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Extending communication-proof equilibrium to infinite games

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Abstract

The concept of Communication-proof equilibrium is extended to infinite games. To that end we make use of abstract stable sets as defined by Greenberg in his Theory of Social Situations. © 2001 Elsevier Science B.V. All rights reserved.

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1. Introduction

The concept of Communication-proof equilibrium, *Com-PE*, was proposed in Ferreira (1996) for finite games. Here we present its extension to infinite games using the approach in Greenberg (1989). To understand this definition think of a group of people that arrives at an agreement and then must leave the room. The problem is that, when a player leaves, he may rightly suspect that the remaining players will change their actions taking as given his own strategy. A coalition-proof Nash equilibrium, *Coalition-PNE*, (Bernheim et al., 1987, from now on B,P&W) is an arrangement that will not be changed regardless of the order of exit. To modify the definition for extensive form games one needs to take into consideration not only the usual problems of time consistency but the renegotiation issues as well. In particular, if part of the deviation can be observed, the possibility of a reaction by the opponents cannot be ruled out. Unlike the definition of perfectly coalition-proof Nash equilibrium, *PCoalition-PNE*, provided by B,P&W, the concept of *Com-PE* reflects these points and can be regarded as an extension of both *Coalition-PNE* and renegotiation-proof equilibria.

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2. Notation and definitions

Let Γ be an extensive form game with no chance moves with $N = \{1, \dots\}$ as the set of players. A subset of N will denote a coalition. The (finite) set of pure strategies for player i will be denoted by S_i . Strategy profiles for sets of players are denoted by $S = \prod_{i \in N} S_i$, $S_C = \prod_{i \in C} S_i$ and $S_{-C} = \prod_{i \in -C} S_i$, where $C \subset N$ is a coalition and $-C = N \setminus C$. Their respective typical elements will be s_i , s , s_C and s_{-C} . The vector (s_{-i}, s'_i) denotes the strategy profile s in which s_i has been replaced with s'_i . Let $u_i: S \rightarrow R$ be the outcome function for player i (it will be identified with his von Neumann-Morgenstern expected utility function). The notation $s \succ_C s'$ means that $u_i(s) > u_i(s')$ for every player i in the coalition C . The game that s_C induces on Γ (i.e. the game that coalition C faces when s_C is regarded as fixed) will be denoted by Γ/s_C . A subgame of Γ is indicated by g .

The number of stages in the game is the maximum number of nested subgames in it. Therefore a stage is the play between two consecutive nested subgames. By $s(t)$ we will denote the behavior that strategy s induces at stage t .

The set of histories when the game starts (stage 1) is given by $H_1 = \{0\}$. The set of feasible histories H for a τ stage game is given by $H = \cup_{t=1}^{\tau} H_t$; where H_t is defined recursively by

$$H_{t+1} = \{h_{t+1} \mid \exists h_t \in H_t \text{ s.t. } h_{t+1} \in \{h_t\} \times S^{h_t}\},$$

where $S^{h_t} = \prod_{i=1}^n S_i^{h_t}$ and where $S_i^{h_t}$ is the set of actions for player i in stage t given h_t . Some more notation about histories include: The set of feasible histories following h , H^h ; the subgame induced by history h , g^h ; the strategy induced by s after history h , s^h ; the stage at which history h is observed, $t(h)$ (history h ends at stage $t(h) - 1$) and the set of histories of length $t(h) + 1$ that belong to H^h , h_1 .

When there is no confusion, the short name of an equilibrium (e.g. *SPE*, after subgame perfect equilibrium) will denote the set of those equilibria in a given game.

In what follows, games will be assumed to be of complete information and perfect recall so that only behavioral strategies will be considered.

Definition 1. (Extension of *Coalition-PNE* in B,P&W) (i) In a single player game, s^* is a Coalition-proof Nash equilibrium restricted to $R \subset S$, *Coalition-PNE*(T), if $s^* \in \text{argmax}_{s \in R} u^1(s)$.

(ii) Assume that *Coalition-PNE*(R) has been defined for games with less than n players. Then, in a n -players game;

(a) $s^* \in S$ is Self-enforcing restricted to R , *SE*(R), if for any coalition $C \neq N$, s^* is *Coalition-PNE*(R) in the game Γ/s_{-C}^* ;

(b) $s^* \in S$ is *Coalition-PNE*(R) if it is *SE*(R) and if it does not exist in any other $s \in R$ such that s is *SE*(R) and $u^i(s) > u^i(s^*)$ for all $i \in N$.

Definition 2. (i) In a single stage game Γ , s^* is a Communication-proof equilibrium, *Com-PE*, if it is a *Coalition-PNE*.

(ii) Let $t > 1$ and assume that *Com-PE* has been defined for games with $r < t$ stages. Then, in a game Γ with t stages, s^* is *Com-PE* if it is a *Coalition-PNE*(S^1) where $S^1 = \{s \in S \mid s \text{ induces a } \text{Com-PE} \text{ in proper subgames of } \Gamma\}$.

3. Extending Com-PE to infinite games

This section follows the approach by Greenberg (1989) where the stable sets are used to extend recursive definitions to the infinite case.

A von Neumann and Morgenstern abstract system (AS) is a pair $(D, >)$ where D is an abstract set and $>$ is a dominance relation ($f > d$ will be read as ‘ f dominates d ’). Let $(D, >)$ be an abstract system, and let $f \in D$. The dominion of f , denoted by $\Delta(f)$, is the set $\Delta(f) = \{d \in D \mid f > d\}$. That is, $\Delta(f)$ consists of all elements of D that f dominates, according to the dominance relation $>$. Similarly, for a subset $F \subset D$, the dominion of F , denoted by $\Delta(F)$, is the set $\Delta(F) = \cup \{\Delta(f) \mid f \in F\}$. That is, an element d in D belongs to $\Delta(F)$ if it is dominated by some element in F . A set $F \subset D$ is a von Neumann and Morgenstern abstract stable set (ASS) for the system $(D, >)$ if $F = D \setminus \Delta(F)$.

Let Γ be a multi-stage game. Inspired by Definition 2 an abstract system $(D, >)$ is introduced. Let the elements of the abstract set consist of a coalition, a subgame and a strategy profile in this subgame,

$$D = \{(C, g^h, s) \mid C \subset N, C \neq \emptyset, h \in H, s \in S^h\}$$

where S^h is the set of strategy profiles in the subgame after h ; and let the domination relation be defined as follows: $(B, g^k, y) > (C, g^h, s)$ if and only if either

1. $k \in H^h$ and $k \neq h$: $B \subset N, s_{-B} = y_{-B}, y^k >_{BS^k}$ or
2. $k = h$: $B \subset C, s_{-B} = y_{-B}, y^k >_{BS^k}$.

The next proposition (the proof of which is just the proof of the lemmas after it) relates the ASS of $(D, >)$ with the definition of *Com-PE* for finite games and allows for a definition applicable to infinite games.

Proposition 1. *Let K be an ASS of $(D, >)$; then, for finite games, we have that for all $h \in H, C \subset N$ and $s \in S$; $(C, g^h, s) \in K$ if and only if $(s_C(t(h)), s^{h_1})$ is a *Com-PE* in the subgame $g^h/s_{-C}(t(h))$. In particular, when $C = N$ and $h = \emptyset$, $\text{Com-PE}(\Gamma) = \{s \mid (N, \Gamma, s) \in K\}$.*

Lemma 1. For all $h \in H, C \subset N$ and $s \in S$; if $(s_C(t(h)), s^{h_1})$ is a *Com-PE* in $g^h/s_{-C}(t(h))$ then there are no B, k and x such that $(x_B(t(k)), x^{k_1})$ is a *Com-PE* in $g^k/x_{-B}(t(k))$ and neither of the following is satisfied:

- (i) $k \in H^h, B \subset N, s_{-B} = x_{-B}, x^k >_{BS^k}$
- (ii) $k = h, T \subset C, s_{-T} = x_{-T}, x^k >_{TS^k}$

Proof. The proof is by induction in the number of stages t . If $t = 1$, the proof reduces to show that for all $C \subset N, s_C$ is a *Com-PE* (= *Coalition-PNE*) in Γ/s_{-C} implies that no $B \subset N$ and $x \in S$ exist such that x_B is *Coalition-PNE* in Γ/x_{-B} and $B \subset C, x_{-B} = s_{-B}, x >_{BS}$; but this comes from Lemma 1 in Greenberg (1989).

Assume that the Lemma has been proved for games with strictly less than t stages and prove it now

for $t > 1$. If $(s_C(t(h)), s^{h_1})$ is a *Com-PE* in the restricted game $g^h/s_{-C}(t(h))$ and $k \in H^h$ and if $k \neq h$ then s^k is a *Com-PE* in g^k by definition of *Com-PE*. Apply the induction hypothesis to get that no $B \subset N$ and $x \in S$ exist such that $(x_B(t(k)), x^{k_1})$ is a *Com-PE* in $g^h/x_{-B}(t(k))$ with $x_{-B} = s_{-B}$ and $x^k >_B s^k$. Finally, for the case $k = h$, suppose that there exist a coalition $B \subset N$ and a strategy profile $x \in S$ such that $(x_B(t(h)), x^{h_1})$ is a *Com-PE* in $g^h/x_{-B}(t(h))$, then x^1 (the restriction of x in g^{h_1}) is a *Com-PE* in g^{h_1} ; if $B \subset C$, $s_{-B} = x_{-B}$ and $x^k >_B s^k$ then s was not a *Coalition-PNE* restricted to $S' = \{s \in S \mid s \text{ induces a } Com-PE \text{ in proper subgames of } \Gamma\}$. \square

Lemma 2. For all $h \in H$, $C \subset N$ and $s \in S$, if $(s_C(t(h)), s^{h_1})$ is not a *Com-PE* then there exist $B \subset N$, $k \in H$ and $x \in S$ such that $(x_B(t(k)), x^{k_1})$ is a *Com-PE* in $g^h/s_{-B}(t(k))$ and either

1. (i) $k \in H^h$, $k \neq h$, $s_{-B} = x_{-B}$ and $x^k >_B s^k$ or
2. (ii) $k = h$, $B \subset C$, $s_{-B} = x_{-B}$ and $x^k >_B s^k$.

Proof. If $t = 1$, it comes from Lemma 2 in Greenberg (1989).

If $t > 1$, that $(s_C(t(h)), s^{h_1})$ is not a *Com-PE* in $g^h/s_{-C}(t(h))$ means that either
 (i') the restriction of s in g^k (s^1) is not a *Com-PE* in g^k with $k \neq h$, $k \in H^h$, or
 (ii') s^1 is a *Com-PE* in g^{h_1} but there exist $B \subset C$ and $x_B \in S_B$ such that $(x_B, s_{-B})^k >_B s^k$ and (x_B, s_{-B}) is a *Com-PE* in $g^h/s_{-B}(t(h))$.
 (i') implies (i) by induction hypothesis and (ii') implies (ii) by setting $x = (x_B, s_{-B})$. \square

Since this characterization of a *Com-PE* is not based on backward recursion, it can be used to formulate a general definition, covering both finite and infinite games (in the number of stages and players).

Definition 3. Consider a multi-stage game Γ with a finite or infinite number of players. A strategy profile s is said to be a *Com-PE* of Γ if and only if there is an ASS, F , for the associated system $(D, >)$ such that $(N, \Gamma, s) \in F$.

Proposition 2. Let L and M be abstract stable sets of the abstract set $(D, >)$ associated with a finite horizon game, as defined above, then $L = M$.

Proof. Let $\{(C_i, g^{h_i}, s_i)\}_i$ be an infinite sequence with $(C_{i+1}, g^{h_{i+1}}, s_{i+1}) > (C_i, g^{h_i}, s_i)$ for all i . By finiteness of the sets of coalitions and histories and because the domination relation requires that either $C_{i+1} \subset C_i$ or $h_{i+1} \in H^{h_i}$, we have that, in the tail of the sequence, $C_{i+1} = C_i$ and $h_{i+1} = h_i$. This tail is clearly transitive. According to corollary 4b in Arce and Kahn (1991), this is a sufficient condition for the uniqueness of the abstract stable set if it exists. \square

Proposition 2 cannot be generalized to infinite horizon games. Perhaps the simplest example to show this is the following. Consider a game consisting of a set of $\{1, 2, \dots\}$ lions, in this order. Each lion can eat the lion before it (except lion 1, that can eat an antelope) or not. If he does not, the game ends, but if he ate the lion before him (or the antelope), the next lion plays. Each lion prefers first to eat the antelope (directly or by eating the lion immediately before him if he ate the lion that ate the lion \dots that ate the antelope) and not to be eaten, second to do nothing and third to be eaten (there is no use in eating a lion that did not eat the antelope directly or indirectly). If there are n lions, there is

a *SPE* that depends on whether n is even or odd. If n is even, odd numbered lions decide not to eat and even numbered ones eat the previous lion only if he ate the lion that ate the lion . . . that ate the antelope. If n is odd, it is the odd numbered who eat and the even numbered do not eat. If the number of lions is infinite, the two equilibria just described are two *SPE* (and *Com-PE*) belonging to different abstract stable sets.

Finally, we compare Definition 3 with that of *PCoalition-PNE*. For that define a new abstract system $(D, >^*)$ with D as before and with the domination relation $>^*$ defined by $(B, g^k, y) >^*(C, g^h, s)$ if and only if

$$(3) \quad k \in H^h, B \subset C, s_{-B} = y_{-B} \text{ and } y^k >_{B^k} s^k.$$

Definition 4. (Asheim, 1991) Let Γ be a multi-stage game. A strategy $s \in S$ is said to be a *PCoalition-PNE* of Γ if there exists an abstract stable set, F , for the associated system $(D, >^*)$ such that $(N, \Gamma, s) \in F$.

The differences between *Com-PE* and *PCoalition-PNE* are now clearer. Condition (3) is equivalent to condition (2) when restricted to the stage where the deviation takes place. After the deviation is known ($k \neq h$), 3 is replaced with (1), the condition that allows for reactions against the deviation by any subset of players.

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