# 5 FORECASTING

Uncertainty is a fact of life for both individuals and organizations. Forecasting is essential for planning and operation control in a variety of areas such as production management, inventory systems, quality control, financial planning, and investment analysis. In this chapter we develop the minimum mean square error forecasts for the stationary and nonstationary time series models introduced in Chapters 3 and 4. These models can also be used to update forecasts when new information becomes available. We also discuss the implication of the constructed time series model in terms of its eventual forecast function.

## 5.1 INTRODUCTION

One of the most important objectives in the analysis of a time series is to forecast its future values. Even if the final purpose of time series modeling is for the control of a system, its operation is usually based on forecasting. The term *forecasting* is used more frequently in recent time series literature than the term *prediction*. However, most forecasting results are derived from a general theory of linear prediction developed by Kolmogorov (1939, 1941), Wiener (1949), Kalman (1960), Yaglom (1962), and Whittle (1983), among others.

Consider the general ARIMA($p$, $d$, $q$) model

$$\phi(B)(1-B)^d Z_t = \theta(B)a_t \tag{5.1.1}$$

where $\phi(B) = (1-\phi_1 B - \cdots - \phi_p B^p)$, $\theta(B) = (1-\theta_1 B - \cdots - \theta_q B^q)$, and the series $a_t$ is a Gaussian $N(0, \sigma_a^2)$ white noise process. The deterministic trend parameter $\theta_0$ is omitted for simplicity but no loss of generality. Equation (5.1.1) is one of the most commonly used models in forecasting applications. We discuss the minimum mean square error forecast of this model for both cases when $d = 0$ and $d \neq 0$.



## 5.2 MINIMUM MEAN SQUARE ERROR FORECASTS

### 5.2.1 Minimum Mean Square Error Forecasts for ARMA Models

To derive the minimum mean square error forecasts, we first consider the case when $d = 0$, i.e., the stationary ARMA model

$$\phi(B)Z_t = \theta(B)a_t. \tag{5.2.1}$$

Because the model is stationary, we can rewrite it in a moving average representation,

$$\begin{aligned} Z_t &= \psi(B)a_t \\ &= a_t + \psi_1 a_{t-1} + \psi_2 a_{t-2} + \cdots \end{aligned} \tag{5.2.2}$$

where

$$\psi(B) = \sum_{j=0}^{\infty} \psi_j B^j = \frac{\theta(B)}{\phi(B)} \tag{5.2.3}$$

and $\psi_0 = 1$. For $t = n + l$, we have

$$Z_{n+l} = \sum_{j=0}^{\infty} \psi_j a_{n+l-j}. \tag{5.2.4}$$

Suppose at time $t = n$ we have the observations $Z_n, Z_{n-1}, Z_{n-2}, \ldots$ and wish to forecast $l$-step ahead of future value $Z_{n+l}$ as a linear combination of the observations $Z_n, Z_{n-1}, Z_{n-2}, \ldots$. Since $Z_t$ for $t = n, n-1, n-2, \ldots$ can all be written in the form of (5.2.2), we can let the minimum mean square error forecast $\hat{Z}_n(l)$ of $Z_{n+l}$ be

$$\hat{Z}_n(l) = \psi_l^* a_n + \psi_{l+1}^* a_{n-1} + \psi_{l+2}^* a_{n-2} + \cdots \tag{5.2.5}$$

where the $\psi_j^*$ are to be determined. The mean square error of the forecast is

$$E(Z_{n+l} - \hat{Z}_n(l))^2 = \sigma_a^2 \sum_{j=0}^{l-1} \psi_j^2 + \sigma_a^2 \sum_{j=0}^{\infty} \left[\psi_{l+j} - \psi_{l+j}^*\right]^2,$$

which is easily seen to be minimized when $\psi_{l+j}^* = \psi_{l+j}$. Hence,

$$\hat{Z}_n(l) = \psi_l a_n + \psi_{l+1} a_{n-1} + \psi_{l+2} a_{n-2} + \cdots. \tag{5.2.6}$$

But using (5.2.4) and the fact that

$$E(a_{n+j} \mid Z_n, Z_{n-1}, \ldots) = \begin{cases} 0, & j > 0, \\ a_{n+j}, & j \le 0, \end{cases}$$

we have

$$E(Z_{n+l} \mid Z_n, Z_{n-1}, \ldots) = \psi_l a_n + \psi_{l+1} a_{n-1} + \psi_{l+2} a_{n-2} + \cdots.$$

Thus, the minimum mean square error forecast of $Z_{n+l}$ is given by its conditional expectation. That is,

$$\hat{Z}_n(l) = E(Z_{n+l} \mid Z_n, Z_{n-1}, \ldots). \tag{5.2.7}$$

$\hat{Z}_n(l)$ is usually read as the $l$-step ahead of the forecast of $Z_{n+l}$ at the forecast origin $n$.

The forecast error is

$$e_n(l) = Z_{n+l} - \hat{Z}_n(l) = \sum_{j=0}^{l-1} \psi_j a_{n+l-j}. \tag{5.2.8}$$

Because $E(e_n(l) \mid Z_t, t \le n) = 0$, the forecast is unbiased with variance

$$\mathrm{Var}(\hat{Z}_n(l)) = \mathrm{Var}(e_n(l)) = \sigma_a^2 \sum_{j=0}^{l-1} \psi_j^2. \tag{5.2.9}$$

For a normal process, the $(1-\alpha)100\%$ forecast limits are

$$\hat{Z}_n(l) \pm N_{\frac{\alpha}{2}} \left[1 + \sum_{j=1}^{l-1} \psi_j^2\right]^{1/2} \sigma_a \tag{5.2.10}$$

where $N_{\alpha/2}$ is the standard normal deviate such that $P(N > N_{\alpha/2}) = \alpha/2$.

The forecast error $e_n(l)$ as shown in (5.2.8) is a linear combination of the future random shocks entering the system after time $n$. Specifically, the one-step ahead forecast error is

$$e_n(1) = Z_{n+1} - \hat{Z}_n(1) = a_{n+1}. \tag{5.2.11}$$

Thus, the one-step ahead forecast errors are independent. This implies that $\hat{Z}_n(1)$ is indeed the best forecast of $Z_{n+1}$. Otherwise, if one-step ahead forecast errors are correlated, then one can calculate the forecast $\hat{a}_{n+1}$ of $a_{n+1}$ from the available errors $a_n, a_{n-1}, a_{n-2} \ldots$ and hence improve the forecast of $Z_{n+1}$ by simply using $\hat{Z}_n(1) + \hat{a}_{n+1}$ as the forecast. However, the forecast errors for longer lead times are correlated. This is true for the forecast errors

$$e_n(l) = Z_{n+l} - \hat{Z}_n(l) = a_{n+l} + \psi_1 a_{n+l-1} + \cdots + \psi_{l-1} a_{n+1} \tag{5.2.12}$$

and

$$e_{n-j}(l) = Z_{n+l-j} - \hat{Z}_{n-j}(l) = a_{n+l-j} + \psi_1 a_{n+l-j-1} + \cdots + \psi_{l-1} a_{n-j+1}, \tag{5.2.13}$$

which are made at the same lead time $l$ but different origins $n$ and $n-j$ for $j < l$. It is also true for the forecast errors for different lead times made from the same time origin. For example,

$$\mathrm{Cov}[e_n(2), e_n(1)] = E[(a_{n+2} + \psi_1 a_{n+1})(a_{n+1})] = \psi_1 \sigma_a^2. \tag{5.2.14}$$



### 5.2.2 Minimum Mean Square Error Forecasts for ARIMA Models

We now consider the general nonstationary ARIMA($p$, $d$, $q$) model with $d \neq 0$, i.e.,

$$\phi(B)(1-B)^d Z_t = \theta(B) a_t, \tag{5.2.15}$$

where $\phi(B) = (1 - \phi_1 B - \cdots - \phi_p B^{p})$ is a stationary AR operator and $\theta(B) = (1 - \theta_1 B - \cdots - \theta_q B^q)$ is an invertible MA operator, respectively. It is interesting to note that although for this process the mean and the second order moments such as the variance and the autocovariance functions vary over time, as shown in Chapter 4, the complete evolution of the process is completely determined by a finite number of fixed parameters. Hence, we can view the forecast of the process as the estimation of a function of these parameters and obtain the minimum mean square error forecast using a Bayesian argument. It is well known that using this approach with respect to the mean square error criterion, which corresponds to a squared loss function, when the series is known up to time $n$, the optimal forecast of $Z_{n+l}$ is given by its conditional expectation $E(Z_{n+l} \mid Z_n, Z_{n-1}, \ldots)$. The minimum mean square error forecast for the stationary ARMA model discussed earlier is, of course, a special case of the forecast for the ARIMA($p$, $d$, $q$) model with $d = 0$.

To derive the variance of the forecast for the general ARIMA model, we rewrite the model at time $t + l$ in an AR representation that exists because the model is invertible. Thus,

$$\pi(B) Z_{t+l} = a_{t+l}, \tag{5.2.16}$$

where

$$\pi(B) = 1 - \sum_{j=1}^{\infty} \pi_j B^j = \frac{\phi(B)(1-B)^d}{\theta(B)}, \tag{5.2.17}$$

or equivalently

$$Z_{t+l} = \sum_{j=1}^{\infty} \pi_j Z_{t+l-j} + a_{t+l}. \tag{5.2.18}$$

Following Wegman (1986), we apply the operator

$$1 + \psi_1 B + \cdots + \psi_{l-1} B^{l-1}$$

to (5.2.18) and obtain

$$\sum_{j=0}^{\infty} \sum_{k=0}^{l-1} \pi_j \psi_k Z_{t+l-j-k} + \sum_{k=0}^{l-1} \psi_k a_{t+l-k} = 0 \tag{5.2.19}$$

where $\pi_0 = -1$ and $\psi_0 = 1$. It can be easily shown that

$$\begin{aligned}\sum_{j=0}^{\infty}\sum_{k=0}^{l-1} \pi_j \psi_k Z_{t+l-j-k} \\ = \pi_0 Z_{t+l} + \sum_{m=1}^{l-1}\sum_{i=0}^{m} \pi_{m-i}\psi_i Z_{t+l-m} + \sum_{j=1}^{\infty}\sum_{i=0}^{l-1} \pi_{l-1+j-i}\psi_i Z_{t-j+1}.\end{aligned} \tag{5.2.20}$$

Choosing $\psi$ weights so that

$$\sum_{i=0}^{m} \pi_{m-i}\psi_i = 0, \qquad \text{for } m = 1, 2, \ldots, l-1, \tag{5.2.21}$$

we have

$$Z_{t+l} = \sum_{j=1}^{\infty} \pi_j^{(l)} Z_{t-j+1} + \sum_{i=0}^{l-1} \psi_i a_{t+l-i} \tag{5.2.22}$$

where

$$\pi_j^{(l)} = \sum_{i=0}^{l-1} \pi_{l-1+j-i}\psi_i. \tag{5.2.23}$$

Thus, given $Z_t$, for $t \le n$, we have

$$\begin{aligned}\hat{Z}_n(l) &= E(Z_{n+l} \mid Z_t, t \le n) \\ &= \sum_{j=1}^{\infty} \pi_j^{(l)} Z_{t-j+1},\end{aligned} \tag{5.2.24}$$

because $E(a_{n+j} \mid Z_t, t \le n) = 0$, for $j > 0$. The forecast error is

$$\begin{aligned}e_n(l) &= Z_{n+l} - \hat{Z}_n(l) \\ &= \sum_{j=0}^{l-1} \psi_j a_{n+l-j}\end{aligned} \tag{5.2.25}$$

where the $\psi_j$ weights, by (5.2.21), can be calculated recursively from the $\pi_j$ weights as follows:

$$\psi_j = \sum_{i=0}^{j-1} \pi_{j-i}\psi_i, \qquad j = 1, \ldots, l-1. \tag{5.2.26}$$

Note that (5.2.25) is exactly the same as (5.2.8). Hence, the results given in (5.2.7) through (5.2.14) hold for both stationary and nonstationary ARMA models.

For a stationary process, $\lim_{l\to\infty} \sum_{j=0}^{l-1} \psi_j^2$ exists. Hence, from (5.2.10), the eventual forecast limits approach two horizontally parallel lines as shown in Figure 5.1(a). For a nonstationary process, $\lim_{l\to\infty} \sum_{j=0}^{l-1} \psi_j^2$ does not exist. In fact, $\sum_{j=0}^{l-1} \psi_j^2$ increases and becomes unbounded as $l \to \infty$. Thus, the forecast limits in this case become wider and wider as the forecast lead $l$ becomes larger and larger, as shown in Figure 5.1(b). The practical implication of the latter case is that the forecaster becomes less certain about the result as the forecast lead time gets larger. For more discussion on the properties of the mean square error forecasts, see Shaman (1983).



## 5.3 COMPUTATION OF FORECASTS

From the result that the minimum mean square error forecasts $\hat{Z}_n(l)$ of $Z_{n+l}$ at the forecast origin $n$ is given by the conditional expectation

$$\hat{Z}_n(l) = E(Z_{n+l} \mid Z_n, Z_{n-1}, \ldots),$$

we can easily obtain the actual forecasts by directly using the difference equation form of the model. Let

$$\Psi(B) = \phi(B)(1-B)^d = (1 - \Psi_1 B - \cdots - \Psi_{p+d}B^{p+d}).$$

The general ARIMA($p$, $d$, $q$) model (5.2.15) can be written as the following difference equation form:

$$(1 - \Psi_1 B - \cdots - \Psi_{p+d}B^{p+d})Z_t = (1 - \theta_1 B - \cdots - \theta_q B^q)a_t. \tag{5.3.1}$$

For $t = n + l$, we have

$$\begin{aligned}Z_{n+l} = \Psi_1 Z_{n+l-1} + \Psi_2 Z_{n+l-2} + \cdots + \Psi_{p+d} Z_{n+l-p-d} \\ + a_{n+l} - \theta_1 a_{n+l-1} - \cdots - \theta_q a_{n+l-q}.\end{aligned} \tag{5.3.2}$$

Taking the conditional expectation at time origin $n$, we obtain

$$\begin{aligned}\hat{Z}_n(l) = \Psi_1 \hat{Z}_n(l-1) + \cdots + \Psi_{p+d}\hat{Z}_n(l-p-d) \\ + \hat{a}_n(l) - \theta_1 \hat{a}_n(l-1) - \cdots - \theta_q \hat{a}_n(l-q)\end{aligned} \tag{5.3.3}$$

where

$$\begin{aligned}&\hat{Z}_n(j) = E(Z_{n+j} \mid Z_n, Z_{n-1}, \ldots), && j \ge 1, \\ &\hat{Z}_n(j) = Z_{n+j}, && j \le 0, \\ &\hat{a}_n(j) = 0, && j \ge 1,\end{aligned}$$

and

$$\hat{a}_n(j) = Z_{n+j} - \hat{Z}_{n+j-1}(1) = a_{n+j}, j \le 0.$$

**Example 5.1** To illustrate the above results, we consider the $l$-step ahead forecast $\hat{Z}_n(l)$ of $Z_{n+l}$ for the following ARIMA(1, 0, 1) or ARMA(1, 1) model:

$$(1 - \phi B)(Z_t - \mu) = (1 - \theta B)a_t. \tag{5.3.4}$$

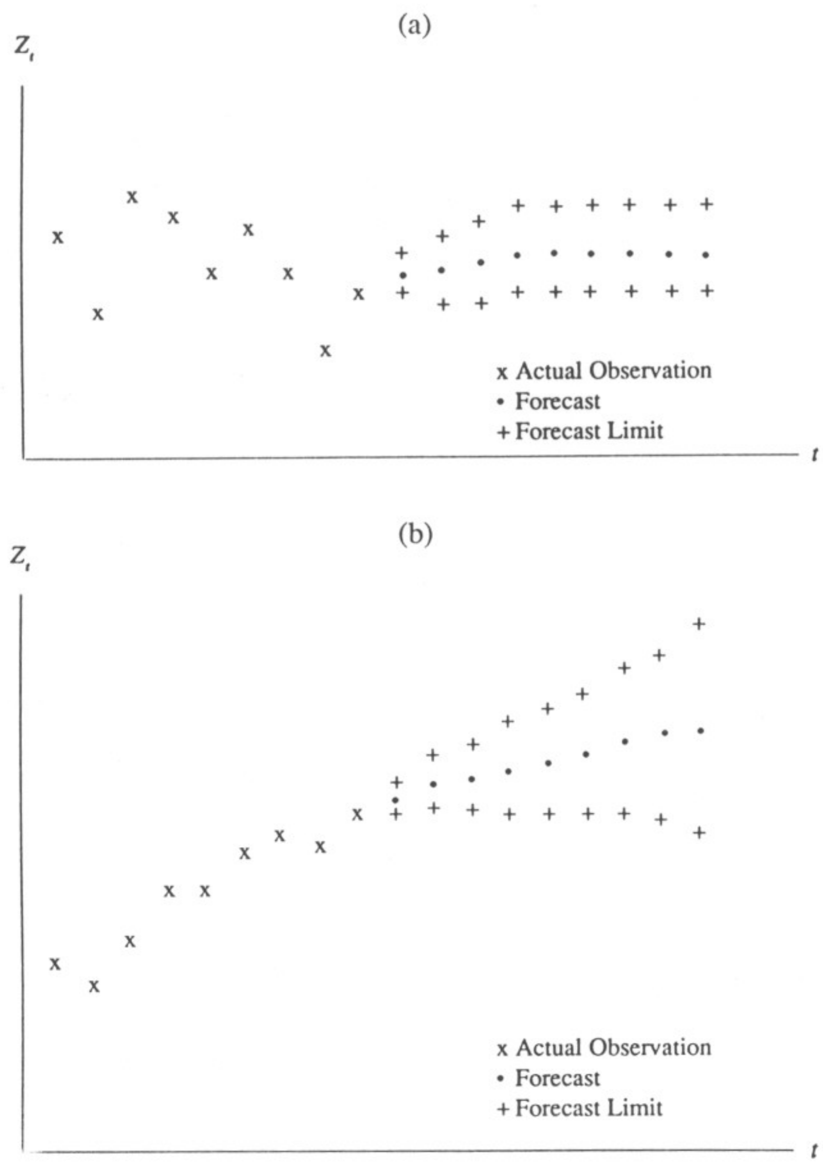


Fig. 5.1 (a) Forecasts for stationary processes. (b) Forecasts for nonstationary processes.



1. Calculate the forecast $\hat{Z}_n(l)$ as the conditional expectation from the difference equation form.

   For $t = n + l$, we can rewrite the above model in (5.3.4) as

$$Z_{n+l} = \mu + \phi(Z_{n+l-1} - \mu) + a_{n+l} - \theta a_{n+l-1}. \qquad (5.3.5)$$

   Hence

$$\hat{Z}_n(1) = \mu + \phi(Z_n - \mu) - \theta a_n \qquad (5.3.6a)$$

   and

$$\begin{aligned}\hat{Z}_n(l) &= \mu + \phi[\hat{Z}_n(l-1) - \mu] \\ &= \mu + \phi^l(Z_n - \mu) - \phi^{l-1}\theta a_n, \qquad l \geq 2.\end{aligned} \qquad (5.3.6b)$$

2. Calculate the forecast variance $\text{Var}(\hat{Z}_n(l)) = \sigma_a^2 \sum_{j=0}^{l-1} \psi_j^2$.

   When $|\phi| < 1$, we can calculate the $\psi$ weights from the moving average representation (5.2.2) with $\phi(B) = (1 - \phi B)$ and $\theta(B) = (1 - \theta B)$. That is,

$$(1 - \phi B)(1 + \psi_1 B + \psi_2 B^2 + \cdots) = (1 - \theta B). \qquad (5.3.7)$$

   Equating the coefficients of $B^j$ on both sides gives

$$\psi_j = \phi^{j-1}(\phi - \theta), \qquad j \geq 1. \qquad (5.3.8)$$

   Hence, the forecast variance becomes

$$\text{Var}(\hat{Z}_n(l)) = \sigma_a^2 \left\{ 1 + \sum_{j=1}^{l-1} [\phi^{j-1}(\phi - \theta)]^2 \right\}, \qquad (5.3.9)$$

   which converges to $\sigma_a^2[1 + (\phi - \theta)^2/(1 - \phi^2)]$.

When $\phi = 1$, which corresponds to taking the first difference, the model in (5.3.4) becomes an IMA(1, 1) process

$$(1 - B)Z_t = (1 - \theta B)a_t, \qquad (5.3.10)$$

where we note that $(1 - B)\mu = 0$. To calculate the $\psi$ weights needed for the forecast variance, since the MA representation does not exist, we first rewrite (5.3.10) in an AR form when time equals $t + l$, i.e.,

$$\pi(B)Z_{t+l} = a_{t+l}$$

where

$$\pi(B) = 1 - \pi_1 B - \pi_2 B^2 - \cdots = \frac{(1 - B)}{(1 - \theta B)}$$

or

$$(1 - B) = 1 - (\pi_1 + \theta)B - (\pi_2 - \pi_1\theta)B^2 - (\pi_3 - \pi_2\theta)B^3 - \cdots$$

Equating the coefficients of $B^j$ on both sides gives

$$\pi_j = (1 - \theta)\theta^{j-1}, \qquad j \geq 1. \qquad (5.3.11)$$

Now, applying (5.2.26) we obtain

$$\psi_1 = \pi_1 = (1-\theta),$$
$$\psi_2 = \pi_2 + \pi_1\psi_1 = (1-\theta)\theta + (1-\theta)^2 = (1-\theta),$$
$$\vdots$$

That is, we have $\psi_j = (1-\theta)$, $1 \le j \le l-1$. Thus, the variance of $\hat{Z}_n(l)$, from (5.2.9), becomes

$$\operatorname{Var}(\hat{Z}_n(l)) = \sigma_a^2[1 + (l-1)(1-\theta)^2], \tag{5.3.12}$$

which approaches $+\infty$ as $l \to \infty$.

As expected, (5.3.12) is the limiting case of (5.3.9) when $\phi \to 1$. Thus, when $\phi$ is close to 1, the choice between a stationary ARMA(1, 1) model and a nonstationary IMA(1, 1) model has very different implications for forecasting. This can be seen even more clearly from the limiting behavior of the $l$-step ahead forecast $\hat{Z}_n(l)$ in (5.3.6b). For $|\phi| < 1$, $\hat{Z}_n(l)$ approaches the mean, $\mu$, of the process as $l \to \infty$. When $\phi \to 1$, the first equation of (5.3.6b) implies that $\hat{Z}_n(l) = \hat{Z}_n(l-1)$ for all $l$. That is, the forecast is free to wander, with no tendency for the values to remain clustered around a fixed level.

## 5.4 THE ARIMA FORECAST AS A WEIGHTED AVERAGE OF PREVIOUS OBSERVATIONS

Recall that we can always represent an invertible ARIMA model in an autoregressive representation. In this representation, $Z_t$ is expressed as an infinite weighted sum of previous observations plus a random shock, i.e.,

$$Z_{n+l} = \sum_{j=1}^{\infty} \pi_j Z_{n+l-j} + a_{n+l}, \tag{5.4.1}$$

or equivalently

$$\pi(B)Z_{n+l} = a_{n+l},$$

where

$$\pi(B) = 1 - \sum_{j=1}^{\infty} \pi_j B^j = \frac{\phi(B)(1-B)^d}{\theta(B)}. \tag{5.4.2}$$

Thus,

$$\hat{Z}_n(l) = \sum_{j=1}^{\infty} \pi_j \hat{Z}_n(l-j), \qquad l \ge 1. \tag{5.4.3}$$



By repeated substitutions, we see that $\hat{Z}_n(l)$ can be expressed as a weighted sum of current and past observations $Z_t$, for $t \le n$. For example,

$$\begin{aligned}
\hat{Z}_n(1) &= \pi_1 Z_n + \pi_2 Z_{n-1} + \pi_3 Z_{n-2} + \cdots \\
&= \sum_{j=1}^{\infty} \pi_j Z_{n+1-j} \\
\hat{Z}_n(2) &= \pi_1 \hat{Z}_n(1) + \pi_2 Z_n + \pi_3 Z_{n-1} + \cdots \\
&= \pi_1 \sum_{j=1}^{\infty} \pi_j Z_{n+1-j} + \sum_{j=1}^{\infty} \pi_{j+1} Z_{n+1-j} \\
&= \sum_{j=1}^{\infty} \pi_j^{(2)} Z_{n+1-j}
\end{aligned}$$

where

$$\pi_j^{(2)} = \pi_1\pi_j + \pi_{j+1}, \qquad j \ge 1.$$

More generally, it can be shown by successive substitutions that

$$\hat{Z}_n(l) = \sum_{j=1}^{\infty} \pi_j^{(l)} Z_{n+1-j}, \tag{5.4.4}$$

where

$$\pi_j^{(l)} = \pi_{j+l-1} + \sum_{i=1}^{l-1} \pi_i \pi_j^{(l-i)}, \qquad l > 1, \tag{5.4.5}$$

and

$$\pi_j^{(1)} = \pi_j.$$

Thus, many smoothing results, such as moving averages and exponential smoothing, are special cases of ARIMA forecasting. ARIMA models provide a very natural and optimal way to obtain the required weights for forecasting. The user does not have to specify either the number or the form of weights as required in the moving average method and the exponential smoothing method. It should also be noted that the ARIMA forecasts are minimum mean square error forecasts. This optimal property is not shared in general by the moving average and the exponential smoothing methods.

For an invertible process, the $\pi$ weights in (5.4.3) or (5.4.4) form a convergent series. This implies that for a given degree of accuracy, $\hat{Z}_n(l)$ depends only on a finite number of recent observations. The associated $\pi$ weights provide very useful information for many important managerial decisions.

**Example 5.2** For the ARMA(1, 1) model in (5.3.4) with $|\theta| < 1$, we have from (5.4.2) with $d = 0$

$$(1-\phi B) = (1-\pi_1 B - \pi_2 B^2 - \cdots)(1-\theta B) \tag{5.4.6}$$

or

$$(1-\phi B) = 1-(\pi_1+\theta)B-(\pi_2-\pi_1\theta)B^2-(\pi_3-\pi_2\theta)B^3-\cdots.$$

Equating the coefficients or $B^j$ on both sides gives

$$\pi_j = (\phi-\theta)\theta^{j-1}, \qquad j \ge 1. \tag{5.4.7}$$

Assuming that $\mu = 0$, from (5.4.3), we have

$$\hat{Z}_n(l) = \sum_{j=1}^{\infty}(\phi-\theta)\theta^{j-1}\hat{Z}_n(l-j). \tag{5.4.8}$$

When $l = 1$, we get

$$\hat{Z}_n(1) = (\phi-\theta)\sum_{j=1}^{\infty}\theta^{j-1}Z_{n+1-j}. \tag{5.4.9}$$

For $l = 2$, from (5.4.4) and (5.4.5),

$$\begin{aligned}\hat{Z}_n(2) &= \sum_{j=1}^{\infty}\pi_j^{(2)}Z_{n+1-j}\\ &= \sum_{j=1}^{\infty}[\pi_{j+1}+\pi_1\pi_j]Z_{n+1-j}\\ &= \sum_{j=1}^{\infty}[(\phi-\theta)\theta^{j}+(\phi-\theta)^2\theta^{j-1}]Z_{n+1-j}\\ &= \phi(\phi-\theta)\sum_{j=1}^{\infty}\theta^{j-1}Z_{n+1-j}.\end{aligned} \tag{5.4.10}$$

Again $\hat{Z}_n(2)$ is a weighted average of the previous observations with the $\pi_j^{(2)}$ weights given by $\pi_j^{(2)} = \phi(\phi-\theta)\theta^{j-1}$ for $j \ge 1$. Note that comparing (5.4.9) and (5.4.10), we see that

$$\hat{Z}_n(2) = \phi\hat{Z}_n(1), \tag{5.4.11}$$

which as expected, agrees with (5.3.6b) with $l = 2$ and $\mu = 0$.

To see the implications of these weights, let us examine (5.4.9) more carefully. For $|\theta| < 1$, $\pi_j = \theta^{j-1}\ (\phi-\theta)$ converges to 0 as $j \to 0$. This implies that the more recent observations have a greater influence on the forecast. For $|\theta| \ge 1$, although the model is still stationary, its AR representation does not exist. To see the trouble, note that if $|\theta| > 1$, the $\pi$ weights rapidly approach $-\infty$ or $+\infty$ as $j$ increases. This implies that the more remote past observations have a much greater influence on the forecast. When $|\theta| = 1$, the $\pi$ weights become $\pi_j = (\phi - 1)$ for $\theta = 1$, and $\pi_j = (-1)^j(1+\phi)$ for $\theta = -1$, which have the same absolute value for all $j$. This means that all past and present observations are equally important in their effect on the forecast. Thus, a meaningful forecast can be derived only from an invertible process. The models corresponding to $|\theta| > 1$ and $|\theta| = 1$ are both noninvertible.



## 5.5 UPDATING FORECASTS

Recall that when a time series $Z_t$ is available for $t \le n$, the $l$-step ahead minimum mean square forecast error for the forecast origin $n$ using the general ARIMA model is obtained in (5.2.25), which, for convenience, is listed again in the following:

$$e_n(l) = Z_{n+l} - \hat{Z}_n(l) = \sum_{j=0}^{l-1}\psi_j a_{n+l-j}. \tag{5.5.1}$$

In particular, the one-step ahead forecast error is

$$e_n(1) = Z_{n+1} - \hat{Z}_n(1) = a_{n+1}. \tag{5.5.2}$$

Clearly, the result can be reversed, giving

$$Z_n - \hat{Z}_{n-1}(1) = a_n. \tag{5.5.3}$$

From (5.5.1), it is clear that

$$e_{n-1}(l+1) = e_n(l) + \psi_l a_n, \tag{5.5.4}$$

where

$$e_{n-1}(l+1) = Z_{n+l} - \hat{Z}_{n-1}(l+1),$$

and

$$e_n(l) = Z_{n+l} - \hat{Z}_n(l).$$

Hence, after substituting, rearranging, and using (5.5.3), we have the following updating equation:

$$\hat{Z}_n(l) = \hat{Z}_{n-1}(l+1) + \psi_l[Z_n - \hat{Z}_{n-1}(1)], \tag{5.5.5}$$

or equivalently

$$\hat{Z}_{n+1}(l) = \hat{Z}_n(l+1) + \psi_l[Z_{n+1} - \hat{Z}_n(1)]. \tag{5.5.6}$$

The updated forecast is obtained by adding to the previous forecast a constant multiple $\psi_l$ of the one-step ahead forecast error $a_{n+1} = Z_{n+1} - \hat{Z}_n(1)$. This is certainly sensible. For example, when the value $Z_{n+1}$ becomes available and is found to be higher than the previous forecast, (resulting in a positive forecast error $a_{n+1} = Z_{n+1} - \hat{Z}_n(1)$), we will naturally modify the forecast $\hat{Z}_n(l+1)$ made earlier by proportionally adding a constant multiple of this error.

## 5.6 EVENTUAL FORECAST FUNCTIONS

Let the ARIMA model be

$$\Psi(B)Z_t = \theta(B)a_t$$

where $\Psi(B) = \phi(B)(1-B)^d$. Recall from Equation (5.3.3) that

$$\hat{Z}_n(l) = \Psi_1\hat{Z}_n(l-1) + \Psi_2\hat{Z}_n(l-2) + \cdots + \Psi_{p+d}\hat{Z}_n(l-p-d) + \hat{a}_n(l) - \theta_1\hat{a}_n(l-1) - \cdots - \theta_q\hat{a}_n(l-q).$$

When $l > q$, $\hat{Z}_n(l)$ becomes

$$\hat{Z}_n(l) = \Psi_1\hat{Z}_n(l-2) + \cdots + \Psi_{p+d}\hat{Z}_n(l-p-d)$$

or

$$\Psi(B)\hat{Z}_n(l) = 0. \tag{5.6.1}$$

That is, $\hat{Z}_n(l)$ for $l > q$ satisfies the homogeneous difference equation of order $(p+d)$. Let $\Psi(B) = \prod_{i=1}^{N}(1-R_iB)^{m_i}$ with $\sum_{i=1}^{N} m_i = (p+d)$. Then from Theorem 2.7.1, the general solution is given by

$$\hat{Z}_n(l) = \sum_{i=1}^{N}\left(\sum_{j=0}^{m_i-1} C_{ij}^{(n)} l^j\right) R_i^l \tag{5.6.2}$$

for $l > (q-p-d)$, where the $C_{ij}^{(n)}$ are constants that depend on time origin $n$. For a given forecast origin $n$, they are fixed constants for all forecast lead times $l$. The constants change when the forecast origin is changed.

The solution in (5.6.2) is called the eventual forecast function because it holds only for $l > (q-p-d)$. When $q < (p+d)$, the eventual forecast function actually holds for all lead times $l > 0$. In general, the function is the unique curve that passes through the $(p+d)$ values given by $\hat{Z}_n(q)$, $\hat{Z}_n(q-1)$, ..., $\hat{Z}_n(q-p-d+1)$, where $\hat{Z}_n(-j) = Z_{n-j}$ for $j \geq 0$. For the ARIMA($p$, $d$, $q$) model with $q = 0$, i.e., the ARIMA($p$, $d$, 0) model, the function passes through the points $Z_n$, $Z_{n-1}$, ..., and $Z_{n-p-d+1}$.

**Example 5.3** For the ARIMA(1, 0, 1) model given in (5.3.4),

$$(1-\phi B)(Z_t - \mu) = (1-\theta B)a_t,$$

the forecast $\hat{Z}_n(l)$ satisfies the difference equation $(1-\phi B)(\hat{Z}_t(l) - \mu) = 0$ for $l > 1$. The eventual forecast function is given by

$$\hat{Z}_n(l) - \mu = C_1^{(n)}\phi^l$$



or

$$\hat{Z}_n(l) = \mu + C_1^{(n)}\phi^l \tag{5.6.3}$$

for $l > (q-p-d) = 0$ and constant $C_1^{(n)}$. The eventual forecast function passes through $\hat{Z}_n(1)$. As $l \to \infty$, $\hat{Z}_n(l)$ approaches the mean $\mu$ of the stationary process as expected.

**Example 5.4** Consider the ARIMA(1, 1, 1) model,

$$(1-\phi B)(1-B)Z_t = (1-\theta B)a_t.$$

The eventual forecast function is the solution of $(1-\phi B)(1-B)\hat{Z}_n(l) = 0$ for $l > 1$, and is given by

$$\hat{Z}_n(l) = C_1^{(n)} + C_2^{(n)}\phi^l \tag{5.6.4}$$

for $l > (q-p-d) = -1$. The function passes through the first forecast $\hat{Z}_n(1)$ and the last observation $Z_n$.

**Example 5.5** Consider the ARIMA(0, 2, 1) model,

$$(1-B)^2 Z_t = (1-\theta B)a_t.$$

The eventual forecast function is the solution of $(1-B)^2\hat{Z}_n(l) = 0$ for $l > 1$, and from (5.6.2) it is given by

$$\hat{Z}_n(l) = C_1^{(n)} + C_2^{(n)} l \tag{5.6.5}$$

for $l > (q-p-d) = -1$. The function is a straight line passing through $\hat{Z}_n(1)$ and $Z_n$.

## 5.7 A NUMERICAL EXAMPLE

As a numerical example, consider the AR(1) model,

$$(1-\phi B)(Z_t - \mu) = a_t$$

with $\phi = .6$, $\mu = 9$, and $\sigma_a^2 = .1$. Suppose that we have the observations $Z_{100} = 8.9$, $Z_{99} = 9$, $Z_{98} = 9$, $Z_{97} = 9.6$, and want to forecast $Z_{101}$, $Z_{102}$, $Z_{103}$, and $Z_{104}$ with their associated 95% forecast limits. We proceed as follows:

1. The AR(1) model can be written as

$$Z_t - \mu = \phi(Z_{t-1} - \mu) + a_t,$$

and the general form of the forecast equation is

$$\begin{aligned}\hat{Z}_t(l) &= \mu + \phi(\hat{Z}_t(l-1) - \mu)\\ &= \mu + \phi^l(Z_t - \mu), \qquad l \geq 1.\end{aligned} \tag{5.7.1}$$

Thus,

$$\hat{Z}_{100}(1) = 9 + .6(8.9 - 9) = 8.94,$$
$$\hat{Z}_{100}(2) = 9 + (.6)^2(8.9 - 9) = 8.964,$$
$$\hat{Z}_{100}(3) = 9 + (.6)^3(8.9 - 9) = 8.9784,$$
$$\hat{Z}_{100}(4) = 9 + (.6)^4(8.9 - 9) = 8.98704.$$

2. To obtain the forecast limits, we obtain the $\psi$ weights from the relationship

$$(1 - \phi B)(1 + \psi_1 B + \psi_2 B^2 + \cdots) = 1. \qquad (5.7.2)$$

That is,

$$\psi_j = \phi^j, \qquad j \geq 0. \qquad (5.7.3)$$

The 95% forecast limits for $Z_{101}$ from (5.2.10) are

$$8.94 \pm 1.96\sqrt{.1} \qquad \text{or} \qquad 8.320 < Z_{101} < 9.560.$$

The 95% forecast limits for $Z_{102}$ are

$$8.964 \pm 1.96\sqrt{1 + (.6)^2}\sqrt{.1} \qquad \text{or} \qquad 8.241 < Z_{102} < 9.687.$$

The 95% forecast limits for $Z_{103}$ and $Z_{104}$ can be obtained similarly. The results are shown in Figure 5.2.

3. Suppose now that the observation at $t = 101$ turns out to be $Z_{101} = 8.8$. Because $\psi_l = \phi^l = (.6)^l$, we can update the forecasts for $Z_{102}$, $Z_{103}$, and $Z_{104}$ by using (5.5.5) as follows.

$$\hat{Z}_{101}(1) = \hat{Z}_{100}(2) + \psi_1[Z_{101} - \hat{Z}_{100}(1)]$$
$$= 8.964 + .6(8.8 - 8.94) = 8.88$$
$$\hat{Z}_{101}(2) = \hat{Z}_{100}(3) + \psi_2[Z_{101} - \hat{Z}_{100}(1)]$$
$$= 8.9784 + (.6)^2(8.8 - 8.94) = 8.928$$
$$\hat{Z}_{101}(3) = \hat{Z}_{100}(4) + \psi_3[Z_{101} - \hat{Z}_{100}(1)]$$
$$= 8.98704 + (.6)^3(8.8 - 8.94) = 8.9568.$$

The earlier forecasts for $Z_{102}$, $Z_{103}$, and $Z_{104}$ made at $t = 100$ are adjusted downward due to the negative forecast error made for $Z_{101}$.

Forecasting discussed above is based on the assumption that the parameters are known in the model. In practice, the parameters are, of course, unknown and have to be estimated from the given observations $\{Z_1, Z_2, \ldots, Z_n\}$. However, with respect to the forecasting origin $t = n$, the estimates are known constants and hence the results remain the same under this conditional sense. Estimation of the model parameters is discussed in Chapter 7.



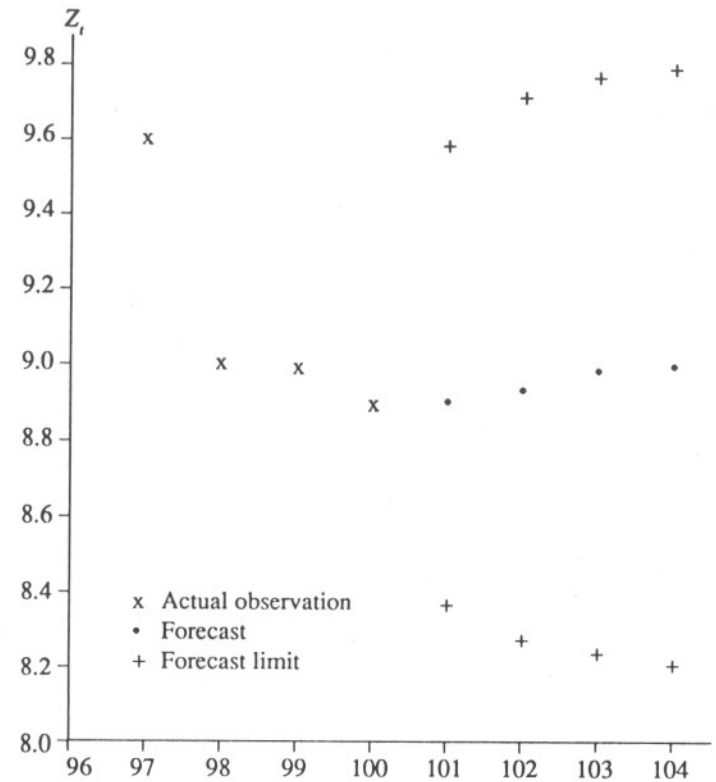


**Fig. 5.2** Forecasts with 95% forecast limits for an AR(1) process.

## Exercises

**5.1** For each of the following models:

(I) $(1 - \phi_1 B)(Z_t - \mu) = a_t$,

(II) $(1 - \phi_1 B - \phi_2 B^2)(Z_t - \mu) = a_t$,

(III) $(1 - \phi_1 B)(1 - B)Z_t = (1 - \theta_1 B)a_t$.

(a) Find the $l$-step ahead forecast $\hat{Z}_n(l)$ of $Z_{n+l}$.

(b) Find the variance of the $l$-step ahead forecast error for $l = 1$, 2, and 3.

**5.2** (a) Show that the covariance between forecast errors from different origins is given by

$$\mathrm{Cov}\left[e_n(l), e_{n-j}(l)\right] = \sigma_a^2 \sum_{i=j}^{l-1} \psi_i \psi_{i-j}, \qquad l > j.$$

(b) Show that the covariance between forecast errors from the same origin but with different lead times is given by

$$\text{Cov}[e_n(l), e_n(l+j)] = \sigma_a^2 \sum_{i=0}^{l-1} \psi_i \psi_{i+j}.$$

5.3 Consider the model

$$(1 - .68B)(1 - B)^2 Z_t = (1 - .75B + .34B^2)a_t.$$

(a) Compute and plot the correlations between the error of the forecast $\hat{Z}_t(5)$ and those of the forecasts $\hat{Z}_{t-j}(5)$ for $j = 1, 2, \ldots, 10$.
(b) Compute and plot the correlations between the error of the forecast $\hat{Z}_t(3)$ and those of $\hat{Z}_t(l)$ for $l = 1, 2, \ldots, 10$.

5.4 A sales series was fitted by the ARIMA(2, 1, 0) model

$$(1 - 1.4B + .7B^2)(1 - B)Z_t = a_t$$

where $\hat{\sigma}_a^2 = 58000$ and the last 5 observations are 560, 580, 640, 770, and 800.
(a) Calculate the forecasts of the next 3 observations.
(b) Find the 95% forecast limits for the forecasts in (a).
(c) Find the eventual forecast function.

5.5 Consider the IMA(1, 1) model

$$(1 - B)Z_t = (1 - \theta B)a_t.$$

(a) Write down the forecast equation that generates the forecasts.
(b) Find the 95% forecast limits produced by this model.
(c) Express the forecasts as a weighted average of previous observations.
(d) Discuss the connection of this model with the simple exponential smoothing method.

5.6 (a) Show that (5.2.23) and (5.4.5) are equivalent.
(b) Illustrate the equivalence of (5.2.23) and (5.4.5) using the model in Exercise 5.5

5.7 Consider an AR(2) model $(1-\phi_1 B-\phi_2 B^2)(Z_t-\mu) = a_t$, where $\phi_1 = 1.2$, $\phi_2 = -.6$, $\mu = 65$, and $\sigma_a^2 = 1$. Suppose we have the observations $Z_{76} = 60.4$, $Z_{77} = 58.9$, $Z_{78} = 64.7$, $Z_{79} = 70.4$, and $Z_{80} = 62.6$.
(a) Forecast $Z_{81}$, $Z_{82}$, $Z_{83}$, and $Z_{84}$.
(b) Find the 95% forecast limits for the forecasts in (a).
(c) Suppose that the observation at $t = 81$ turns out to be $Z_{81} = 62.2$. Find the updated forecasts for $Z_{82}$, $Z_{83}$, and $Z_{84}$.



5.8 Consider the model

$$(1 - .43B)(1 - B)Z_t = a_t$$

and the observations $Z_{49} = 33.4$ and $Z_{50} = 33.9$.
(a) Compute the forecast $\hat{Z}_{50}(l)$, for $l = 1, 2, \ldots, 10$, and their 90% forecast limits.
(b) What is the eventual forecast function for the forecasts made at $t = 50$?
(c) At time $t = 51$, $Z_{51}$ became known and equaled 34.1. Update the forecasts obtained in (a).

5.9 Consider the ARIMA(0, 1, 1) model

$$(1 - B)Z_t = (1 - .8B)a_t.$$

(a) Find the $\pi$ weights for the following AR representation:

$$Z_t = \bar{Z}_{t-1} + a_t$$

where $\bar{Z}_t = \sum_{j=1}^{\infty} \pi_j Z_{t-j}$, and show that $\sum_{j=1}^{\infty} \pi_j = 1$.
(b) Let $\hat{Z}_t(2) = \sum_{j=1}^{\infty} \pi_j^{(2)} Z_{t-j+1}$ be the two-step ahead forecasts of $Z_{t+2}$ at time $t$. Express $\pi_j^{(2)}$ in terms of the weights $\pi_j$.

5.10 Obtain the eventual forecast function for the following models:
(a) $(1 - .6B)Z_t = (1 - .8B + .3B^2)a_t$,
(b) $(1 - .3B)(1 - B) = .4 + a_t$,
(c) $(1 - 1.2B + .6B^2)(Z_t - 65) = a_t$,
(d) $(1 - B)^2 Z_t = (1 - .2B - .3B^2)a_t$,
(e) $(1 - .6B)(1 - B)^2 Z_t = (1 - .75B + .34B^2)a_t$.