Resale in Auctions with Financial Constraints*

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Abstract

This paper analyzes auctions where bidders face financial constraints that may force them to resell part of the property of the good (or subcontract part of a project) at a resale market. First we show that the inefficient speculative equilibria of second-price auctions (Garratt and Tröger, 2006) generalizes to situations with partial resale where only the high value bidder is financially constrained. However, when all players face financial constraints the speculation inefficiency is mitigated and if constraints are severe only efficient equilibria survive. Therefore, for auctioning big facilities or contracts where all bidders are financially constrained and there is a resale market, the second price auction remains a simple and appropriate mechanism to achieve an efficient allocation.

Keywords: auctions, resale, financial constraints, subcontracting

JEL Class. : D44, L1, D82

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Introduction

Competition for acquiring a public firm or winning the allocation of a big facility is often characterized by the presence of a small number of qualified bidders who assign a large value to the good although they may face financial constraints. Because of this, the acquirer can share the property of the good with other buyers. One specific example is the allocation problem of the European Spallation Source that has to be allocated to a single country or location but whose property can be shared after the initial allocation, to alleviate the winner's financial constraints.[1] Similarly, operating licences (e.g., in the telecommunications sector) are awarded to one firm, and (some or all of) the actual services can be subcontracted.[2] Beyond these particular examples, this framework fits privatization processes involving the sale of a public firm to a single buyer meeting the (legal, administrative) requirements or the procurement of large-scale production contracts in the public sector.

Previous work on auctions with resale relies mostly on the potential inefficiencies of the auction allocation mechanism to provide the basis for resale. An inefficient allocation may result from noisy signals at the time of the auction, as in Haile (2000, 2001, 2003), or from asymmetries among bidders when the auction is conducted as first price, as in Gupta and Lebrun (1998) or Hafalir and Krishna (2008). In contrast, in our model the auction is a second price auction and the resale market is justified by the presence of financial constraints which may force the winner of the auction to sell part of the property of the good.[3]

Our paper is related to Garratt and Tröger's (2006), who have shown that a second price auction can result in inefficient allocations in the presence of speculators who only value the object by its resale price. In a similar context,[4] we show that the introduction of financial constraints (players' wealth may be below their use value) and partial resale can bring back the efficiency of the second price auction. However, when the financial constraints are slight, an extension of Garratt and Tröger's inefficiency result is delivered.

[1]Partial resale after an auction or contract award may come from the divestment requirement imposed by antitrust authorities on the winners whenever their market shares increase substantially after the award.

[2]Horizontal subcontracting is a common phenomenon in many industries (see, e.g., Kamien et al., 1989; Spiegel, 1993; and Chen et al., 2004, for further discussion and examples).

[3]Splitting of an auction target was the outcome in the UK brewer Scottish & Newcastle accepted takeover bid from Carlsberg and Heineken. Despite splitting up the bussiness, concerns were raised that Calsberg may struggle to fund the deal.

[4]We also model an auction in which a strong buyer with a private use value competes against a weak buyer whose use value is known to be lower than the rival's; although in our model the weak buyer is not a pure speculator because he values consuming the good, his behavior can be interpreted as speculative.

The literature on auctions with resale has focused on the case of total resale of the good and to the best of our knowledge this is the first paper introducing partial resale in an auction framework under incomplete information. However, partial resale (or horizontal subcontracting) is a common assumption in two-stage contract games under other modes of competition. Kamien, Li and Samet (1989) study a procurement auction for an endogenously determined quantity of a perfectly divisible good with two identical and completely informed bidders. In their setup decreasing returns create a need for subcontracting, as financial constraints do in ours. Whereas we assume that the object to be procured has a fixed size, we relax two other important assumptions such as symmetry and complete information. By considering a double source of asymmetry (in use-values and in wealth) our paper is also close to that of Spiegel (1993) where firms are supposed to compete in quantities rather than prices again under complete information. As in our model, incentives to resale arise from asymmetries, and at the bidding stage firms take advantage of their (relative) strength.

The presence of a resale market affects bidding strategies in several ways. First, it may generate a more aggressive bidding behavior at the first stage as the auction winner can get extra resources from reselling, which affects players' endogenous valuations. Second, first stage bidding can reveal information on the loser's use value which will be taken into account at the post-auction resale market.

The presence of financial constraints also affects the first stage bids as the possibility of default creates a link between the resale price and the auction price.[5] This link can affect equilibrium behavior at the auction stage as it may induce a potential loser to set the auction price so as to fine-tune the winner's resale offer. The loser bidder may have incentives to raise the auction price to make the winner financially constrained.[6] Interestingly, a loser may also have incentives to decrease the auction price to soften his competitor's financial constraint. A buyer with severe financial constraints can behave very aggressively when setting his resale offers as the opportunity cost for a rejected offer -namely, losing his own wealth- is very

[5] Examples of real-life auctions in which financial constraints have played a key role abound. In the privatization of ENTel (an Argentinean telecommunications company) the winner of the ENTel North, Bell Atlantic and Manufacturers Hanover Corporation, failed to obtain the necessary financial resources to meet their bid. ENTel North was then awarded to the next bidder, a consortium of buyers including France Telecom and J. P. Morgan. Similarly, in the European 3G Telecom Auctions some firms faced difficulties in borrowing (see Klemperer, 2002).

[6] This effect is reminiscent of the incentives for bidders in multiple-object auctions to bid aggressively on one object, with the objective of raising the price paid by the rival and depleting his budget, so that other objects may be obtained at a lower price (Benoît and Krishna, 2001).

low. Facing such a rival, a loser buyer may prefer setting a low auction price to soften his competitor's financial constraints.

We show that whenever the weak buyer is not financially constrained the structure of Garratt and Tröger's speculative equilibrium, and its resulting inefficiency, hold true. In equilibrium high-value strong players bid their endogenous valuation or use value and win the auction, while the low-values lose the auction pooling at either the valuation or use value of the lowest type.[7] The weak buyer bids his valuation taking into account that as a winner he will resell optimally to the set of low types who are pooling (cf. Proposition 2). It is worth noting that under financial constraints partial resale is necessary for this result to hold, and that partial resale boosts the auctioneer's revenues.

The crucial assumption driving inefficient speculative equilibria with incomplete information is the absence of financial constraints for the weak buyer. When the weak buyer does not have unlimited wealth, the inefficiency created by speculative behavior disappears if his wealth is below a critical level, even if the weak buyer wins the auction. Furthermore, when all players face the same financial constraints and they are severe enough, the inefficient speculative equilibria disappear (cf. Proposition 3). Similarly, if the weak buyer's budget is sufficiently low then some of the inefficient equilibria disappear as well. Therefore, for second price auctions of big facilities or contracts where all players are financially constrained and there is a resale market, inefficient speculative behavior is not an equilibrium phenomenon.

Another result stemming from our analysis is that financial constraints can bring about efficient collusive-like equilibria in which the strong buyer bids low to soften her rival's financial situation and to fine-tune the resale price she will be offered (cf. Proposition 4). Since the strong bidder is the only player who has private information it is not surprising that efficiency is achieved when she turns out to win the initial auction so that the price-taker has no private information. The novelty here is that efficiency is achieved in the event where the strong bidder turns out to lose and therefore becomes the price-taker. Although collusive equilibria hurt auctioneer's revenues, they result in efficient allocations.

The paper is organized as follows. Section 1 presents the model, which is solved in Section 2 under complete information. In Section 3 we solve the resale stage and the bidding stage

[7]Other related models where partial pooling occurs in equilibrium are Haile (2000) and Jehiel and Moldovanu (2000). Our pooling at the lowest "common" valuation plays a role similar to the pooling at the reserve price in Haile's. In Jehiel and Moldovanu (2000) pooling is the result of the externality that awarding the object to one bidder imposes on the others. As Haile points out, the existence of a resale market imposes a positive externality not only on the auction loser but also on the winner as the option value of selling in the resale market is positive.

under private information on use values and present the main results of the paper. Section 4 is devoted to analyzing the role of the weak buyer's financial situation for the efficiency of the auction. In Section 5 we show that the results hold true when pure speculators participate in the auction, and Section 6 presents some concluding remarks. All proofs are relegated to the Appendix.

1 The model

A government wants to auction the location of a facility, or to assign a big project to one of two potential risk-neutral buyers, bidder A ("she") and bidder B ("he"). The worth of the auctioned good may be large compared to the buyers' wealth, so that default may occur. Each buyer i has a budget or wealth w_i. As in Zheng (2001), a buyer's wealth represents both her liquidity constraint and her liability. Thus, w_A and w_B will set the maximum amount by which buyers can be penalized if they default. We assume that w_A and w_B are known.

Buyer i has use value v_i when i is the solo owner of the good.[8] If i obtains a fraction z of the property of the good then her use value will be zv_i. Use value v_A is private information. It has distribution F with associated density f and support $[\underline{v}_A, \bar{v}_A]$. We will assume that f is log-concave. We will denote by h the hazard rate of F, i.e. $h(x) = \frac{f(x)}{1-F(x)}$ which is non-decreasing by the log-concavity of f (see Bagnoli and Bergstrom (2005)). Monotonicity of the hazard rate implies that the virtual use value ($\psi(v_A) = v_A - \frac{1-F}{f}$) is strictly increasing.[9] Use value v_B is common knowledge, with $v_B \leq \underline{v}_A$.

We will define buyers' ex-ante financial situation by the relationship between their use values and their wealth. We will say that buyer i is ex-ante financially constrained if $v_i > w_i$, and that i is ex-ante unconstrained otherwise.

An important assumption of the model is the inability of the initial seller to prohibit resale. Because of this, buyers participate *de facto* in a two-stage selling game; in the first stage, they compete for the object at an ascending auction, and in the second stage the auction winner can make a take-it-or-leave-it offer to the auction loser for a part of the property or for the entire object. Player i can always guarantee himself w_i by not participating in the

[8] Due to the possibility of resale, and following Haile (2003), we will distinguish between buyers' *use value* of the object, which is exogenously determined, and buyers' *valuation* - the value players attach to winning the auction- which will be endogenously determined.

[9] The hazard rate represents the instantaneous probability that the valuation of buyer i is v_i given that it is not smaller than v_i. A sufficient condition for the hazard rate to be increasing is the log-concavity of f. See Bagnoli and Bergstrom (2005) for the class of log-concave distributions.

selling game.

At the first stage, the good is sold through a second-price auction and assigned to a single buyer. Bids are denoted $b = (b_A, b_B)$. Ties are solved in favor of the player who values the object the most. We will denote by p the auction price. The price will be paid to the auctioneer by the winner at the end of the game, i.e., after resale has taken place. The loser does not pay anything to the auctioneer. We will denote by U_i^{Wj} the utility of player i, $i = A, B$, when the auction winner is j, $j = A, B$. At the end of this first stage the auction price is announced publicly. This bid announcement policy may prevent the existence of a first stage equilibrium in strictly increasing bidding strategies. However, it is consistent with most real life auctions given the prevalence of the English format.

At the second stage, the winner of the first stage auction, i, must decide whether to keep the object or to resell it, and if so, at what price and which fraction. We will assume that the winner has all the bargaining power.[10] Thus, resale takes place via monopoly pricing - the winner of the auction makes an offer to the loser after updating her/his prior beliefs based on the information revealed from winning and from the auction price. A resale offer by bidder i, O_i, is a pair $[r_i, z_i]$ which comprises a resale price r_i and a fraction of the good z_i. Keeping the object is dominated by reselling if the auction winner does not have enough wealth to cover the auction price, i.e., if $w_i < p$.[11] We will denote the option of keeping the object by the offer $O_i = [0, 0]$. If the winner is unable to pay p after resale, she defaults and loses all her wealth. We will denote defaulting by an empty offer, i.e., by $O_i = \varnothing$. Note that if $p > w_A + w_B$ then $O_i = \varnothing$ no matter the identity of the auction winner. The auction loser must decide whether to accept or reject the resale offer. It can be easily verified that the auction loser j will accept buying z_i at a price r_i if and only if the following two conditions simultaneously hold: 1) $r_i \leq v_j$ and 2) $r_i z_i \leq w_j$.

We search for the Perfect Bayesian Equilibria of the selling game (PBE, for short). A strategy for a player must hence specify a first round bid, a second round offer if the player is the auction winner, and a second round acceptance decision if the player is the auction loser. Posterior beliefs are determined by Bayes rule whenever possible, and resale offers must be optimal given the posterior beliefs and the first round bids. Finally, we will only consider

[10] Similar assumption is adopted in Zheng (2002) to characterize the optimal auction with resale, and can also be found in Hafalir and Krishna (2008). In contrast, Pagnozzi (2007) assumes that bidders bargain in the resale market, so that the outcome is given by the Nash bargaining solution.

[11] An alternative interpretation is that the winning bidder can sell equity to finance a portion of his bid as in Rhodes-Kropf and Viswanathan (2000, 2005). But here the equity provider is a bidder and not the equity market.

rationalizable equilibria or equilibria which survive the elimination of (weakly) dominated strategies.

2 Solving the model under complete information

Let us first assume that use values v_A and v_B are both known and that $v_A > v_B$. The opportunity of resale creates a first stage auction with *endogenous common-valuations* among players who have *private-use values.* These *endogenous* valuations will take into account the overall surplus from winning and reselling the object, as well as the bidders' wealth.

To determine bidders' valuations, assume first that bidder A is the auction winner. At the second stage, she will never resell if she has enough wealth to cover the auction price ($p \leq w_A$). Her offer will be $O_A = [r_A, z_A] = [0,0]$, and her utility from winning the first round auction will be $U_A^{WA} = v_A - p + w_A$. In contrast, if her wealth does not suffice to cover the auction price, she will sell, at a price $r_A = v_B$, the minimum fraction needed to fulfill her financial obligation, i.e., $z_A = \frac{p-w_A}{v_B}$. For z_A to be lower than one, it must be the case that $p - w_A \leq v_B$. Similarly, B's total payment cannot exceed his wealth so that $z_A r_A = p - w_A \leq w_B$. Combining both requirements, $p - w_A \leq \min\{v_B, w_B\}$ must hold. Consequently, if $w_A + \min\{v_B, w_B\} \geq p$, the optimal resale offer by A is

$$O_A^*(p) = \begin{cases} [0,0] & \text{if} \quad p - w_A \leq 0 \\ \left[v_B, \frac{p-w_A}{v_B}\right] & \text{if} \quad p - w_A > 0. \end{cases}$$

Finally, if $w_A + \min\{v_B, w_B\} < p$, then A will default and will lose her entire wealth. Summing up, her payoffs from winning are as follows:

$$U_A^{WA} = \begin{cases} v_A - (p - w_A) & \text{if} \quad p \leq w_A \\ v_A\left(1 - \frac{p-w_A}{v_B}\right) & \text{if} \quad w_A + \min\{v_B, w_B\} \geq p > w_A \\ 0 & \text{if} \quad p > w_A + \min\{v_B, w_B\}. \end{cases}$$

Player B's payoff from losing is $U_B^{WA} = w_B$, with or without resale.

If the auction winner is player B, he will resell at a price v_A, the largest fraction that player A can afford with her wealth. Consequently,

$$O_B^* = [r_B, z_B] = \left[v_A, \frac{\min\{v_A, w_A\}}{v_A}\right].$$

Thus, if A is financially unconstrained, player B sells the entire object to A as $\frac{\min\{v_A, w_A\}}{v_A} = \frac{v_A}{v_A} = 1$. Whereas if the overall wealth, $w_A + w_B$, is lower than the auction price, B will default. Player B's payoffs from winning are hence given by

$$U_B^{WB} = \begin{cases} v_B\left(1 - \frac{\min\{v_A, w_A\}}{v_A}\right) + \min\{v_A, w_A\} + w_B - p & \text{if} \quad p \in [0, w_B + \min\{v_A, w_A\}] \\ 0 & \text{if} \quad p > \min\{v_A, w_A\} + w_B, \end{cases}$$

whereas player A gets $U_A^{WB} = w_A$ when B wins.

The difference between the payoff from winning and the payoff from losing will determine players' *endogenous* valuations, $V_i = U_i^{Wi} - U_i^{Wj}$, which are as follows

$$V_A = \begin{cases} v_A - p & \text{if} \quad p \leq w_A \\ \frac{v_A}{v_B}\left[v_B\left(1 - \frac{w_A}{v_A}\right) + w_A - p\right] & \text{if} \quad w_A + \min\{v_B, w_B\} \geq p > w_A \\ -w_A & \text{if} \quad p > w_A + \min\{v_B, w_B\} \end{cases}$$

and

$$V_B = \begin{cases} v_B\left(1 - \frac{\min\{v_A, w_A\}}{v_A}\right) + \min\{v_A, w_A\} - p & \text{if} \quad p \in [0, w_B + \min\{v_A, w_A\}] \\ -w_B & \text{if} \quad p > \min\{v_A, w_A\} + w_B. \end{cases}$$

Bidders' endogenous valuations will in turn determine the first round bids. To see this, let us denote by Λ_i the maximum willingness to pay of player i, that is, the value that would make a player valuation equal to zero ($V_i = 0$ at $p = \Lambda_i$). In what follows we discuss the different values of Λ_i which will depend on players' wealth.

If A is unconstrained, $v_A \leq w_A$, it is weakly dominant for both players to bid up to v_A.[12] Since $\min\{v_A, w_A\} = v_A$ then $v_B\left(1 - \frac{\min\{v_A, w_A\}}{v_A}\right) + \min\{v_A, w_A\} = v_A$, which makes B's valuation identical to that of player A, independently of w_B. Trivially, bidding w_B is weakly dominated by bidding v_B due to the possibility of resale, which, by the same token, is dominated by bidding v_A.[13]

If A is constrained then, in contrast to the previous case, buyer B's wealth affects players' valuations as it can beget default. In particular, if $w_B > v_B$ then for all $p \in [w_A, w_A + v_B]$,

[12] A similar result in obtained by Kamien, Li and Samet (1989) when analyzing Bertrand competition under subcontracting. At the unique equilibrium, firms bid the same price at the first stage and both receive zero profits.

[13] In a static one-round second price auction with budget constraints it is a dominant strategy to bid $\min\{w_i, v_i\}$ (see Che and Gale, 1998). The reason is that if the second highest bid is above the winner's budget, he will renege, will not get the object and will pay the fine, resulting in a negative surplus. With the possibility of resale this argument breaks down if the winner can resell the good and, by doing so, can get more than the auction price. This is the case here as long as the potential buyer at the resale market does not follow dominated strategies.

it remains true that $\Lambda_A = \Lambda_B = \Lambda$, $\Lambda = w_A + v_B\left(1 - \frac{w_A}{v_A}\right)$, even though $V_A \neq V_B$, and, for the same arguments as above, the first stage bids will equal the maximum willingness to pay Λ. Since $\Lambda < v_B + w_A$, no player will default. For higher prices, $p \in [w_A + v_B, w_A + w_B]$, player A will default whereas player B will not. In contrast, if $w_B < v_B$ then the set of first round defaulting prices coincides for both players, namely, $p > w_A + w_B$. Furthermore, if $w_B < \underline{w}_B = v_B\left(1 - \frac{w_A}{v_A}\right)$ it is no longer true that both players will bid Λ in the first stage as bidding Λ and winning begets default $p = w_A + \underline{w}_B > w_A + w_B$. Consequently, either player is better off deviating to $w_A + w_B$.

The discussion above is summarized in the next proposition.

Proposition 1 *(Complete information) Assume* $w_B \geq \underline{w}_B = v_B\left(1 - \frac{w_A}{v_A}\right)$. *Then*

i) If bidder A is financially unconstrained, at any SPNE in (weakly) undominated strategies, players will bid $b = (v_A, v_A)$ *at the first stage auction. The high value bidder, player A, will get the object and she will pay her entire use value* v_A. *In equilibrium, there is no resale at the second stage.*

ii) If bidder A is financially constrained, at any SPNE in (weakly) undominated strategies players will bid $b = (\Lambda, \Lambda)$ *at the first stage auction, where* $\Lambda = w_A + v_B\left(1 - \frac{w_A}{v_A}\right)$, *and will follow the take-it-or-leave-it offers* O_A^* *and* O_B^* *at the second stage. The high value bidder, player A, will get the object and she will pay her valuation. In equilibrium, there is resale at the second stage.*

Corollary 1 *If bidder B were a pure speculator* ($v_B = 0$) *then at any SPNE in (weakly) undominated strategies players will bid* $b = (v_A, v_A)$ *at the first stage auction if A is financially unconstrained, and* $b = (w_A, w_A)$ *otherwise. In equilibrium, there is no resale at the second stage.*

Some features of Proposition 1 are noteworthy. First, the presence of a resale market is beneficial to the seller. In the absence of financial constraints by the strong buyer [*part i)*], the seller's revenue equals v_A which is larger than the revenue when resale is prohibited, v_B. If the strong buyer is constrained while $w_B > \underline{w}_B$, [*part ii)*], the seller would get a revenue equal to $\min\{w_A, v_B\}$ if there is no resale market, which is lower than $w_A + v_B\left(1 - \frac{w_A}{v_A}\right)$, the proceeds under resale. Second, in the presence of resale, financial constraints hurt the seller as when A is constrained seller's revenue decreases from v_A to $w_A + v_B\left(1 - \frac{w_A}{v_A}\right)$. Third, the condition $w_B > \underline{w}_B$ ensures that $p = \Lambda \leq w_A + v_B$ so that in equilibrium there is no default. Finally, these results provide a justification for the assumption that the seller sells the object to a single buyer. The seller has nothing to gain from selling shares of the object, as all the surplus is already extracted. This result depends on the complete information assumption which prevents any informational rents.

As discussed above, the symmetric equilibrium fails to materialize when both players are financially constrained and the auction price is large enough. This is because players' maximal willingness to pay exhibits a discontinuity at $p = w_A + w_B$. At that price the utility from winning is strictly higher than that from losing, while for a price slightly higher the utility from winning drops to zero. Because of this, in equilibrium at least one player must stop bidding at $w_A + w_B$. However, both players stopping at $w_A + w_B$ is not an equilibrium. To see this note that B, with wealth $w_B < \underline{w}_B$, loses when bidding $w_A + w_B$. If he deviates instead and bids higher, his expected utility will be $v_B\left(1 - \frac{w_A}{v_A}\right) + w_A + w_B - p = v_B\left(1 - \frac{w_A}{v_A}\right) = \underline{w}_B > w_B$. It hence follows that in equilibrium one and only one of the bidders will drop at $p = w_A + w_B$. In fact, it is a SPNE to bid $b = (w_A + w_B, \lambda)$, for any $\lambda > w_A + w_B$. In this case the seller also gets more when resale is allowed than when it is prohibited as $w_A + w_B > \min\{w_A, w_B\}$.[14]

In sum, under complete information, resale emerges as a response to financial constraints and not due to speculation. The auctioneer extracts all the surplus, and the outcome is equivalent to that in a market with a discriminating monopoly.[15] Note that there is allocative efficiency since there is no other outcome which provides higher payoffs to the bidders and the auctioneer. We next explore the impact of incomplete information on these results.

3 Incomplete Information

3.1 The resale stage

At the resale stage, the auction winner will set the offer that maximizes her/his expected payoff given her/his posterior beliefs about the loser's use value. A key difference between the strong buyer and the weak buyer behavior when reselling lies in the shares they put up for sale. Whereas buyer A resells the minimum fraction needed to cover the auction price, i.e., $z_A = \min\left\{\frac{p-w_A}{r_A}, 0\right\}$, buyer B may resell the entire object if $w_A \geq v_A^L$. Resale offers will

[14]Nevertheless in these equilibria B is using a dominated strategy since there is always a $\lambda' \in (w_A + w_B, \lambda)$ which is a better response in case the other player's bid lies in (λ', λ), and it is equivalent otherwise.

[15]When player A is unconstrained, a monopolist would sell the object to player A at a price v_A. When A is constrained and $w_B > \underline{w}_B$, the monopolist would sell the fraction $\frac{w_A}{v_A}$ to the strong player at a price v_A, and the rest $\left(1 - \frac{w_A}{v_A}\right)$ to player B at a price v_B; if player B is further constrained and cannot afford to pay v_B then the monopolist would sell him at a price such that B total payment is w_B.

hence depend on the identity of the winner.

If the winner is player A, she will not resell if $p \leq w_A$ and she will always default if $p > \min\{v_B, w_B\} + w_A$. Since buyer A has complete information her optimal offers are identical to those obtained in Section 2.

Lemma 1 *At the resale stage, player A will set $O_A = O_A^*(p)$ if $p - w_A \leq \min\{v_B, w_B\}$, and she will default, $O_A = \varnothing$, otherwise.*

When the winner is player B, and he is not forced to default because $p < \min\left\{w_A, v_A^H\right\} + w_B$, his posteriors will depend on the first round bids. Whenever they are fully revealing (a perfect separating equilibrium), the updated distribution is a point distribution so that his optimal offers will coincide with those described in the previous section as there will be perfect information at the resale stage. When there is incomplete information at the resale stage, his posteriors, which are characterized in the next lemma, will take into account what he learns from the auction price.

Lemma 2 *If b_A is a non-decreasing bidding strategy and the auction price is such that $p = b_A(v_A)$ for all $v_A \in \left[v_A^L, v_A^H\right]$ and $p \neq b_A(v_A)$ for $v_A \notin \left[v_A^L, v_A^H\right]$, then*

i) Buyer B updated beliefs are given by $\widehat{F}(x) = \Pr\left(v_A \leq x \middle| v_A \in \left[v_A^L, v_A^H\right]\right)$, where

$$\widehat{F}(x) = \begin{cases} \frac{F(x)-F(v_A^L)}{F\left(v_A^H\right)-F\left(v_A^L\right)} & \text{if} \quad x \in \left[v_A^L, v_A^H\right] \\ 1 & \text{if} \quad x \geq v_A^H \\ 0 & \text{if} \quad x < v_A^L, \end{cases}$$

with $p = b_A\left(v_A^L\right) = b_A\left(v_A^H\right)$, and $\underline{v}_A \leq v_A^L < v_A^H \leq \bar{v}_A$. If $v_A^L = \underline{v}_A$ and $v_A^H = \bar{v}_A$, then the updated distribution coincides with the prior distribution, i.e., $\widehat{F}(x) = F(x)$. Conversely, if $v_A^L = v_A^H$, then the updated distribution is a point distribution.

ii) If the posterior beliefs generate a left truncation random variable, i.e., $v_A^L > \underline{v}_A$, then the posterior hazard rate $\widehat{h}$ coincides with the prior hazard rate h, whereas if they generate a right truncation random variable, i.e., $v_A^H < \bar{v}_A$, then $\widehat{h} > h$.

Since $\widehat{F}$ is a truncation of F, both $\widehat{h}$ and $\widehat{\psi}$ are non decreasing functions (see Bagnoli and Bergstrom, 2005), with $\widehat{h} = h$ ($\widehat{h} > h$) if the posterior beliefs generate a left (right) truncation random variable.

Buyer B must choose between selling the entire object ($z = 1$) or only a part of it ($z < 1$). When $z = 1$, r must solve

$$\max_r \left\{(w_B - p + r)\left(1 - \widehat{F}(r)\right) + K_B(p)\widehat{F}(r)\right\} \tag{1}$$

$$O_B^o(p \leq w_B) = \begin{cases} w_A < v_A^L & \begin{cases} \left[v_A^L,\ \frac{w_A}{v_A^L}\right] & \text{if } \ v_B \leq \widehat{M}(v_A^L) \\ \left[r^{**},\ \frac{w_A}{r^{**}}\right] & \text{if } \ v_B > \widehat{M}(v_A^L) \end{cases} \\ w_A \geq v_A^L & \begin{cases} \left[v_A^L,\ 1\right] & \text{if } \ v_B \leq \widehat{\psi}(v_A^L) \\ \left[r^{*},\ 1\right] & \text{if } \ v_B \in \left[\widehat{\psi}(v_A^L),\ \widehat{\psi}(w_A)\right] \\ \left[w_A,\ 1\right] & \text{if } \ v_B \in \left[\widehat{\psi}(w_A),\ \widehat{M}(w_A)\right] \\ \left[r^{**},\ \frac{w_A}{r^{**}}\right] & \text{if } \ v_B \geq \widehat{M}(w_A) \end{cases} \end{cases}$$

$$O_B^o(p > w_B) = \begin{cases} w_A < v_A^L & \begin{cases} \left[v_A^L,\ \frac{w_A}{v_A^L}\right] & \text{if } \ p - w_B \in \left[0, L_{\widehat{M}}(w_A)\right] \\ \left[r^{**},\ \frac{w_A}{r^{**}}\right] & \text{if } \ p - w_B \in \left[L_{\widehat{M}}(w_A), w_A\right] \end{cases} \\ w_A \geq v_A^L & \begin{cases} \left[v_A^L,\ 1\right] & \text{if } \ p - w_B \in \left[0, \widehat{\psi}(v_A^L)\right] \\ \left[r^{*},\ 1\right] & \text{if } \ p - w_B \in \left[\widehat{\psi}(v_A^L), \widehat{\psi}(w_A)\right] \\ \left[w_A,\ 1\right] & \text{if } \ p - w_B \in \left[\widehat{\psi}(w_A), L_{\widehat{h}}(w_A)\right] \\ \left[r^{**},\ \frac{w_A}{r^{**}}\right] & \text{if } \ p - w_B \in \left[L_{\widehat{h}}(w_A), w_A\right] \end{cases} \end{cases}$$

Table 1: *B's optimal resale offers, where* $\widehat{M}(x) = \frac{x^2\widehat{h}(x)}{1+x\widehat{h}(x)}$, $L_{\widehat{M}}(w_A) = w_A + v_B\left(1 - \frac{w_A}{\widehat{M}(v_A^L)}\right)$ *and* $L_{\widehat{h}}(w_A) = w_A - \frac{v_B}{w_A\widehat{h}(w_A)}$. *Note that* $\widehat{M}(x) = M(x)$ *when* $\widehat{h}(x) = h(x)$.

whereas when $z < 1$, r must solve

$$\max_r \left\{ \left[w_B - p + w_A + v_B\left(1 - \frac{w_A}{r}\right)\right]\left(1 - \widehat{F}(r)\right) + K_B(p)\widehat{F}(r) \right\} \tag{2}$$

where $K_B(p)$ stands for the payoff when buyer B keeps the object because the resale offer is rejected. It coincides with the payoff in an auction without resale and hence it takes on the positive value $v_B + w_B - p$ when there is no default $(p \leq w_B)$ and 0 when there is default.

The comparison of the expected payoff at $z = 1$ and at $z < 1$ determines B's optimal resale offers, $[r^o,\ z^o]$. They are summarized in Table 1 by using $\widehat{M}(x)$ to denote the increasing, bounded above by x, and positive function $\frac{x^2\widehat{h}(x)}{1+x\widehat{h}(x)}$.

The optimal offers depend on B's use value for the good (v_B) when there is no default risk $(p \leq w_B)$, but on the amount to be covered at resale to avoid bankruptcy when there is default risk $(p > w_B)$. Note that a less wealthy B will behave more aggressively at the resale market than a wealthy one as he has less to lose from bankruptcy. This non-monotonicity stems from the different opportunity costs, losing one's own wealth, associated with a rejected resale offer. For a very low wealth, it is worthy setting a larger resale price, as the gains from getting an offer accepted compensate the losses. This is the case unless w_B is large enough.[16]

Regarding z, when all the A types are financially constrained ($w_A \leq v_A^L$), B will always set $z = \frac{w_A}{r} < 1$. In contrast, when A is wealthy enough ($w_A > \widehat{M}^{-1}(v_B)$), B may find it optimal to sell the entire object. As for r, resale prices satisfy $v_A^L \leq r^* \leq w_A < r^{**}$.

The resale offers by the weak buyer are stated in the following lemma.

Lemma 3 *i) At the resale stage, player B will set $O_B = O_B^o(p \leq w_B)$ if $p \leq w_B$, he will set $O_B^o(p > w_B)$ if $p \in \left(w_B, \min\left\{w_A, v_A^H\right\} + w_B\right]$ and he will default otherwise.*

ii) The optimal resale prices set by B are constant in p for $p \in [0, w_B)$ and non-decreasing in p for $p \in (w_B, \infty)$. However, resale prices may decrease with p at $p = w_B$.

Proof. *See the Appendix.* ■

3.2 The bidding stage

The presence of a resale market affects bidding strategies in several ways, as the existence of the post-auction market allows players to foresee future resale revenues that they will incorporate into their valuations. Similarly, the presence of financial constraints affects the bidding stage, as a strong bidder A facing a wealthy rival may prefer losing with a bid above w_B than with one just below to induce B to set better resale offers (Lemma 3 [*part ii)*]), whereas a strong player A facing a less wealthy rival may prefer setting a low auction price to soften her rival's financial situation.

When financial constraints are absent, a separating truth-telling equilibrium ($b_A = v_A$, $b_B = v_B$) coexists with a continuum of inefficient equilibria (see Garratt and Tröger, 2006). We explore the effect of financial constraints on this result. To disentangle the role of each buyer's wealth, we first suppress any strategic incentive by player A to affect the resale price by assuming that w_B is sufficiently large, while allowing w_A to vary.[17] As we saw in the previous section, the wealth of the strong buyer is crucial for determining whether a winner weak buyer will sell the entire object or just a part of it. Player B sells only a fraction of the good as long as $w_A < \underline{v}_A$ and also when $w_A \geq \underline{v}_A$ as long as buyer B's value is sufficiently large (see $O_B^o(p \leq w_B)$ in Table 1).

[16] In a model without resale Zheng (2001) has shown that the symmetric Bayes Nash equilibrium bidding strategies are also not monotonic as a function of the bidder's budget.

[17] In Section 4 we analyze how results change when wealths are identical, $w_A = w_B$, and also when A is wealthier than B.

Since B's optimal resale price depends on the information he obtains from the bidding stage, whenever $[v_A^L, v_A^H] = [\underline{v}_A, v]$ we will write r_v^{**} to denote that optimal price when $z < 1$ and r_v^* to denote the optimal price when $z = 1$. In either case, for any $x \in [\underline{v}_A, v]$ we will write the updated distribution as $\widehat{F}_v(x)$, $\widehat{F}_v(x) = F(x)/F(v)$, with $\widehat{F}_{\tilde{v}_A}(x) = F(x)$. Finally, to save on notation, let us denote $\Lambda_x = w_A + v_B\left(1 - \frac{w_A}{x}\right)$, with $\Lambda_{\underline{v}_A} = \underline{\Lambda}$ and $\Lambda_{\bar{v}_A} = \overline{\Lambda}$ for short. The following proposition presents the main result of this section.

Proposition 2 *(Inefficient equilibria) Let $w_B \geq \overline{\Lambda}$. The second price auction with resale has a family of Perfect Bayesian equilibria in (weakly) undominated strategies parameterized by $\tilde{v}_A$, $\tilde{v}_A \in [\underline{v}_A, \bar{v}_A]$, with equilibrium bidding functions given by*

$$b_A(v_A) = \begin{cases} \min\{\underline{v}_A, \underline{\Lambda}\} & \text{if } \; v_A \in [\underline{v}_A, \tilde{v}_A] \\ \min\left\{v_A, w_A + v_B\left(1 - \frac{w_A}{v_A}\right)\right\} & \text{if } \; v_A \in (\tilde{v}_A, \bar{v}_A] \end{cases}$$

and

$$b_B = \begin{cases} w_A + v_B\left(1 - \frac{w_A}{r_{\tilde{v}_A}^{**}}\right) & \text{if } \; w_A \in [0, M^{-1}(v_B)) \text{ and } v_B \geq \widehat{M}_{\tilde{v}_A}(w_A) \\ r_{\tilde{v}_A}^{*} & \text{if } \; w_A \geq M^{-1}(v_B) \text{ or } \left\{w_A \in [\underline{v}_A, M^{-1}(v_B)) \text{ and } v_B < \widehat{M}_{\tilde{v}_A}(w_A)\right\} \end{cases}$$

At this equilibrium, resale offers by the strong buyer are given by $O_A^*(p)$ whereas resale offers by the weak buyer are given by $O_B^o(p < w_B)$ and are optimal given B's posterior beliefs

$$\widehat{F}(x/p; \tilde{v}_A) = \begin{cases} \min\left\{\widehat{F}_{\tilde{v}_A} = \frac{F(x)}{F(\tilde{v}_A)}, 1\right\} & \text{if } \quad p \leq b_A(\underline{v}_A) \\ \mathbf{1}_{v_A \geq \tilde{v}_A} & \text{if } \; p \in \left(b_A(\underline{v}_A), \min\left\{\tilde{v}_A, w_A + v_B\left(1 - \frac{w_A}{\tilde{v}_A}\right)\right\}\right] \\ \mathbf{1}_{v_A \geq b_A^{-1}(p)} & \text{if } \; p \in \left(\min\left\{\tilde{v}_A, w_A + v_B\left(1 - \frac{w_A}{\tilde{v}_A}\right)\right\}, b_A(\bar{v}_A)\right] \end{cases}$$

i.e., $r_{\tilde{v}_A}^{**}$ solves (1) *and $r_{\tilde{v}_A}^{*}$ solves* (2) *for $v_A \sim \widehat{F}_{\tilde{v}_A}$, $v_A \in [\underline{v}_A, \tilde{v}_A]$.*

Proof. See the Appendix. ■

Corollary 2 *If $v_B = \underline{v}_A = 0$ as in Garratt and Tröger's (2006) then $w_A \geq \underline{v}_A$ will always hold. Bidder A will bid $\underline{v}_A$ if $v_A \in [\underline{v}_A, \tilde{v}_A]$ and $\min\{w_A, v_A\}$ otherwise, whereas bidder B will bid $r_{\tilde{v}_A}^{*}$ (note that $M(w_A) \geq v_B = 0$).*

Proposition 2 identifies a continuum of equilibria in undominated strategies, in which high-value A players, $v_A > \tilde{v}_A$, bid their valuation and win the auction, while the low-values lose the auction bidding either the valuation of the lowest type $\underline{\Lambda}$ when $w_A < \underline{v}_A$, or the lowest use value $\underline{v}_A$ when $w_A \geq \underline{v}_A$. Player B bids his valuation taking into account that as a winner he will resell optimally to the set of low types who are pooling (see $O^o_B(p \leq w_B)$). When he wins at a price $\min\{\underline{v}_A, \underline{\Lambda}\}$ he infers that $v_A \in [\underline{v}_A, \tilde{v}_A]$ and resells accordingly. When he wins at a price above $\min\left\{\tilde{v}_A, w_A + v_B\left(1 - \frac{w_A}{\tilde{v}_A}\right)\right\}$ he infers the type of his rival and he resells under complete information. Finally, given an off-equilibrium price, he believes that $v_A = \tilde{v}_A$.[18] Incentives to hide information from the auction winner create pooling at the bottom.

Consider first the case $w_A \leq \underline{v}_A$. In this case the low A types are constrained and they all bid the valuation of the lowest type, $\underline{\Lambda}$, since $\min\{\underline{v}_A, \underline{\Lambda}\} = \underline{\Lambda}$. Since all the high types will have to resell part of the good, their valuation is lower than v_A. By contrast, B's valuation is higher than v_B as when B wins he may resell part of the good. For $\tilde{v}_A = \bar{v}_A$ the strategies detailed above generate a perfect pooling equilibrium in which all A types bid $\underline{\Lambda}$, whereas for $\tilde{v}_A = \underline{v}_A$ they generate a perfect separating equilibrium with all A bidders bidding their valuation truthfully while B bids $\underline{\Lambda}$. From the seller's viewpoint both extremes in the family of equilibria generate the same expected revenue, namely $\underline{\Lambda}$, lower than in any of the other equilibria. When there is pooling and player B wins (he wins over the low types of buyer A if and only if at the resale market he sets a price above the infimum of the strong buyer's values, i.e., iff $r^{**}_{\tilde{v}_A} > \underline{v}_A$), there is the possibility that B's resale offer is rejected and an allocative inefficiency is generated. Therefore, from an efficiency viewpoint, the best equilibrium is $\tilde{v}_A = \underline{v}_A$ (all A types bid their valuation) as the risk of no resale is zero. The tension between efficiency and the auctioneer's revenues is hence present.

Consider next that $w_A \geq \underline{v}_A$. In this case the low A types are unconstrained and they all bid the use value of the lowest type, $\underline{v}_A$. When most types are constrained ex-ante, $w_A < M^{-1}(v_B)$, the optimal reselling strategy by B entails $z < 1$. Note that in this case the set of $\tilde{v}_A$ for which the aforementioned strategies constitute an equilibrium is smaller as compared to the previous case.[19] In contrast, when $w_A > M^{-1}(v_B)$ there is an equilibrium for any $\tilde{v}_A \in [\underline{v}_A, \bar{v}_A]$ with the weak buyer optimal reselling strategy entailing $z = 1$ so that B bids the resale price. For any $\tilde{v}_A > \underline{v}_A$ such that B wins, the equilibria result in some inefficiency as B's resale offers are rejected with positive probability, whereas for $\tilde{v}_A = \underline{v}_A$ the equilibrium is efficient as the strong buyer wins, pays $p = \underline{v}_A$ and consumes the entire good.

Despite the potential strategic behavior induced by the presence of financial constraints,

[18] Equilibria in Proposition 2 are supported by other off-equilibrium beliefs as long as they lead to a resale price not lower than the one set by B when he wins and observes $p = b_A(\underline{v}_A)$.

[19] To further illustrate this point assume that $v_A \sim U(3,5)$ with $v_B = 3$. Since $r^{**}_{\tilde{v}_A} > w_A$ requires $w_A < \hat{M}^{-1}_{\tilde{v}_A}(v_B) = \sqrt{3\tilde{v}_A}$ to hold, then for $w_A < 3.5$ there is a PBE for any $\tilde{v}_A \geq 4.083$, whereas for $w_A = 3.87$ the only equilibrium in the family entails $\tilde{v}_A = \bar{v}_A = 5$.

perfectly separating strategies in which the strong buyer bids her true valuation, namely $\min\left\{v_A, w_A + v_B\left(1 - \frac{w_A}{v_A}\right)\right\}$, constitute an efficient PBE of the second price auction with resale. The possibility of partial resale is crucial for a fully separating equilibrium to emerge under financial constraints. If partial resale were banned, then bidding above $\min\{v_A, w_A\}$ would be a dominated strategy so that financial constraints would give rise to pooling for the high A types (the same holds true if $v_B = 0$). In our case, if the high A types pooled at w_A, the weak buyer might find it profitable to deviate so as to win, pay $p = w_A$ and resell a fraction of the good at a price above w_A. This would be the case, for instance, if w_B is large enough whereas $w_A \in (\underline{v}_A, M^{-1}(v_B))$.

Proposition 2 is an extension of Garratt and Tröger's (2006) speculative equilibria for the case of partial resale and financial constraints for player A. High A types bid their valuation considering that they may have to resell a fraction of the good to be able to meet their financial constraints; player B also bids his valuation taking into account the resources of player A. When limited, he will find it optimal to resell only a fraction and get in return all A's resources, w_A. As in Garratt and Tröger, low A types pool at the lowest valuation (which may coincide with the use value when the low type is not financially constrained, $w_A > \underline{v}_A$).

This result shows that the inefficiency in G&T's speculative equilibria may also be obtained when player B is not a pure speculator in the sense that here he has a positive use value for the good, $v_B > 0$, and in the presence of financial constraints and partial resale.[20] In the equilibria of Proposition 2 player B is acting as a speculator, that is, his valuation is determined by the returns he can get at the resale stage, although his optimal resale offer depends on his value v_B. In the next section, we will see that it is necessary for B to play this role that he be financially unconstrained. When both players have access to the same financial resources ($w_A = w_B$) and these are scarce, the second price auction leaves no room for inefficient equilibria of the family shown in Proposition 2. In other words, we will show that what is essential to the speculative equilibria, and the resulting inefficiency, is not the fact that the strong player is financially unconstrained or that the speculator has no use value for the good, but the fact that the speculator is not financially constrained.

4 The financial situation of the weak buyer

We have provided a description of the PBE when B has enough wealth. We now assess the role played by buyer B's financial situation in the previous equilibria. In order to uncover the

[20] Garratt and Tröger (2006) show that there are inefficient speculative equilibria also for $v_B > \underline{v}_A = 0$.

role of the weak buyer's wealth for our results, we first analyze the case of identical budgets. We will end the section by considering a less wealthy weak buyer.

4.1 Equal wealth

Arguably, whenever bidders' resources come from the financial market, buyers' financial constraints might be similar so that $w_B = w_A = w$ may prevail. We show next that with equal wealth, the inefficient equilibria disappear below a certain level of w_B, leaving only efficient equilibria.

Consider first wealth levels in $[\underline{w}_B, \underline{v}_A]$ with $\underline{w}_B = v_B(1 - \frac{w}{\underline{v}_A})$.[21] Since $w < \underline{v}_A$ then for any $\tilde{v}_A \geq \underline{v}_A$ the strategies in Proposition 2 imply an auction price $p > w$. When $r^{**}_{\tilde{v}_A} = \underline{v}_A$, $b_B = \underline{\Lambda}$ so that B loses and $p = \underline{\Lambda}$; when $r^{**}_{\tilde{v}_A} > \underline{v}_A$, $b_B = \Lambda_{r^{**}_{\tilde{v}_A}} > \underline{\Lambda}$; if B wins then $p = \underline{\Lambda} > w$ and if he loses then $p = \Lambda_{r^{**}_{\tilde{v}_A}} > w$. Note that an inefficient outcome emerges only when there is a possibility of no resale, i.e., when $b_B > \underline{\Lambda}$ since in that case B wins against the low value A types, and he may set a resale price $r^{**}_{\tilde{v}_A} > \underline{v}_A$ such that some low value A types will refuse to buy. Since for $r^{**}_{\tilde{v}_A} > \underline{v}_A$, it is necessary that $p - w > L_{\widehat{M}_{\tilde{v}_A}}(w)$ (see $O^o_B(p > w_B)$ in Table 1), the inefficient equilibria would disappear as low-types of player A would not bid $\underline{\Lambda}$. By bidding lower (but above w) they can get the resale price $\underline{v}_A$ instead of $r^{**}_{\tilde{v}_A}$, and a higher fraction of the good. Thus, with equal wealth, if $w < \underline{v}_A$, the strategies in Proposition 2 would never give rise to an inefficient outcome.

Consider next $w \in [\underline{v}_A, M^{-1}(v_B))$ and $z < 1$. For these levels of wealth either the equilibrium is efficient ($r^{**}_{\tilde{v}_A} = \underline{v}_A$), or if $r^{**}_{\tilde{v}_A} > \underline{v}_A$ there is a profitable deviation for low-types of player A, who by bidding higher than $\underline{v}_A$ (slightly higher than w but lower than b_B) would remain losers but they would get the resale price $\underline{v}_A$ instead of $r^{**}_{\tilde{v}_A}$ (see $O^o_B(p > w_B)$).

In sum, whenever the strategies in Proposition 2 imply that B is bidding above w, either there is a profitable deviation by low value A types to fine-tune the resale offer they get, so that the strategies are no longer an equilibrium, or they produce an efficient outcome. Any of the surviving equilibria in Proposition 2 result in an efficient allocation.

Finally, for higher wealth levels, with either $\left\{ w \in [\underline{v}_A, M^{-1}(v_B)) \text{ and } v_B < \widehat{M}_{\tilde{v}_A} \right\}$ or $w \geq M^{-1}(v_B)$, at any of the equilibria in Proposition 2 when bidder B wins with $b_B = r^*_{\tilde{v}_A} > \underline{v}_A$, he pays the auction price $p = \underline{v}_A$, lower than his wealth. Moreover, since $r^*_{\tilde{v}_A} \leq w$, the low

[21] Note that if w were even lower, financial constraints would be so severe that the entire family of strategies in Proposition 2 would result in default; since buyers would go bust, these strategies cannot be an equilibrium. As with complete information, equilibria when $w < \underline{w}_B$ may require one buyer to use a weakly dominated strategy.

value A types losing the auction cannot affect the resale price as it only depends upon v_B (see $O_B^o(p \leq w_B)$). Consequently, all equilibria in Proposition 2 remain equilibria for these wealth levels.

The discussion above is summarized in the next proposition.

Proposition 3 *(Equal wealth) Assume $w_B = w_A = w$. If either $w < \underline{v}_A$ or $w \in [\underline{v}_A, M^{-1}(v_B))$ with $v_B \geq \widehat{M}_{\tilde{v}_A}(w_A)$, the strategies in Proposition 2 are a PBE only if they result in an efficient outcome.*

4.2 A less affluent weak buyer

When the good to be auctioned-off is a big facility or an important procurement contract, buyers might not afford their use values, with bidders/firms facing severe financial constraints becoming the right modelling assumption for these problems. We will study these environments by keeping the strong buyer's wealth fixed at some $w_A \leq \underline{v}_A$. In order to reveal the role of the weak buyer's wealth we will gradually decrease w_B starting from the wealth level considered in Proposition 2, $w_B = \overline{\Lambda}$.

We first note that $w_B \geq \overline{\Lambda}$ is more stringent than needed for Proposition 2 to hold. Since the family of strategies played by A (parameterized by $\tilde{v}_A$) is such that the posterior generates a right truncation random variable, B's resale prices are increasing in $\tilde{v}_A$ with $r_{\tilde{v}_A}(p) \in \left[r^{**}_{\underline{v}_A}, r^{**}_{\tilde{v}_A}\right] = \left[\underline{v}_A, r^{**}_{\tilde{v}_A}\right]$. This implies that the entire family of equilibria described in Proposition 2 would remain equilibria if B is financially unconstrained at his bid, i.e., $w_B > \Lambda_{r^{**}_{\tilde{v}_A}}$ suffices for the result.

Consider next wealth levels $w_B \in \left[\underline{\Lambda}, \Lambda_{r^{**}_{\tilde{v}_A}}\right]$. If $v_B \leq \widehat{M}_{\tilde{v}_A}(\underline{v}_A) = M(\underline{v}_A)$ holds, then the strategies in Proposition 2 result in efficient equilibria as $r^{**}_{\tilde{v}_A} = \underline{v}_A$ (see $O_B^o(p \leq w_B)$). By contrast, if $v_B > M(\underline{v}_A)$ then there is $\tilde{v}'_A$ with $v_B = \widehat{M}_{\tilde{v}'_A}(\underline{v}_A)$ such that for all $\tilde{v}_A \leq \tilde{v}'_A$ it remains true that they generate efficient equilibria, whereas for $\tilde{v}_A > \tilde{v}'_A$ they are equilibrium strategies iff $w_B \geq b_B$ holds, as, otherwise, it is profitable for loser A types to deviate so as **to raise the auction price**. Note that if $\tilde{v}_A > \tilde{v}'_A$ then buyer B wins against the low types and resells at a price $r^{**}_{\tilde{v}_A} > \underline{v}_A$ as $v_B > \widehat{M}_{\tilde{v}_A}(\underline{v}_A)$ holds. If A bids $\underline{\Lambda}$ then B has enough wealth to pay the auction price, whereas by bidding higher, $b'_A \in (w_B, b_B)$, A can make B a financially constrained winner and get a resale price equal to $\underline{v}_A$ what constitutes a profitable deviation. Thus, for wealth levels in $w_B \in \left[\underline{\Lambda}, \Lambda_{r^{**}_{\tilde{v}_A}}\right]$ only a subset of the family of equilibria in Proposition 2 survive (those with lower $\tilde{v}_A$), with the subset becoming smaller as w_B decreases.

When B has even lower wealth ($w_B < \underline{\Lambda}$) while $w_A \in [M(\underline{v}_A), \underline{v}_A)$, two new concerns arise. First, high A-type bidders can no longer bid their valuation as, otherwise, their bid could be larger than total wealth, a weakly dominated strategy. Second, deviations by low A-types to bids below $\underline{\Lambda}$ are no longer dominated.[22] To the contrary, it may be optimal for them to bid low in an attempt **to lower the auction price**. To further clarify this issue, note that if the weak buyer turns the auction winner, his resale price depends upon the sign of $\underline{\Lambda} - w_B - L_{\widehat{M}_v}(w_A)$ (see $O_B^o(p > w_B)$) with $L_{\widehat{M}_v}(w_A)$ decreasing in v. If $w_B \geq \frac{v_B w_A}{\underline{v}_A^2 h(\underline{v}_A)}$ then $p - w_B \leq L_{\widehat{M}_v}(w_A)$ holds for any v and the strategies in Proposition 2 constitute an efficient equilibrium for any $\tilde{v}_A$. The same holds true if $w_B \in \left[0, \frac{v_B w_A}{\underline{v}_A^2 \widehat{h}_{\tilde{v}_A}(\underline{v}_A)}\right)$ and $\tilde{v}_A \leq \tilde{v}'_A$ where $\tilde{v}'_A$ is such that $w_B = \widehat{h}_{\tilde{v}'_A}(\underline{v}_A)$. In either case, low types of buyer A win the auction ($b_B = \Lambda_{r^{**}_{\tilde{v}_A}} = \underline{\Lambda}$ given that $r^{**}_{\tilde{v}_A} = \underline{v}_A$). By contrast, if $w_B \in \left[0, \frac{v_B w_A}{\underline{v}_A^2 \widehat{h}_{\tilde{v}_A}(\underline{v}_A)}\right)$ and $\tilde{v}_A > \tilde{v}'_A$, low types of buyer A will profit from deviating to a lower bid. Note that by bidding $b'_A \in [w_B, b_A)$ such that $b'_A - w_B < L_{\widehat{M}_v}(w_A)$ holds, they remain as losers but they get a resale price equal to $\underline{v}_A$ lower than the resale price they are offered when bidding at $\underline{\Lambda}$. As a consequence, when B's wealth is scant, loser low A-types will not bid $\underline{\Lambda}$ as they are better-off deviating to a lower bid so as to induce a lower resale price by softening B's financial situation. Thus, the strategies considered in Proposition 2 may no longer be an equilibrium if B has low wealth but wins the auction.

An equilibrium exists when B's wealth is very low (alternatively, when use values are high enough) at which B bids total wealth ($b_B = w_A + w_B$) and all A types pool at any bid that makes B financially constrained and ensures for them a resale price equal to $\underline{v}_A$. Any equilibrium in this family is outcome-equivalent: B wins and resells part of the object at a resale price equal to $\underline{v}_A$. Note that these equilibria may be perceived as "collusive-like" as A sets a low auction price to get a better offer at the resale stage.[23] As mentioned before, this equilibrium requires $w_B \geq \underline{w}_B$ as, otherwise, buyer A prefers to win by bidding total wealth.

For completeness we end by discussing situations in which some strong buyers are financially unconstrained so that $w_A > \underline{v}_A$. As in the previous case, Proposition 2 holds true for any $w_B \geq b_B$. In contrast, if $w_B \in (\underline{v}_A, b_B)$ then for any $\tilde{v}_A$ either there is an efficient equilibrium ($v_B \leq \widehat{\psi}_{\tilde{v}_A}(\underline{v}_A)$) or loser A types will deviate to $b'_A = w_B$ to induce $r^*_{\tilde{v}_A} = \underline{v}_A$. Similarly, if

[22]This is not true if $w_A \leq M(\underline{v}_A)$ as shown in Lemma 5 (in the Appendix). We have ruled out here such low levels of wealth for the sake of a clearer exposition.

[23]As in Pagnozzi (2007), the strong bidder may prefer to drop out of the auction before the price has reached her valuation, and acquire the good in the aftermarket. However, in our case the strong bidder does so to soften her rival's financial constraint, while in Pagnozzi (2007) she does so to gain a better bargaining position in the aftermarket.

$w_B < p = \underline{v}_A$ then loser A types may now be better-off by softening B's financial constraint. More precisely, if $w_B h(\underline{v}_A) < 1$ holds (see $O^o_B(p > w_B)$) then any inefficient equilibria in Proposition 2 will disappear as loser A players will deviate by lowering their bids to ensure themselves a resale price equal to $\underline{v}_A$.[24]

As before, several efficient "collusive" equilibria emerge.[25] For instance, if $w_B h(\underline{v}_A) < 1$ holds, it is an equilibrium for all A types to bid $w_B + \psi(\underline{v}_A)$ and for buyer B to bid w_A. The weak buyer wins and resells the entire object to A at a price equal to $\underline{v}_A$. As in Proposition 2 resale offers by the strong buyer are given by $O^*_A(p)$ whereas resale offers by the weak buyer are given by $O^o_B(p)$ and are optimal given B's posterior beliefs. Other bidding strategies also result in an equilibrium as the next proposition states.

Proposition 4 *(Collusive equilibria) i) Assume* $w_A \leq \underline{v}_A$. *If* $w_B \in \left[\underline{w}_B, \underline{w}_B + \frac{v_B}{\underline{v}_A h(\underline{v}_A)}\right]$, *there is an efficient PBE equilibrium in which all A types pool at any* $b_A \in \left(w_B, w_B + w_A - \frac{v_B}{\underline{v}_A h(\underline{v}_A)}\right)$ *and* B *bids* $b_B = w_A + w_B$.

ii) Assume $w_A > \underline{v}_A$. *There is a family of efficient PBE equilibria parameterized by* $\tilde{v}_A$, $\tilde{v}_A \in [\underline{v}_A, \bar{v}_A]$, *at which buyer A bids*

$$b_A(v_A) = \begin{cases} w_B + \widehat{\psi}_{\tilde{v}_A}(\underline{v}_A) & \text{if } v_A \in [\underline{v}_A, \tilde{v}_A] \\ \min\{v_A, w_A + v_B\left(1 - \frac{w_A}{v_A}\right)\} & \text{if } v_A \in (\tilde{v}_A, \bar{v}_A] \end{cases}$$

and buyer B *bids* $b_B = \underline{v}_A$.

Proof. See the Appendix. ■

Results in this section reinforce our previous claim. What is essential for the inefficiency associated to *speculative-like* equilibria, is not the fact that the strong player is financially unconstrained or that the speculator has no use value for the good, but the fact that the speculator is not financially constrained. When the speculator's wealth is low, his default cost may be manipulated by the other bidder during the initial auction so that bidder B upon winning has to resell the good at $\underline{v}_A$, therefore eliminating the possibility of no resale and the inefficiency created by the speculator reselling to a privately informed buyer.[26]

[24] Since $\widehat{h}_v(x)$ is decreasing in v, it follows that $w_B \widehat{h}_{\tilde{v}_A}(\underline{v}_A) = w_B h(\underline{v}_A) < 1$ ensures that the result holds true for any $\tilde{v}_A$.

[25] Following Garrat, Tröger and Zheng (2009), it is a "collusive equilibrium" as all bidders' types are better off than in the bid-your-value equilibrium.

[26] A biding-to-lose argument is also present in Panozzi (2007) but under complete information and a Nash bargaining solution for the resale market.

As with complete information, even at the "collusive" equilibria, the seller benefits from the possibility of resale. He gets at least $w_B + \widehat{\psi}_{\tilde{v}_A}(\underline{v}_A)$, which is larger than w_B, the price he would receive in the absence of a resale market.

5 Adding a pure speculator

We study next the robustness of the results to the participation in the auction of a pure speculator, i.e., a player with a zero use value for the object and a high wealth. We show that the presence of pure speculators would neither alter the allocation of the good at the auction (nor the bids) as long as one player with positive use value is unconstrained. Thus, a pure speculator cannot profit from participating. We present this result for two specific resale rules but we believe the result holds more generally.

At the first stage a pure speculator buyer C with $v_C = 0$ and $w_C > \bar{v}_A$ participates in the auction together with buyers A and B whose wealth levels are $w_A < \bar{v}_A$ and $w_B > \bar{v}_A$. After the auction stage, all bids are made public and the auction winner has the possibility of reselling. Two resale rules are analyzed, which differ in the timing of the resale offers. Under rule 1, resale comprises a single stage so that the winner makes an offer to the other two players, who may accept or reject and then the game ends. Under rule 2, the winner makes a resale offer to one of the losers, either with $z = 1$ or $z < 1$. That loser (loser 1) responds. If either $z < 1$ or the offer is rejected, the winner may then make a new offer to the other loser (loser 2). No further resale offer by the winner is allowed. Loser 1 can then make an offer to loser 2 if he has become the owner of the entire good. This offer may be accepted or rejected. After that, the resale game ends.[27]

Under either rule, the resale offers by players A and B are not affected by the presence of C. Notice that there is no point in trying to resell to the speculator. Player C has no use value, so that A and B may only want to sell to C as an intermediary (i. e., as loser 1 if rule 2 is employed), but as an intermediary he is in a worse position (he is as wealthy as B but less willing to take risks with the resale price since he has no use value, which means that C's valuation -when facing player A as a buyer- is strictly lower than B's). This implies that player A prefers selling to player B, and B would prefer selling to player A.

The next lemma characterizes C's behavior at the resale market. We will further assume in this subsection that $\underline{v}_A h(\underline{v}_A) < 1$ holds as, otherwise, buyer C will always set a resale price

[27] This timing tries to prevent signaling (rejecting an offer to send a signal of low value so that the next offer is better). Thus, each player receives at most one offer at the resale stage.

equal to v_A^L and the results would follow trivially. Notice that C behaves at the resale market *as if* he were player B with $v_B = 0$. Thus, when reselling the entire object r_C^* solves (1) for $v_A \sim \hat{F}(x)$ and $v_B = 0$.

Lemma 4 *Let C be the auction winner,*

i) If $w_A < \underline{v}_A$, under rule 1 C will split the object between the two buyers, selling $z = \frac{w_A}{v_A^L}$ at a price v_A^L to player A, and $1 - z$ at a price v_B to player B. Under rule 2, C will sell the entire object to player B (as loser 1).

ii) If $w_A \geq \underline{v}_A$, under rule 1 C will sell $z = 1$ at a price r_C^* to player A,where r_C^* solves (1) for $v_A \sim \hat{F}(x)$ and $v_B = 0$. Under rule 2, C will sell $z = 1$ at a price r_B^* to player B (as loser 1) where r_B^* solves (1) for $v_A \sim \hat{F}(x)$.

Proof. See the Appendix. ■

The lemma above provides the arguments for our claim: As long as one player with positive use value is unconstrained, there is no role for a pure speculator.

Proposition 5 *A pure speculator would never (profitably) win the auction as long as w_B is high enough.*

Proof. See the Appendix. ■

6 Concluding Remarks

We have analyzed the impact of financial constraints on second-price auctions with resale. The financial situation of the weak buyer shapes his resale offers and hence the players' optimal bids. When he has enough resources an extension of Garratt and Tröger's (2006) speculative equilibria is obtained, which includes the possibility of financial constraints for player A and partial resale. High A types bid their valuation considering that they will have to resell a fraction of the good to be able to meet their financial constraints; low A types pool at the lowest valuation, while player B bids his valuation taking into account how much he will resell and the extent of his rival's resources. As in Garratt and Tröger, there are inefficient equilibria where the weak buyer *acts as* a speculator, that is, his valuation is determined by the returns he can get at the resale stage and he puts up for sale as much as A can afford with her wealth w_A.

When the weak buyer's wealth is low, the Garrat and Tröger speculation inefficiency is mitigated. The reason is the link between the resale price and the auction price introduced by the presence of financial constraints. Such a link induces a potential loser to modify the auction price so as to fine-tune the winner's resale offer, which may require forcing the winner to be financially constrained, or, to the contrary, to soften his/her financial constraint. As the weak buyer's wealth gets reduced inefficient equilibria disappear and when his financial constraints are severe, only efficient equilibria survive. The reason behind this efficiency result is that the cost for the speculator from not being able to resell the good is made sufficiently high (default and forfeiting one's wealth) so that he does not raise the price even though he faces a privately informed buyer.

In the context of our motivating example, a government auctioning off a big facility or contract, severe financial constraints seem to be the right modelling approach with use values above the financial resources of a single bidder. When this is the case, the second price auction remains a simple and appropriate mechanism to achieve an efficient allocation.

7 References

Bagnoli, Mark and Ted Bergstrom, 2005. "Log-concave probability and its applications," *Economic Theory*, vol. 26 (2), pages 445-469.

Benoît, Jean-Pierre and Vijay Krishna, 2001. "Multiple-Object Auctions with Budget Constrained Bidders," *Review of Economic Studies,* vol. 68 (1), 155-179.

Che, Yeon-Koo and Gale, Ian, 1998. "Standard Auctions with Financially Constrained Bidders," *Review of Economic Studies*, vol. 65 (1), pages 1-21.

Chen, Yongmin, Ishikawa, Jota and Yu, Zhihao , 2004. "Trade liberalization and strategic outsourcing," *Journal of International Economics*, vol. 63 (2), pages 419-436.

Garratt, Rod and Thomas Tröger, 2006. "Speculation in Standard Auctions with Resale," *Econometrica,* vol. 74 (3), pages 753-769.

Garratt, Rod, Thomas Tröger and Charles Z. Zheng, 2009. "Collusion via Resale," *Econometrica,* vol. 77 (4), pages 1095-1136.

Gupta, Madhurima and Bernard Lebrun, 1999. "First Price Auctions with Resale," *Economics Letters,* vol. 64 (2), pages 181-185.

Hafalir, Isa Emin and Krishna, Vijay, 2008. "Asymmetric Auctions with Resale," *American Economic Review,* vol. 98 (1), pages 87-112.

Haile, Philip A., 2000. "Partial Pooling at the Reserve Price in Auctions with Resale Opportunities," *Games and Economic Behavior*, vol. 33 (2), pages 231-248.

Haile, Philip A., 2001. "Auctions with Resale Markets: An Application to U.S. Forest Service Timber Sales," *American Economic Review*, vol. 91 (3), pages 399-427.

Haile, Philip A., 2003. "Auctions with private uncertainty and resale opportunities," *Journal of Economic Theory*, vol. 108 (1), pages 72-110.

Jehiel, Philippe and Benny Moldovanu, 2000. "Auctions with Downstream Interaction among Buyers," *RAND Journal of Economics*, 31 (4), pages 768-791.

Kamien, Morton I., Lode Li and Dov Samet, 1989. "Bertrand competition with subcontracting," *RAND Journal of Economics*, vol. 20 (4), pages 553-567.

Klemperer, Paul, 2002. "How (not) to run auctions: The European 3G telecom auctions," *European Economic Review*, vol. 46 (4-5), pages 829-845.

Meland, Fride and Odd Straume, 2007. "Outsourcing in contests," *Public Choice*, Springer, vol. 131 (3), pages 315-331.

Pagnozzi, Marco, 2007. "Bidding to lose? Auctions with resale," *RAND Journal of Economics*, vol. 38 (4), pages 1090-1112.

Rhodes-Kropf, M. and S. Viswanathan, 2000. “Corporate Reorganizations and Non-Cash Auctions,” *Journal of Finance*, vol. 55 (4), pages 1807-1854.

Rhodes-Kropf, M. and S. Viswanathan, 2005. “Financing Auction Bids,” *RAND Journal of Economics*, vol. 36 (4), pages 789-815.

Spiegel, Yossi, 1993. “Horizontal subcontracting,” *RAND Journal of Economics,* vol. 24 (4), pages 570-590.

Zheng, Charles, 2001. “High Bids and Broke Winners,” *Journal of Economic Theory*, vol. 100 (1), pages 129-171.

Zheng, Charles, 2002. “Optimal Auction With Resale,” *Econometrica,* vol. 70, pages 2197-2224.

8 Appendix A.1

Proof of Lemma 3

Denoting by $[r^o, z^o]$ B's optimal resale offer, we first show some of its properties.
i) $v_A^H \geq r^o \geq v_A^L$ and $r^o z^o \leq w_A$.
Since buyer A will never pay more than her use value, $v_A^H \geq r^o$ follows, and since any type of player A will pay at least v_A^L then $r^o \geq v_A^L$. Finally, buyer A total payment cannot exceed her wealth, $r^o z^o \leq w_A$.
ii) If $r^o z^o < w_A$, then $z^o = 1$.
If $r^o z^o < w_A$ it is possible for B to increase his payoff by increasing z^o while keeping r^o constant, so that $r^o z^o < w_A$ still holds. Since his utility increases (note that $r^o \geq v_A^L \geq v_B$ and the probability of A accepting the resale offer does not change) it must be the case that $z^o = 1$ whenever $r^o z^o < w_A$. For the same arguments next property follows:
iii) If $z^o < 1$, then $r^o z^o = w_A$.
iv) If $w_A < v_A^L$, then $z^o < 1$.
Assume not, $z^o = 1$. Then $r^o = r^o z^o \leq w_A < v_A^L$, which contradicts *i)*.

Using the results above we show next part i). When bidder B sets $z = 1$, his optimal resale price $r^o \in [v_A^L, \min\{w_A, v_A^H\}]$ solves

$$\max_r \left\{ (w_B - p + r)\left(1 - \widehat{F}(r)\right) + K_B(p)\widehat{F}(r) \right\}$$

In an interior solution, the optimal resale price when $z = 1$, denoted $r^*(p)$, solves

$$r - v_B = \frac{1}{\widehat{h}(r)} \quad \text{if } p \leq w_B \tag{3}$$

$$r - (p - w_B) = \frac{1}{\widehat{h}(r)} \quad \text{if } p > w_B \tag{4}$$

These equations yield the optimal resale price, for any value of p, when the solution satisfies (a) $r^* \geq v_A^L$ (property i)), which requires the LHS to be larger than the RHS when evaluated at the minimum possible price v_A^L and (b) $r^* \leq w_A$ (from $z = 1$ and property i)), which requires the LHS to be larger than the RHS when evaluated at w_A. Note also that, from property iv, $z = 1$ may only be optimal when $w_A \geq v_A^L$. Finally, there are corner solutions with resale offers $[v_A^L, 1]$ if (a) fails, and $[w_A, 1]$ if (b) fails.

Alternatively, B can set $z < 1$, so that $rz = w_A$ (property iii)). When $z < 1$, so that $z = w_A/r$ (by property iii)), resale price r solves

$$\max_r \left\{ \left[w_B - p + w_A + v_B\left(1 - \frac{w_A}{r}\right)\right] \left(1 - \widehat{F}(r)\right) + K_B(p)\widehat{F}(r) \right\}$$

In an interior solution, the optimal resale price when $z < 1$, denoted $r^{**}(p)$, is implicitly defined by

$$\frac{r}{v_B}(r - v_B) = \frac{1}{\widehat{h}(r)} \quad \text{if } p \leq w_B \tag{5}$$

$$\frac{r}{v_B}\left[\frac{r}{w_A}(w_A + w_B - p) + v_B\left(\frac{r - w_A}{w_A}\right)\right] = \frac{1}{\widehat{h}(r)} \quad \text{if } p > w_B \tag{6}$$

with $z^{**} = \frac{w_A}{r^{**}}$. Each equation for r^{**} yields the optimal price when the solution satisfies $r^{**} \geq \max\{v_A^L, w_A\}$ (from properties i) and iii)). Corner solutions with resale offers $\left[v_A^L,\, \frac{w_A}{v_A^L}\right]$ and $[w_A,\, 1]$ emerge when the aforementioned conditions ($r^{**} \geq v_A^L$ and $r^{**}z^{**} = w_A$) fail.

To determine the optimality of $z = 1$ or $z < 1$, we compare the two alternatives for $p \leq w_B$ and $p > w_B$.

1). Assume first that $p \leq w_B$ so that we need to compare the solution to (3) with the solution to (5):

1.1). If $w_A < v_A^L$ and $p \leq w_B$ then the offer $z < 1$ dominates any offer with $z = 1$ by property iv). The optimal resale price is hence the solution to (5). Using $\widehat{M}(x) = \frac{x^2\widehat{h}(x)}{1+x\widehat{h}(x)}$, resale offers are

$$\begin{array}{ll} \left[v_A^L,\, \frac{w_A}{v_A^L}\right] & \text{if} \quad v_B \leq \widehat{M}(v_A^L) \\ \left[r^{**},\, \frac{w_A}{r^{**}}\right] & \text{if} \quad v_B > \widehat{M}(v_A^L) \end{array}$$

where $v_B > \widehat{M}(v_A^L)$ ensures that $r^{**} > v_A^L$.

1.2). If $w_A \geq v_A^L$ and $p \leq w_B$ then $r^* > r^{**}$ (the LHS of (3) is steeper than the LHS in (5) while they are both equal to zero at $r = v_B$). The optimal offer entails r^{**} only if the solution to (3) is the corner solution $r^* = w_A$. Consequently, the resale offer $[r^* < w_A,\, 1]$ dominates the offer $[r^{**},\, z < 1]$, whereas $[r^{**} > w_A,\, z < 1]$ dominates the offer $[r^* = w_A,\, 1]$ as B gets the same resources with both of them, namely w_A, but with the former he also gets to consume part of the good. The optimal offers are hence

$$\begin{array}{ll} [v_A^L,\, 1] & \text{if} \quad v_B \leq \widehat{\psi}(v_A^L) \\ [r^*,\, 1] & \text{if} \quad v_B \in \left[\widehat{\psi}(v_A^L),\; \widehat{\psi}(w_A)\right] \\ [w_A,\, 1] & \text{if} \quad v_B \in \left[\widehat{\psi}(w_A),\; \widehat{M}(w_A)\right] \\ [r^{**},\, z < 1] & \text{if} \quad v_B \geq \widehat{M}(w_A) \end{array}$$

where the condition $v_B > \widehat{\psi}(v_A^L)$ ensures $r^* > v_A^L$ when $z = 1$, and $r^{**} > w_A$.

2). Assume next that $p > w_B$, so that we need to compare (4) and (6):

2.1). If $w_A < v_A^L$ and $p > w_B$ then selling only part of the object dominates any offer with $z_B = 1$ for the same arguments given above (both yield w_A and with the former B gets to consume part of the good). Resale offers are:

$$\begin{array}{lll} \left[v_A^L,\, \frac{w_A}{v_A^L}\right] & \text{if} & p-w_B \in \left[0, w_A + v_B\left(1-\frac{w_A}{\widehat{M}(v_A^L)}\right)\right] \\ \left[r^{**},\, \frac{w_A}{r^{**}}\right] & \text{if} & p-w_B \in \left[w_A + v_B\left(1-\frac{w_A}{\widehat{M}(v_A^L)}\right), w_A\right] \end{array}$$

2.2). If $w_A \geq v_A^L$ and $p > w_B$ the optimal offers are:

$$\begin{array}{lll} \left[v_A^L,\, 1\right] & \text{if} & p-w_B \in \left[0, \widehat{\psi}(v_A^L)\right] \\ \left[r^{*},\, 1\right] & \text{if} & p-w_B \in \left[\widehat{\psi}(v_A^L), \widehat{\psi}(w_A)\right] \\ \left[w_A,\, 1\right] & \text{if} & p-w_B \in \left[\widehat{\psi}(w_A), w_A - \frac{v_B}{w_A\widehat{h}(w_A)}\right] \\ \left[r^{**},\, z<1\right] & \text{if} & p-w_B \in \left[w_A - \frac{v_B}{w_A\widehat{h}(w_A)}, w_A\right] \end{array}$$

with $r^{**} > w_A > r^*$.

By defining $L(p) = p + K_B(p) - w_B$, $L_{\widehat{h}}(w_A) = w_A - \frac{v_B}{w_A\widehat{h}(w_A)}$ and $L_{\widehat{M}}(w_A) = w_A + v_B\left(1-\frac{w_A}{\widehat{M}(v_A^L)}\right)$, resale offers for the cases $p \leq w_B$ and $p > w_B$ stated in Table 1 in the main text follow.

We show next part *ii).* Since $L(p) = v_B$ at $p = w_B$ and $L(p) = p - w_B$ at $p > w_B$, it follows from $O_B^o(p \leq w_B)$ and $O_B^o(p > w_B)$ that the resale price decreases in p at $p = w_B$ if $v_B > p - w_B$ holds. ■

Proof of Proposition 2

Assume first that $w_A \leq \underline{v}_A$ so that $b_A(v_A) = w_A + v_B\left(1-\frac{w_A}{v_A}\right) \leq v_A$ as $v_A - b_A(v_A) = (v_A - w_A)\left(1-\frac{v_B}{v_A}\right) \geq 0$. Since $w_B \geq \overline{\Lambda}$, when B wins he pays $p < w_B$ and resells according to $O_B^o(p \leq w_B)$. Any type $v_A > \tilde{v}_A$ is indifferent between losing, by bidding $b_A(\underline{v}_A)$, or winning, by bidding $b_A(v_A)$, as

$$U_A^{WA} = v_A\left(1-\frac{b_B - w_A}{v_B}\right) = v_A\frac{w_A}{r_{\tilde{v}_A}^{**}} = \max\left\{w_A, v_A\left(\frac{w_A}{r_{\tilde{v}_A}^{**}}\right)\right\} = U_A^{WB}$$

where the second equality follows from the fact that $p = b_B = w_A + v_B\left(1-\frac{w_A}{r_{\tilde{v}_A}^{**}}\right)$, and the third equality follows from $r_{\tilde{v}_A}^{**} < v_A^H(b_A(\underline{v}_A)) = \tilde{v}_A$. Any other deviation either increases the resale price when losing given the off-equilibrium beliefs, or it is equally profitable when winning. Consequently, the high value A's types will not deviate.

Types in $[\underline{v}_A, r_{\tilde{v}_A}^{**}]$ strictly prefer losing and refusing to buy at resale as winning yields a payoff $v_A\left(1-\frac{b_B - w_A}{v_B}\right) = v_A\frac{w_A}{r_{\tilde{v}_A}^{**}}$ which is lower than the payoff from losing and refusing to buy, w_A. Types in $[r^{**}_{\tilde{v}_A}, \tilde{v}_A]$ are indifferent between waiting for resale (by bidding $b_A(\underline{v}_A)$) or overbidding player B as they get the same payoff in any event. If they deviate to a bid in $(b_A(\underline{v}_A), b_B)$ they are worse off by the off-equilibrium beliefs.[28]

Focus next on player B. Trivially any bid above $b_A(\underline{v}_A)$ is payoff-equivalent to b_B, either he remains a winner against the low types, at the same price, and a loser against the high types (when he loses the resale price is always v_B so B has no incentive to change the auction price or the fraction that A resells), or he also wins over some interval of high A types to which B will resell at the auction price. Deviations to tie, $b'_B = b_A(\underline{v}_A)$, yield lower payoffs than b_B. Given that there is no profitable deviation for either player, the result follows for the case $w_A < \underline{v}_A$.

When $w_A \in [\underline{v}_A, M^{-1}(v_B))$, the proof follows the same steps. Nevertheless, the following remarks are in order. First, low types of buyer A now bid $\underline{v}_A$ (note that $\underline{\Lambda}$ is strictly increasing in w_A with $\underline{\Lambda} = \underline{v}_A$ for $w_A = \underline{v}_A$, and $\underline{\Lambda} > \underline{v}_A$ for $w_A > \underline{v}_A$). Second, the set of $\tilde{v}_A$ for which $b_B = w_A + v_B\left(1 - \frac{w_A}{r^{**}_{\tilde{v}_A}}\right)$ is part of a PBE is reduced to $\tilde{v}_A \in (T(w_A), \bar{v}_A]$, where $T(w_A)$ is the value that $\tilde{v}_A = v^H_A$ must reach for the condition $v_B \geq \widehat{M}_{\tilde{v}_A}(w_A)$ to hold.[29] Note that if $\tilde{v}_A$ were lower than $T(w_A)$ so that $v_B < \widehat{M}_{\tilde{v}_A}(w_A)$ would hold, then $r^{**}_{\tilde{v}_A} < w_A$, $z = 1$ and $b_B = r^*_{\tilde{v}_A}$.

Consider next that $w_A \geq M^{-1}(v_B)$ so that $b_B = r^*_{\tilde{v}_A}$ for any $\tilde{v}_A \in [\underline{v}_A, \bar{v}_A]$. For $\tilde{v}_A \leq w_A$ the strong buyer's bids are $b_A(v_A) = \underline{v}_A$ for $v_A \leq \tilde{v}_A$, $b_A(v_A) = v_A$ for $v_A \in (\tilde{v}_A, w_A]$ and $b_A(v_A) = w_A + v_B\left(1 - \frac{w_A}{v_A}\right)$ for $v_A \geq w_A$; whereas for $\tilde{v}_A > w_A$ her bids are $b_A(v_A) = \underline{v}_A$ for $v_A \leq \tilde{v}_A$ and $b_A(v_A) = w_A + v_B\left(1 - \frac{w_A}{v_A}\right)$ for $v_A \geq \tilde{v}_A$. We first show that there is no profitable deviation for the weak buyer. His bid b_B wins against the low types, $v_A \in [\underline{v}_A, \tilde{v}_A]$; in that case he makes positive payoffs as he buys at $\underline{v}_A$ and resells at $r^*_{\tilde{v}_A} > \underline{v}_A$. Any bid in $(\underline{v}_A, b_A(\bar{v}_A)]$ would yield the same payoffs than b_B: B remains a winner against the low types, reselling the entire object at the same price $r^*_{\tilde{v}_A}$; if he now wins against any type larger than $\tilde{v}_A$ he breaks even (either buys at v_A and resells at v_A, or he buys at $w_A + v_B\left(1 - \frac{w_A}{v_A}\right)$ resells $z_B = \frac{w_A}{v_A}$ at a price $v_A = v_A$ getting payoffs equal to the auction price).

[28] If B considers that such deviation is equally likely to have come from either type in $[\underline{v}_A, \tilde{v}_A]$ (i.e., if we alter the off-equilibrium beliefs), it would remain true that deviations to bid in $(b_A(\underline{v}_A), b_B)$ are unprofitable as they will not affect either the probability of winning or the resale price set by B given that $w_B \geq b_A(\overline{v}_A)$ holds.

[29] In a right truncation, $v_A \in [\underline{v}_A, v)$, the condition $v_B \geq \widehat{M}_v(w_A)$ can be alternatively rewritten as $F(v) \geq F(w_A) + w_A f(w_A)\left(\frac{w_A}{v_B} - 1\right)$ so that $T(w_A) = F^{-1}\left(F(w_A) + w_A f(w_A)\left(\frac{w_A}{v_B} - 1\right)\right)$. Note that $T(w_A)$ is an increasing function with $T(w_A) > w_A$ if $w_A > \underline{v}_A$, $T(w_A) = \underline{v}_A$ when $w_A \leq \underline{v}_A$ and $T(w_A) = \bar{v}_A$ when $w_A \geq M^{-1}(v_B)$.

Focus next on the strong buyer. For low types in $[\underline{v}_A, \tilde{v}_A]$ the losing bid $\underline{v}_A$ is preferable to any other losing bid by the off-equilibrium beliefs. Types $v_A \in [\underline{v}_A, r^*_{\tilde{v}_A})$ strictly prefer losing to winning as when losing they get payoff w_A whereas when winning they get $v_A - r^*_{\tilde{v}_A} + w_A < w_A$. Types $v_A \in [r^*_{\tilde{v}_A}, \tilde{v}_A]$ are indifferent between winning and losing as in either event they get $v_A - r^*_{\tilde{v}_A} + w_A$. Finally, types in $(\tilde{v}_A, \overline{v}_A]$ win and get payoffs $v_A - b_B + w_A = v_A - r^*_{\tilde{v}_A} + w_A$. Since no other bid allows them to get higher payoffs, there is no profitable deviation for these types either.

Finally, appealing to Lemma 5 in Appendix A.2, it follows that no player is using a weakly dominated strategy. ■

Proof of Proposition 4

i) At the equilibrium candidate B wins and resells at $\underline{v}_A$ the fraction $\frac{w_A}{\underline{v}_A}$ as

$$p - w_B = w_A - \frac{v_B}{\underline{v}_A h(\underline{v}_A)} = w_A + v_B\left(1 - \frac{w_A}{M(\underline{v}_A)}\right)$$

Buyers expected payoffs are given by

$$U_B^{WB} = v_B\left(1 - \frac{w_A}{\underline{v}_A} + \frac{1}{\underline{v}_A h(\underline{v}_A)}\right) \text{ and } U_A^{WB} = v_A\left(\frac{w_A}{\underline{v}_A}\right)$$

The only payoff relevant deviation by buyer B is to lose so that he would obtain w_B. Since $U_B^{WB} = \underline{w}_B + \frac{v_B}{\underline{v}_A h(\underline{v}_A)} \geq w_B$ buyer B will not deviate. Consider next deviations by buyer A. Deviations to bids below w_B will leave B unconstrained and are hence unprofitable. Deviations to bids $b'_A \in (w_B, b_A)$ are payoff equivalent to b_A while deviations to any $b'_A \in (b_A, b_B)$ will trigger a higher resale price, and are hence unprofitable. If she deviates to win, by bidding, for instance, $b'_A = b_B$ then her expected utility will be $U_A^{WA} = v_A\left(1 - \frac{w_B}{v_B}\right)$. Since $w_B \geq \underline{w}_B = v_B\left(1 - \left(\frac{w_A}{\underline{v}_A}\right)\right)$ those deviations are unprofitable too.

ii) At the equilibrium candidate if $w_B + \widehat{\psi}_{\tilde{v}_A}(\underline{v}_A) < \underline{v}_A$ holds then B wins against low types and resells the entire object at $r^*_{\tilde{v}_A} = \underline{v}_A$ given that $w_A > \underline{v}_A$ and $p = w_B + \widehat{\psi}_{\tilde{v}_A}(\underline{v}_A)$ (see $O^o_B(p > w_B)$). His expected payoffs are given by

$$U_B^{WB} = w_B - w_B - \widehat{\psi}_{\tilde{v}_A}(\underline{v}_A) + \underline{v}_A > w_B = U_B^{WA}.$$

As $r^*_{\tilde{v}_A} = \underline{v}_A$, low A types do not find it profitable to deviate neither lo larger losing bids (they would trigger a higher resale price) nor to larger winning bids. Regarding buyer B, deviations so as to win to the high A types are unprofitable as B will resell at the auction price whereas deviations to lose against the low types give lower payoffs, w_B. Finally, any other deviation by B is payoff equivalent to b_B.

If $w_B + \widehat{\psi}_{\tilde{v}_A}(\underline{v}_A) \geq \underline{v}_A$ then B loses and prefers to do so as $U_B^{WB} = \widehat{\psi}_{\tilde{v}_A}(\underline{v}_A) + \underline{v}_A \leq w_B = U_B^{WA}$. Regarding buyer A, deviations to either larger winning bids or to losing bids are payoff equivalent to b_A.■

Proof of Lemma 4

i) Focus first on rule 1. In this case, if $w_A < \underline{v}_A$ player C would sell $z = \frac{w_A}{v_A^L}$ at a price v_A^L to player A (C's optimal offers to A are the same as B's for $v_B = 0$, see $O_B^o(p \leq w_B)$), and $z = 1 - \frac{w_A}{v_A^L}$ at a price v_B to player B, getting payoffs from the resale market equal to $\Lambda_{v_A^L} = w_A + v_B\left(1 - \frac{w_A}{v_A^L}\right)$. Note that by reselling the entire object to a single buyer, the most C can get is $\max\{w_A, v_B\}$ which is lower than $\Lambda_{v_A^L}$.

Under rule 2, selling the entire object to A is dominated by selling the entire object to B. If selling to A, the resale price must equal w_A , whereas when selling to B the resale price will equal $v_B(1 - z_B) + w_A$ (B will consume part of the good and will resell the remaining part to A in return of w_A). Compare next selling the entire object to B with selling part of the good to each buyer. If selling $z = 1$ to B, the resale price would equal B's valuation so that C gains $w_A + v_B\left(1 - \frac{w_A}{r_B^{**}}\right)$. If C were to sell to A (as loser 1) then C will set $r = v_A^L$ and $z_C = \frac{w_A}{v_A^L}$ (see $O_B^o(p \leq w_B)$ for $v_B = 0$) and he will sell the remaining fraction to B at a price equal to B's use value. Proceeds accrued by C would be $\Lambda_{v_A^L}$, which are strictly lower than $w_A + v_B\left(1 - \frac{w_A}{r_B^{**}}\right)$ whenever $v_B > M(v_A^L)$, i.e. when $r_B^{**} > v_A^L$ and are equal otherwise. Thus, the result follows.

ii) Under rule 1, selling the entire object to A dominates selling it to B as $\underline{v}_A \geq v_B$. When selling only to A the optimal offer when $w_A \geq v_A^L$ is $[r_C, z_C] = [v_A^L, 1]$ (see $O_B^o(p \leq w_B)$ for $v_B = 0$). Similarly, selling to A with $z_C = 1$ and $r_C = v_A^L$ dominates selling to both buyers as $v_A^L > v_B$, and v_B is the maximum price that C can charge to B.

Under rule 2, C will approach first buyer B. The reason is simple, C gains from the resale market less than B when dealing with buyer A as $r_C^* \leq r_B^*$ (see $O_B^o(p \leq w_B)$ for $v_B = 0$ and recall that the resale price is non-decreasing in v_B). Thus, it is best for C to sell the entire object to B at a price r_B^*. ■

Proof of Proposition 5

Assume, by way of contradiction, that C wins at a price p.

i) Let $w_A < \underline{v}_A$ hold. Since under either rule he can get at most B's valuation from the resale market, then $p < w_A + v_B\left(1 - \frac{w_A}{r_B^{**}}\right)$ must hold for C to make profits from participating. Since B bids $w_A + v_B\left(1 - \frac{w_A}{r_B^{**}}\right)$, C cannot profitably win the auction.

ii) Let $w_A \geq \underline{v}_A$ hold. Since C's proceeds from the resale market are bounded above by r_B^* (r_B^* under rule 2 and v_A^L under rule 1, with $v_A^L \leq r_B^*$), then $p < r_B^*$ must hold for C to participate. Since B bids at least r_B^*, C cannot profitably win the auction. ■

Appendix A.2

The following lemma presents some results on weakly dominated strategies.

Lemma 5 *i) If $w_A < \underline{v}_A$, bids below $\underline{\Lambda} = w_A + v_B\left(1 - \frac{w_A}{\underline{v}_A}\right)$ are weakly dominated for player A if any of the following two conditions hold: (a) $w_B \geq \underline{\Lambda}$ or (b) $w_A \leq \widehat{M}(v_A^L)$.*
ii) If $\underline{v}_A < \min\{w_A, w_B\}$, then

1) Bidding below $\underline{v}_A$ is weakly dominated for buyer B.

2) If $v_A = \underline{v}_A$ then the bid $\underline{v}_A$ constitutes a weakly dominant strategy for player A.

3) If $w_B \geq \bar{v}_A$, bidding above $\min\{v_A, w_A + v_B(1 - \frac{w_A}{v_A})\}$, is a weakly dominated strategy for buyer A.

Proof. *i)* Compare payoffs from $b_A = \underline{\Lambda}$ with those obtained with a lower bid b'_A. Trivially, if A wins with either bid they are equally profitable. When losing with both of them, payoffs will depend on B's resale offers. If $w_B \geq \underline{\Lambda}$, resale prices are independent of p so that both bids yield the same expected payoff, whereas if $w_B \in (b'_A,\ \underline{\Lambda}]$ while $w_A \leq \widehat{M}(v_A^L)$ resale prices will be lower if $p = b_A = \underline{\Lambda}$ making A better-off. Finally, if $b_B \in (b'_A,\ b_A)$ then A wins with the former and loses with the latter. Since A is better off when winning the result follows. Note that $U_A^{WA} = v_A\left(1 - \frac{b_B - w_A}{v_B}\right)$ and $U_A^{WB} = \max\left\{w_A,\ v_A\left(\frac{w_A}{r}\right)\right\}$ so that

$$U_A^{WA} = v_A\left(1 - \frac{b_B - w_A}{v_B}\right) > v_A\left(\frac{w_A}{\underline{v}_A}\right) \geq U_A^{WB}$$

where the inequality is deduced from $b_B < \underline{\Lambda}$. Since $U_A^{WA} \geq U_A^{WB}$ the claim follows.

We show next part *ii)*.

1) Since buyer B can always get $\underline{v}_A$ at the resale market by setting $r = \underline{v}_A$, a bid below $\underline{v}_A$ is weakly dominated by bidding $\underline{v}_A$.

2) For buyer A, payoffs with the strategies $b_A = \underline{v}_A$ and $b'_A < \underline{v}_A$ are equal when either losing, since $r \geq \underline{v}_A$, or winning; when winning with the former while losing with the latter it

must be the case that $p = b_B < \underline{v}_A$ while $r \geq \underline{v}_A$ and therefore $b_A = \underline{v}_A$ yields higher payoffs. Similarly, payoffs with the strategies $b_A = \underline{v}_A$ and $b'_A > \underline{v}_A$ only differ when losing with the former while winning with the latter. Since any resale offer satisfies $r \geq \underline{v}_A$, her utility when losing equals w_A. When winning the price must be $b_B > \underline{v}_A$ (recall that player A wins with $b'_A > \underline{v}_A$and loses with $b_A = \underline{v}_A$) so that the utility is lower than w_A. As losing yields larger payoffs than winning, the result follows.

3) Since $w_B \geq \bar{v}_A$, resale offers by buyer B only depend upon his use value. Consequently, payoffs with the strategies $b_A = \min\{v_A, w_A + v_B(1 - \frac{w_A}{v_A})\}$ and $b'_A > b_A$ only differ when losing with the former while winning with the latter.

Assume first $\min\{v_A, w_A + v_B(1 - \frac{w_A}{v_A})\} = v_A$ so that $b_A = v_A$. When winning with b'_A the utility is lower than w_A as $p > v_A$ (since $b_A = v_A$ would be a losing bid against b_B) and the resale price is $v_B < v_A$. As losing yields larger payoffs (at least w_A) than winning, the result follows.

Assume next $\min\{v_A, w_A + v_B(1 - \frac{w_A}{v_A})\} = w_A + v_B(1 - \frac{w_A}{v_A})$. For any $b_B \in (b_A, b'_A)$, when winning by bidding b'_A buyer A has to resell as $p > w_A$ (note that $b_A = w_A + v_B(1 - \frac{w_A}{v_A})$ would be a losing bid against b_B), consequently her utility is lower than w_A as the most she can get after resale is $w_A + v_B(1 - \frac{w_A}{v_A})$ which is lower than the auction price. Since when losing her expected payoff is at least w_A, losing yields larger payoffs than winning, and the result follows. ■