Theory of the Firm

Production Technology

The Firm

What is a *firm*?

In reality, the concept firm and the reasons for the existence of firms are complex.

Here we adopt a simple viewpoint: a firm is an economic agent that produces some goods (outputs) using other goods (inputs).

Thus, a firm is characterized by its *production technology*.

The Production Technology

A production technology is defined by a subset Y of $\Re^L$.

A production plan is a vector $y = (y_1, \ldots, y_L) \in \Re^L$ where positive numbers denote outputs and negative numbers denote inputs.

Example: Suppose that there are five goods (L=5). If the production plan $y = (-5, 2, -6, 3, 0)$ is feasible, this means that the firms can produce 2 units of good 2 and 3 units of good 4 using 5 units of good 1 and 6 units of good 3 as inputs. Good 5 is neither an output nor an input.

The Production Technology

In order to simplify the problem, we consider a firm that produces a single output (Q) using two inputs (L and K).

A single-output technology may be described by means of a production function $F(L,K)$, that gives the maximum level of output Q that can be produced using the vector of inputs $(L,K) \geq 0$.

The production set may be described as the combinations of output Q and inputs (L,K) satisfying the inequality

$$Q \leq F(L,K).$$

The function $F(L,K)=Q$ describes the frontier of Y.

Production Technology

$$\boldsymbol{Q = F(L,K)}$$

Q = output

L = labour

K = capital

$F_L = \partial F / \partial L > 0$ (marginal productivity of labour)

$F_K = \partial F / \partial K > 0$ (marginal productivity of capital)

Example: Production Function

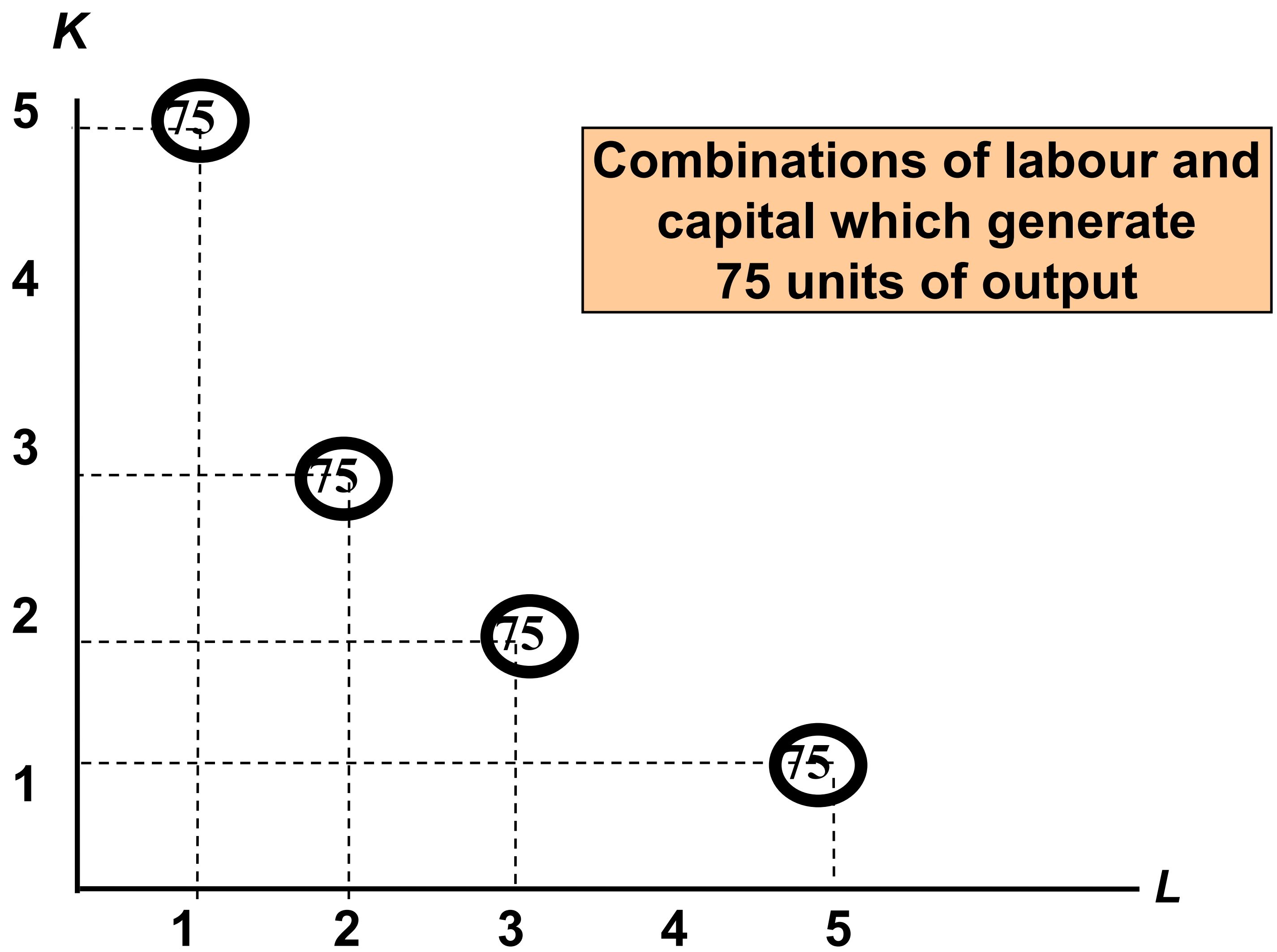
Quantity of capital	Quantity of labour 1	2	3	4	5
1	20	40	55	65	75
2	40	60	75	85	90
3	55	75	90	100	105
4	65	85	100	110	115
5	75	90	105	115	120

Isoquants

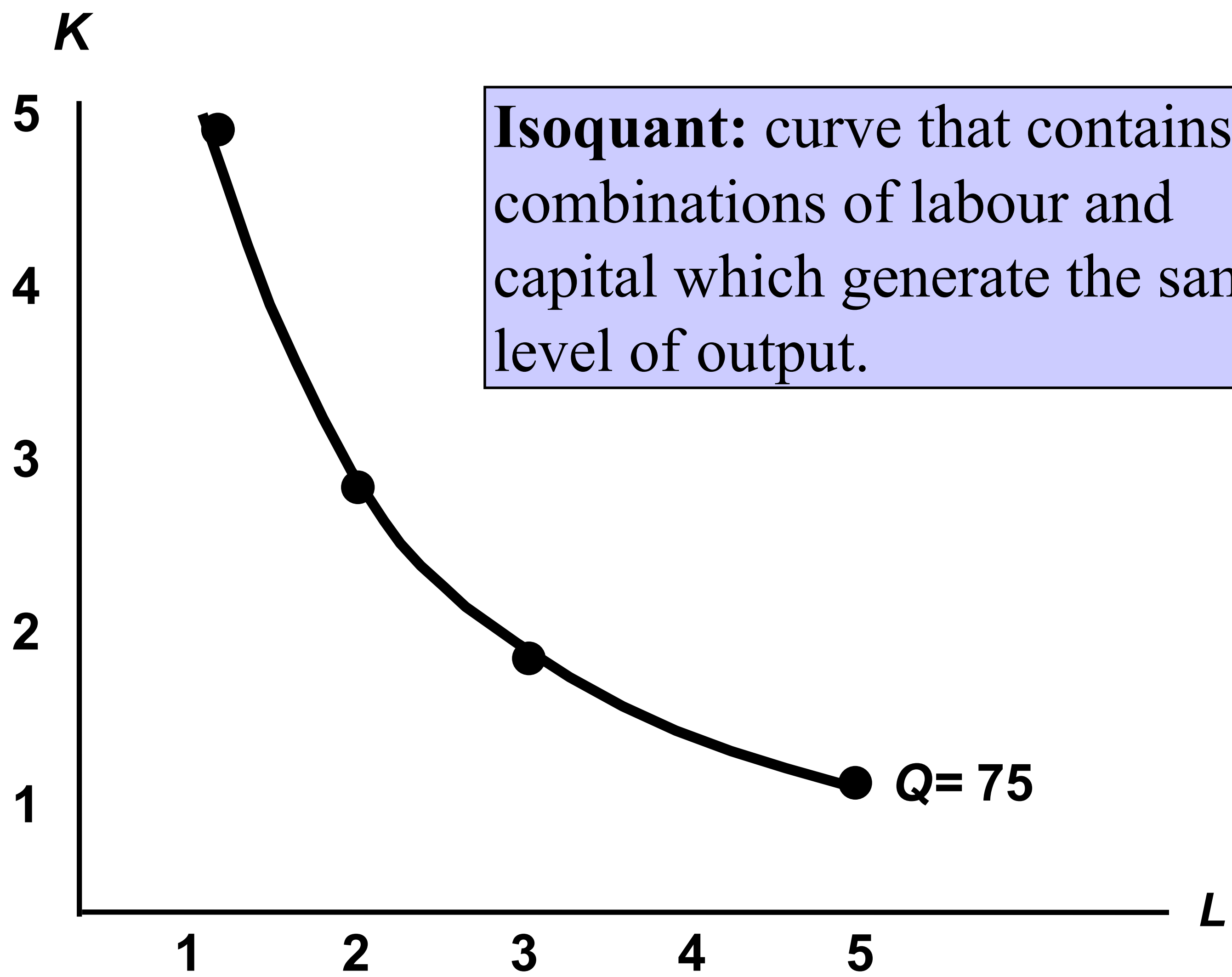
The production function describes also the set of inputs vectors *(L,K)* that allow to produce a certain level of output *Q*.

Thus, one may use technologies that are either relatively labour-intensive, or relatively capital-intensive.

Isoquants

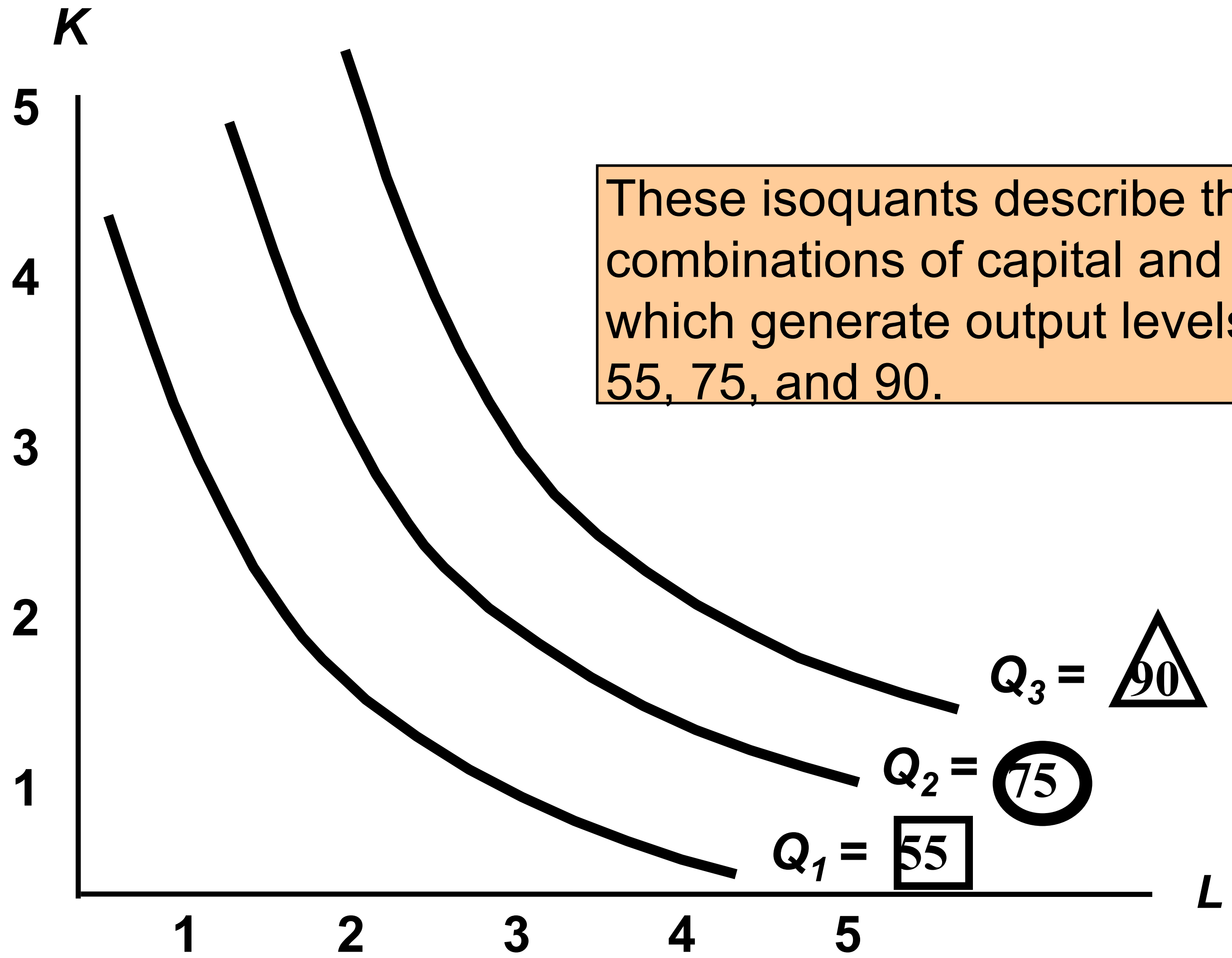

K
5
4
3
2
1
75
75
75
75
L
1
2
3
4
5
Combinations of labour and capital which generate 75 units of output

Isoquants



Isoquant: curve that contains all combinations of labour and capital which generate the same level of output.

Isoquant Map



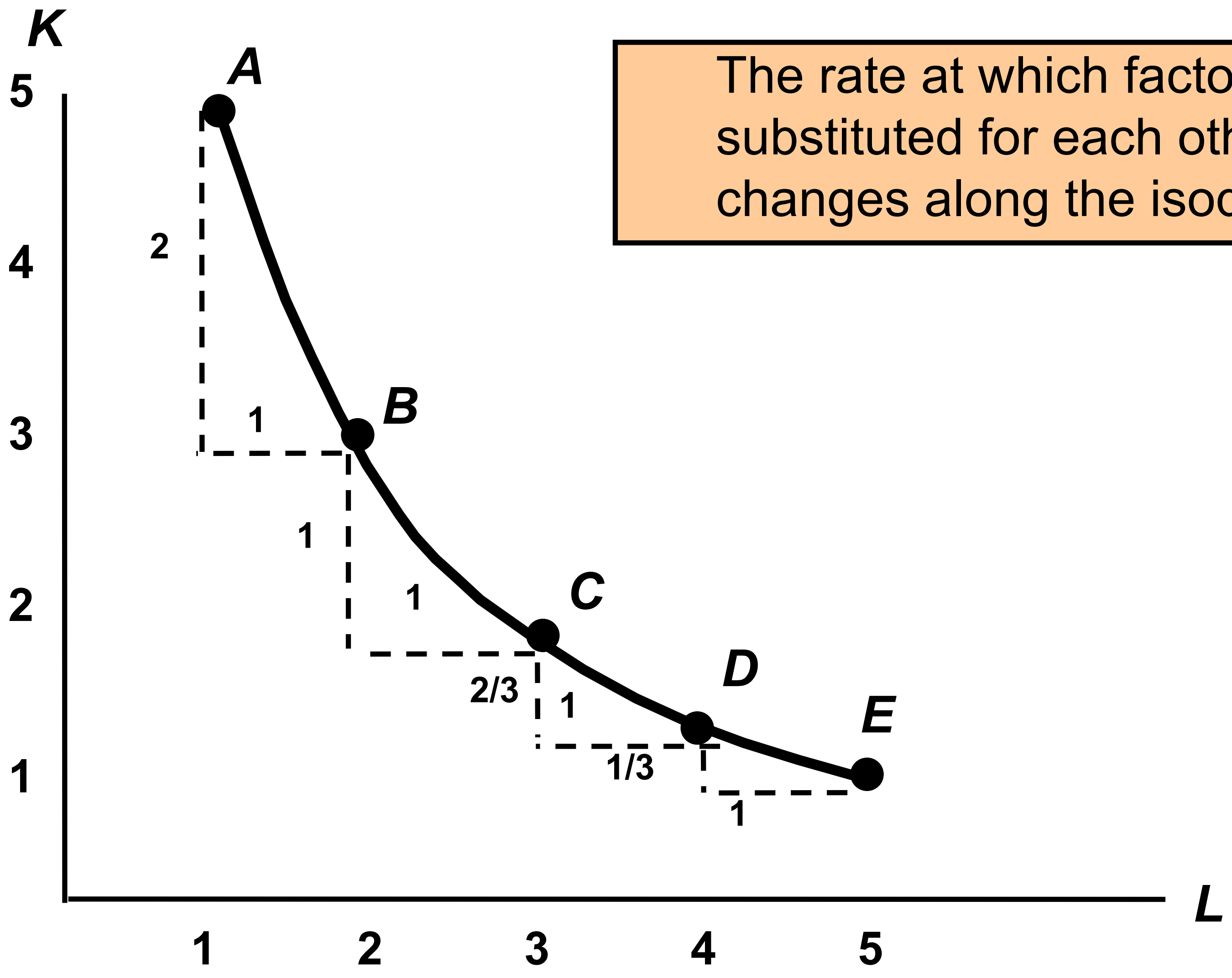
These isoquants describe the combinations of capital and labor which generate output levels of 55, 75, and 90.

Information Contained in Isoquants

Isoquants show the firm's possibilities for substituting inputs without changing the level of output.

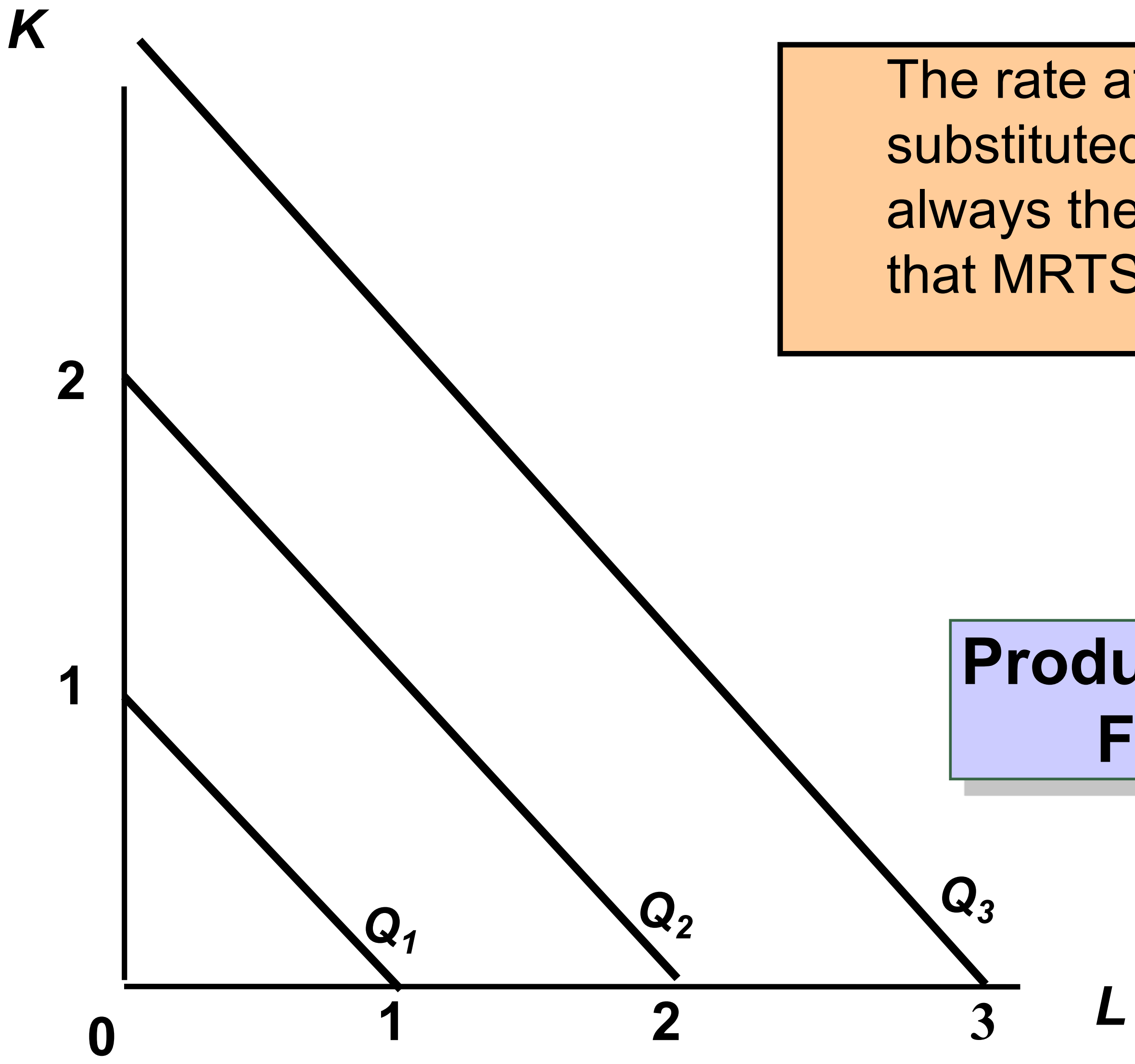
These possibilities allow the producer to react to changes in the prices of inputs.

Production with Imperfect Substitutes and Complements



The rate at which factors are substituted for each other changes along the isoquant.

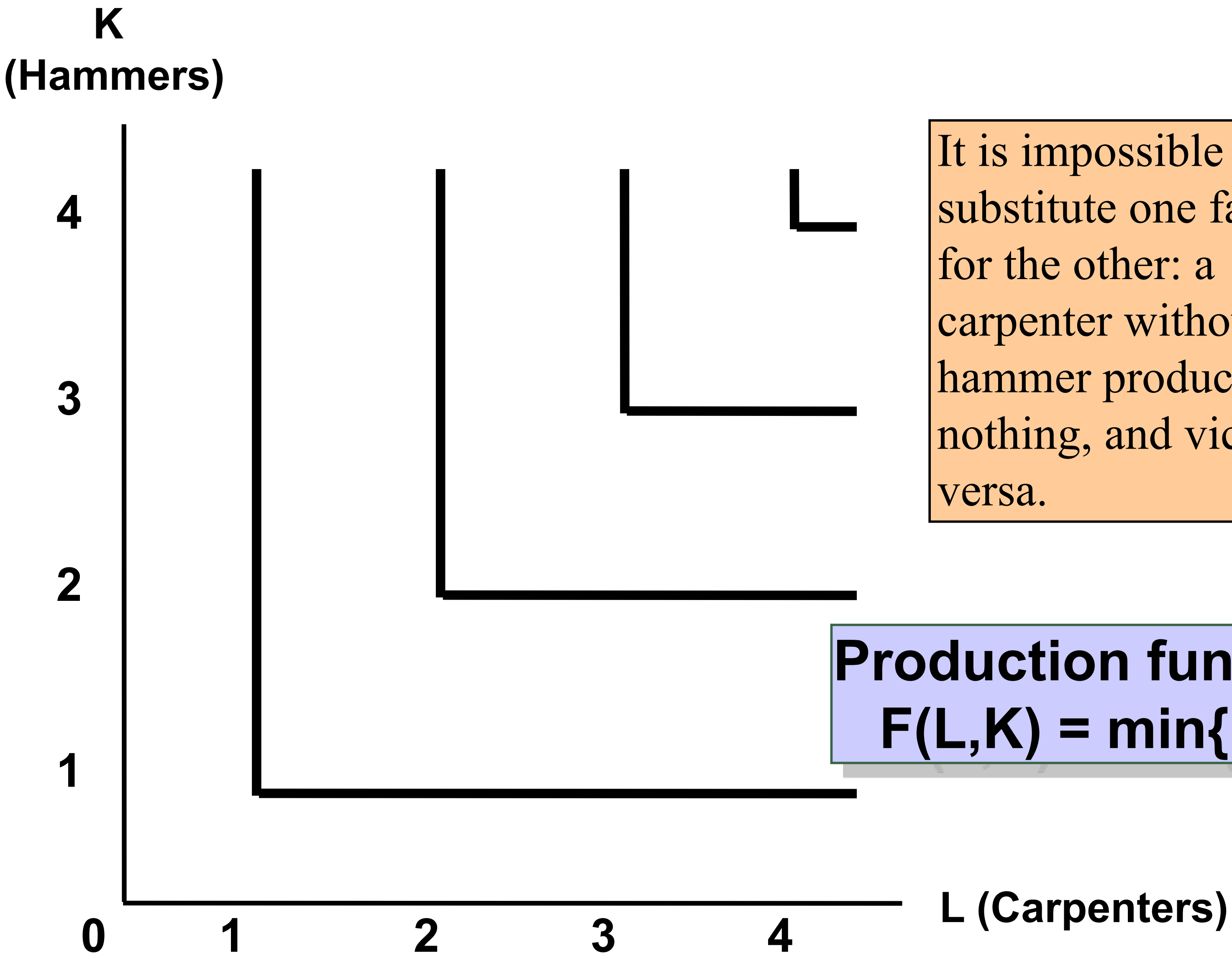
Production with Perfect Substitutes



The rate at which factors are substituted for each other is always the same (we will see that MRTS is a constant).

Production function: F(L,K) = L+K

Production with Perfect Complements



It is impossible to substitute one factor for the other: a carpenter without a hammer produces nothing, and vice versa.

Production function:
$F(L,K) = \min\{L,K\}$

Production: One Variable Input

Suppose the quantity of all but one input are fixed, and consider how the level of output depends on the variable input:

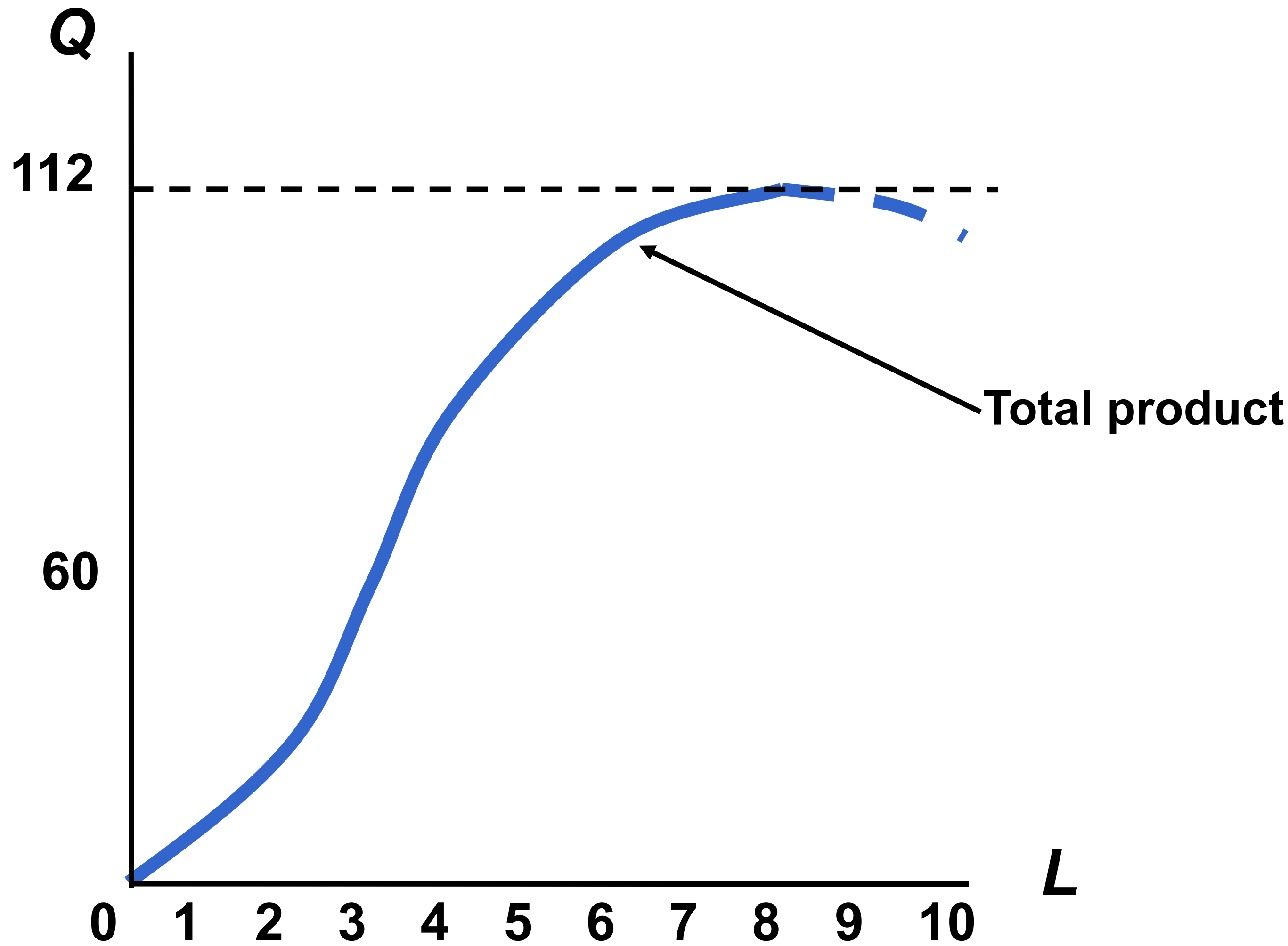
$$\boxed{Q = F(L, K_0) = f(L).}$$

Numerical Example: One variable input

Labour (L)	Capital (K)	Output (Q)
0	10	0
1	10	10
2	10	30
3	10	60
4	10	80
5	10	95
6	10	108
7	10	112
8	10	112
9	10	108
10	10	100

Assume that capital is fixed and labour is variable.

Total Product Curve



Average Productivity

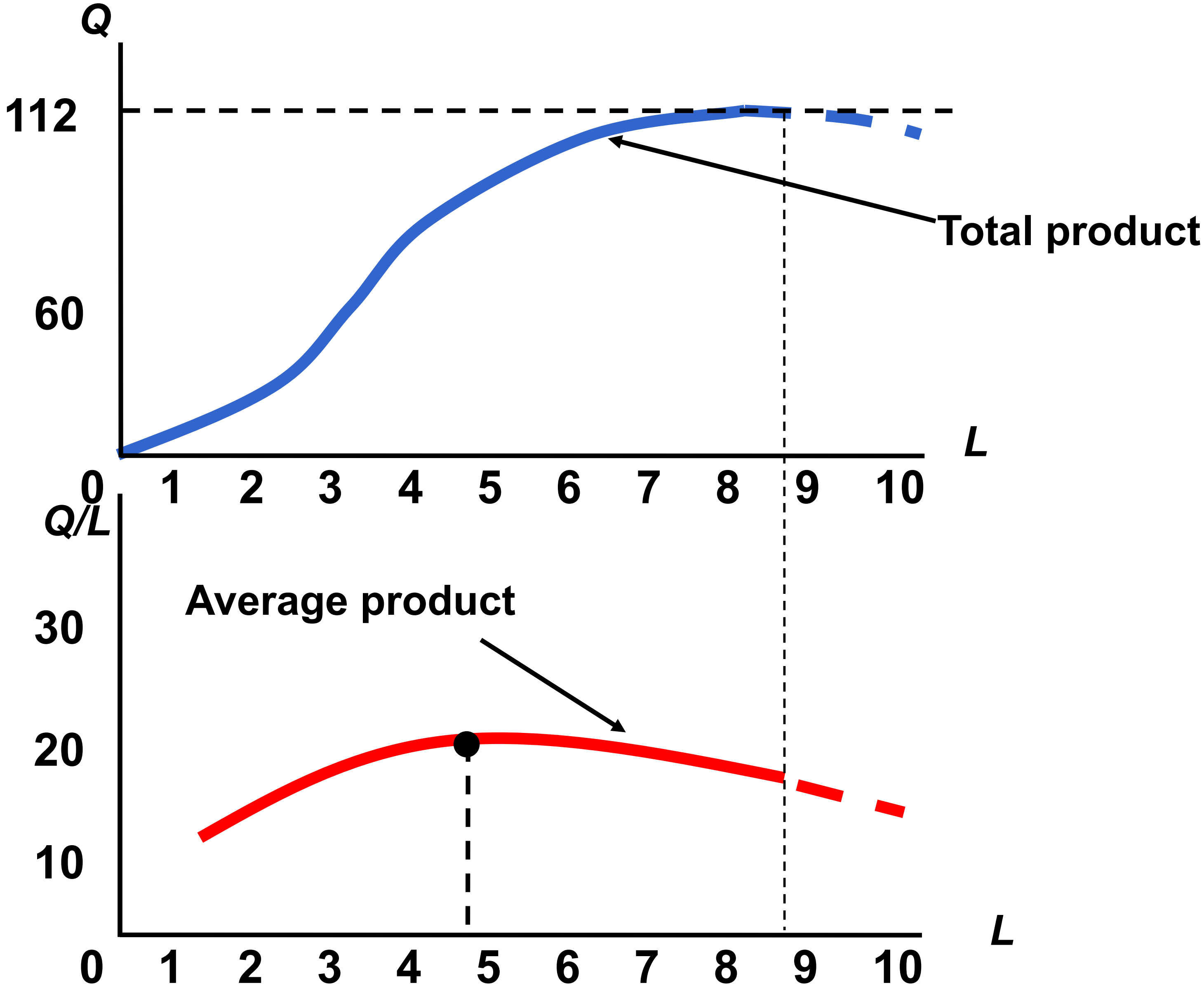
We define the average productivity of labour (AP_L) as the produced output per unit of labour.

$$\boxed{AP_L = Q / L}$$

Numerical Example: Average productivity

Labour (*L*)	Capital (*K*)	Output (*Q*)	Average product
0	10	0	0
1	10	10	10
2	10	30	15
3	10	60	20
4	10	80	20
5	10	95	19
6	10	108	18
7	10	112	16
8	10	112	14
9	10	108	12
10	10	100	10

Total Product and Average Productivity



Marginal Productivity

The marginal productivity of labour (MP_L) is defined as the additional output obtained by increasing the input labour in one unit

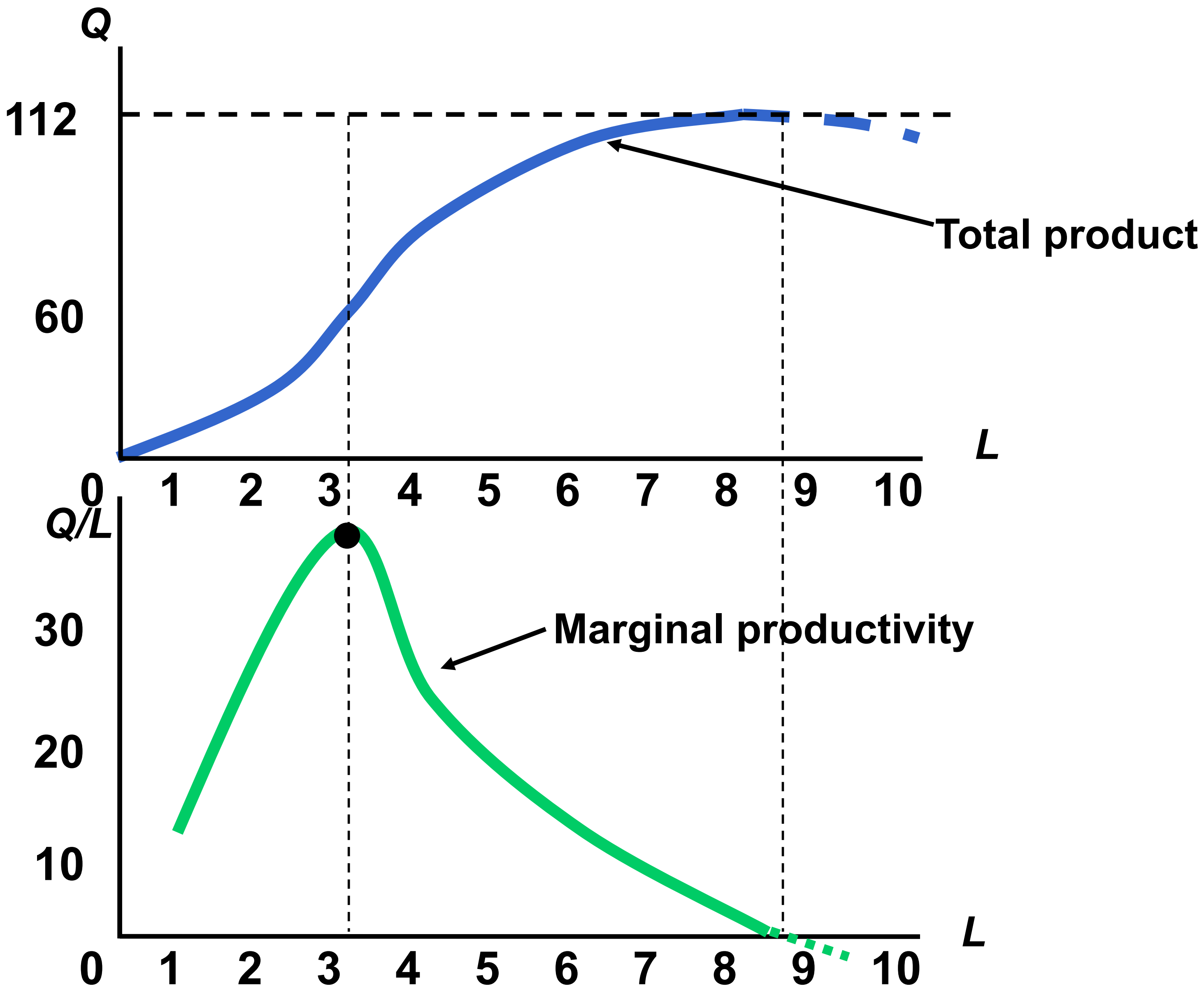
$$MP_L = \frac{\Delta Q}{\Delta L}$$

$$MP_L = \frac{dQ}{dL}$$

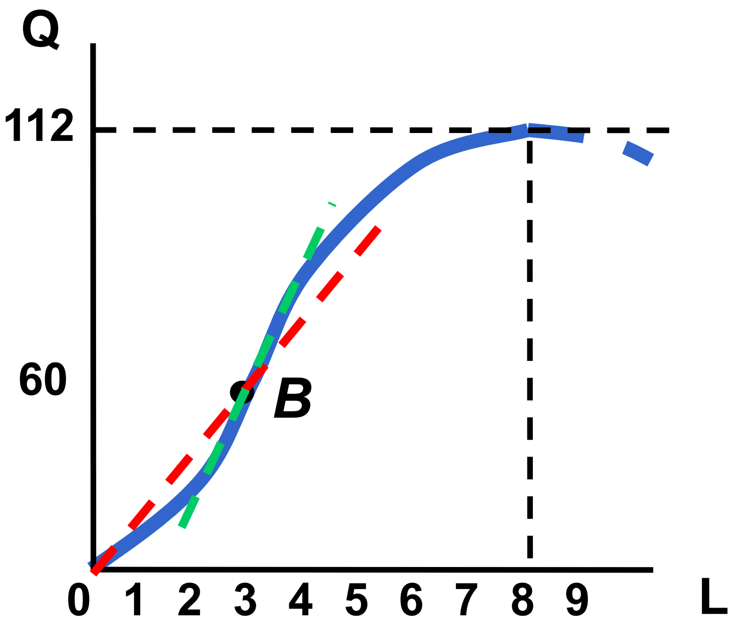
Numerical Example: Marginal Productivity

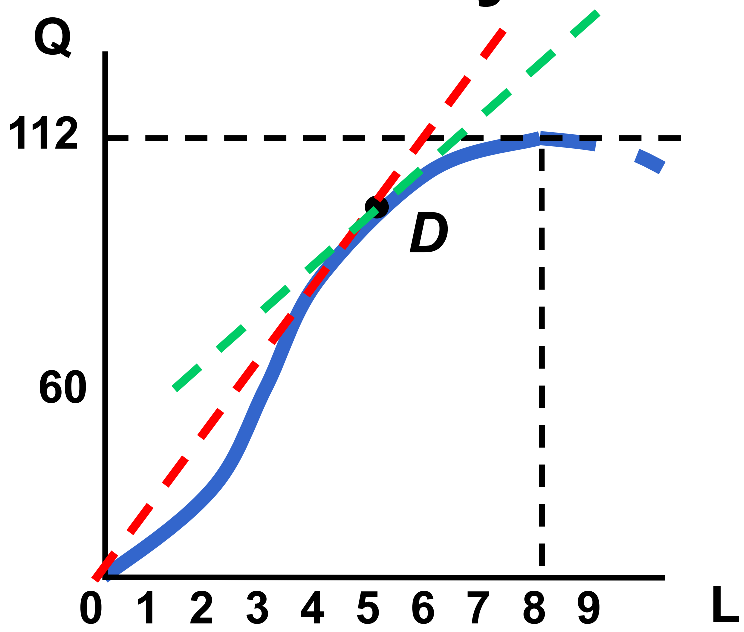
Labour (L)	Capital (K)	Output (Q)	Average product	Marginal product
0	10	0	0	---
1	10	10	10	10
2	10	30	15	20
3	10	60	20	30
4	10	80	20	20
5	10	95	19	15
6	10	108	18	13
7	10	112	16	4
8	10	112	14	0
9	10	108	12	-4
10	10	100	10	-8

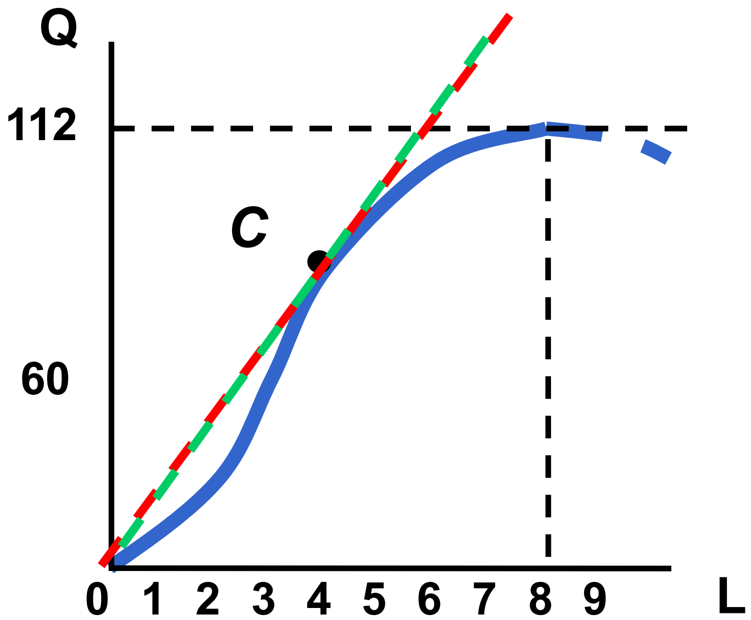
Total Product and Marginal Productivity



Average and Marginal Productivity







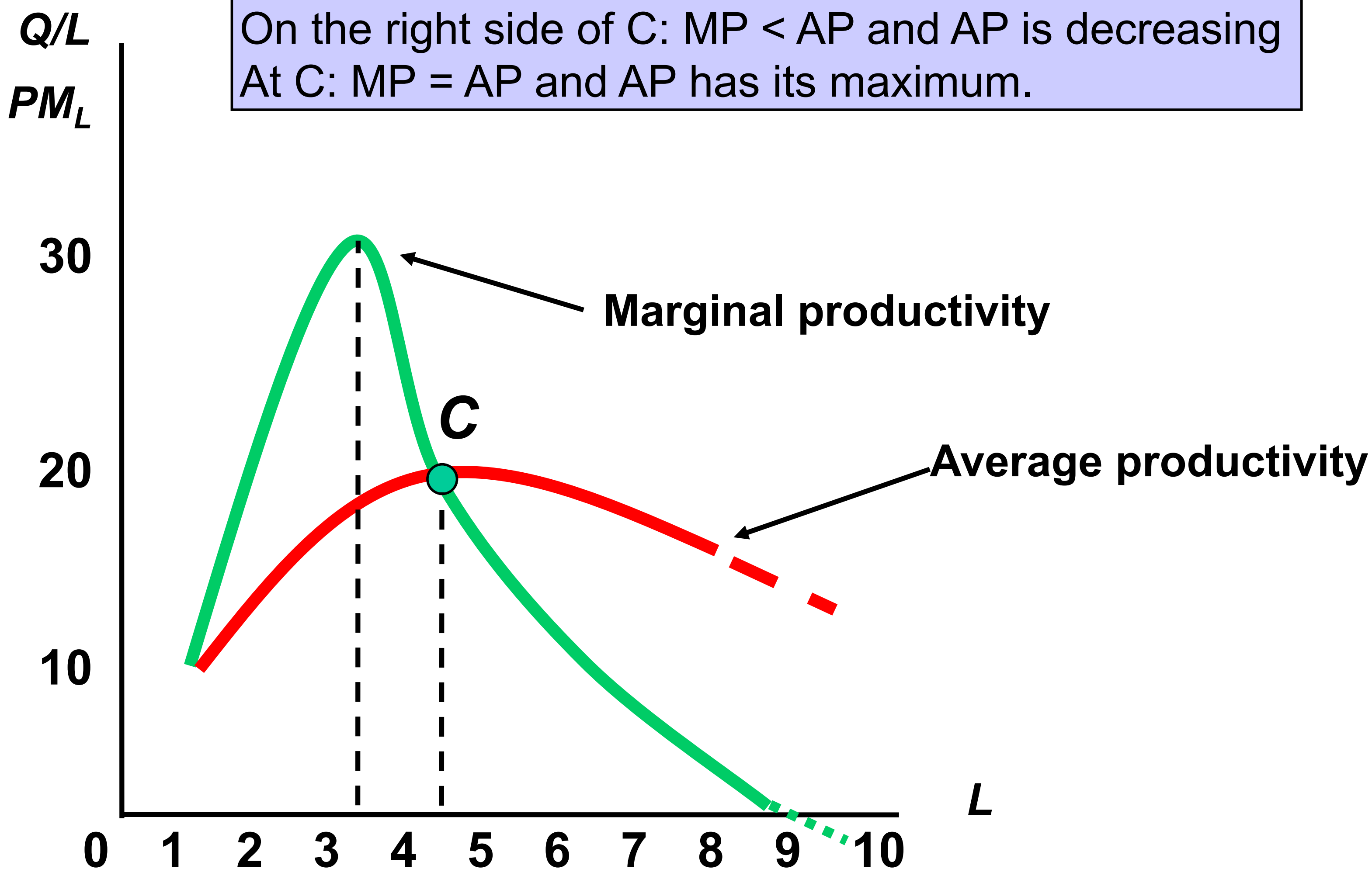
B → Q/L < dQ/dL

C → Q/L = dQ/dL

D → Q/L > dQ/dL

Average and Marginal Productivity

On the left side of C: MP > AP and AP is increasing
On the right side of C: MP < AP and AP is decreasing
At C: MP = AP and AP has its maximum.



Marginal Rate of Technical Substitution

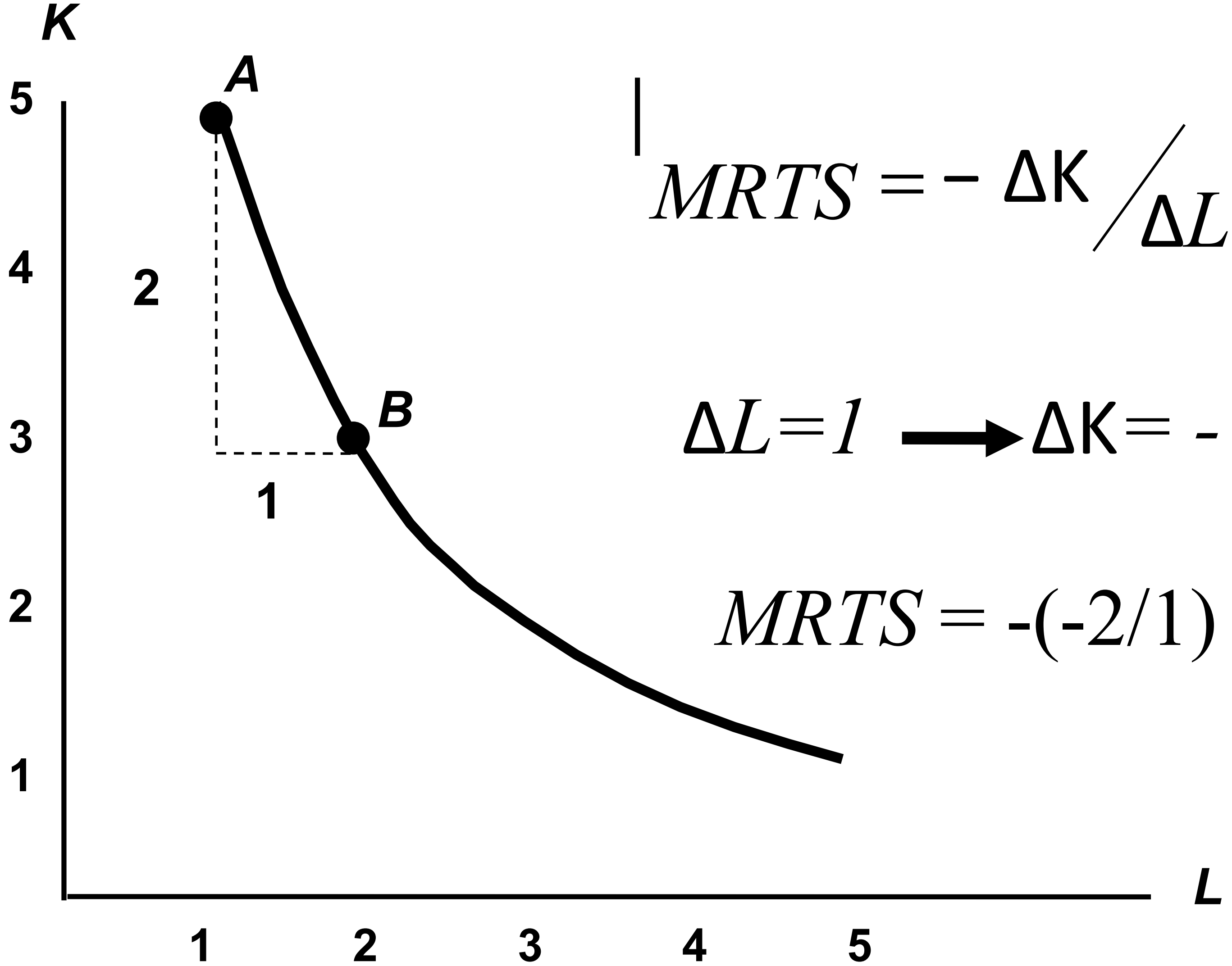
The Marginal Rate of Technical Substitution (*MRTS*) shows the rate at which inputs may be substituted while the output level remains constant.

Defined as

$$MRTS = \left| -F_L / F_K \right| = F_L / F_K$$

measures the additional amount of capital that is needed to replace one unit of labour if one wishes to maintain the level of output.

Marginal Rate of Technical Substitution

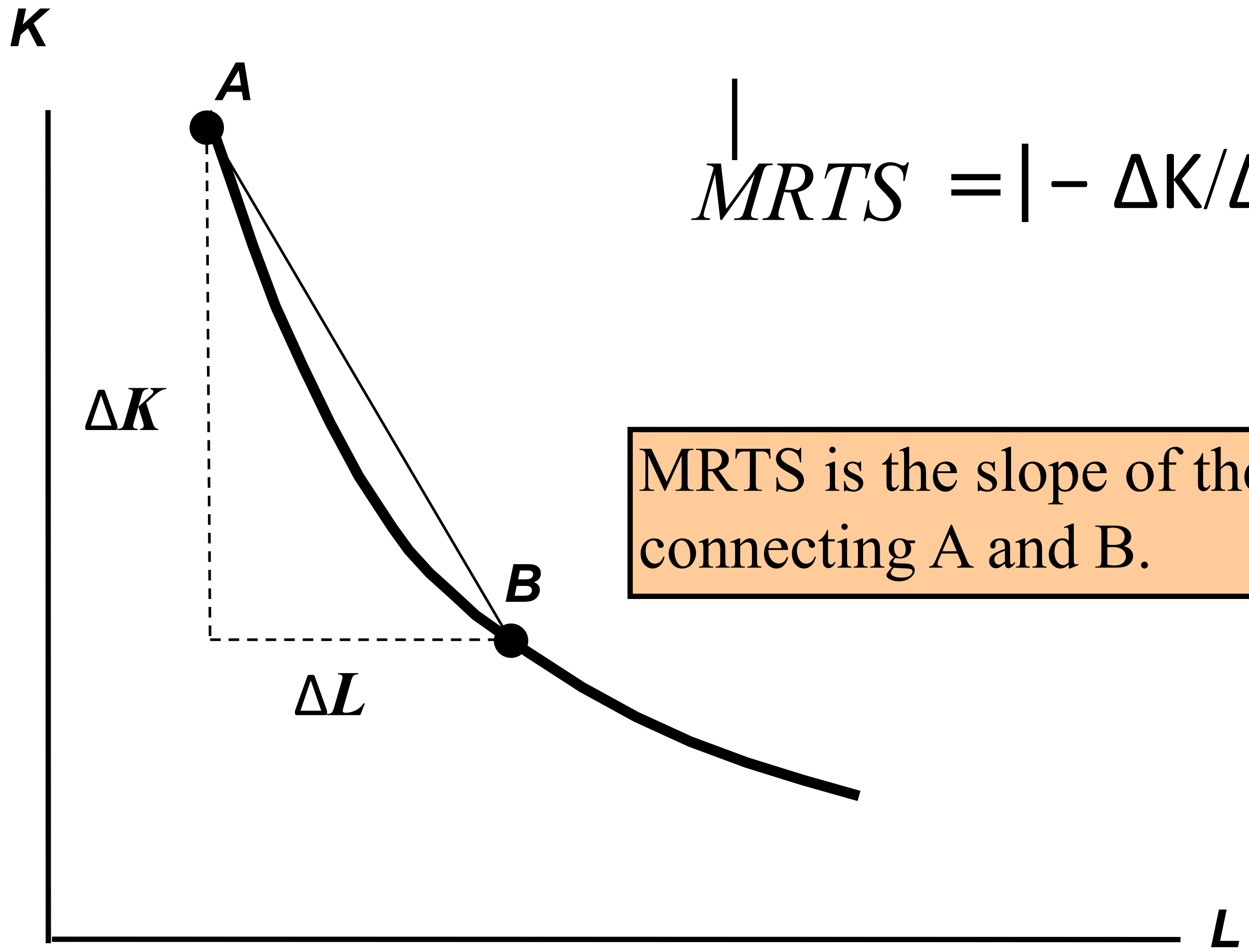


$$MRTS = -\Delta K \big/ \Delta L$$

$$\Delta L = 1 \longrightarrow \Delta K = -2$$

$$MRTS = -(-2/1) = 2$$

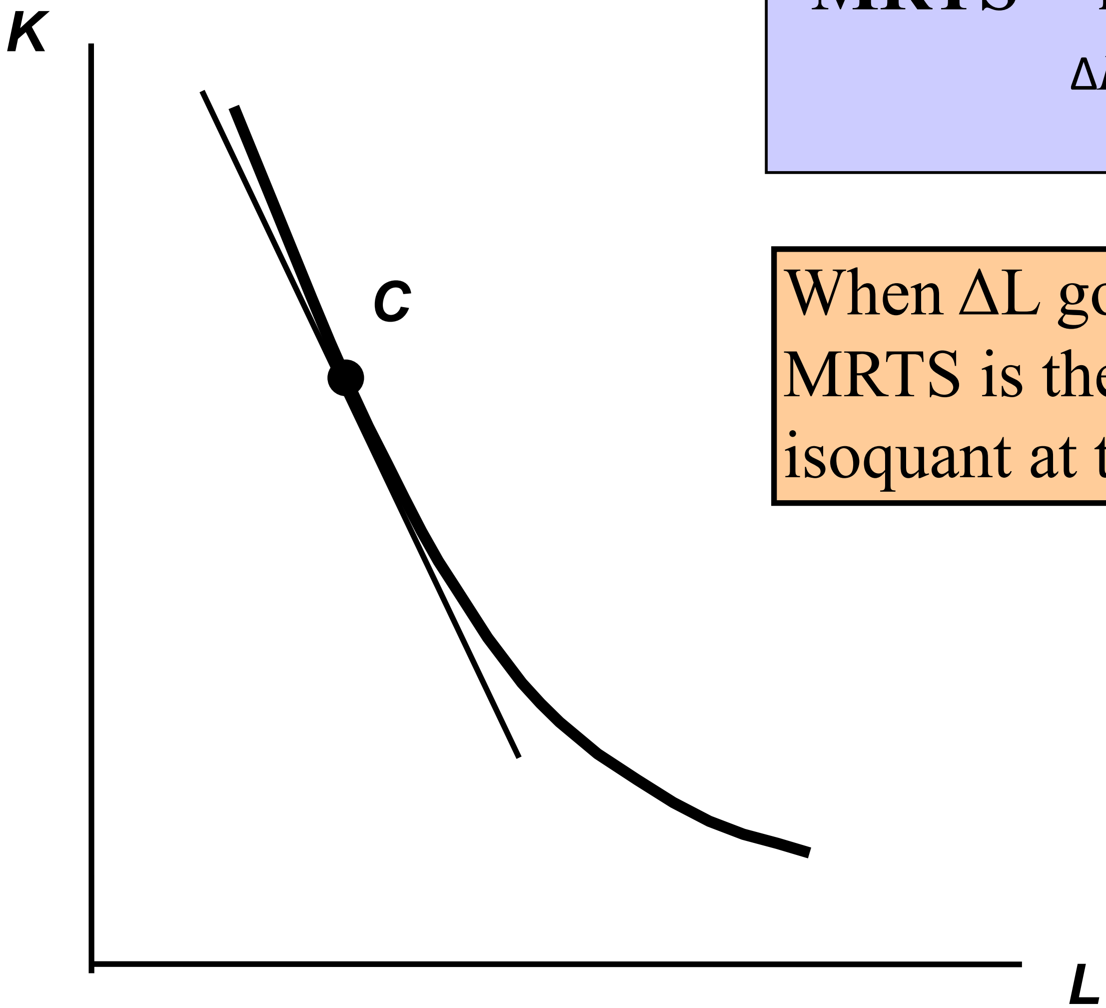
Marginal Rate of Technical Substitution



$$MRTS = \left| -\Delta K / \Delta L \right|$$

MRTS is the slope of the line connecting A and B.

Marginal Rate of Technical Substitution



$$\text{MRTS} = \lim_{\Delta L \to 0} -\Delta K / \Delta L$$

When ΔL goes to zero, the MRTS is the slope of the isoquant at the point C.

Calculating the MRTS

As we did in the utility functions' case, we can calculate the MRTS as a ratio of marginal productivities using the Implicit Function Theorem:

$$F(L,K)=Q_0 \qquad (*)$$

where $Q_0=F(L_0,K_0)$.

Taking the total derivative of the equation (*), we get

$$F_L\, dL+ F_K\, dK= 0.$$

Hence, the derivative of the function defined by (*) is

$$dK/dL= -F_L/F_K\,.$$

We can evaluate the MRTS at any point of the isoquant

Example: Cobb-Douglas Production Function

Let $Q = F(L,K) = L^{3/4}K^{1/4}$. Calculate the MRTS

Solution:

$PM_L = 3/4\ (K / L)^{1/4}$

$PM_K = 1/4\ (L / K)^{3/4}$

$MRST = F_L / F_K = 3\ K / L$

Example: Perfect Substitutes

Let $Q = F(L,K) = L + 2K$. Calculate the MRTS

Solution:

$PM_L = 1$

$PM_K = 2$

$MRST = F_L / F_K = 1/2$ *(constant)*

Returns to Scale

We are interested in studying how the production changes when we modify the scale; that is, when we multiply the inputs by a constant, thus maintaining the proportion in which they are used; e.g., $(L,K) \rightarrow (2L,2K)$.

Returns to scale: describe the rate at which output increases as one increases the scale at which inputs are used.

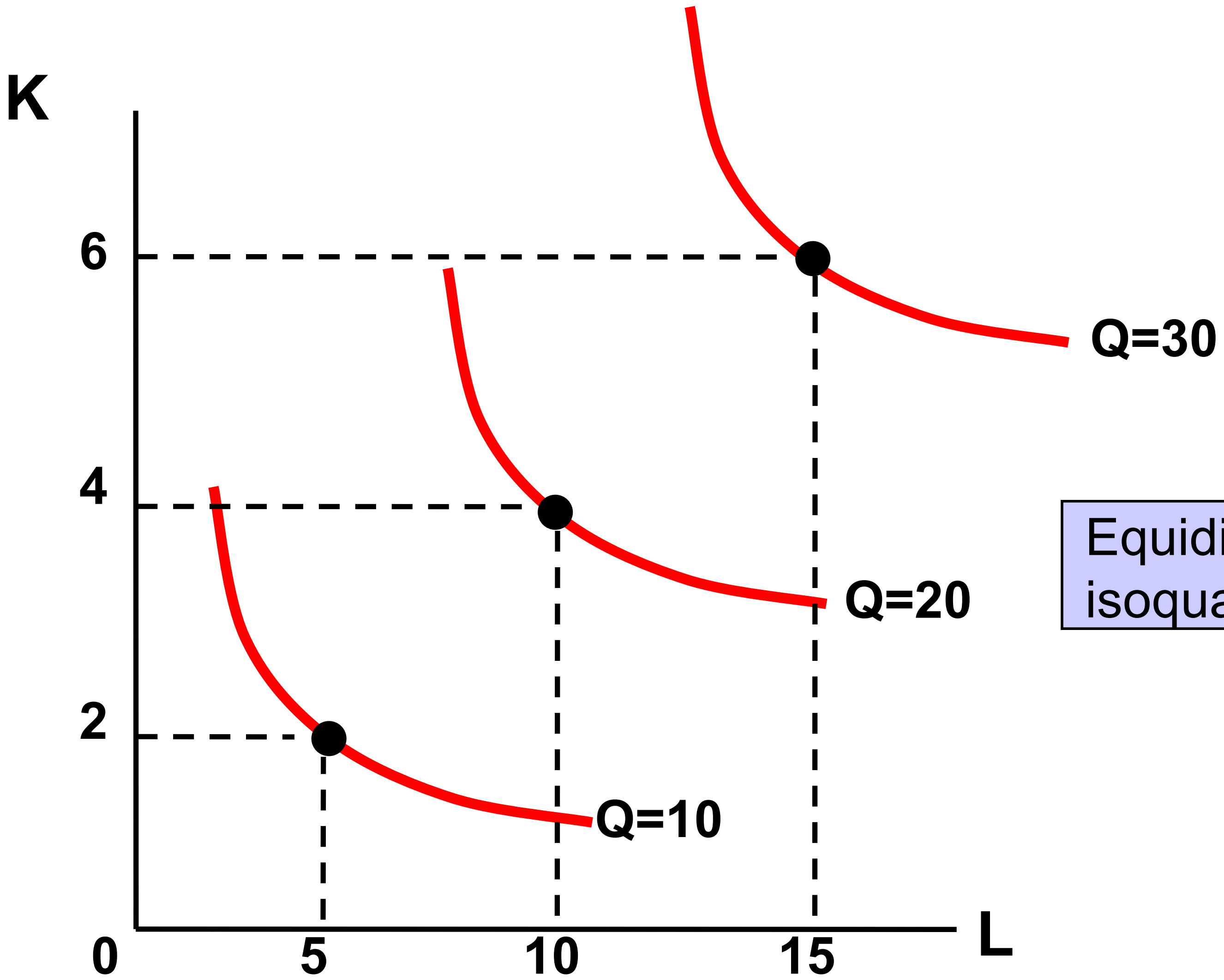
Returns to Scale

Let us consider an increase of scale by a factor $r > 1$; that is, $(L, K) \rightarrow (rL, rK)$.

We say that there are

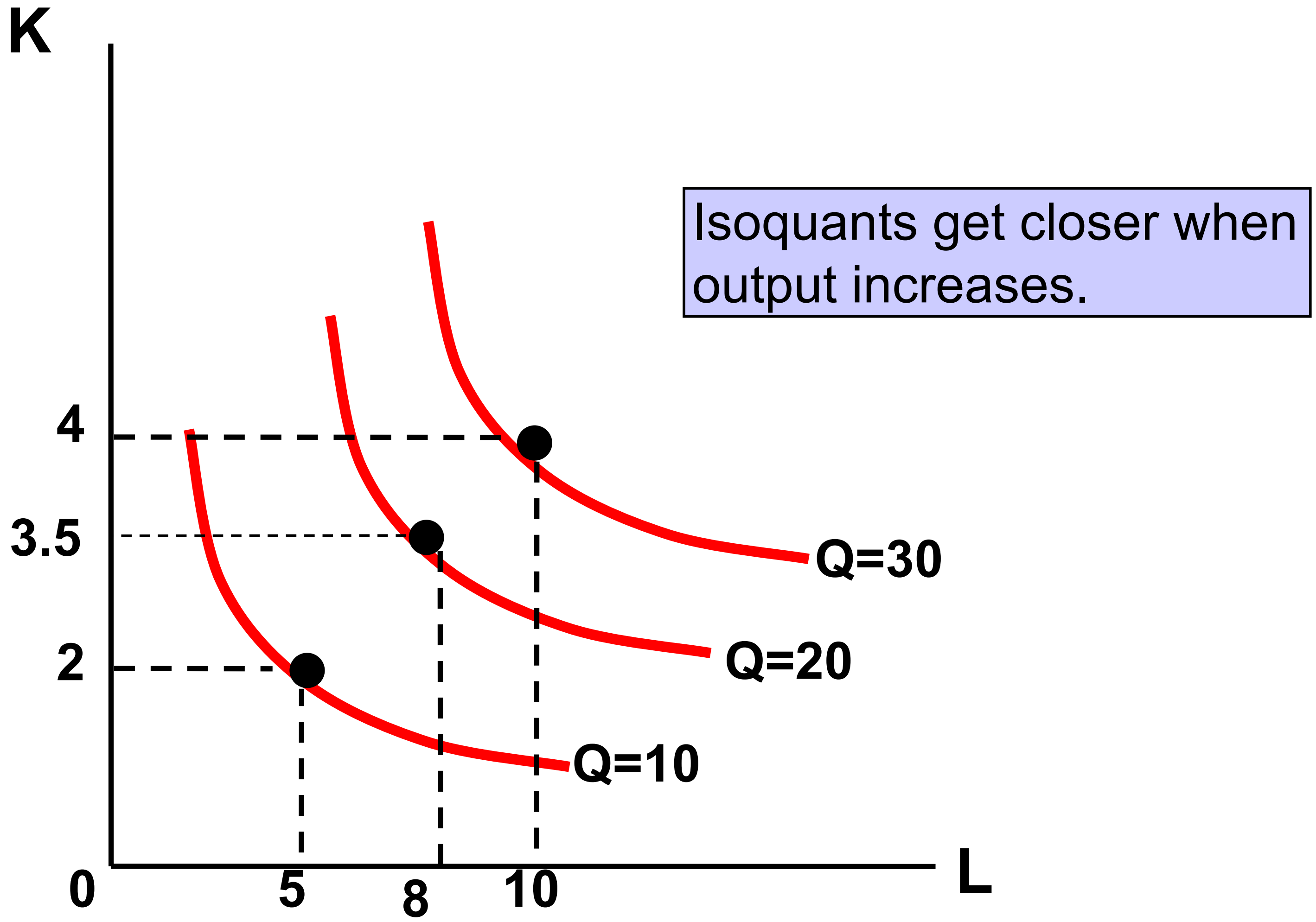
- increasing returns to scale if
$$F(rL, rK) > r\,F(L,K)$$
- constant returns to scale if
$$F(rL, rK) = r\,F(L,K)$$
- decreasing returns to scale if
$$F(rL, rK) < r\,F(L,K).$$

Example: Constant Returns to Scale

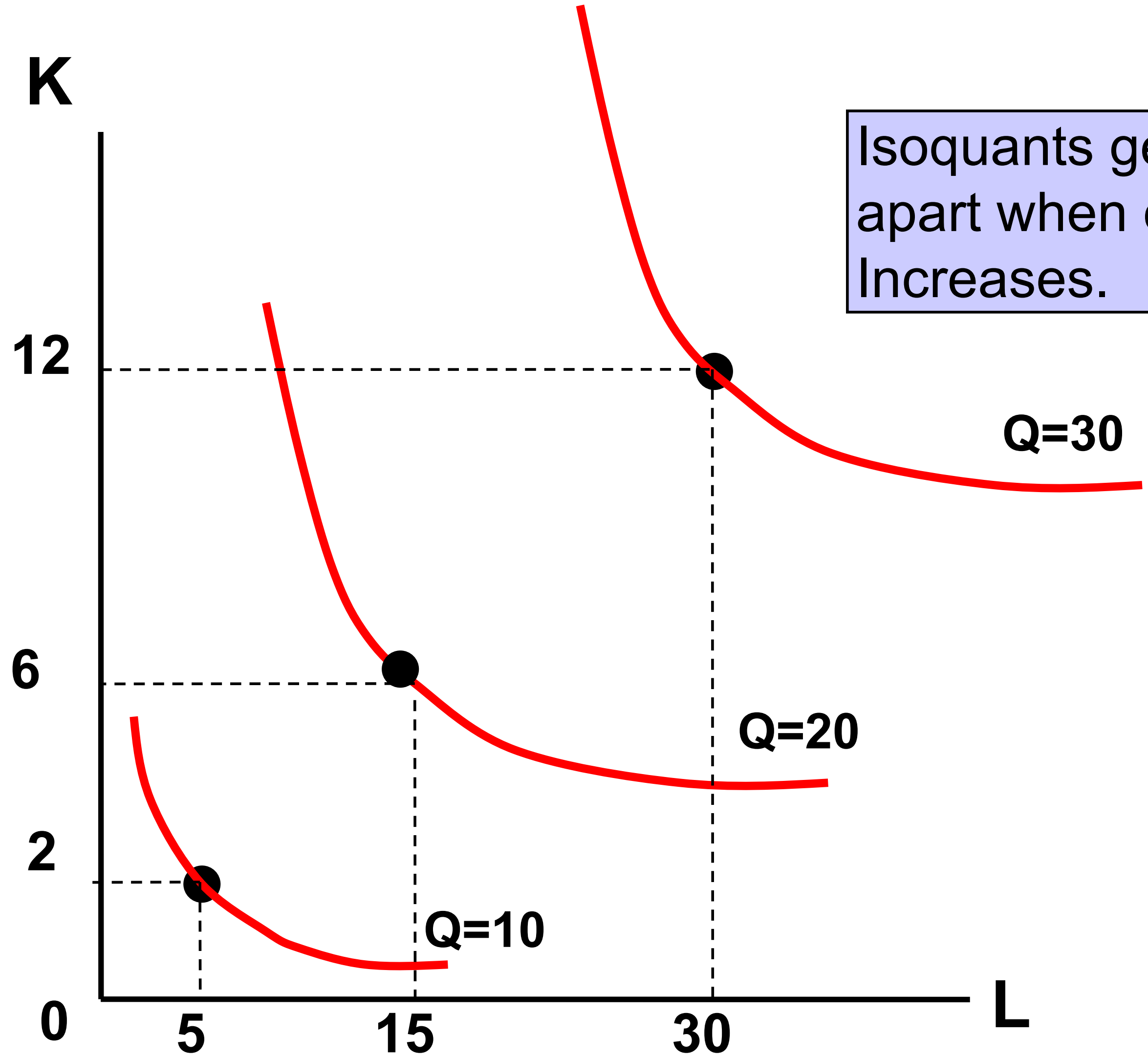


Equidistant isoquants.

Example: Increasing Returns to Scale



Example: Decreasing Returns to Scale



Example: Returns to Scale

What kind of returns to scale exhibits the production function $Q = F(L,K) = L + K$?

Solution: Let $r > 1$. Then

$$\begin{aligned} F(rL, rK) &= (rL) + (rK) \\ &= r\,(L+K) \\ &= r\,F(L,K). \end{aligned}$$

Therefore F has constant returns to scale.

Example: Returns to Scale

What kind of returns to scale exhibits the production function $Q = F(L,K) = LK$?

Solution: Let $r > 1$. Then

$$\begin{aligned} F(rL,rK) &= (rL)(rK) \\ &= r^2\,(LK) \\ &= r\,F(L,K). \end{aligned}$$

Therefore F has increasing returns to scale.

Example: Returns to Scale

What kind of returns to scale exhibits the production function $Q = F(L,K) = L^{1/5}K^{4/5}$?

Solution: Let $r > 1$. Then

$$\begin{aligned} F(rL,rK) &= (rL)^{1/5}(rK)^{4/5} \\ &= r(L^{1/5}K^{4/5}) \\ &= r\,F(L,K). \end{aligned}$$

Therefore F has constant returns to scale.

Example: Returns to Scale

What kind of returns to scale exhibits the production function $Q = F(L,K) = min\{L,K\}$?

Solution: Let $r > 1$. Then

$$\begin{aligned} F(rL,rK) &= min\{rL,rK\} \\ &= r\, min\{L,K\} \\ &= r\, F(L,K). \end{aligned}$$

Therefore F has constant returns to scale.

Example: Returns to Scale

Be the production function $Q = F(L,K_0) = f(L) = 4L^{1/2}$.

Are there increasing, decreasing or constant returns to scale?

Solution: Let $r > 1$. Then

$$\begin{aligned} f(rL) &= 4\,(rL)^{1/2} \\ &= r^{1/2}\,(4L^{1/2}) \\ &= r^{1/2} f(L) \\ &< r\,f(L) \end{aligned}$$

There are decreasing returns to scale

Production Functions: Monotone Transformations

Contrary to utility functions, production functions are not an ordinal, but cardinal representation of the firm's production set.

If a production function F_2 is a monotonic transformation of another production function F_1 then they represent different technologies.

For example, $F_1(L,K) = L + K$, and $F_2(L,K) = F_1(L,K)^2$. Note that F_1 has constant returns to scale, but F_2 has increasing returns to scale.

However, the *MRTS* is invariant to monotonic transformations.

Production Functions: Monotone Transformations

Let us check what happen with the returns to scale when we apply a monotone transformation to a production function:

$F(L,K) = LK;\ G(L,K) = (LK)^{1/2} = L^{1/2}K^{1/2}$

For $r > 1$ we have

$$F(rL,rK) = r^2 LK = r^2 F(L,K) > rF(L,K) \rightarrow \text{IRS}$$

and

$$G(rL,rK) = r(LK)^{1/2} = rF(L,K) \rightarrow \text{CRS}$$

Thus, monotone transformations modify the *returns to scale*, but not the *MRTS*:

$MRTS_F(L,K) = K/L;$

$MRTS_G(L,K) = (1/2)L^{(-1/2)}K^{1/2}/[(1/2)L^{(1/2)}K^{(-1/2)}] = K/L.$