence to zero in probability of this expression we show first that (43) is upper bounded, when $n \to \infty$, by

$$\sup_{\tau \in \Omega} \left| \frac{1}{\sqrt{n}} \sum_{t=1}^{n} \left[(\tau - \widehat{\varepsilon}_t)^{\gamma} - (\tau - \varepsilon_t)^{\gamma} \right] \left[I(\widehat{\varepsilon}_t \leq \tau) - I(\varepsilon_t \leq \tau) \right] \right|. \tag{44}$$

For this, it is sufficient that

$$\lim_{n \to \infty} \left(\frac{1}{\sqrt{n}} \sum_{t=1}^{n} (\tau - \varepsilon_t)^{\gamma} [I(\widehat{\varepsilon}_t \leq \tau) - I(\varepsilon_t \leq \tau)] \right) \leq \lim_{n \to \infty} \left(\frac{1}{\sqrt{n}} \sum_{t=1}^{n} [(\tau - \varepsilon_t)^{\gamma} - (\tau - \widehat{\varepsilon}_t)^{\gamma}] I(\varepsilon_t \leq \tau) \right). \tag{45}$$

Expression (45) follows from Koul and Ling (2006, theorem 4.1 and lemma 4.1) that shows

$$\sup_{\tau \in \Omega} \left| \frac{1}{\sqrt{n}} \sum_{t=1}^{n} \left[I(\widehat{\varepsilon}_t \leq \tau) - I(\varepsilon_t \leq \tau) \right] \right| \xrightarrow{p} 0. \tag{46}$$

Now, using the properties of the supremum in (44) the following inequality is obtained:

$$\sup_{\tau \in \Omega} \left| \frac{1}{\sqrt{n}} \sum_{t=1}^{n} \left[(\tau - \widehat{\varepsilon}_t)^{\gamma} - (\tau - \varepsilon_t)^{\gamma} \right] \left[I(\widehat{\varepsilon}_t \leq \tau) - I(\varepsilon_t \leq \tau) \right] \right| \leq$$

$$\sup_{\tau \in \Omega} \left| \frac{1}{\sqrt{n}} \sum_{t=1}^{n} \left[(\tau - \widehat{\varepsilon}_t)^{\gamma} - (\tau - \varepsilon_t)^{\gamma} \right] \right| \sup_{\tau \in \Omega} \left| \frac{1}{\sqrt{n}} \sum_{t=1}^{n} \left[I(\widehat{\varepsilon}_t \leq \tau) - I(\varepsilon_t \leq \tau) \right] \right|. \tag{47}$$

By assumption A.6, $\frac{1}{\sqrt{n}} \sum_{t=1}^{n} (\widehat{\varepsilon}_t - \varepsilon_t)$ and powers of it are bounded in probability. Therefore, to derive the convergence of expression (47) to zero we only need to show that the second right term is $o_P(1)$. This follows from (46).

TABLES

$\nu = 5$	Method	$\gamma = 0$			$\gamma = 1$		
$\rho = 0$		10%	5%	1%	10%	5%	1%
$n = 50$	Gp-value	0.118	0.060	0.008	0.140	0.070	0.018
	p-value	0.110	0.054	0.004	0.088	0.050	0.012
$n = 100$	Gp-value	0.112	0.060	0.010	0.104	0.046	0.012
	p-value	0.108	0.064	0.010	0.110	0.052	0.016
$n = 500$	Gp-value	0.078	0.036	0.002	0.112	0.062	0.016
	p-value	0.108	0.044	0.008	0.086	0.048	0.004
$\rho = 0.4$		10%	5%	1%	10%	5%	1%
$n = 50$	Gp-value	0.116	0.058	0.008	0.142	0.080	0.014
	p-value	0.090	0.046	0.006	0.098	0.042	0.020
$n = 100$	Gp-value	0.114	0.066	0.010	0.122	0.070	0.006
	p-value	0.104	0.054	0.018	0.106	0.044	0.016
$n = 500$	Gp-value	0.098	0.046	0.008	0.094	0.046	0.008
	p-value	0.116	0.058	0.006	0.094	0.048	0.006
$\rho = 0.8$		10%	5%	1%	10%	5%	1%
$n = 50$	Gp-value	0.096	0.038	0.010	0.126	0.060	0.014
	p-value	0.094	0.038	0.004	0.094	0.042	0.018
$n = 100$	Gp-value	0.098	0.048	0.012	0.120	0.062	0.008
	p-value	0.114	0.050	0.014	0.090	0.044	0.012
$n = 500$	Gp-value	0.104	0.066	0.008	0.108	0.048	0.010
	p-value	0.094	0.046	0.020	0.114	0.054	0.006

TABLE 1. Empirical size for $H_{0,\gamma}$, $\gamma = 0, 1$ for a standardized bivariate Student-t with $\nu = 5$ degrees of freedom and correlation parameter ρ. Gp : asymptotic p-value, p : multiplier method p-value. n sample size. $\mathbb{B} = 1000$ Monte-Carlo simulations to approximate the exact finite-sample distribution. $mc = 500$ Monte-Carlo iterations to approximate the nominal size. $m = 100$ partitions of the real line to generate observations from the estimated asymptotic Gaussian process with covariance function (20).

$\nu = 10$	Method	$\gamma = 0$			$\gamma = 1$		
$\rho = 0$		10%	5%	1%	10%	5%	1%
$n = 50$	Gp-value	0.106	0.042	0.012	0.162	0.098	0.032
	p-value	0.094	0.046	0.002	0.086	0.044	0.016
$n = 100$	Gp-value	0.112	0.056	0.014	0.126	0.064	0.022
	p-value	0.112	0.062	0.008	0.116	0.066	0.016
$n = 500$	Gp-value	0.098	0.048	0.008	0.122	0.086	0.024
	p-value	0.114	0.038	0.008	0.102	0.040	0.004
$\rho = 0.4$		10%	5%	1%	10%	5%	1%
$n = 50$	Gp-value	0.092	0.054	0.006	0.142	0.094	0.026
	p-value	0.088	0.044	0.010	0.082	0.042	0.010
$n = 100$	Gp-value	0.114	0.062	0.012	0.114	0.074	0.020
	p-value	0.116	0.062	0.010	0.118	0.060	0.014
$n = 500$	Gp-value	0.104	0.052	0.006	0.108	0.060	0.016
	p-value	0.114	0.048	0.008	0.112	0.050	0.004
$\rho = 0.8$		10%	5%	1%	10%	5%	1%
$n = 50$	Gp-value	0.080	0.032	0.004	0.120	0.076	0.010
	p-value	0.088	0.040	0.002	0.098	0.042	0.012
$n = 100$	Gp-value	0.114	0.066	0.012	0.130	0.066	0.022
	p-value	0.108	0.048	0.002	0.090	0.042	0.012
$n = 500$	Gp-value	0.146	0.062	0.012	0.110	0.058	0.008
	p-value	0.102	0.044	0.012	0.104	0.050	0.010

TABLE 2. Empirical size for $H_{0,\gamma}$, $\gamma = 0, 1$ for a standardized bivariate Student-t with $\nu = 10$ degrees of freedom and correlation parameter ρ. Gp : asymptotic p-value, p : multiplier method p-value. n sample size. $\mathbb{B} = 1000$ Monte-Carlo simulations to approximate the exact finite-sample distribution. $mc = 500$ Monte-Carlo iterations to approximate the nominal size. $m = 100$ partitions of the real line to generate observations from the estimated asymptotic Gaussian process with covariance function (20).

$\nu = 5$	Method	$\gamma = 0$			$\gamma = 1$		
$\rho = 0$		10%	5%	1%	10%	5%	1%
$n = 200$	Gp-value	0.080	0.048	0.006	0.098	0.048	0.008
$(n_u \approx 50)$	p-value	0.102	0.050	0.002	0.098	0.044	0.006
$n = 400$	Gp-value	0.094	0.038	0.010	0.124	0.064	0.010
$(n_u \approx 100)$	p-value	0.098	0.048	0.008	0.086	0.050	0.012
$n = 2000$	Gp-value	0.078	0.040	0.006	0.116	0.064	0.012
$(n_u \approx 500)$	p-value	0.092	0.042	0.008	0.098	0.052	0.010
$\rho = 0.4$		10%	5%	1%	10%	5%	1%
$n = 200$	Gp-value	0.092	0.040	0.008	0.130	0.062	0.010
$(n_u \approx 50)$	p-value	0.112	0.056	0.008	0.082	0.048	0.012
$n = 400$	Gp-value	0.088	0.036	0.004	0.122	0.072	0.022
$(n_u \approx 100)$	p-value	0.078	0.042	0.006	0.092	0.040	0.012
$n = 2000$	Gp-value	0.104	0.044	0.010	0.144	0.086	0.020
$(n_u \approx 500)$	p-value	0.114	0.064	0.018	0.112	0.052	0.006
$\rho = 0.8$		10%	5%	1%	10%	5%	1%
$n = 200$	Gp-value	0.110	0.052	0.004	0.110	0.070	0.018
$(n_u \approx 50)$	p-value	0.098	0.054	0.006	0.092	0.050	0.010
$n = 400$	Gp-value	0.106	0.050	0.016	0.098	0.050	0.014
$(n_u \approx 100)$	p-value	0.120	0.066	0.008	0.102	0.044	0.004
$n = 2000$	Gp-value	0.084	0.028	0.002	0.108	0.050	0.012
$(n_u \approx 500)$	p-value	0.078	0.030	0.006	0.092	0.038	0.010

TABLE 3. Empirical size for $H_{0,\gamma,u}$, $\gamma = 0, 1$, $u = 0$, for a standardized bivariate Student-t with $\nu = 5$ degrees of freedom and correlation parameter ρ. Gp : asymptotic p-value, p : multiplier method p-value. n is length of original sample (n_u observations available for the tests). $\mathbb{B} = 1000$ Monte-Carlo simulations to approximate the exact finite-sample distribution. $mc = 500$ Monte-Carlo iterations to approximate the nominal size. $m = 100$ partitions of the real line to generate observations from the estimated asymptotic Gaussian process with covariance function (28).

$\nu = 10$	Method	$\gamma = 0$			$\gamma = 1$		
$\rho = 0$		10%	5%	1%	10%	5%	1%
$n = 200$	Gp-value	0.100	0.052	0.014	0.128	0.086	0.012
$(n_u \approx 50)$	p-value	0.100	0.046	0.008	0.110	0.044	0.006
$n = 400$	Gp-value	0.102	0.050	0.006	0.128	0.074	0.020
$(n_u \approx 100)$	p-value	0.118	0.056	0.008	0.100	0.058	0.018
$n = 2000$	Gp-value	0.096	0.054	0.012	0.116	0.064	0.020
$(n_u \approx 500)$	p-value	0.096	0.054	0.006	0.100	0.040	0.006
$\rho = 0.4$		10%	5%	1%	10%	5%	1%
$n = 200$	Gp-value	0.088	0.040	0.004	0.132	0.074	0.018
$(n_u \approx 50)$	p-value	0.106	0.044	0.006	0.106	0.052	0.012
$n = 400$	Gp-value	0.078	0.040	0.008	0.090	0.046	0.014
$(n_u \approx 100)$	p-value	0.088	0.040	0.010	0.088	0.044	0.014
$n = 2000$	Gp-value	0.090	0.044	0.010	0.114	0.050	0.008
$(n_u \approx 500)$	p-value	0.094	0.048	0.010	0.124	0.062	0.010
$\rho = 0.8$		10%	5%	1%	10%	5%	1%
$n = 200$	Gp-value	0.112	0.048	0.016	0.160	0.092	0.014
$(n_u \approx 50)$	p-value	0.100	0.054	0.008	0.088	0.052	0.008
$n = 400$	Gp-value	0.088	0.024	0.008	0.096	0.044	0.006
$(n_u \approx 100)$	p-value	0.114	0.072	0.010	0.116	0.046	0.010
$n = 2000$	Gp-value	0.146	0.062	0.012	0.110	0.058	0.008
$(n_u \approx 500)$	p-value	0.092	0.048	0.010	0.092	0.036	0.006

TABLE 4. Empirical size for $H_{0,\gamma,u}$, $\gamma = 0, 1$, $u = 0$, for a standardized bivariate Student-t with $\nu = 10$ degrees of freedom and correlation parameter ρ. Gp : asymptotic p-value, p : multiplier method p-value. n is length of original sample (n_u observations available for the tests). $\mathbb{B} = 1000$ Monte-Carlo simulations to approximate the exact finite-sample distribution. $mc = 500$ Monte-Carlo iterations to approximate the nominal size. $m = 100$ partitions of the real line to generate observations from the estimated asymptotic Gaussian process with covariance function (28).

$\nu = 5$, $\alpha = 0.05$	$\gamma = 0$			$\gamma = 1$		
$\rho = 0$ / $c =$	0.5	1	5	0.5	1	5
$n = 50$	0.102	0.172	0.980	0.162	0.230	0.912
$n = 100$	0.136	0.300	1.000	0.178	0.286	0.988
$n = 500$	0.336	0.768	1.000	0.240	0.522	1.000
$\rho = 0.4$ / $c =$	0.5	1	5	0.5	1	5
$n = 50$	0.110	0.196	0.996	0.134	0.216	0.978
$n = 100$	0.136	0.332	1.000	0.182	0.344	1.000
$n = 500$	0.408	0.890	1.000	0.290	0.692	0.618
$\rho = 0.8$ / $c =$	0.5	1	5	0.5	1	5
$n = 50$	0.226	0.448	1.000	0.264	0.464	1.000
$n = 100$	0.232	0.554	1.000	0.284	0.570	1.000
$n = 500$	0.714	0.998	1.000	0.578	0.986	1.000

TABLE 5. Empirical power for $H_{0,\gamma}$, $\gamma = 0, 1$. The family of alternative hypotheses are $F^A(\tau) = F^B(\tau) + \frac{cf^B(\tau)}{\sqrt{n}}$ with F^B and f^B a Student-t distribution and density function with $\nu = 5$ and $c = 0.5, 1, 5$. The correlation parameter is ρ, α denotes significance level and n sample size. $\mathbb{B} = 1000$ Monte-Carlo simulations to approximate the exact finite-sample distribution. $mc = 500$ Monte-Carlo iterations to approximate the nominal size. $m = 100$ partitions of the real line to generate observations from the estimated asymptotic Gaussian process with covariance function (20).

$\nu = 10$, $\alpha = 0.05$		$\gamma = 0$			$\gamma = 1$	
$\rho = 0$ / $c =$	0.5	1	5	0.5	1	5
$n = 50$	0.056	0.106	0.926	0.198	0.268	0.988
$n = 100$	0.134	0.276	0.798	0.108	0.218	0.980
$n = 500$	0.070	0.150	0.952	0.074	0.168	0.984
$\rho = 0.4$ / $c =$	0.5	1	5	0.5	1	5
$n = 50$	0.166	0.276	0.990	0.114	0.192	0.974
$n = 100$	0.132	0.290	1.000	0.164	0.308	0.996
$n = 500$	0.324	0.792	1.000	0.274	0.662	1.000
$\rho = 0.8$ / $c =$	0.5	1	5	0.5	1	5
$n = 50$	0.212	0.406	1.000	0.226	0.430	1.000
$n = 100$	0.202	0.480	1.000	0.244	0.528	1.000
$n = 500$	0.580	0.990	1.000	0.570	0.986	1.000

TABLE 6. Empirical power for $H_{0,\gamma}$, $\gamma = 0, 1$. The family of alternative hypotheses are $F^A(\tau) = F^B(\tau) + \frac{cf^B(\tau)}{\sqrt{n}}$ with F^B and f^B a Student-t distribution and density function with $\nu = 10$ and $c = 0.5, 1, 5$. The correlation parameter is ρ, α denotes significance level and n sample size. $\mathbb{B} = 1000$ Monte-Carlo simulations to approximate the exact finite-sample distribution. $mc = 500$ Monte-Carlo iterations to approximate the nominal size. $m = 100$ partitions of the real line to generate observations from the estimated asymptotic Gaussian process with covariance function (20).

$\nu = 5,\ \alpha = 0.05$	$\gamma = 0$			$\gamma = 1$		
$\rho = 0\ /\ c =$	0.5	1	5	0.5	1	5
$n = 200$	0.044	0.052	0.230	0.070	0.040	0.070
$n = 400$	0.100	0.282	1.000	0.386	0.548	0.996
$n = 2000$	0.826	1.000	1.000	0.492	0.870	1.000
$\rho = 0.4\ /\ c =$	0.5	1	5	0.5	1	5
$n = 200$	0.246	0.556	1.000	0.178	0.360	1.000
$n = 400$	0.302	0.722	1.000	0.218	0.472	1.000
$n = 2000$	0.864	1.000	1.000	0.526	0.950	1.000
$\rho = 0.8\ /\ c =$	0.5	1	5	0.5	1	5
$n = 200$	0.328	0.752	1.000	0.246	0.536	1.000
$n = 400$	0.470	0.954	1.000	0.284	0.704	1.000
$n = 2000$	0.984	1.000	1.000	0.852	1.000	1.000

TABLE 7. Empirical power for $H_{0,\gamma,u}$, $\gamma = 0, 1$, $u = 0$. The family of alternative hypotheses are $F^A(\tau) = F^B(\tau) + \frac{cf^B(\tau)}{\sqrt{n}}$ with F^B and f^B a Student-t distribution and density function with $\nu = 5$ and $c = 0.5, 1, 5$. The correlation parameter is ρ, α denotes significance level and n is length of original sample ($n_u \approx n/4$ observations available for the tests). $\mathbb{B} = 1000$ Monte-Carlo simulations to approximate the exact finite-sample distribution. $mc = 500$ Monte-Carlo iterations to approximate the nominal size. $m = 100$ partitions of the real line to generate observations from the estimated asymptotic Gaussian process with covariance function (28).

$\nu = 10,\ \alpha = 0.05$	$\gamma = 0$			$\gamma = 1$		
$\rho = 0\ /\ c =$	0.5	1	5	0.5	1	5
$n = 200$	0.144	0.350	1.000	0.046	0.124	0.968
$n = 400$	0.200	0.606	1.000	0.152	0.338	1.000
$n = 2000$	0.684	1.000	1.000	0.318	0.834	1.000
$\rho = 0.4\ /\ c =$	0.5	1	5	0.5	1	5
$n = 200$	0.150	0.372	1.000	0.164	0.294	1.000
$n = 400$	0.236	0.610	1.000	0.142	0.378	1.000
$n = 2000$	0.766	1.000	1.000	0.532	0.944	1.000
$\rho = 0.8\ /\ c =$	0.5	1	5	0.5	1	5
$n = 200$	0.296	0.714	1.000	0.264	0.566	1.000
$n = 400$	0.406	0.920	1.000	0.256	0.712	1.000
$n = 2000$	0.944	1.000	1.000	0.808	1.000	1.000

TABLE 8. Empirical power for $H_{0,\gamma,u}$, $\gamma = 0, 1$, $u = 0$. The family of alternative hypotheses are $F^A(\tau) = F^B(\tau) + \frac{cf^B(\tau)}{\sqrt{n}}$ with F^B and f^B a Student-t distribution and density function with $\nu = 10$ and $c = 0.5, 1, 5$. The correlation parameter is ρ, α denotes significance level and n is length of original sample ($n_u \approx n/4$ observations available for the tests). $\mathbb{B} = 1000$ Monte-Carlo simulations to approximate the exact finite-sample distribution. $mc = 500$ Monte-Carlo iterations to approximate the nominal size. $m = 100$ partitions of the real line to generate observations from the estimated asymptotic Gaussian process with covariance function (28).

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