Unit Roots and Co-integration

Topic 2: Deterministic Trends

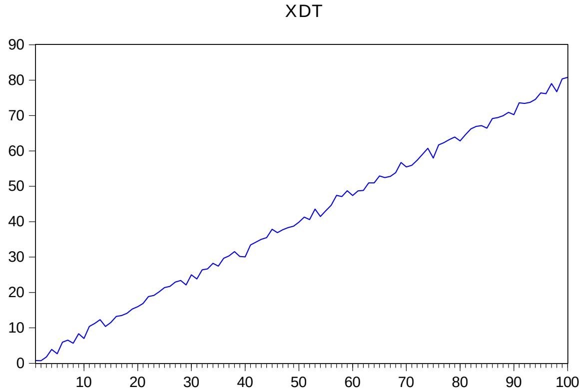
by Vanessa Berenguer-Rico

University of Oxford

January 2014

Deterministic Trends

$$x_t = \alpha + \delta t + \varepsilon_t; \quad \varepsilon_t \sim i.i.d.\,(0,1)$$



Deterministic Trends: Estimation and Testing

- The coefficients of regression models involving deterministic time trends are typically estimated by OLS
- However, the asymptotic distribution cannot be calculated in the same way as are those for regression models involving stationary variables
- Among other peculiarities, the estimates of the different parameters will in general have different asymptotic rates of convergence
- Nevertheless, the usual t and F statistics have the same asymptotic distributions as they do for stationary regressions

Deterministic Trends: Estimation

The model

$$y_t = \alpha + \delta t + \varepsilon_t,$$

can be written in standard regression model form as follows

$$y_t = x_t' \beta + \varepsilon_t,$$

where

$$\underset{(1\times 2)}{x_t'} \equiv \begin{bmatrix} 1 & t \end{bmatrix},$$

and

$$\underset{(2\times 1)}{\beta} \equiv \begin{bmatrix} \alpha \\ \delta \end{bmatrix}.$$

Deterministic Trends: Estimation

Let b_T denote the OLS estimate of β based on a sample of size T

$$b_T \equiv \begin{bmatrix} \hat{\alpha}_T \\ \hat{\delta}_T \end{bmatrix} = \left[\sum_{t=1}^{T} x_t x_t' \right]^{-1} \left[\sum_{t=1}^{T} x_t y_t \right].$$

It is simple to see that

$$(b_T - \beta) = \left[\sum_{t=1}^{T} x_t x_t' \right]^{-1} \left[\sum_{t=1}^{T} x_t \varepsilon_t \right],$$

or equivalently

$$\begin{bmatrix} \hat{\alpha}_T - \alpha \\ \hat{\delta}_T - \delta \end{bmatrix} = \begin{bmatrix} \sum_{t=1}^{T} 1 & \sum_{t=1}^{T} t \\ \sum_{t=1}^{T} t & \sum_{t=1}^{T} t^2 \end{bmatrix}^{-1} \begin{bmatrix} \sum_{t=1}^{T} \varepsilon_t \\ \sum_{t=1}^{T} t\varepsilon_t \end{bmatrix}$$

Notice that

$$\left[\sum_{t=1}^{T} x_t x_t'\right] = \begin{bmatrix} \sum_{t=1}^{T} 1 & \sum_{t=1}^{T} t \\ \sum_{t=1}^{T} t & \sum_{t=1}^{T} t^2 \end{bmatrix}$$

$$= \begin{bmatrix} T & T\left(T+1\right)/2 \\ T\left(T+1\right)/2 & T\left(T+1\right)\left(2T+1\right)/6 \end{bmatrix}$$

Deterministic Trends: Estimation

Now, let

$$\Upsilon_T = \begin{bmatrix} T^{1/2} & 0 \\ 0 & T^{3/2} \end{bmatrix},$$

and notice that

$$\begin{aligned} \Upsilon_T^{-1}\left[\sum_{t=1}^{T} x_t x_t'\right]\Upsilon_T^{-1} &= \begin{bmatrix} T^{-1}\sum_{t=1}^{T} 1 & T^{-2}\sum_{t=1}^{T} t \\ T^{-2}\sum_{t=1}^{T} t & T^{-3}\sum_{t=1}^{T} t^2 \end{bmatrix} \\ &= \begin{bmatrix} T^{-1}T & T^{-2}T\left(T+1\right)/2 \\ T^{-2}T\left(T+1\right)/2 & T^{-3}T\left(T+1\right)\left(2T+1\right)/6 \end{bmatrix} \\ &\longrightarrow \begin{bmatrix} 1 & 1/2 \\ 1/2 & 1/3 \end{bmatrix} \equiv Q. \end{aligned}$$

Deterministic Trends: Estimation

Recall

$$\Upsilon_T = \begin{bmatrix} T^{1/2} & 0 \\ 0 & T^{3/2} \end{bmatrix}.$$

If $\varepsilon_t \sim i.i.d.\left(0, \sigma^2\right)$ and $E\left[\varepsilon_t^4\right] < \infty$, then

$$\Upsilon_T^{-1}\left[\sum_{t=1}^{T} x_t \varepsilon_t\right] = \begin{bmatrix} \left(1/\sqrt{T}\right)\sum_{t=1}^{T} \varepsilon_t \\ \left(1/\sqrt{T}\right)\sum_{t=1}^{T} \left(t/T\right)\varepsilon_t \end{bmatrix} \xrightarrow{d} N\left(0, \sigma^2 Q\right).$$

Deterministic Trends: Estimation

Therefore,

$$\left[\begin{array}{c} T^{1/2}\left(\hat{\alpha}_T - \alpha\right) \\ T^{3/2}\left(\hat{\delta}_T - \delta\right) \end{array} \right] \xrightarrow{d} N\left(0, \left[Q^{-1}\sigma^2 Q Q^{-1}\right]\right) = N\left(0, \sigma^2 Q^{-1}\right).$$

Deterministic Trends: Estimation

Theorem

Let $y_t = \alpha + \delta t + \varepsilon_t$ *where* ε_t *is i.i.d.* $\left(0, \sigma^2\right)$ *and* $E\left[\varepsilon_t^4\right] < \infty$*. Then,*

$$\left[\begin{array}{c} T^{1/2}\left(\hat{\alpha}_T - \alpha\right) \\ T^{3/2}\left(\hat{\delta}_T - \delta\right) \end{array}\right] \xrightarrow{d} N\left(\left[\begin{array}{c} 0 \\ 0 \end{array}\right], \sigma^2 \left[\begin{array}{cc} 1 & 1/2 \\ 1/2 & 1/3 \end{array}\right]^{-1}\right).$$

Remark: $\hat{\delta}_T$ is superconsistent!

Deterministic Trends: Testing

- Null Hypothesis

$$H_o : \alpha = \alpha_0$$

- Test statistic

$$t_\alpha = \frac{\hat{\alpha}_T - \alpha_0}{\left\{ s_T^2 \begin{bmatrix} 1 & 0 \end{bmatrix} \left[\sum_{t=1}^T x_t x_t' \right]^{-1} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right\}^{1/2}},$$

where

$$s_T^2 = \frac{1}{T-2} \sum_{t=1}^T \left(y_t - \hat{\alpha}_T - \hat{\delta}_T t \right)^2$$

Deterministic Trends: Testing

- Asymptotic Distribution

$$t_{\alpha} = \frac{\hat{\alpha}_T - \alpha_0}{\left\{ s_T^2 \begin{bmatrix} 1 & 0 \end{bmatrix} \left[\sum_{t=1}^{T} x_t x_t' \right]^{-1} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right\}^{1/2}} \xrightarrow{d} N(0,1)$$

Deterministic Trends: Testing

- Null Hypothesis

$$H_o : \delta = \delta_0$$

- Test statistic

$$t_\delta = \frac{\hat{\delta}_T - \delta_0}{\left\{ s_T^2 \begin{bmatrix} 0 & 1 \end{bmatrix} \left[\sum_{t=1}^T x_t x_t' \right]^{-1} \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}^{1/2}},$$

where again

$$s_T^2 = \frac{1}{T-2} \sum_{t=1}^{T} \left(y_t - \hat{\alpha}_T - \hat{\delta}_T t \right)^2$$

Deterministic Trends: Testing

- Asymptotic Distribution

$$t_{\delta} = \frac{\hat{\delta}_T - \delta_0}{\left\{ s_T^2 \begin{bmatrix} 0 & 1 \end{bmatrix} \left[\sum_{t=1}^{T} x_t x_t' \right]^{-1} \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}^{1/2}} \xrightarrow{d} N(0,1)$$

- $\hat{\alpha}_T$ and $\hat{\delta}_T$ converge at different rates

- The corresponding standard errors also incorporate different orders of T

- Hence, the usual OLS t tests are asymptotically valid