Theory of the firm

Profit Maximization

and

Cost Minimization

The Problem of the Firm

We consider a firm producing a single good *Q,* using labour (L) and capital (K), and a technology described by the *production function,* $F(L,K)$.

The firm is a *price taker* competitive in the labour and capital markets, in which the prices are w and r, respectively. (This assumption is reasonable if the labour and capital markets are large relative to the firm's output market.)

Let p denote the market price of good Q.

Problem of the Firm

The firm's profit maximization problem is:

$$\max \; pQ - wL - rK$$

s.t.

$$F(L,K) \geq Q$$

$$Q \geq 0,\; L \geq 0,\; K \geq 0.$$

Here pQ is the firm's *revenue*, and $wL+rK$ is the cost of the inputs used by the firm.

What are the firm's decision variables?

Q, L, K, p?

Problem of the Firm

In a competitive market, the supply of a single firm is very small compared to the market supply. In this case, a single firm has a negligible impact on the market price, p, and is thefefore reasonable to assume that the firm acts as a *price-taker*.

But if the firm's output is large relative to the market supply, that is, if the firm has *market power*, then assuming that the firm acts as a price taker would be a mistake.

Cost Minimization

For now, let us postpone the profit-maximization problem and let us treat the “internal” problem of the firm taking the production level as given: Q_0 .

Fixing Q_0 , then the objective of maximizing profits implies, as an intermediate objective, minimizing the cost of producing the level Q_0.

Cost Minimization

There are several types of cost concepts:

Accounting cost: purchase price net of depreciation.

Opportunity cost: value of the best alternative use.

Sunk Cost: unrecoverable costs asociated with past decisions.

Cost Minimization

From an economic point of view, relevant costs are opportunity cost. Sunk costs are irrelevant in making optimal decisions.

Example: A firm owns a building that is not being used in the production process. As the firm does not pay any rent, there is no accounting cost. Nevertheless, the opportunity cost is greater than zero – the building may be put up for rent.

Cost Minimization

Short Run and Long Run Cost

Long run: all inputs are variable.

Short run: some inputs are fixed – capital, for example.

Fixed and Variable Costs

The variable cost is the cost of the inputs that may be varied in the short run depending on the desired level of output, whereas the fixed cost is the cost of those inputs that are fixed in the short independently of the level of output.

Cost Minimization

$$Max_{L\geq 0,K\geq 0}\ pQ_0 - wL - rK$$

Since pQ_0 is a constant, this problem is equivalent to:

$$Max_{L\geq 0,K\geq 0}\ -(wL + rK)$$

Which in turns is equivalent to:

$$Min_{L\geq 0,K\geq 0}\ wL + rK$$

Cost Minimization: Short Run

In our framework there are only two inputs, labour and capital. If we assume capital is fixed K_0 in the short run, then the short run cost-minimization problem is

$$\min\ wL - rK_0$$

$$\text{s.t.}\ \ F(L,K_0) \geq Q,\ L \geq 0.$$

Here rK_0 is the fixed cost (FC).

Cost Minimization: Short Run

The solution to this problem involves using amount of labor (the only variable input) that solves the equation

$$F(L, K_0) = Q.$$

That is, the solution to the cost minimization consist of choosing the minimum amount of labor that allows for producing Q units of the good given that we have K_0 units of capital.

By solving this equation, we find the short run conditional labor demand

$$L^* = L\,(K_0, Q).$$

Cost Minimization in the Short Run

Example. A firm's production function is

$$F(L,K)=(LK)^{1/2}$$

Its capital is fixed in the short run to $K_0 = 36$. Hence its short run production function is

$$F(L,36)=(L36)^{1/2} = 6L^{1/2}.$$

Therefore its short run conditional demand of labor is

$$L(Q) = Q^2/36.$$

Cost Minimization: Long Run

In the long run, both inputs, labour and capital, are variable. Thus, the cost-minimization problem can be written as

$$\min_{L\geq 0, K\geq 0} wL + rK$$

$$F(L,K) \geq Q$$

Solving the problem, we find the conditional demand functions of inputs:

$$L^* = L(w,r,Q) \text{ and } K^* = K(w,r,Q)$$

Cost Minimization: Long Run

As in the consumer theory, the cost-minimization problem may have interior and/or corner solutions, depending on the features of the production function.

(a) Interior solution

$$MRTS(L,K) = \frac{w}{r}$$

$$F(L,K) = Q$$

Cost Minimization: Long Run

(b) Corner solution

(b1) Only capital is used (L*=0)

$$MRTS(L, K) \leq \frac{w}{r}$$

$$F(0, K) = Q$$

(b2) Only labour is used (K*=0)

$$MRTS(L, K) \geq \frac{w}{r}$$

$$F(L, 0) = Q$$

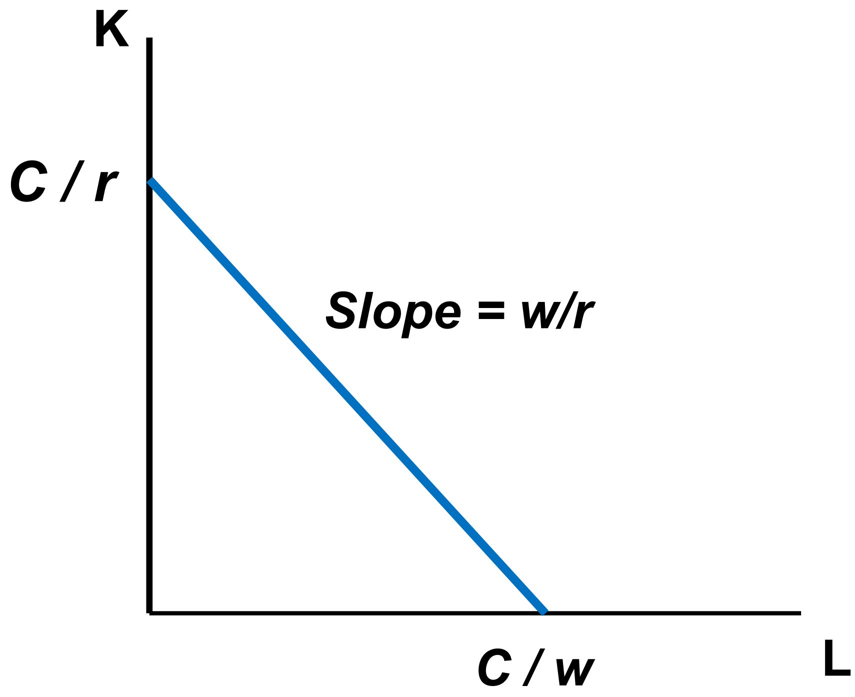
Cost Minimization: Long Run

To solve the problem graphically, we need to use a new concept: the isocost line.

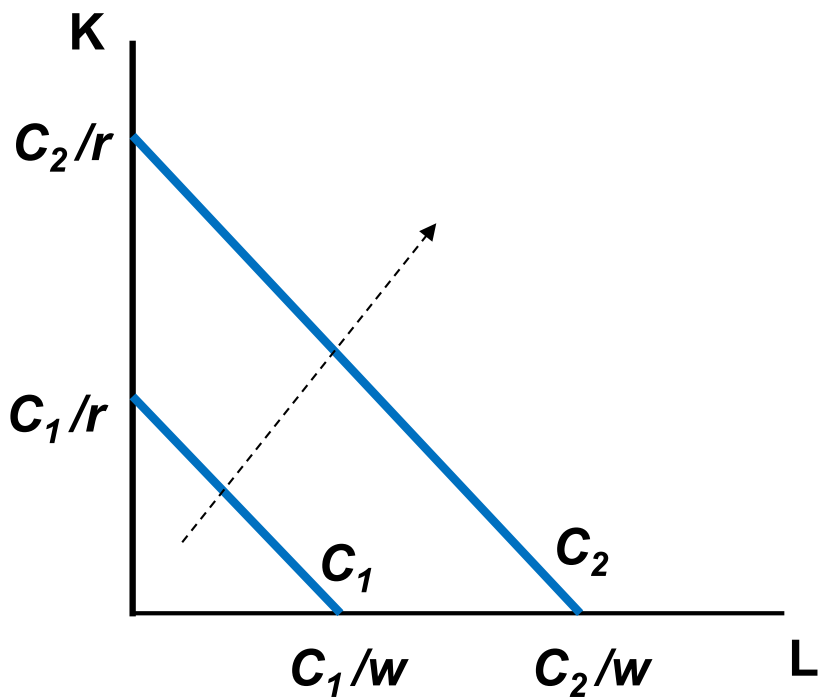
The isocost line represents the input combinations that cost the same.

Cost Minimization: Long Run

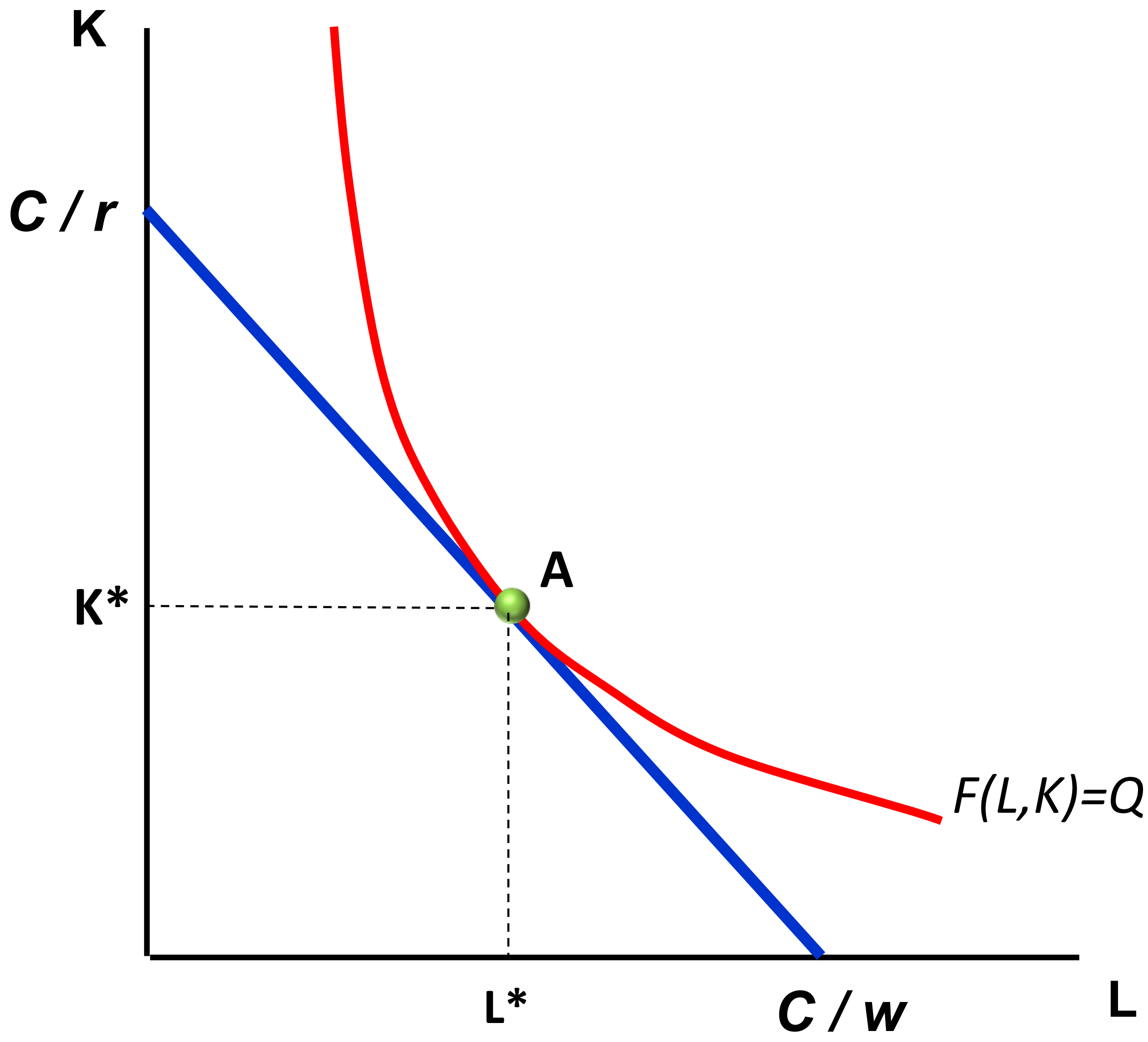
$wL + rK = C$



The cost increases along all the northeast directions: $C_1 < C_2$

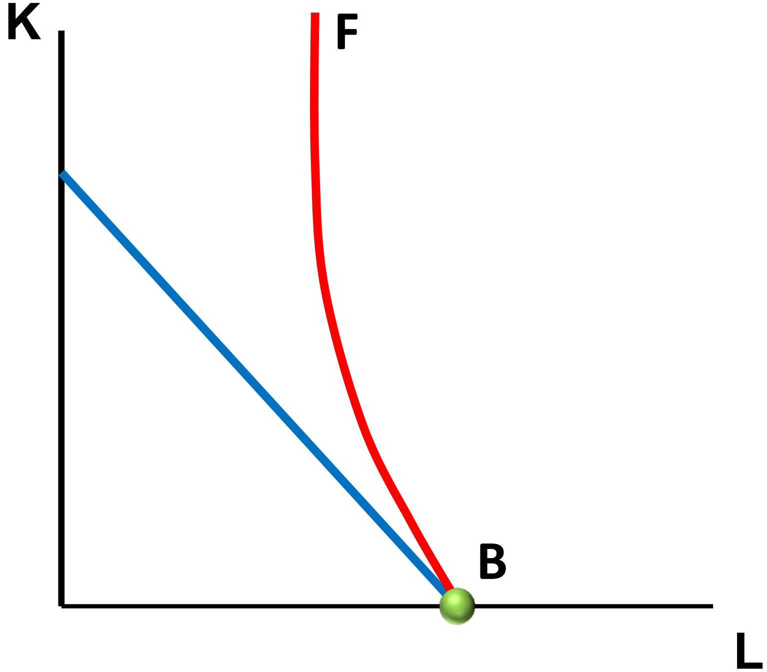


Cost Minimization: Long Run

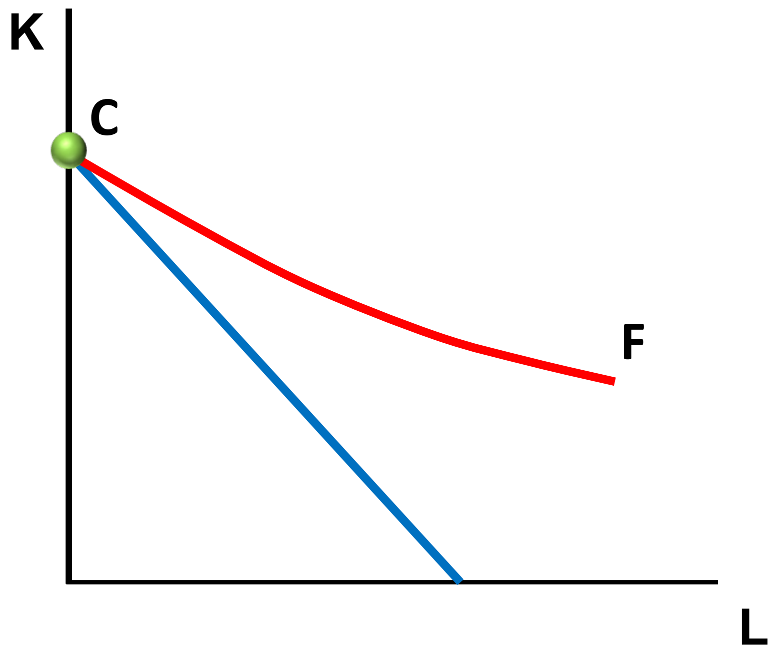


Cost Minimization: Long Run

K= 0*

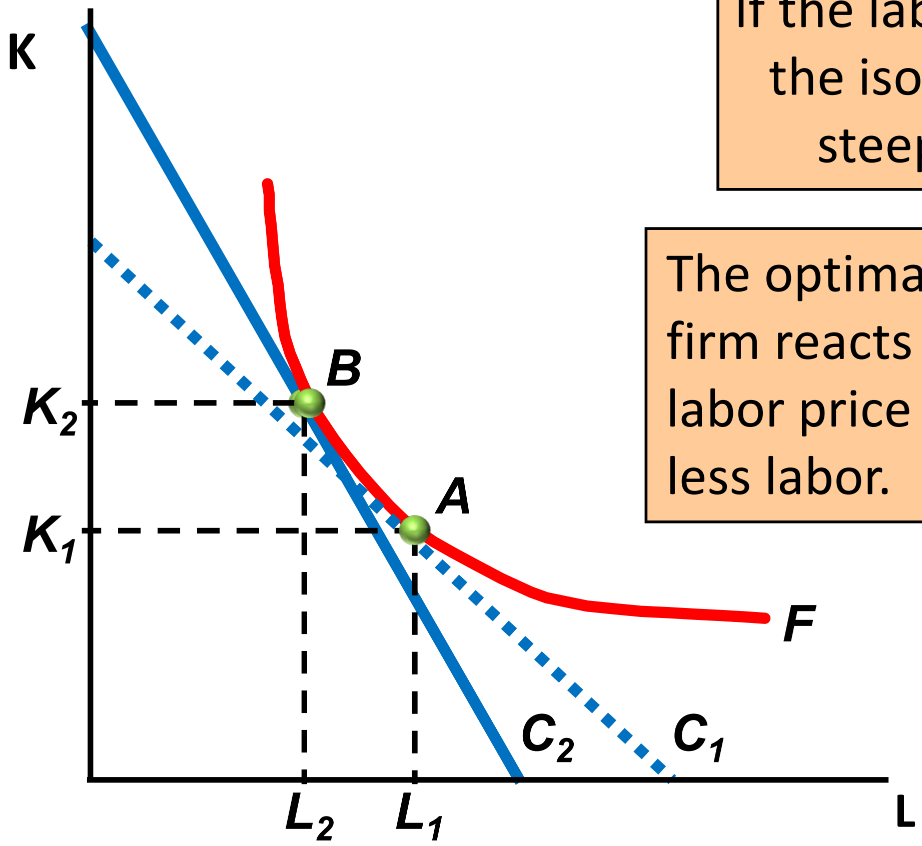


L= 0*



Cost Minimization: Long Run

Input substitution when the price of one of the inputs changes



If the labor price increases, the isocost line becomes steeper. Slope = w / r

The optimal choice now is B. The firm reacts to the increase of the labor price using more capital and less labor.

Cost Minimization in the Long Run: Examples

$(a)\ F(L,K) = LK$

Interior solution:

We solve the system formed by

$$MRTS(L,K) = w/r \Rightarrow K/L = w/r$$
$$F(L,K) = Q \Rightarrow LK = Q$$

And we obtain the conditional demands of inputs:

$$L^* = \sqrt{\frac{r}{w}Q}\,; \quad K^* = \sqrt{\frac{w}{r}Q}$$

Cost Minimization in the Long Run: Examples

$(b)\ F(L,K)=\sqrt{LK}$

Interior solution:

We solve the system formed by

$$MRTS(L,K)=w/r \Rightarrow K/L=w/r$$
$$F(L,K)=Q \Rightarrow \sqrt{LK}=Q$$

And we obtain the conditional demands of inputs:

$$L^{*}=Q\sqrt{\frac{r}{w}}\ ;\ \ K^{*}=Q\sqrt{\frac{w}{r}}$$

Cost Minimization in the Long Run: Examples

$(c)\ F(L,K)=\sqrt[3]{LK}$

Interior solution: We solve the system formed by

$$MRTS(L,K)=w/r \Rightarrow K/L=w/r$$

$$F(L,K)=Q \Rightarrow \sqrt[3]{LK}=Q$$

The conditional demands of inputs are:

$$L^*=Q^{3/2}\sqrt{\frac{r}{w}};\ \ K^*=Q^{3/2}\sqrt{\frac{w}{r}}$$

Cost Minimization in the Long Run: Examples

$$(d)\ F(L,K) = \min\{2L, K\}$$

Interior solution:

We solve the system formed by

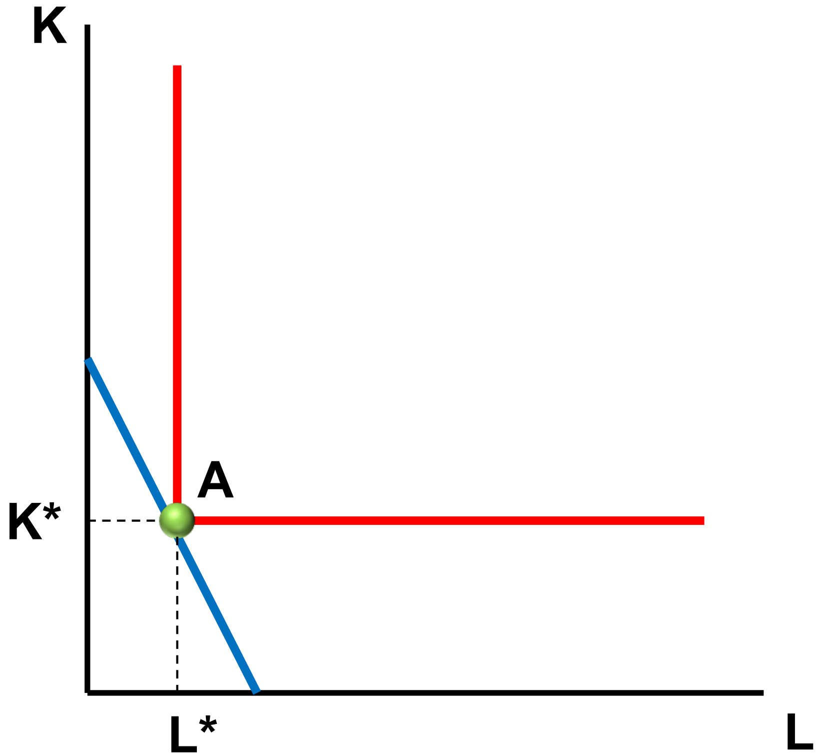
$$2L = K$$

$$F(L,K) = Q \Rightarrow \min\{2L, K\} = Q$$

And we obtain the conditional demands of inputs:

$$L^* = \frac{Q}{2};\ \ K^* = Q$$

Cost Minimization in the Long Run: Examples



Cost Minimization in the Long Run: Examples

$(e)\ F(L,K) = L + 2K$

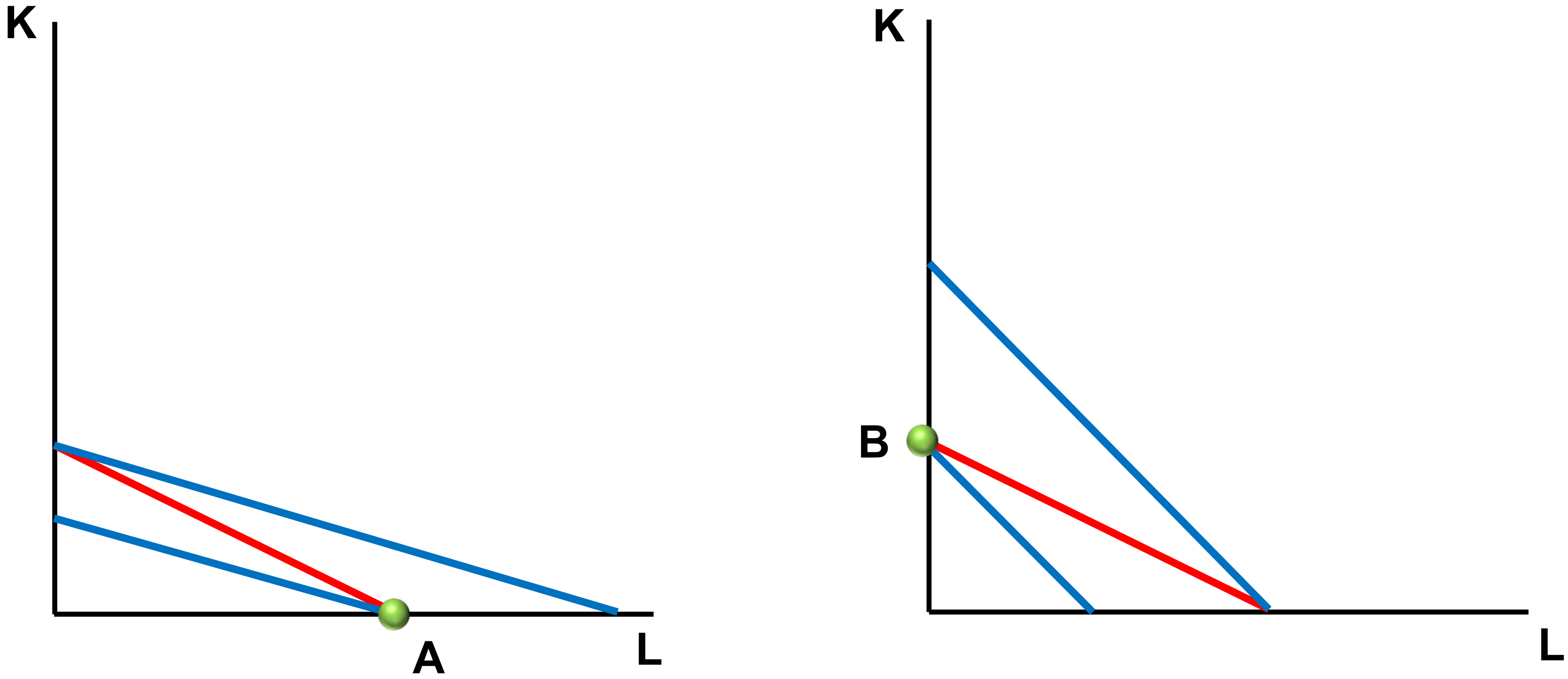
Corner solution: In this case the conditional demands of inputs are:

$$L^* = \begin{cases} Q \ \ if \ \ w/r < 1/2 \\ 0 \ \ if \ \ w/r > 1/2 \\ \gamma \in [0,Q] \ \ if \ \ w/r = 1/2 \end{cases}$$

$$K^* = \begin{cases} 0 \ if \ \ w/r < 1/2 \\ Q/2 \ \ if \ \ w/r > 1/2 \\ (Q-\gamma)/2 \ \ if \ \ w/r = 1/2 \end{cases}$$

Cost Minimization in the Long Run: Examples

F(·)=L+2K, w=1, r=3 *F(·)=L+2K, w=r=1*



Cost functions

The total cost function is the minimum cost of production for a given level of output *Q*,and input prices w and r:

$$C(Q,w,r) = wL(Q,w,r) + rK(Q,w,r).$$

For given input prices, the long run total cost is less than or equal to the total cost in the short run – why?

The total cost may be decomposed as the sum of the variable cost (the cost of the variable inputs), *VC(Q,w,r),* and the fix cost (the cost of the fix inputs), *FC*, which is independent of the level of output.

$$C(Q,w,r) = VC(Q,w,r) + FC = wL_0(Q,w) + rK_0$$

In the long run the total and variable cost coincide.

Cost functions

The average (total) cost measures average cost of production,

$$AC(Q,w,r) = C(Q,w,r)/Q.$$

For given input prices, the long run average cost is less than or equal to the short run average cost.

Likewise, the average variable cost is

$$AVC(Q,w,r) = VC(Q,w,r)/Q.$$

In the long run the average total and variable costs coincide.

The average total cost can be decompose as the sum of the average variable cost and the average fixed cost

$$AC(Q,w,r) = AVC(Q,w,r) + FC/Q\ .$$

Cost functions

The marginal cost measures the cost increase due to a marginal (infinitesimal) increase of output,

$$MC(Q,w,r) = dC(Q,w,r)/dQ.$$

For given input prices the long run marginal cost cost may be larger or smaller than the the short run marginal cost.

Economies of Scale

Economies of scale: cost increases less than proportionally with the level of output; that is, for $\lambda > 1$,

$$C(\lambda Q) < \lambda C(Q).$$

Equivalently, the average cost decreases with the level of output; that is,

$$dAC(Q,w,r)/dQ < 0.$$

If the firm's technology exhibits increasing returns to scale, then the firm has economies of scale.

Economies of Scale

Diseconomies of scale: cost increases more than proportionally with the level of output; that is, for $\lambda > 1$,

$$C(\lambda Q) > \lambda C(Q).$$

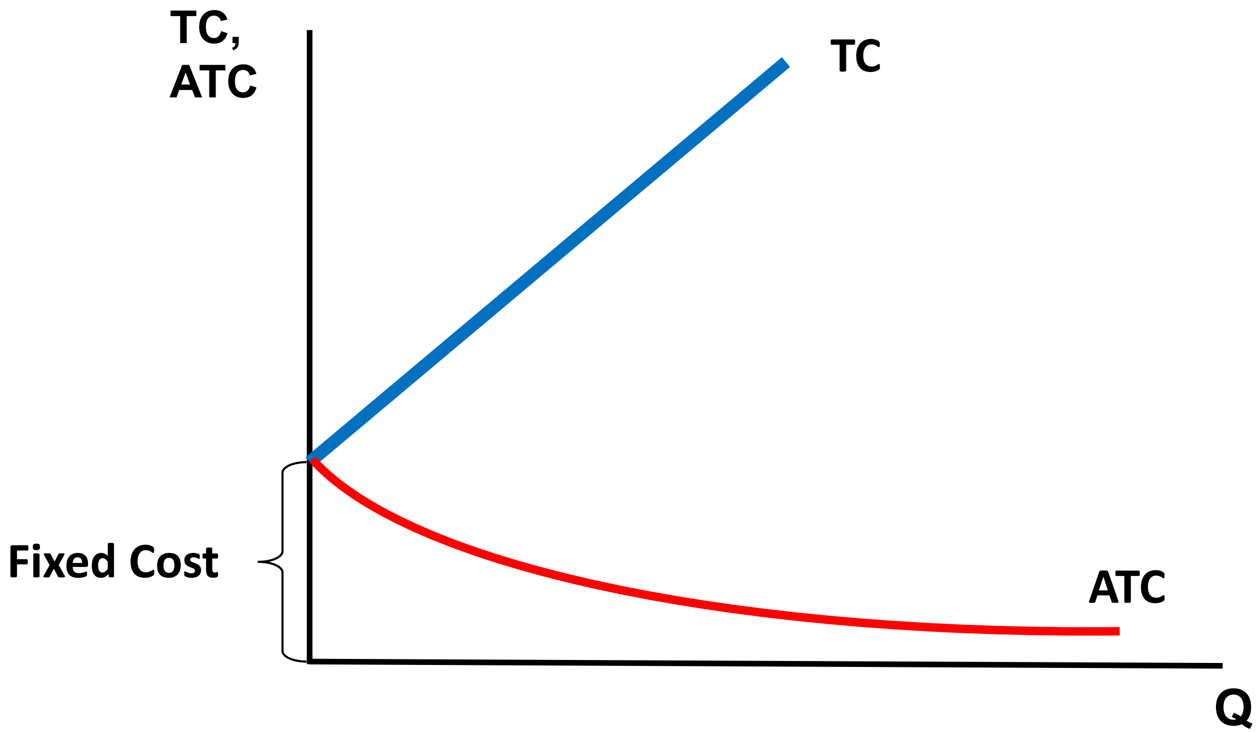
Equivalently, the average cost increases with the level of output; that is,

$$dAC(Q,w,r)/dQ > 0.$$

A the firm's technology exhibits decreasing returns to scale, the firm has diseconomies of scale.

Economies of Scale

Economies of scale may result from the existence of fixed costs, for example.



Economies of Scale

Constant economies of scale: cost increases proportionally with the level of output; that is, for $\lambda > 1$,

$$C(\lambda Q) = \lambda C(Q).$$

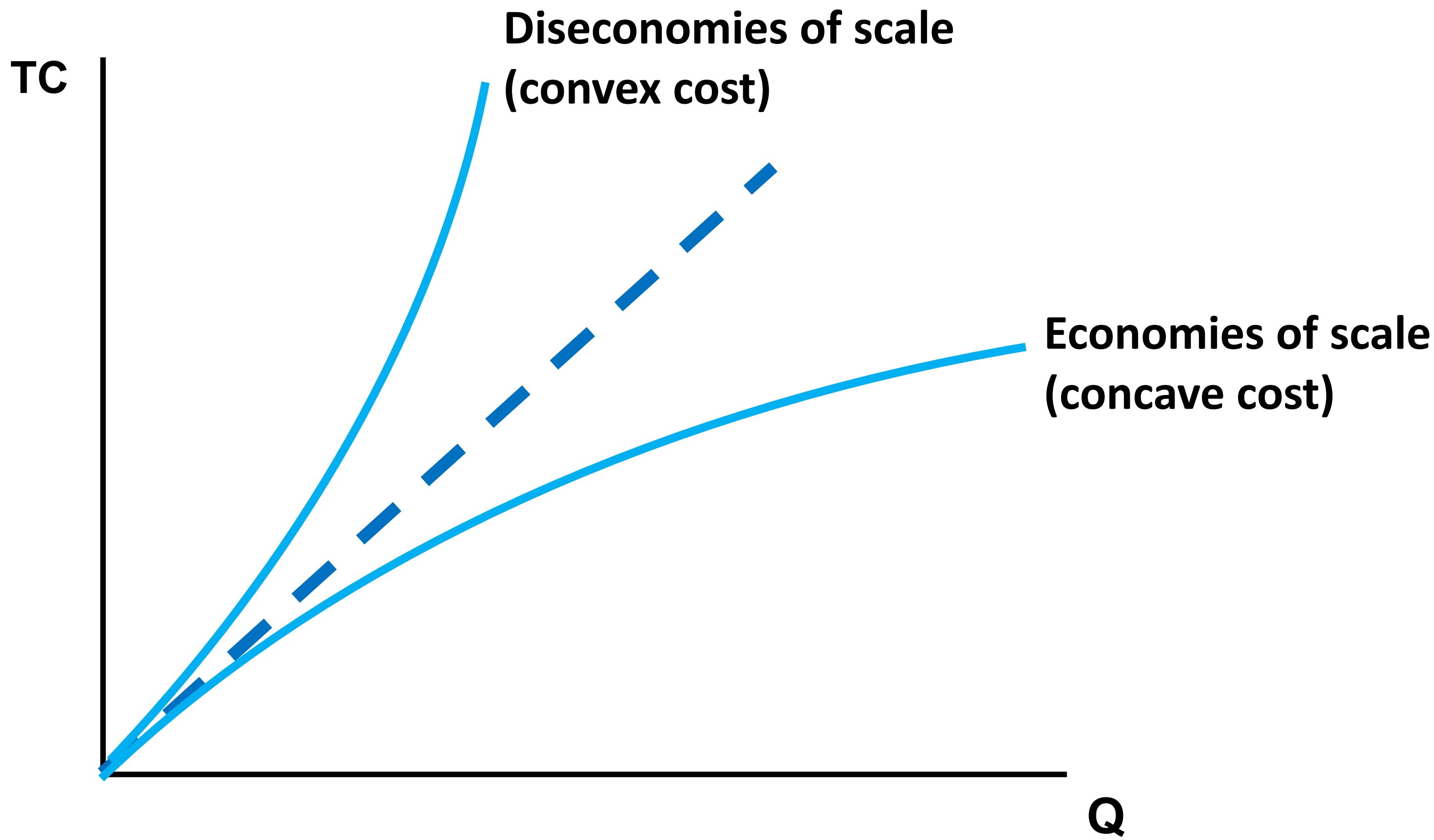
Equivalently, the average cost decreases is constant; i.e.,

$$dAC(Q,w,r)/dQ = 0.$$

In the firm's technology exhibits constant returns to scale, then the firm has constant economies of scale.

Economies of Scale

Economies and diseconomies of scale WITHOUT fixed costs

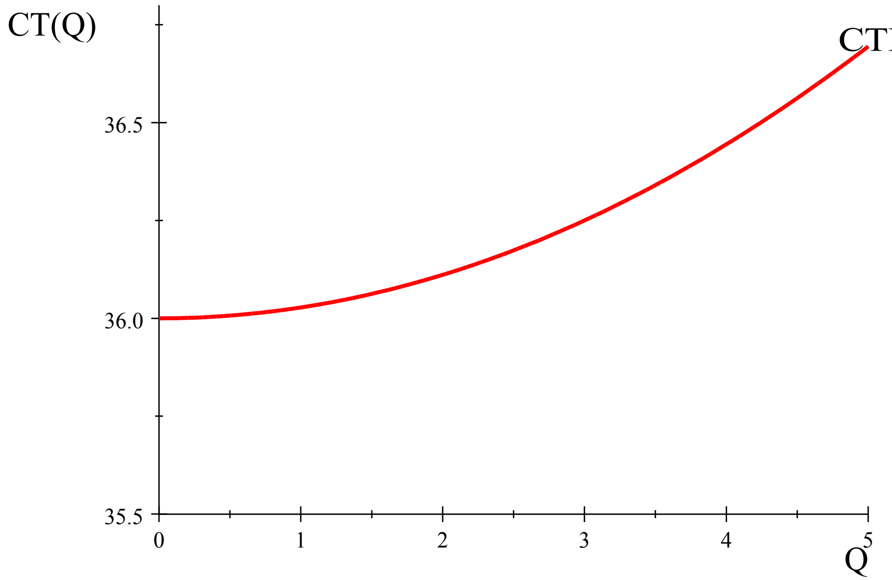


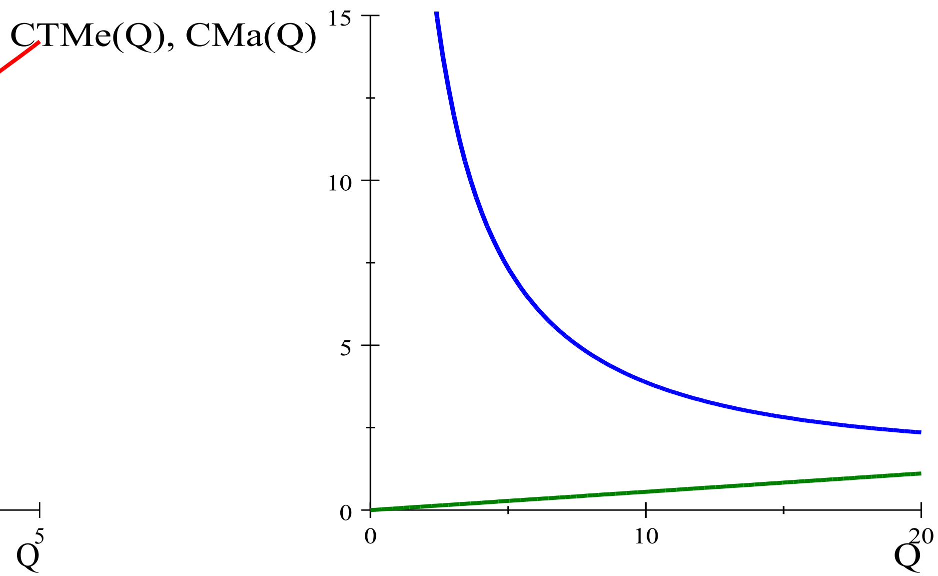
Costs and Economies of Scale: Examples

In the example above, for *w=1* and *r=1* we have:

$$F(L,K) = \sqrt{LK_0}$$

$$TC(Q) = L^*(Q) + 36 = \frac{Q^2}{36} + 36; \quad ATC(Q) = \frac{Q}{36} + \frac{36}{Q}; \quad MC(Q) = \frac{Q}{18}$$





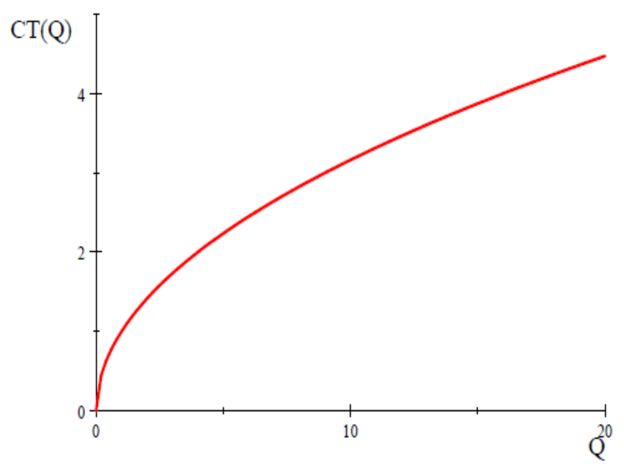
Costs and Economies of Scale: Examples

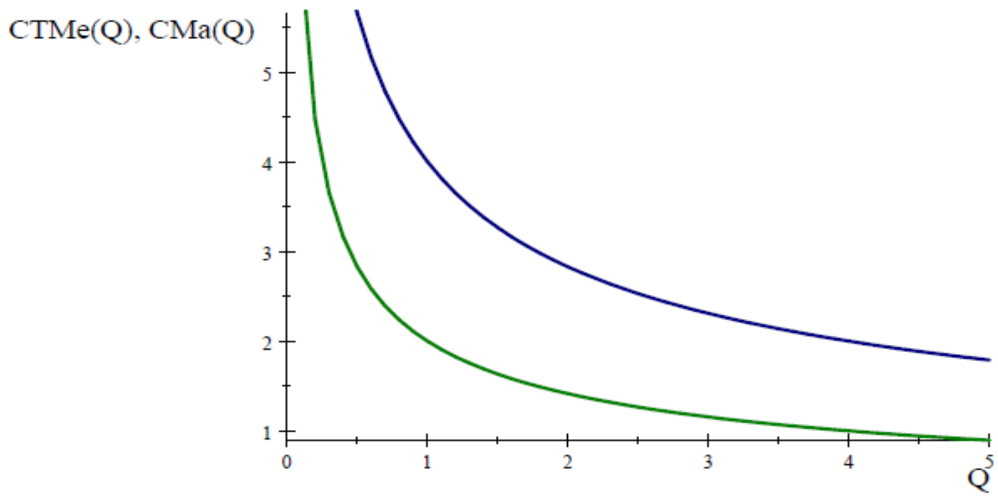
In the examples above, for *w=1* and *r=4* we have:

(a) $F(L,K) = LK$

$$TC(Q) = L^*(Q,1,4) + 4K^*(Q,1,4) = 4\sqrt{Q}$$

$$ATC(Q) = 4/\sqrt{Q}; \quad MC(Q) = 2/\sqrt{Q}$$



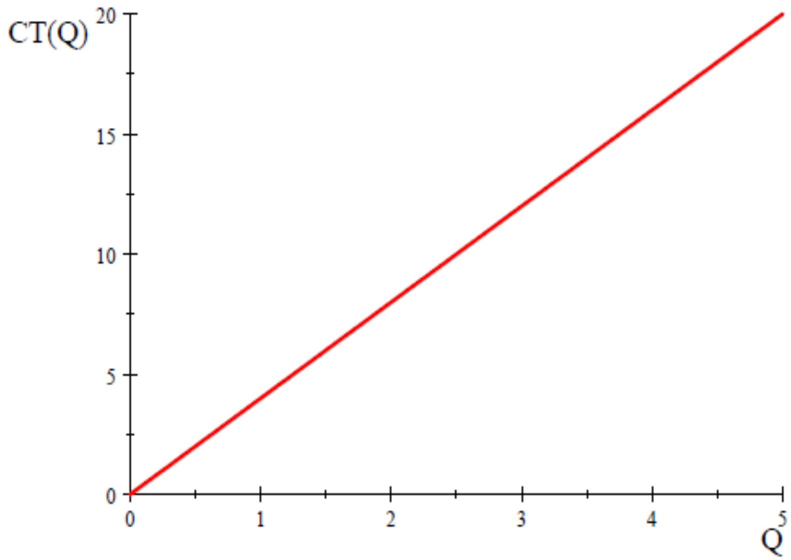


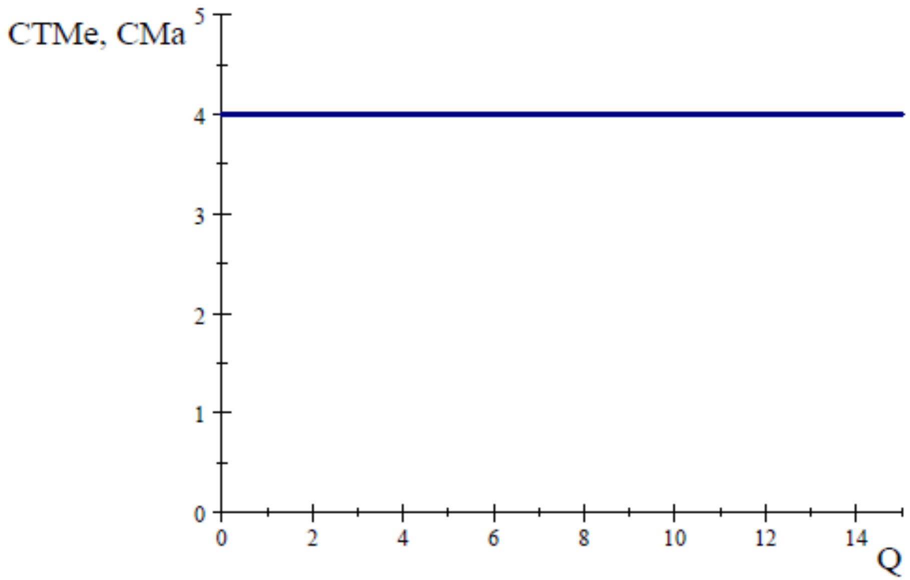
Costs and Economies of Scale: Examples

(b) $F(L,K) = \sqrt{LK}$

$$TC(Q) = L^*(Q,1,4) + 4K^*(Q,1,4) = 4Q$$

$$ATC(Q) = 4; \quad MC(Q) = 4$$



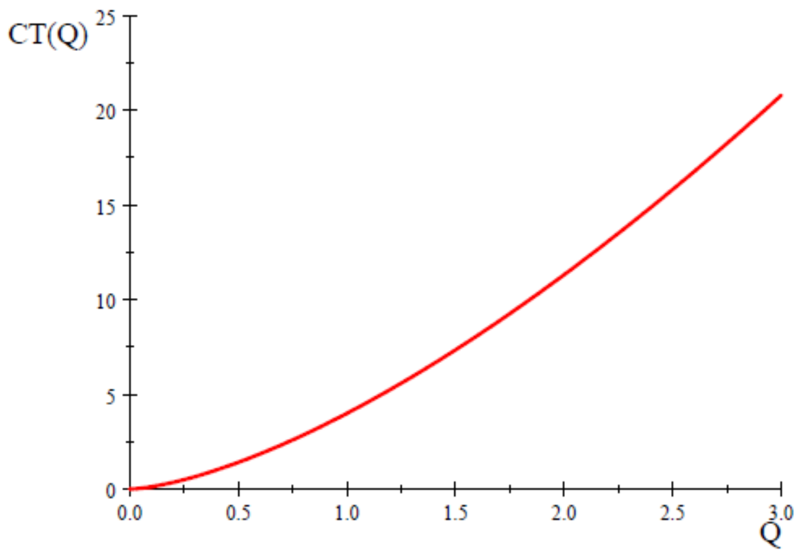


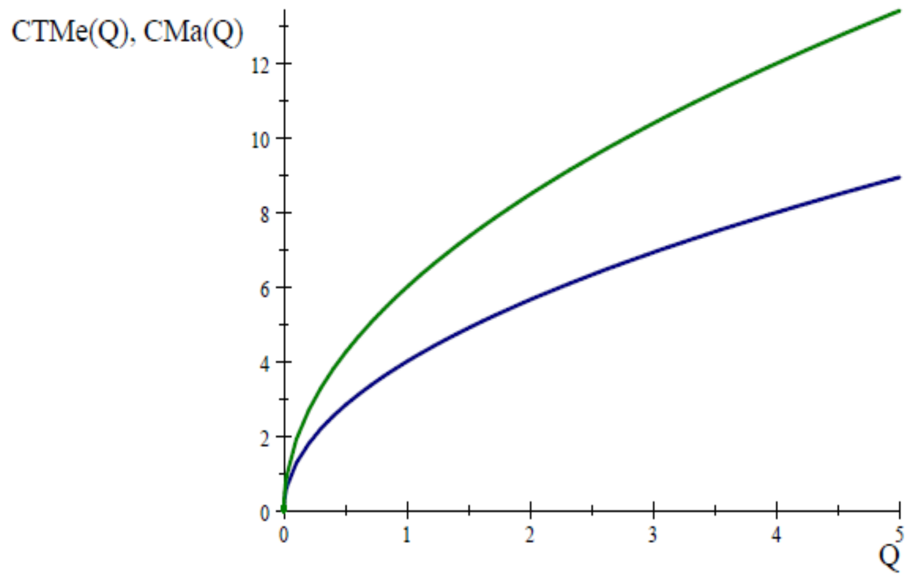
Costs and Economies of Scale: Examples

(c) $F(L,K) = \sqrt[3]{LK}$

$$TC(Q) = L^*(Q,1,4) + 4K^*(Q,1,4) = 4Q^{3/2}$$

$$ATC(Q) = 4Q^{1/2}; \quad MC(Q) = 6Q^{1/2}$$



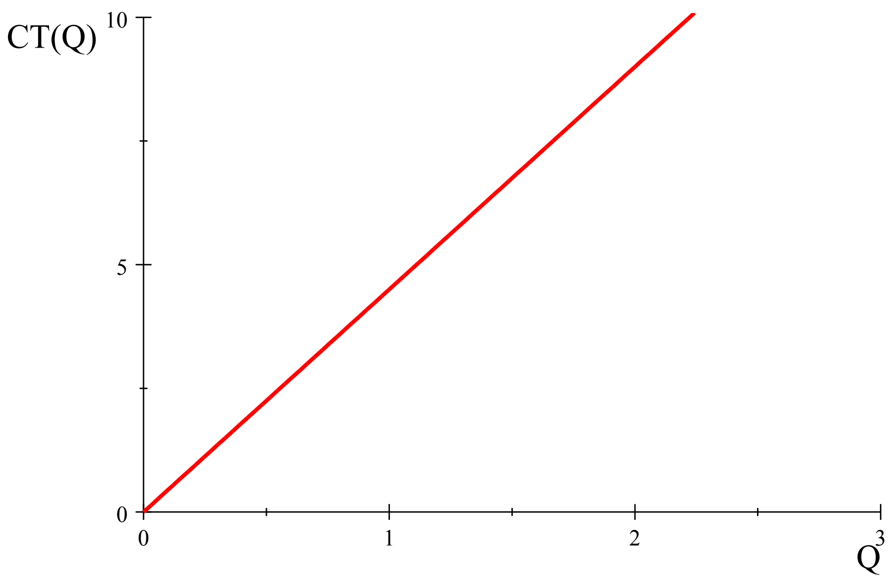


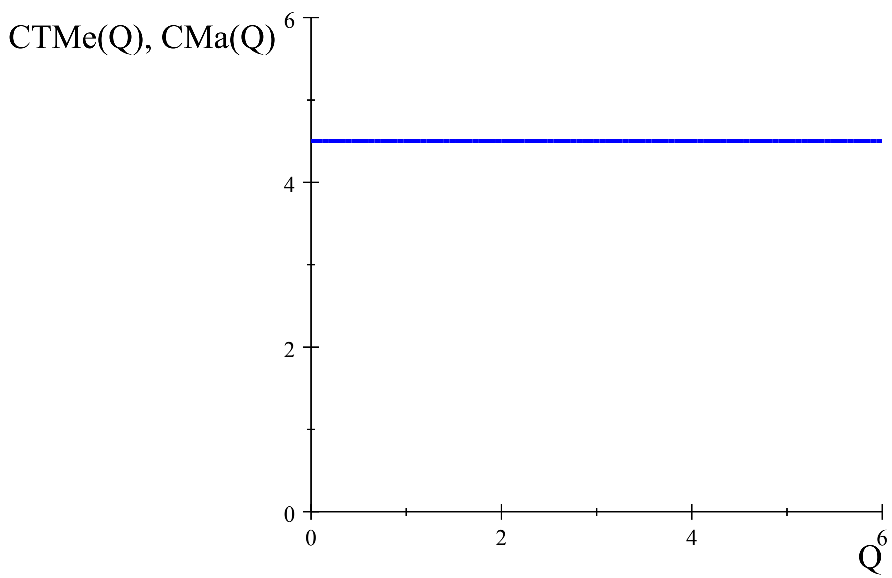
Costs and Economies of Scale: Examples

(d) $F(L,K) = \min\{2L, K\}$

$$TC(Q) = L^*(Q,1,4) + 4K^*(Q,1,4) = 4.5Q$$

$$ATC(Q) = 4.5; \quad MC(Q) = 4.5$$



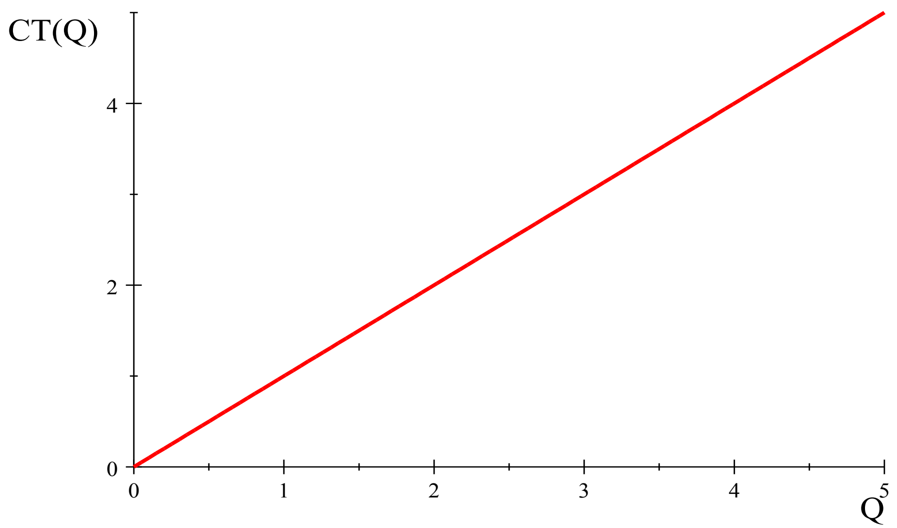


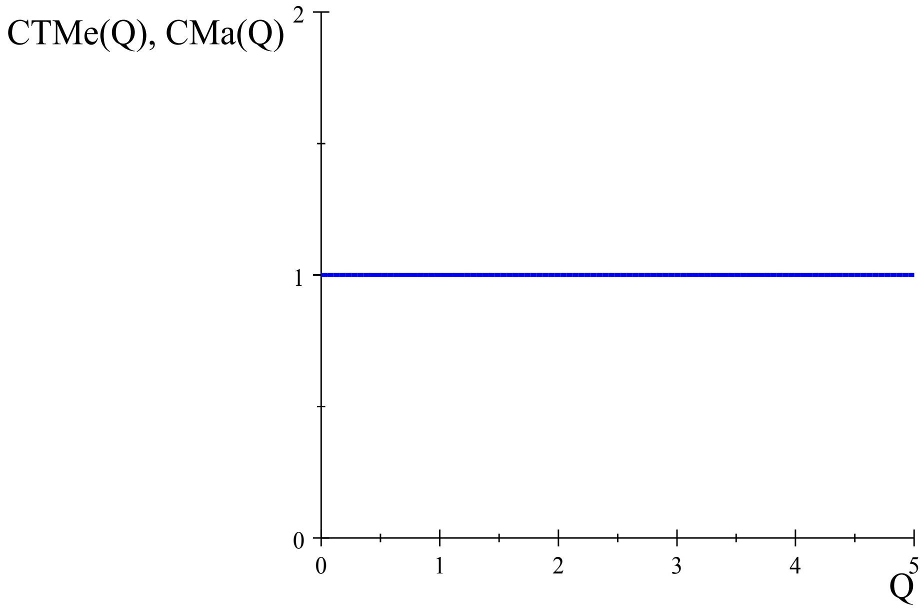
Costs and Economies of Scale: Examples

(e) $F(L,K) = L + 2K$

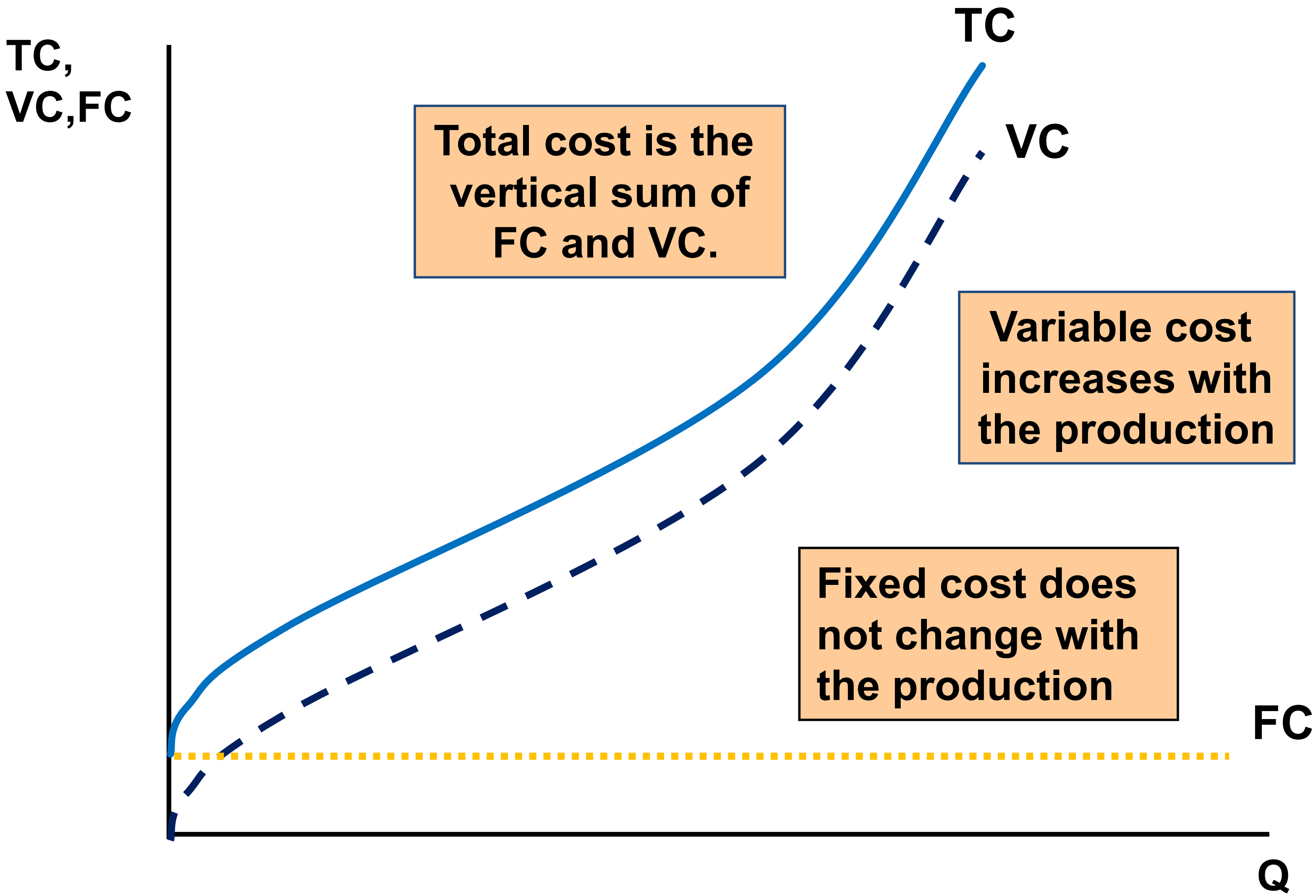
$$TC(Q) = L^*(Q,1,4) + 4K^*(Q,1,4) = Q$$

$$ATC(Q) = 1; \quad MC(Q) = 1$$

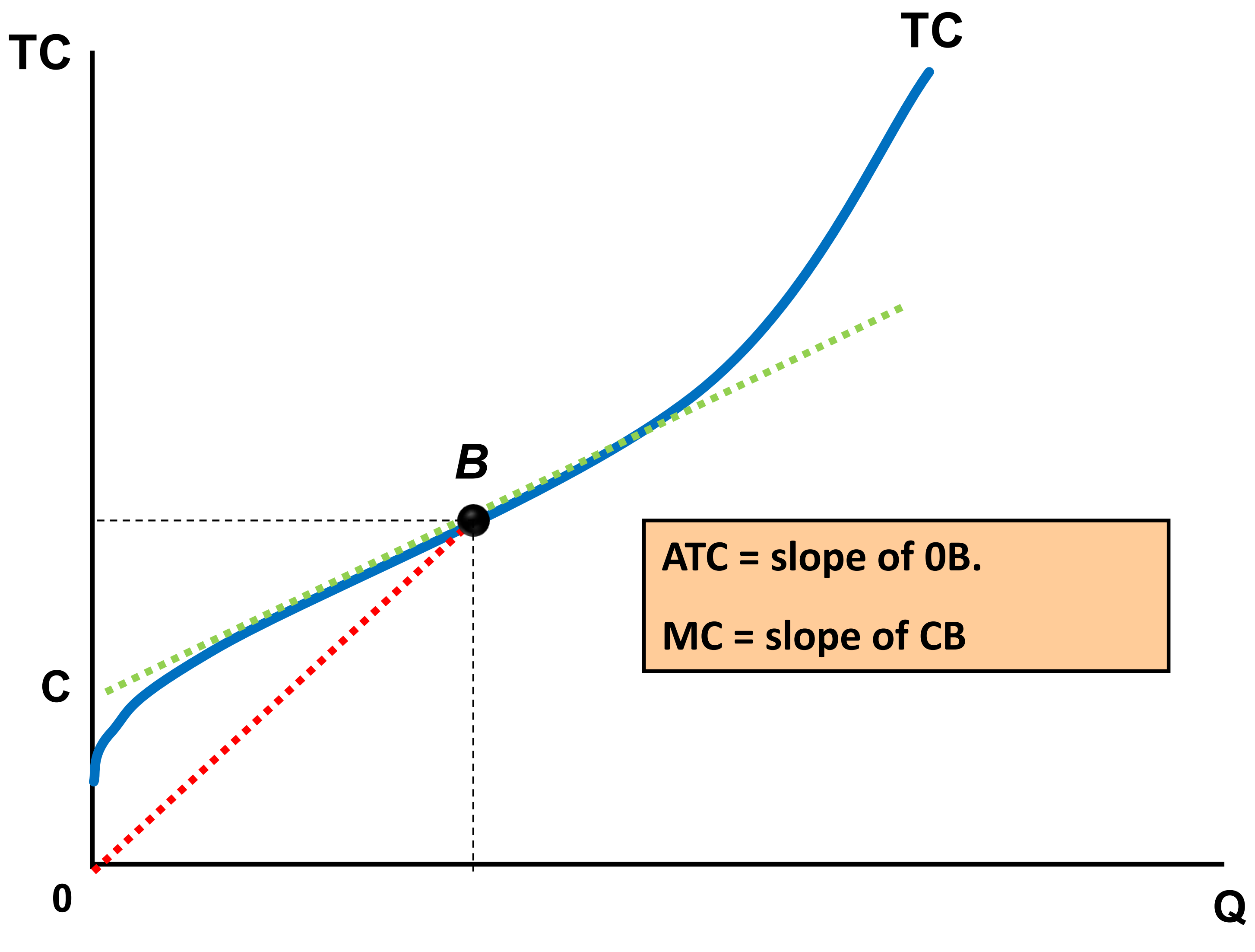




Cost Curves in the Short Run



Cost Curves in the Short Run



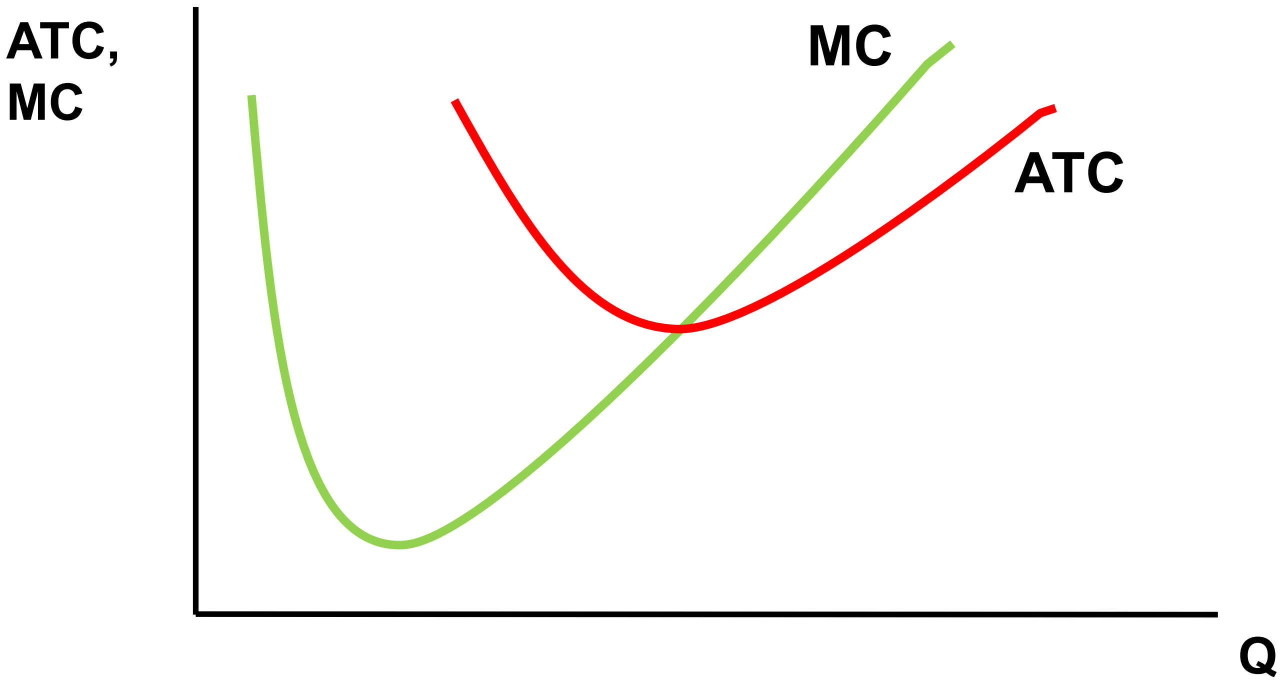
Cost Curves in the Short Run

ATC minimization: $dATC(Q)/dQ = 0$

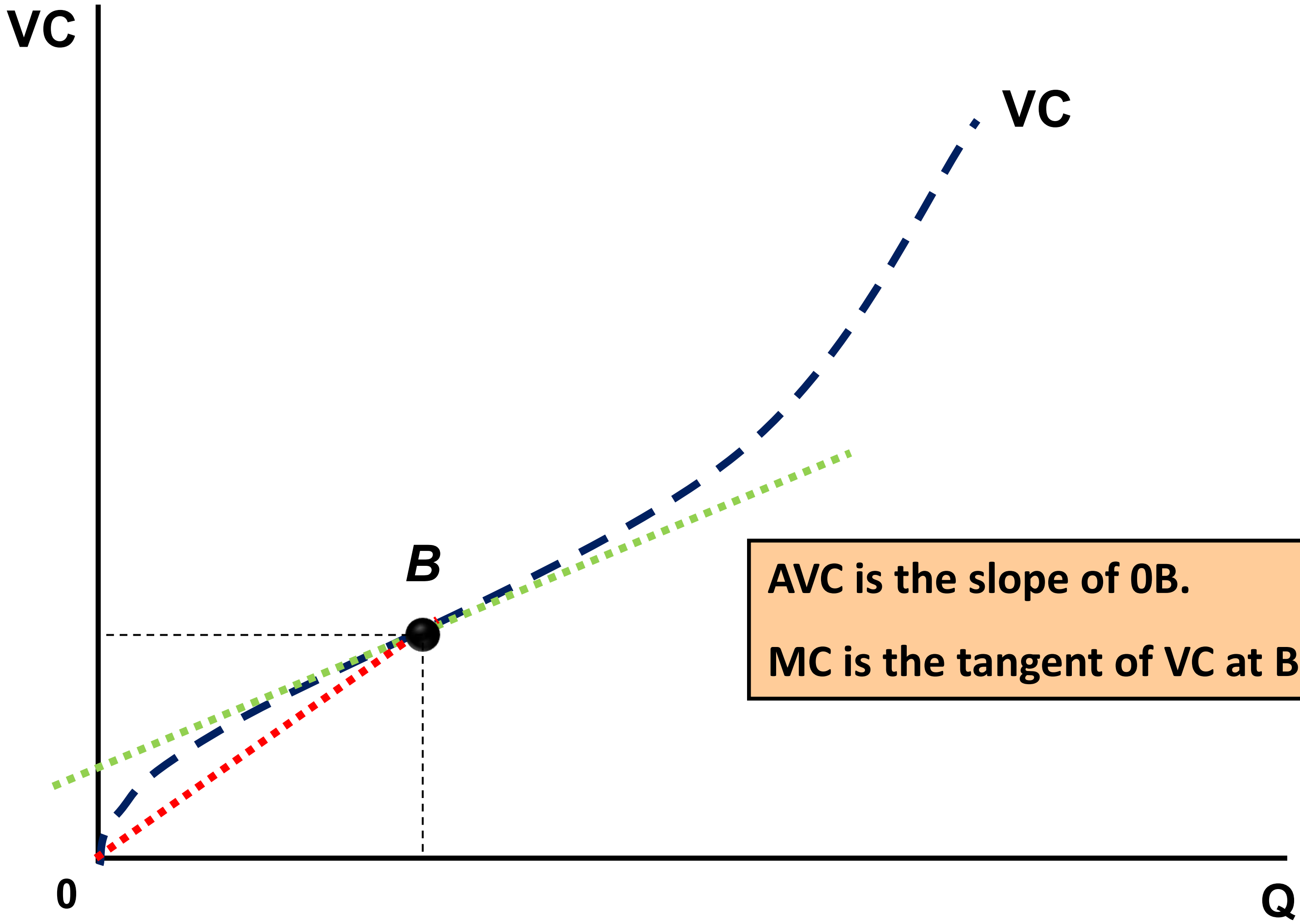
$$d(TC/Q)/dQ = (1/Q)(dTC/dQ) - TC/Q^2 = 0$$

Therefore, at the minimum point of the ATC, it holds that:

$$ATC = MC$$



Cost Curves in the Short Run



AVC is the slope of 0B.

MC is the tangent of VC at B

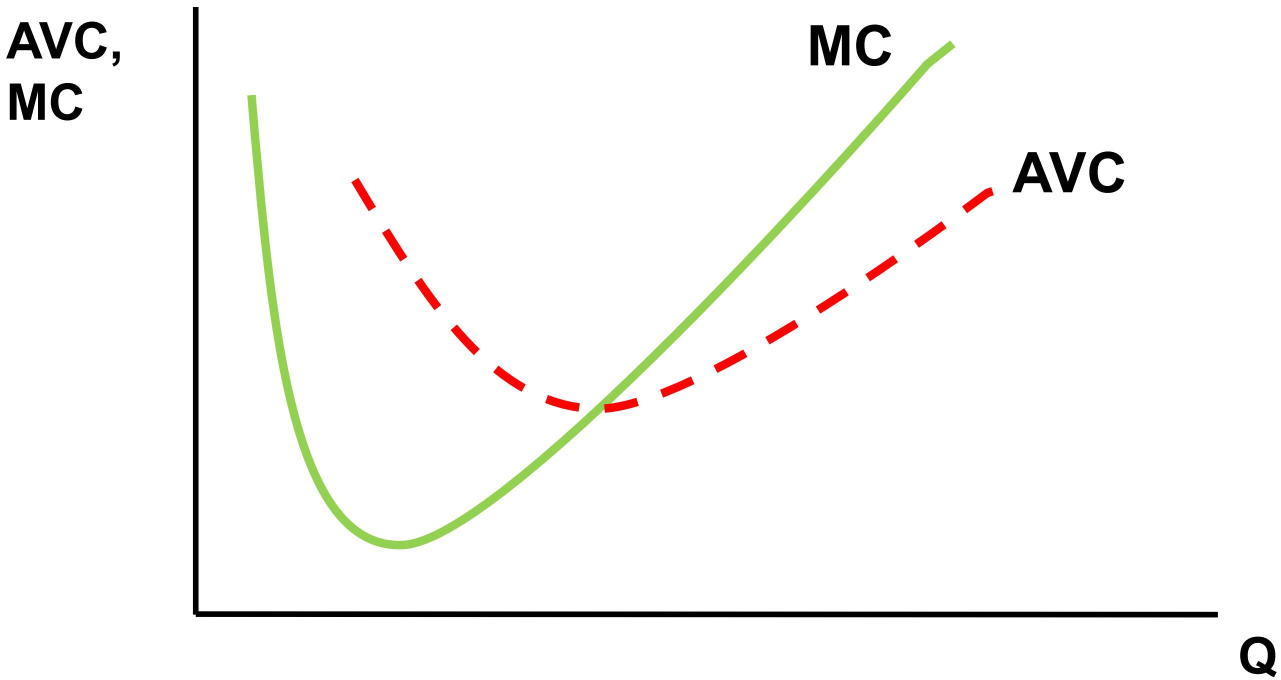
Cost Curves in the Short Run

AVC minimization: $dAVC(Q)/dQ = 0$

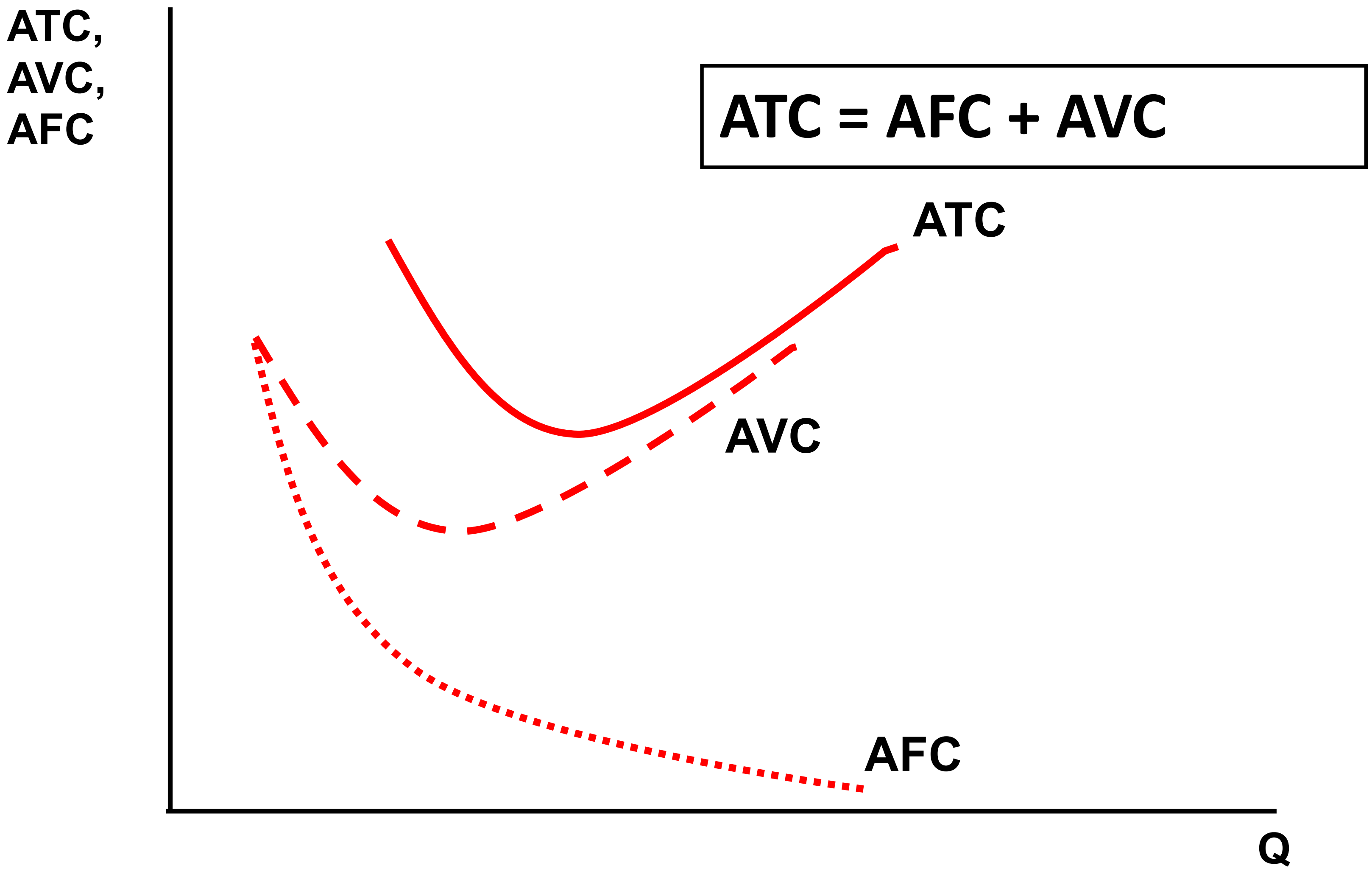
$d(VC/Q)/dQ = (1/Q)(dVC/dQ) - VC/Q^2 = 0$

Therefore, at the minimum point of the AVC, it holds that:

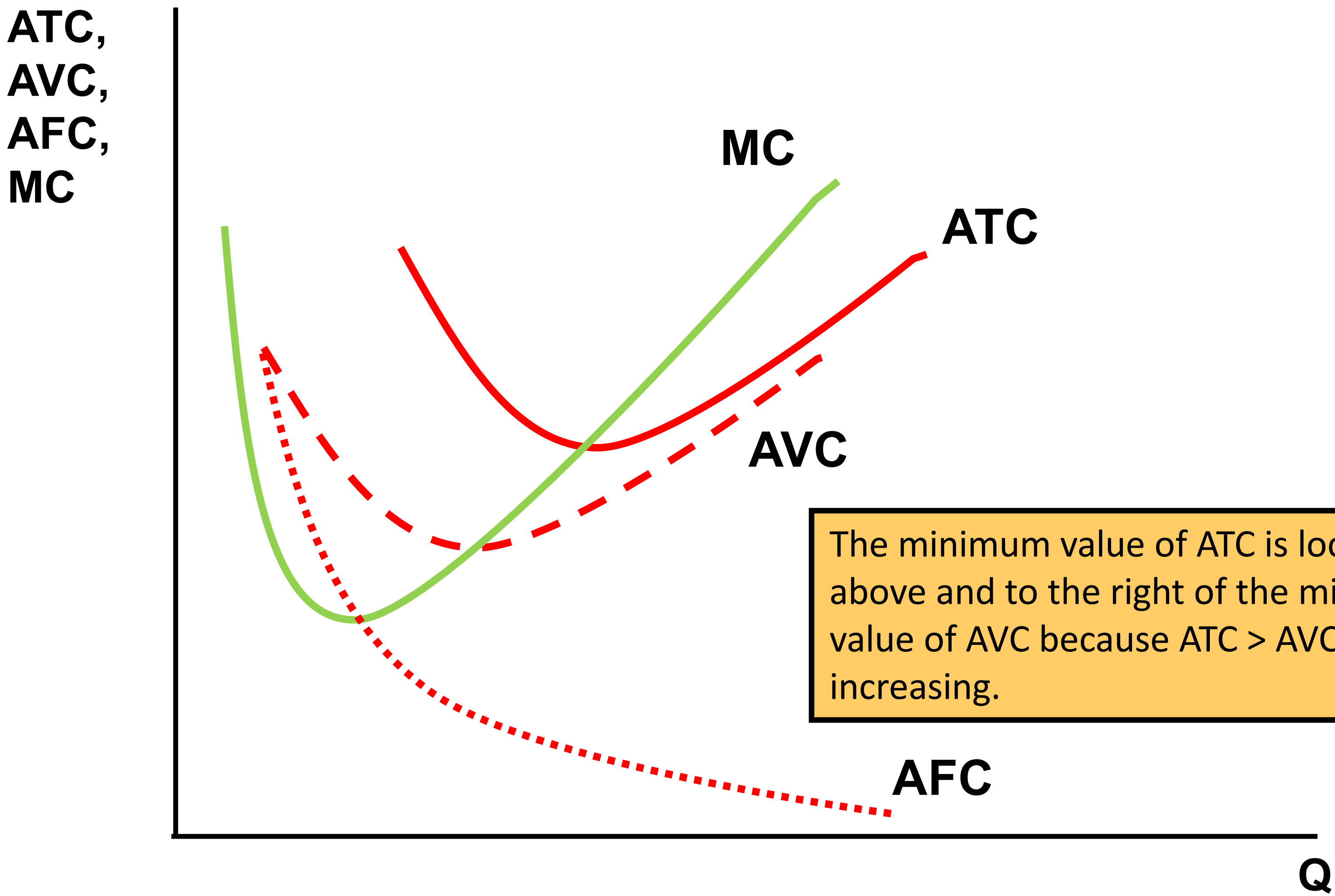
AVC = MC



Cost Curves in the Short Run



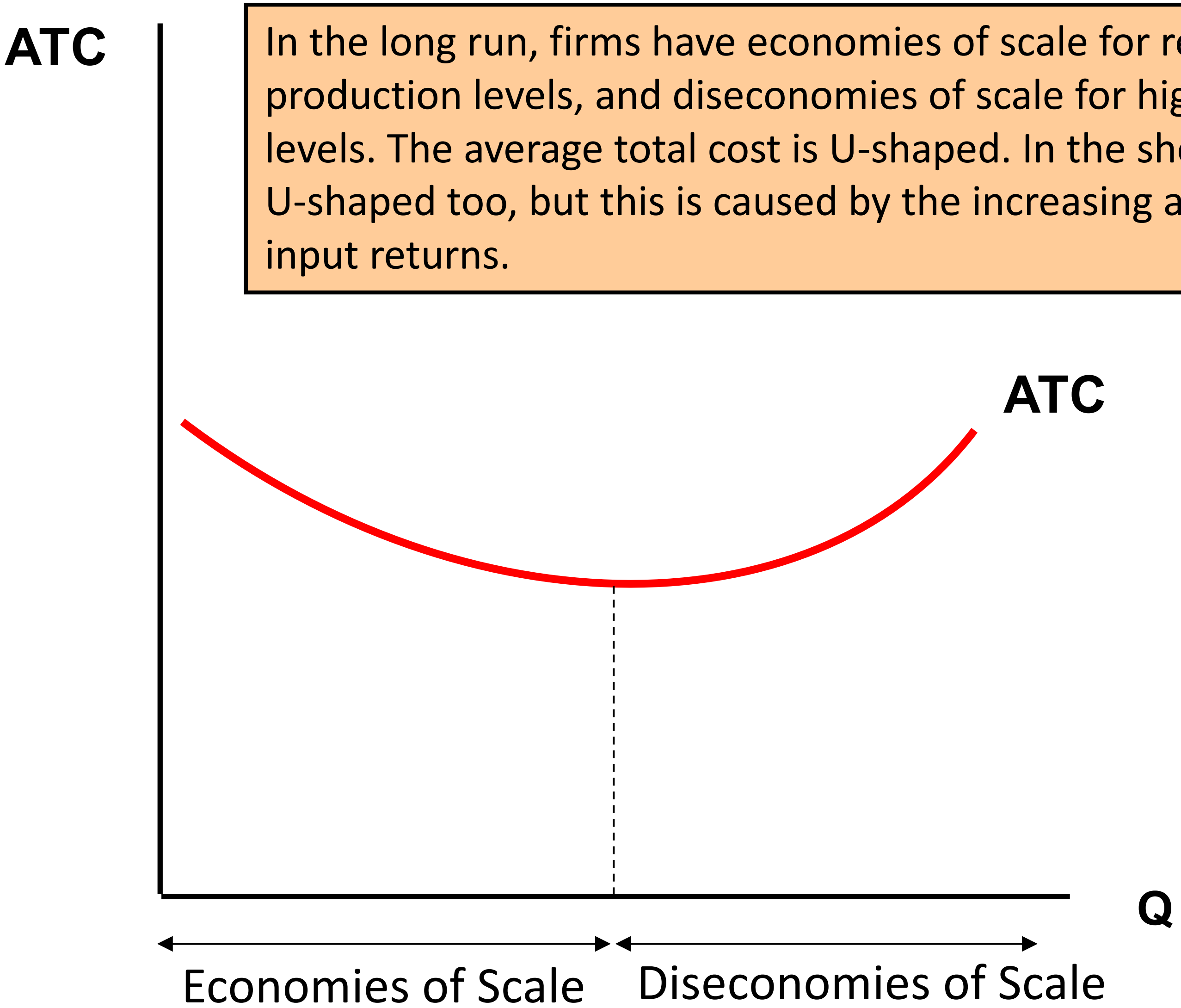
Cost Curves in the Short Run



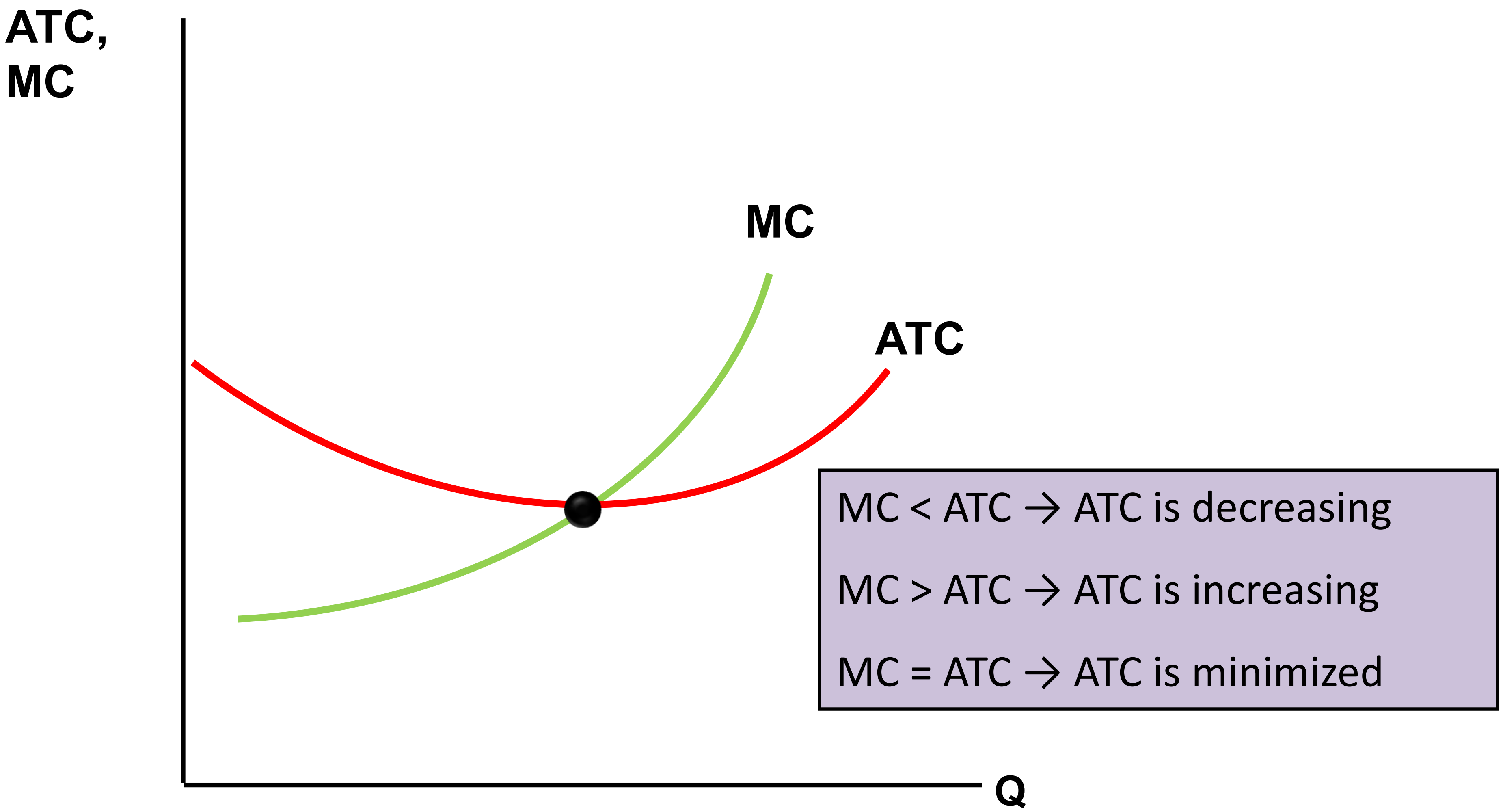
The minimum value of ATC is located above and to the right of the minimum value of AVC because ATC > AVC and MC is increasing.

Cost Curves in the Long Run

In the long run, firms have economies of scale for relatively low production levels, and diseconomies of scale for high production levels. The average total cost is U-shaped. In the short run, ATC is U-shaped too, but this is caused by the increasing and decreasing input returns.

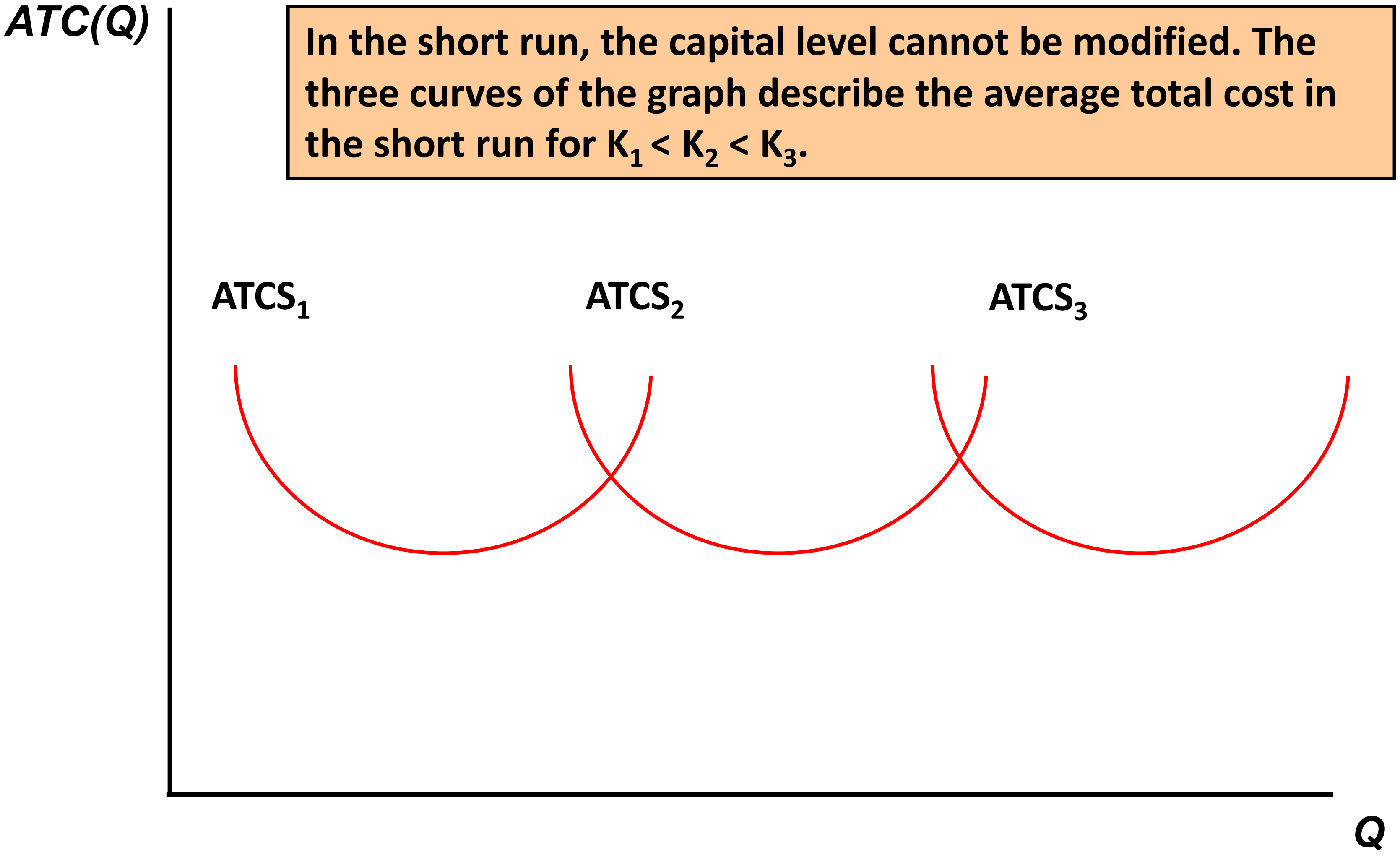


Cost Curves in the Long Run



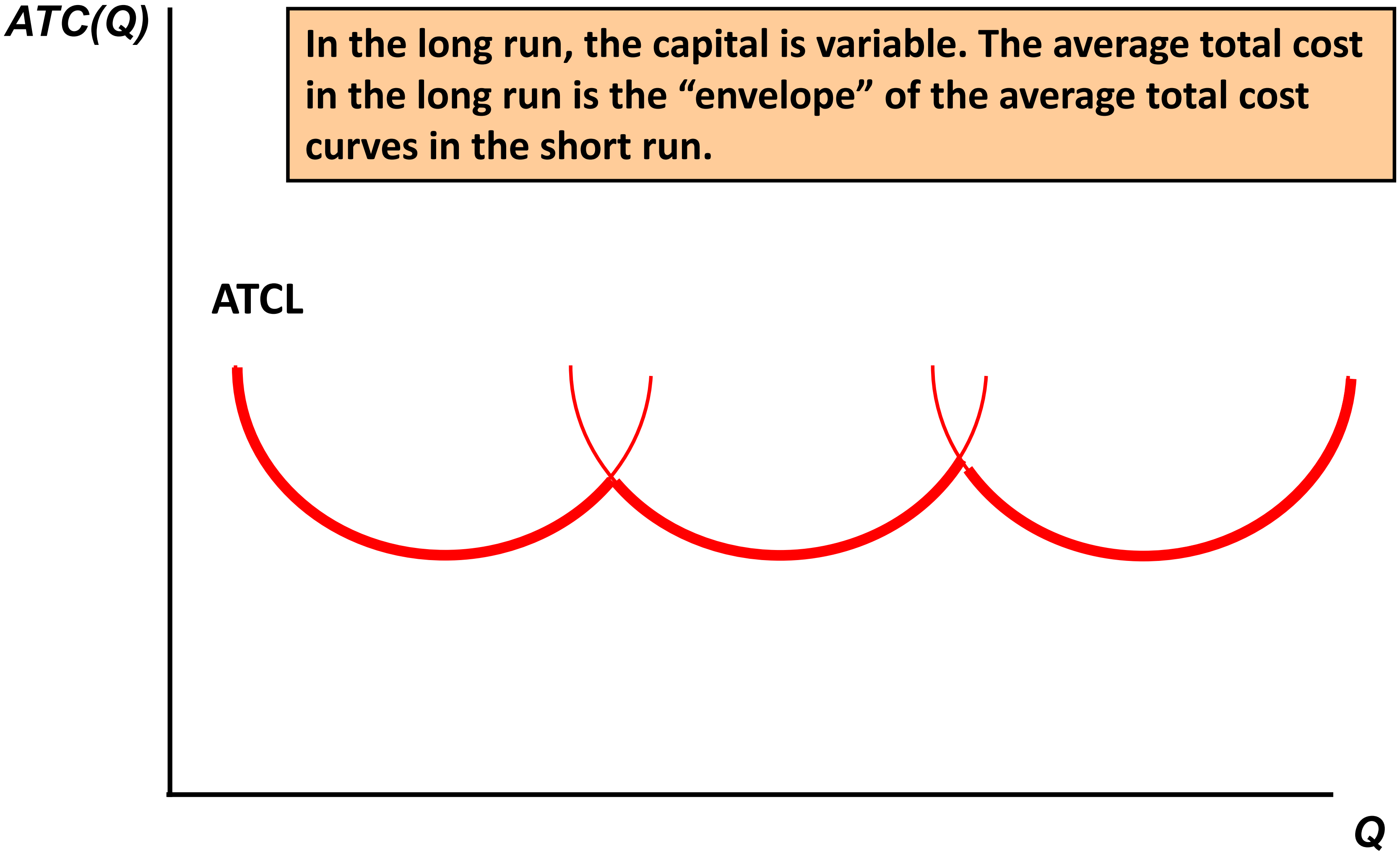
Cost Curves in the Short and the Long Run

In the short run, the capital level cannot be modified. The three curves of the graph describe the average total cost in the short run for $K_1 < K_2 < K_3$.

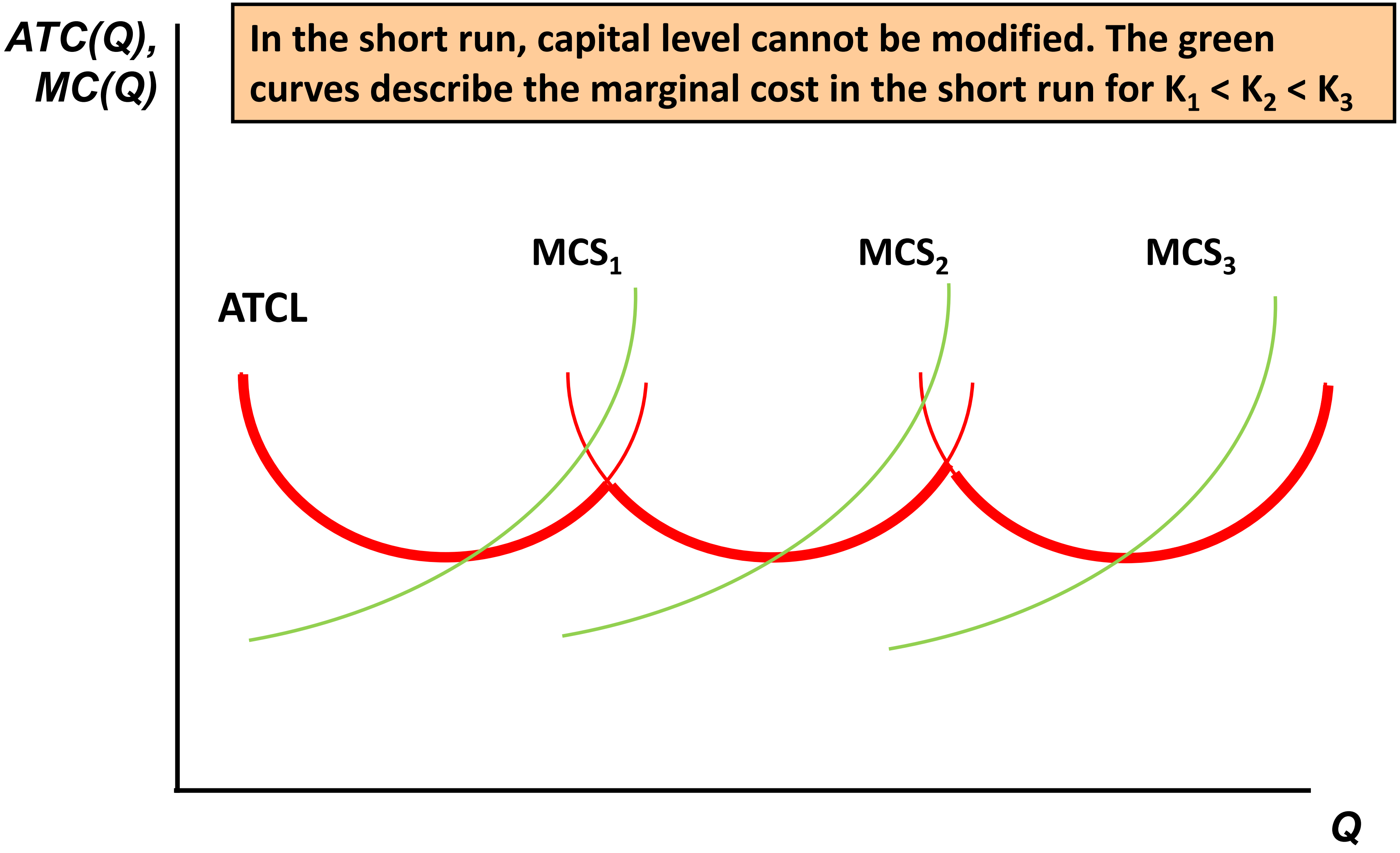


Cost Curves in the Short and the Long Run

In the long run, the capital is variable. The average total cost in the long run is the "envelope" of the average total cost curves in the short run.

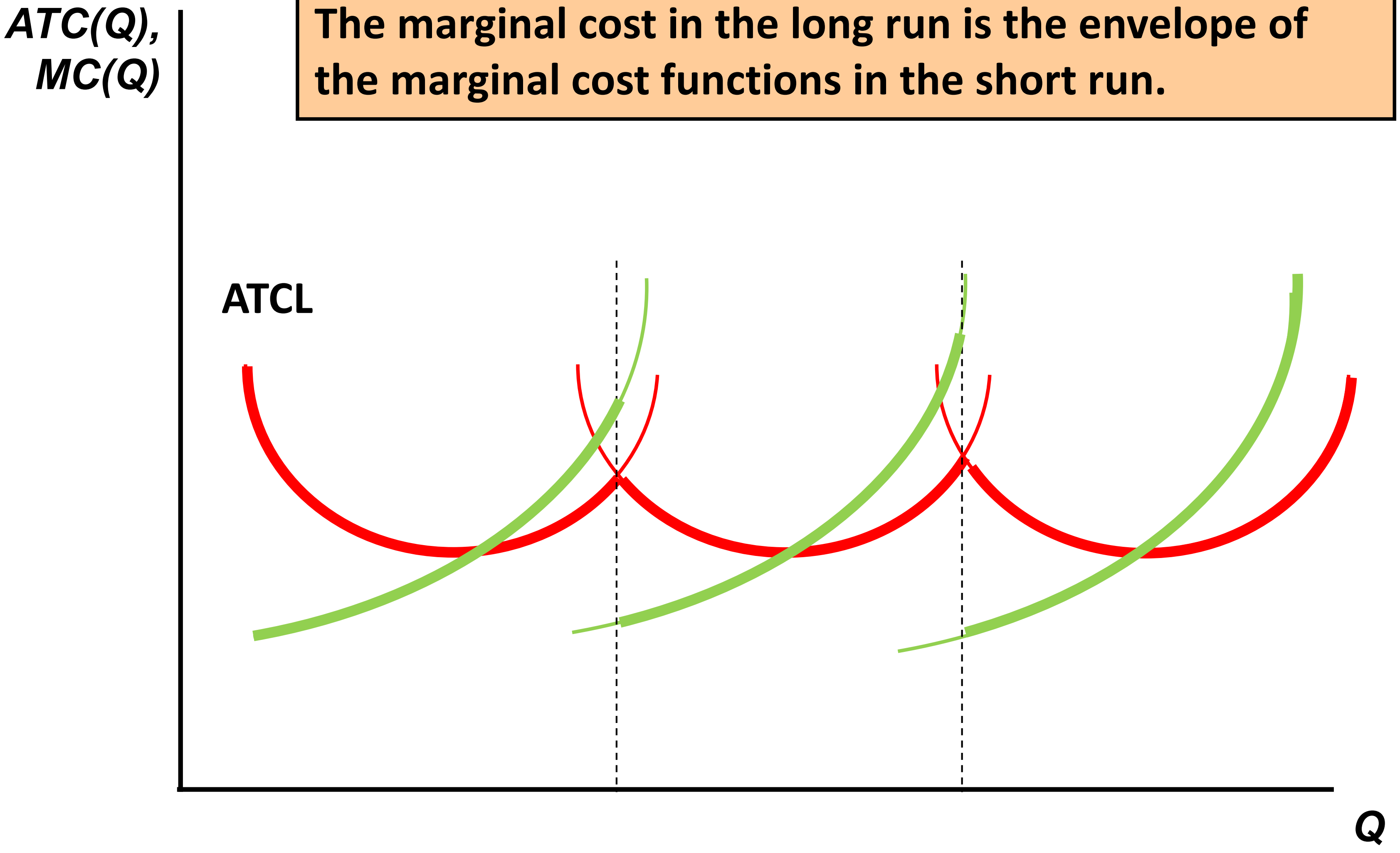


Cost Curves in the Short and the Long Run

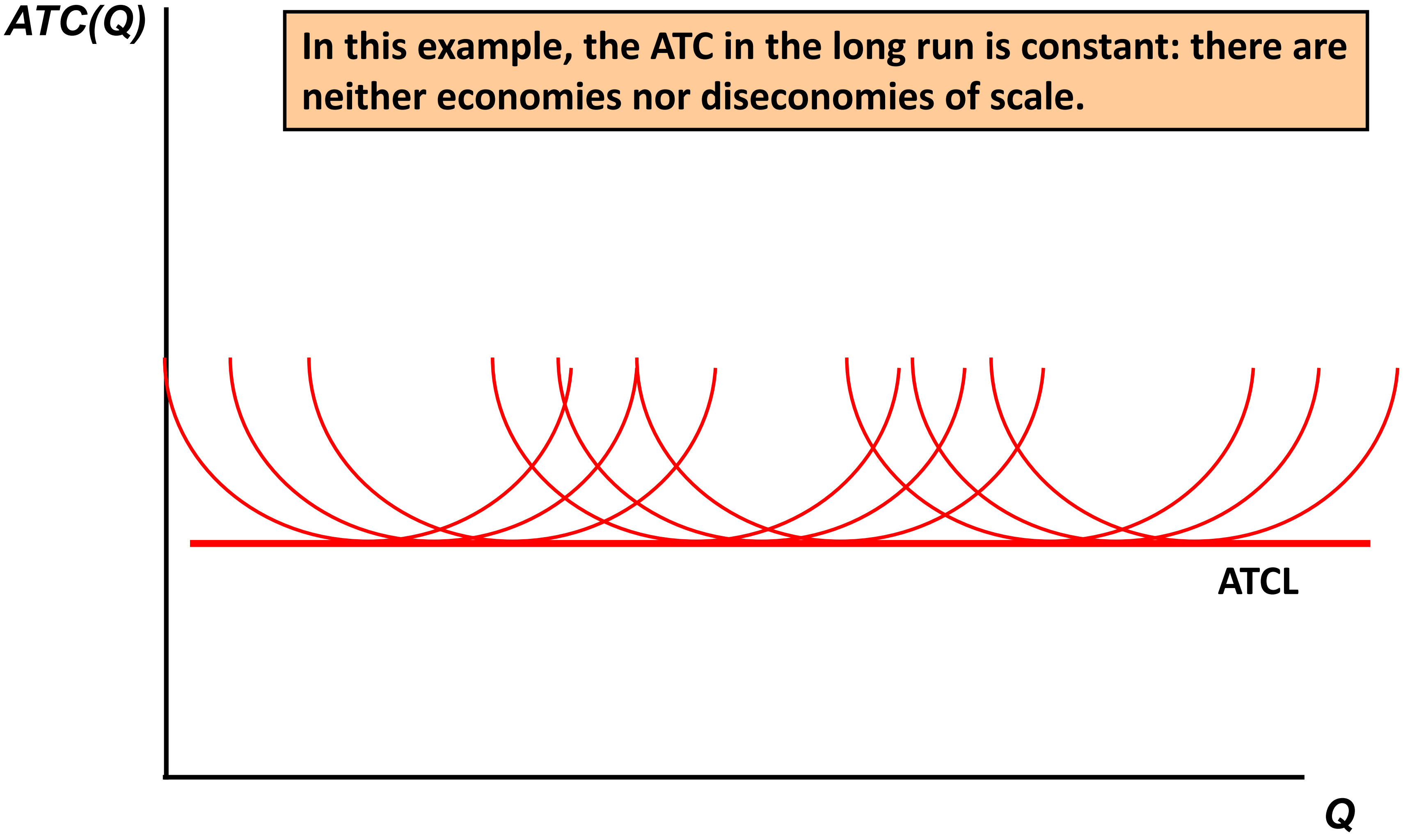


Cost Curves in the Short and the Long Run

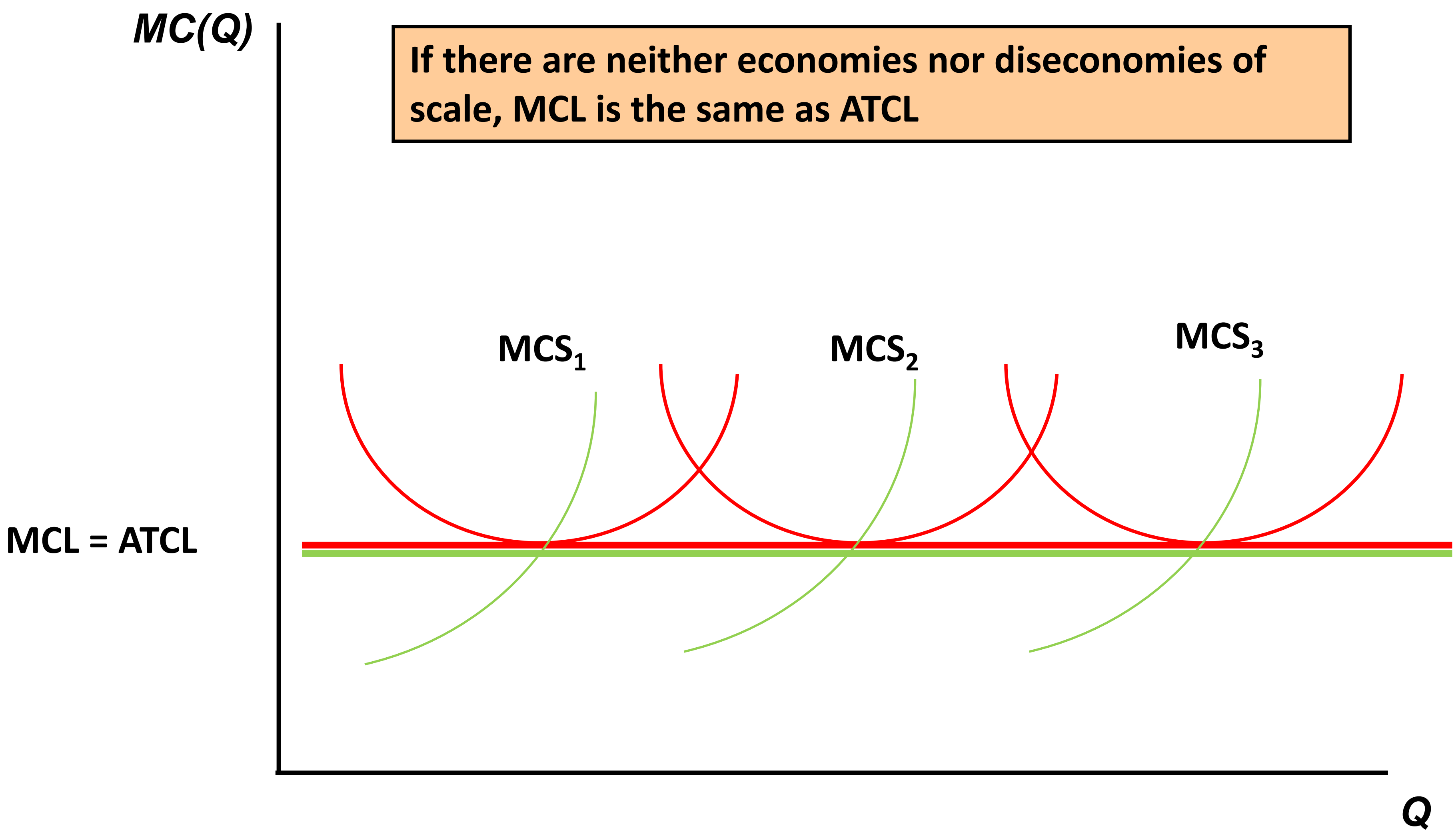
The marginal cost in the long run is the envelope of the marginal cost functions in the short run.



Cost Curves in the Short and the Long Run



Cost Curves in the Short and the Long Run

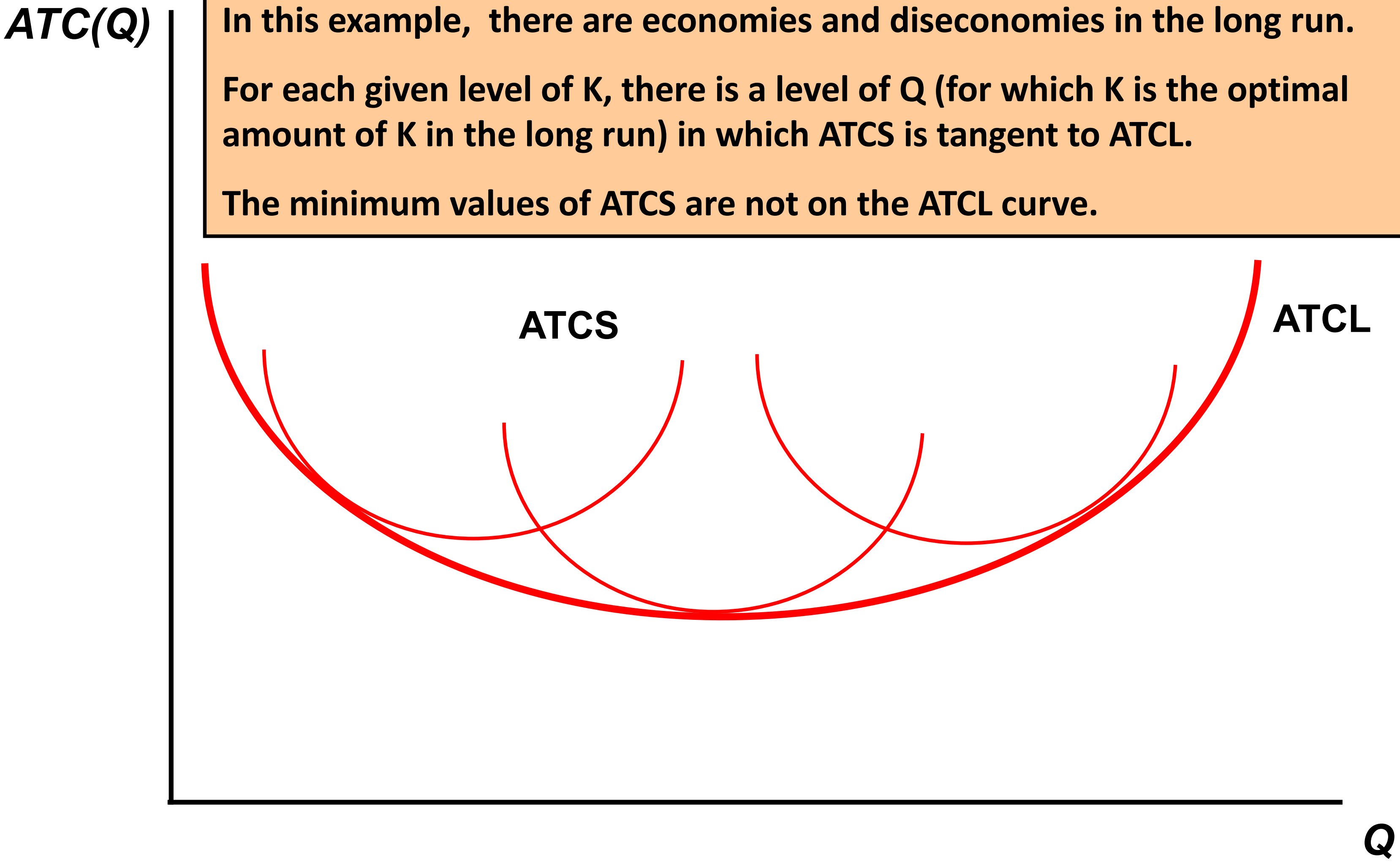


Cost Curves in the Short and the Long Run

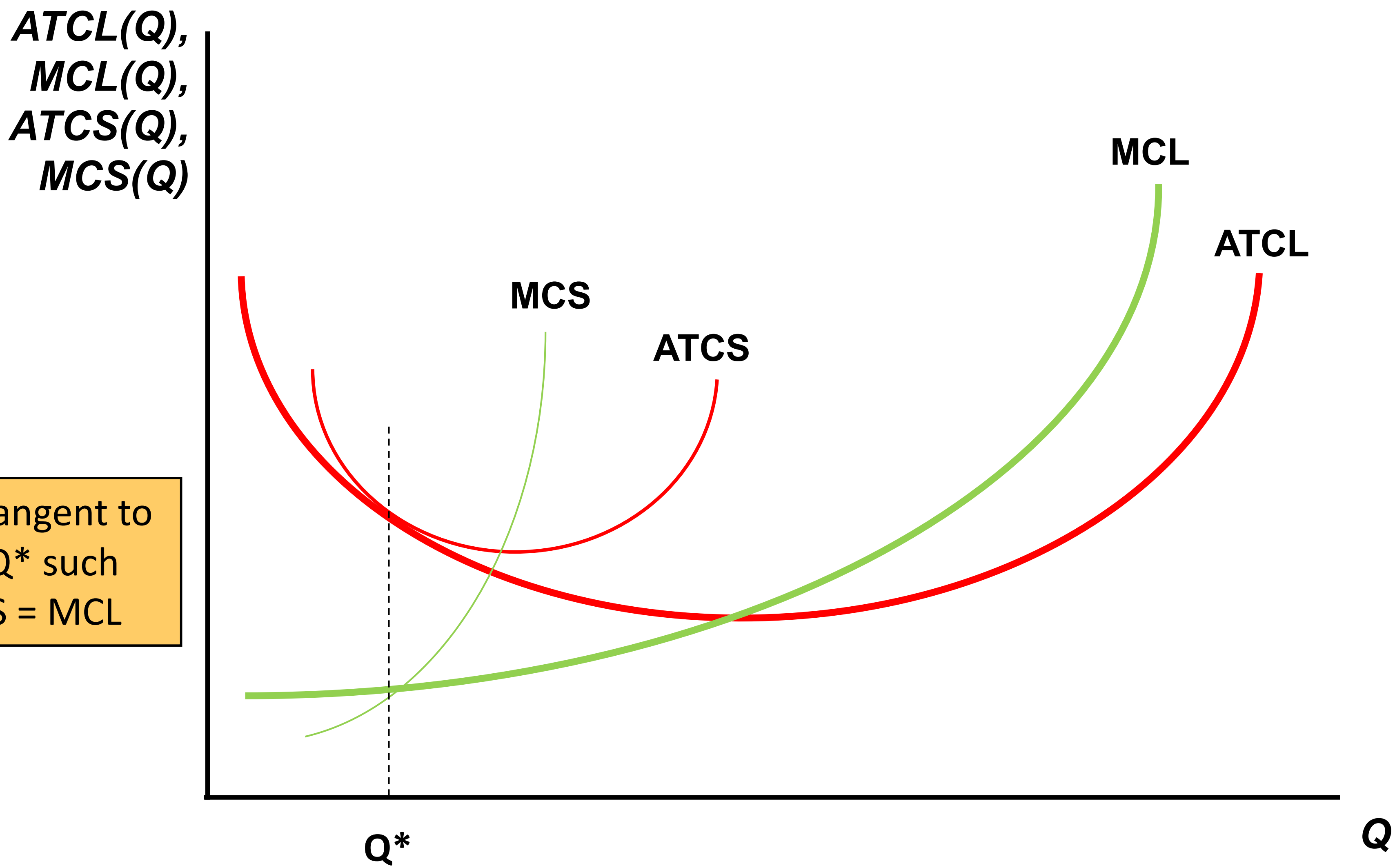
In this example, there are economies and diseconomies in the long run.

For each given level of K, there is a level of Q (for which K is the optimal amount of K in the long run) in which ATCS is tangent to ATCL.

The minimum values of ATCS are not on the ATCL curve.



Cost Curves in the Short and the Long Run



ATCS is tangent to ATCL at Q* such that MCS = MCL

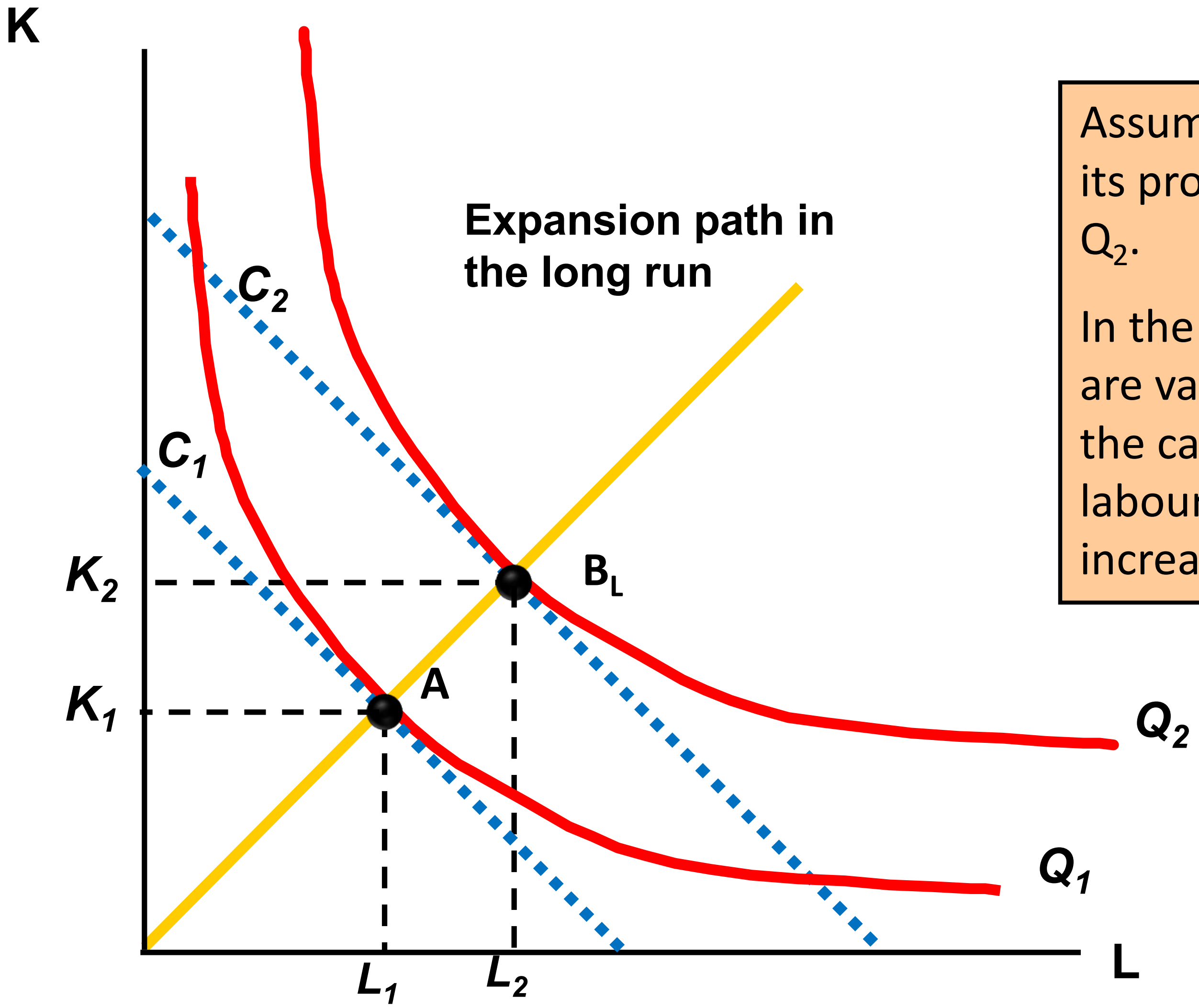
Short Run and Long Run

- Every fixed input in the SR represents the results of the LR decisions made previously by firms. These previous LR decisions are a function of the forecast about the amount of good that would be profitable to produce.

- Decisions made in the SR and in the LR are very different.

- The period to differentiate between the SR and the LR depends on the sector.

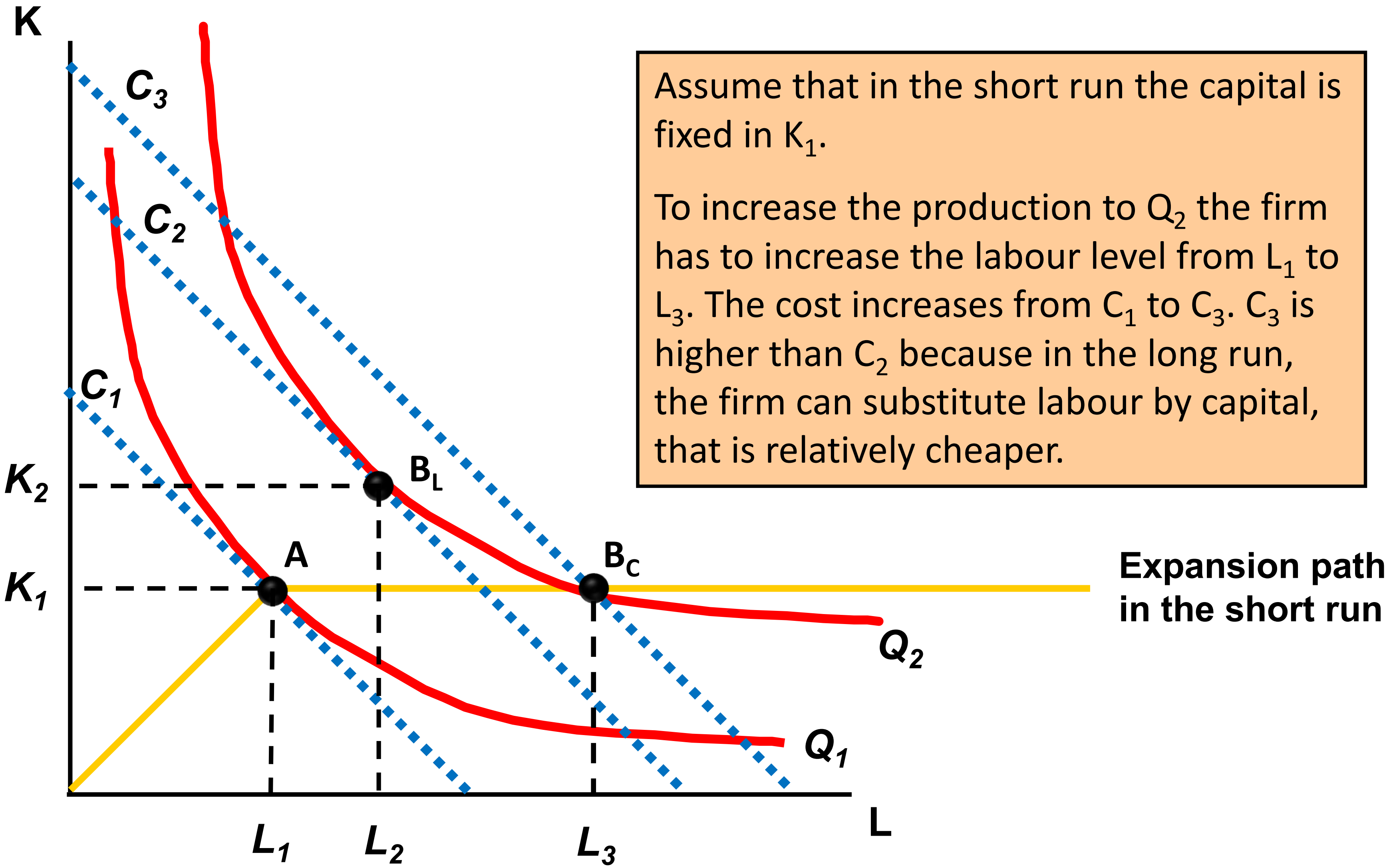
Short Run and Long Run: Expansion Path



Assume a firm wants to increase its production level from Q_1 to Q_2.

In the long run, all the factors are variable. The firm increases the capital from K_1 to K_2 and the labour from L_1 to L_2. The cost increases from C_1 to C_2.

Short Run and Long Run: Expansion Path



Assume that in the short run the capital is fixed in K_1.

To increase the production to Q_2 the firm has to increase the labour level from L_1 to L_3. The cost increases from C_1 to C_3. C_3 is higher than C_2 because in the long run, the firm can substitute labour by capital, that is relatively cheaper.