Unit Roots and Co-integration

Topic 1: Trends in Time Series Data

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What is a Time Series?

- A time series is a realization of a stochastic process

- A stochastic process is a family of random variables $\{X_t(\omega), t \in T, \omega \in \Omega\}$ defined in a probability space $\{\Omega, \mathcal{F}, P\}$

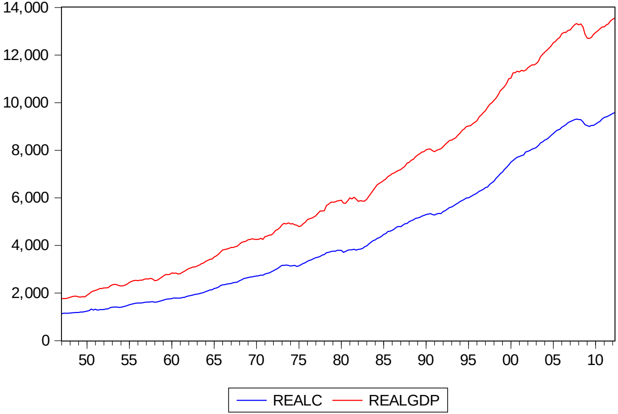
- Examples: i.i.d., heteroscedastic, ARMA(p,q)

- This course is about: **Trended Time Series**

- *"No one understands trends. Everyone sees them in data"* (Phillips, 2010)

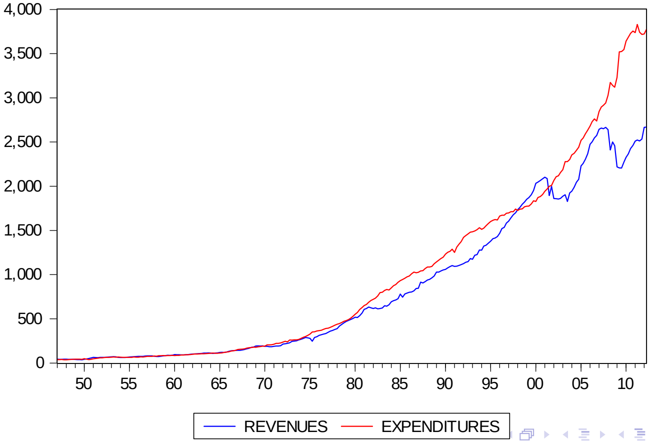
Trends in Time Series Data

- Real Consumption and Real GDP
- U.S. Quarterly Data from the Federal Reserve Bank of St. Louis: 1947Q1-2012Q2



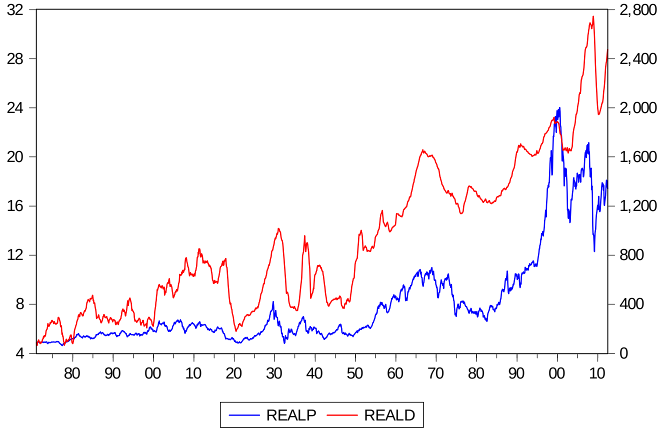
Trends in Time Series Data

- Government Expenditures and Revenues
- U.S. Quarterly Data from the Federal Reserve Bank of St. Louis: 1947Q1-2012Q2



Trends in Time Series Data

- Real Stock Prices and Real Dividends
- U.S. Monthly Data from Robert Shiller: 1871m1-2012m6



Modelling Trends

- No trends. Example: i.i.d., ARMA(p,q)

$$u_t \sim i.i.d.\,(0,1)\,; \quad x_t = \phi x_{t-1} + u_t; \quad |\phi| < 1$$

- Deterministic trends. Example: Linear time trend

$$x_t = \mu + \beta t + u_t; \quad u_t \sim i.i.d.\,(0,1)$$

- Stochastic trend. Example: Random Walk

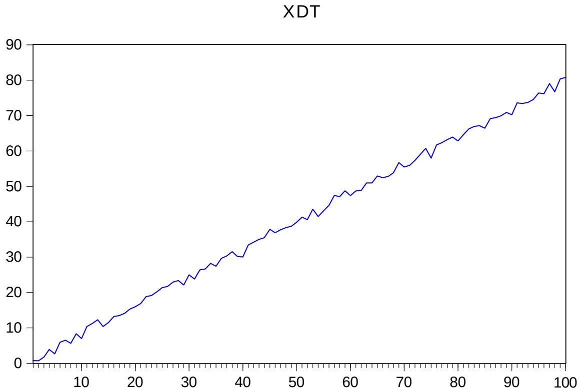
$$x_t = x_{t-1} + u_t; \quad u_t \sim i.i.d.\,(0,1)\,; \quad x_0 = 0$$

- Deterministic & Stochastic trend. Example: Random Walk with Drift

$$x_t = \alpha + x_{t-1} + u_t; \quad u_t \sim i.i.d.\,(0,1)\,; \quad x_0 = 0$$

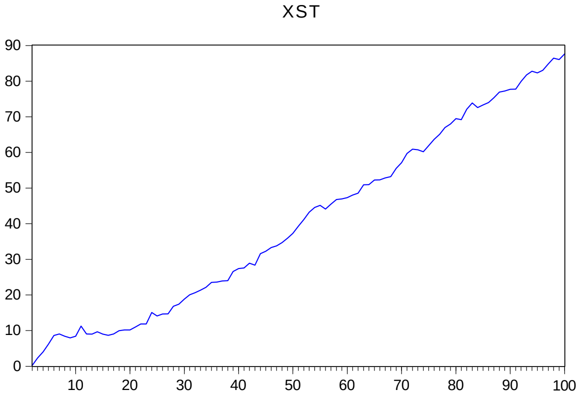
Modelling Trends

$$x_t = \mu + \beta t + u_t; \quad u_t \sim i.i.d.N\,(0, 1)$$



Modelling Trends

$$x_t = \alpha + x_{t-1} + u_t; \;\; u_t \sim i.i.d.N\left(0,1\right); \;\; x_0 = 0$$



Modelling Trends: Stochastic Properties

- Let $u_t \sim i.i.d. \left(0, \sigma^2\right)$
- No trends. Example:

$$x_t = \mu + u_t$$

- Stochastic Properties

$$E\left[x_t\right] = \mu$$

$$V\left[x_t\right] = \sigma^2$$

$$Cov\left[x_t, x_s\right] = 0$$

Modelling Trends: Stochastic Properties

- Let $u_t \sim i.i.d.\left(0, \sigma^2\right)$
- No trends. Example:

$$x_t = \mu + \phi x_{t-1} + u_t; \quad |\phi| < 1$$

- Stochastic Properties

$$E\left[x_t\right] = \frac{\mu}{(1-\phi)}$$

$$V\left[x_t\right] = \frac{\sigma^2}{\left(1-\phi^2\right)}$$

$$Cov\left[x_t, x_s\right] = \phi^{|t-s|} \frac{\sigma^2}{\left(1-\phi^2\right)}$$

Modelling Trends: Stochastic Properties

- Let $u_t \sim i.i.d. \left(0, \sigma^2\right)$
- Deterministic trends. Example:

$$x_t = \mu + \beta t + u_t$$

- Stochastic Properties

$$E\left[x_t\right] = \mu + \beta t$$

$$V\left[x_t\right] = \sigma^2$$

$$Cov\left[x_t, x_s\right] = 0$$

Modelling Trends: Stochastic Properties

- Let $u_t \sim i.i.d.\left(0, \sigma^2\right)$
- Deterministic trends. Example:

$$x_t = \mu + \beta t + \phi x_{t-1} + u_t; \quad |\phi| < 1$$

- Stochastic Properties

$$E\left[x_t\right] = \frac{\mu}{(1-\phi)} - \frac{\phi\beta}{(1-\phi)^2} + \frac{\beta}{(1-\phi)}t$$

$$V\left[x_t\right] = \frac{\sigma^2}{\left(1-\phi^2\right)}$$

$$Cov\left[x_t, x_s\right] = \phi^{|t-s|}\frac{\sigma^2}{\left(1-\phi^2\right)}$$

Modelling Trends: Stochastic Properties

- Let $u_t \sim i.i.d.\ (0, \sigma^2)$
- Stochastic Trend. Example:

$$x_t = x_{t-1} + u_t; \quad x_0 = 0$$

- Solving Backwards

$$\begin{aligned} x_t &= x_{t-1} + u_t \\ &= x_{t-2} + u_{t-1} + u_t \\ &= x_{t-3} + u_{t-2} + u_{t-1} + u_t \\ &= ... \\ &= x_0 + \sum_{j=1}^{t} u_j \end{aligned}$$

Modelling Trends: Stochastic Properties

- Stochastic Trend. Example:

$$x_t = x_{t-1} + u_t; \quad u_t \sim i.i.d.\,(0,1)\,; \quad x_0 = 0$$

$$x_t = x_0 + \sum_{j=1}^{t} u_j$$

- Stochastic Properties

$$E\left[x_t\right] = x_0 = 0$$

$$V\left[x_t\right] = \sigma^2 t$$

$$Cov\left[x_t, x_s\right] = \min\left\{t, s\right\}\sigma^2$$

Modelling Trends: Stochastic Properties

- Let $u_t \sim i.i.d.\left(0, \sigma^2\right)$
- Deterministic & Stochastic Trend. Example:

$$x_t = \beta + x_{t-1} + u_t; \quad x_0 = 0$$

- Solving Backwards

$$\begin{aligned} x_t &= \beta + x_{t-1} + u_t \\ &= 2\beta + x_{t-2} + u_{t-1} + u_t \\ &= 3\beta + x_{t-3} + u_{t-2} + u_{t-1} + u_t \\ &= \ldots \\ &= t\beta + x_0 + \sum_{j=1}^{t} u_j \end{aligned}$$

Modelling Trends: Stochastic Properties

- Deterministic & Stochastic Trend. Example:

$$x_t = \beta + x_{t-1} + u_t; \quad u_t \sim i.i.d.\,(0,1)$$

$$x_t = x_0 + \beta t + \sum_{j=1}^{t} u_j$$

- Stochastic Properties

$$E\left[x_t\right] = x_0 + \beta t$$

$$V\left[x_t\right] = \sigma^2 t$$

$$Cov\left[x_t, x_s\right] = \min\left\{t, s\right\} \sigma^2$$

Stationary vs Non-stationary World

- Two notions of stationarity
- **Strict Stationarity:** The time series $\{X_t, t \in \mathbb{Z}\}$ is said to be strictly stationary if the joint distributions of $(X_{t_1}, ..., X_{t_k})'$ and $(X_{t_{1+h}}, ..., X_{t_{k+h}})'$ are the same for all positive integers k and for all $t_1, ..., t_k, h \in \mathbb{Z}$.
- **Weak Stationarity**: The time series $\{X_t, t \in \mathbb{Z}\}$ is said to be weakly stationary if:

  (i) $E[X_t] = m$ for all t
  (ii) $E[X_t^2] < \infty$ for all t
  (iii) $Cov(X_t, X_s) = Cov(X_{t+h}, X_{s+h})$ for all $t, s, h \in \mathbb{Z}$

Stationary vs Non-stationary World

- Non-stationarities: Examples
- **Non-statinary in mean:**

$$x_t = \mu + \beta t + u_t; \quad u_t \sim i.i.d.N\left(0,1\right) \;\; \text{or} \;\; x_t = \begin{cases} u_t, & t < k \\ \mu + u_t, & t \geq k \end{cases}$$

- **Non-statinary in variance**:

$$x_t = x_{t-1} + u_t; \quad u_t \sim i.i.d.\left(0,1\right)$$

- **Non-statinary in mean and variance**:

$$x_t = \beta + x_{t-1} + u_t; \quad u_t \sim i.i.d.\left(0,1\right)$$