Lecture7: Dynamic Regression Models

Outline

Short overview

(1) Introduction: Dynamic models
(2) Distributed Lag Models
(3) Autoregressive Distributed Lag Models
(4) Economic Models Leading to ARDL Specifications
(5) Estimation of Dynamic Regression Models

Introduction: Dynamic models

- We have concentrated on the statistical approach of fitting a time series model (from the class of ARMA models) to an observed series.

- Now, we will study the multivariate time series and focus on the analysis of the causal relationships between different economic variables.

- In economics we often face situations in which dynamic relationships are relevant (e.g. for policy makers), so it is important to study the specification and estimation of dynamic regression models.

Example

Dynamic relationship between: interest rate ad Gross Domestic product, inflation and interest rate, stock return, dividend, etc.

Introduction: Dynamic models

- Models are dynamic if we assume that:
  - The effect of a change in an independent variable does not just affect the current value of the dependent variable, but affects it for several periods (distributed lag models).
  - The dependent variable is also affected by its own past lags (models with lagged dependent variables).

- These dynamic regression models give rise to particular specification and estimation problems.

Distributed Lag Models

Definition

A Distributed lag model of order r, denoted $DL(r)$, is given by:

$$\begin{aligned} y_t &= \sum_{i=0}^{r} \beta_i x_{t-i} + \epsilon_t \\ &= B(L)x_t + \epsilon_t, \end{aligned}$$

where $B(L) = \beta_0 + \beta_1 L + ... + \beta_r L^r$.

- If $r = \infty$ we have an infinite lag model $DL(\infty)$.

Distributed Lag Models: Dynamic Multipliers

Definition

- The short-run multiplier of x on y (or short-run impact of x on y), denoted $B(0)$, is given by:

$$B(0) = \beta_0$$

- The long-run (equilibrium) multiplier of x on y (or long-run impact of x on y), denoted $B(1)$, is given by:

$$B(1) = \sum \beta_i.$$

Distributed Lag Models

- The model can be rewritten as:

$$y_t = B(1)\sum_{i=0}^{r} w_i x_{t-i} + \epsilon_t,$$

where $w_i = \beta_i / B(1)$ are the lag weights (relative impact of each lag observation).

- Consequently, we can define the following useful measures:

$$\begin{aligned}\text{Mean lag} &= \sum i w_i = \sum i\beta_i / B(1)\\ \text{Median lag} &= min\{q \mid \textstyle\sum_{i=0}^{q} w_i \geq 0.5\}.\end{aligned}$$

Autoregressive Distributed Lag Models-ARDL(p,r)

Definition

An autoregressive distributed lag model of order p and r, denoted ARDL(p,r), is given by:

$$y_t = \alpha + \sum_{i=1}^{p} \gamma_i y_{t-i} + \sum_{i=0}^{r} \beta_i x_{t-i} + \epsilon_t.$$

- It also can be written in the following form:

$$C(L)y_t = B(L)x_t + \epsilon_t,$$

where

$$C(L) = 1 - \gamma_1 L - ... - \gamma_p L^p,$$
$$B(L) = \beta_0 + \beta_1 L + ... + \beta_r L^r.$$

Autoregressive Distributed Lag Models

Remark

- *If* $r = 0 \Rightarrow ARDL(p, r)$ *simplifies to* $ARDL(p, 0)$ *(we still have contemporaneous effect).*
- *If* $p = 0 \Rightarrow ARDL(p, r)$ *simplifies to* $DL(r)$.

Remark

We will consider the case where x_t *as a single regressor, but the analysis can be generalized to vector of regressors.*

Autoregressive Distributed Lag Models

- As long as the lag polynomial $C(L)$ is invertible, the model has a DL(∞) representation:

$$y_t = C(L)^{-1}B(L)x_t + C(L)^{-1}\epsilon_t = D(L)x_t + \eta_t = \sum_{i=0}^{\infty} \delta_i x_{t-i} + \eta_t$$

where the δ_i's can be retrieved with the usual method of matching coefficients:

$$B(L) = C(L)D(L).$$

- The invertibility of $C(L)$ (i.e., the condition that all its characteristics roots lie outside the unit circle) is also called the stability condition.

ARDL - Dynamic Multipliers

Definition (Dynamic Multipliers for ARDL)

- The short-run multiplier (also impact or contemporaneous) of x on y, denoted m_0, is given by:

$$m_0 \equiv \partial y_t / \partial x_t = D(0) = \frac{B(0)}{C(0)} = \delta_0.$$

- The dynamic multiplier of order j of x on y, denoted m_j,:

$$m_j \equiv \frac{\partial y_t}{\partial x_{t-j}} = \delta_j. \text{ (note } \neq \beta_j)$$

- The long-run multiplier (also total) is given by:

$$m_T \equiv \sum_{j=0}^{\infty} m_j = D(1) = \sum_{j=0}^{\infty} \delta_j = \frac{B(1)}{C(1)}.$$

ARDL - Dynamic Multipliers

Example

- Consider the following $ARDL(1,1)$ model:

$$y_t = \alpha + \gamma_1 y_{t-1} + \beta_0 x_t + \beta_1 x_{t-1} + \epsilon_t$$

- The short-run multiplier in this model is:

$$m_0 = \partial y_t / \partial x_t = \beta_0$$

ARDL - Dynamic Multipliers

Example (cont.)

- The dynamic multipliers are:

$$
\begin{aligned}
m_1 &= \frac{\partial y_t}{\partial x_{t-1}} = \gamma_1 \frac{\partial y_{t-1}}{\partial x_{t-1}} + \beta_1 = \gamma_1 \beta_0 + \beta_1 \\
m_2 &= \frac{\partial y_t}{\partial x_{t-2}} = \gamma_1 \frac{\partial y_{t-1}}{\partial x_{t-2}} = \gamma_1 (\gamma_1 \beta_0 + \beta_1) \\
m_k &= \frac{\partial y_t}{\partial x_{t-k}} = \gamma_1 \frac{\partial y_{t-1}}{\partial x_{t-k}} = \gamma_1^{k-1} (\gamma_1 \beta_0 + \beta_1)
\end{aligned}
$$

Remark: As long as $|\gamma_1| < 1$,

$$m_k \to 0 \ \text{ as } \ k \to \infty.$$

ARDL - Dynamic Multipliers

Example (cont.)

- The long-run multiplier in this model is given by:

$$
\begin{aligned}
m_T &= \sum_{i=0}^{\infty} m_i = \frac{\partial y_t}{\partial x_t} + \frac{\partial y_{t+1}}{\partial x_t} + \frac{\partial y_{t+2}}{\partial x_t} + ... \\
&= \beta_0 + (\gamma_1\beta_0 + \beta_1) + \gamma_1(\gamma_1\beta_0 + \beta_1) + \gamma_1^2(\gamma_1\beta_0 + \beta_1) + ... \\
&= \beta_0(1 + \gamma_1 + \gamma_1^2 + ...) + \beta_1(1 + \gamma_1 + \gamma_1^2 + ...)
\end{aligned}
$$

Under $|\gamma_1| < 1$,

$$m_T = \frac{\beta_0}{1-\gamma_1} + \frac{\beta_1}{1-\gamma_1} = \frac{\beta_0 + \beta_1}{1-\gamma_1}.$$

ARDL - Dynamic Multipliers

Remark

This is also called the long-run multiplier (equilibrium), since

$$E(y_t) = \frac{\alpha}{1-\gamma_1} + \frac{\beta_0 + \beta_1}{1-\gamma_1} E(x_t) = \phi_0 + \phi_1 E(x_t).$$

From the above result, we can rewrite the $ARDL(1,1)$ *model in its error-correction representation:*

$$\Delta y_t = \beta_0 \Delta x_t - (1-\gamma_1)(y_{t-1} - \phi_0 - \phi_1 x_{t-1}) + \epsilon_t.$$

which links the variation in y *to its deviation from the equilibrium level* $(\phi_0 + \phi_1 x)$.

ARDL - Mean and Medial Lag

Definition

The mean and median lags are given by:

- Mean lag: $\frac{\sum_{j=0}^{\infty} j\ m_j}{m_T} = B'(1)/B(1) - C'(1)/C(1)$
- Median lag: $min\{q \mid \frac{\sum_{j=0}^{q} m_j}{m_T} \geq 0.5\}$.

ARDL - Mean and Medial Lag

Example

Find the above multipliers and lags in the following model:

$$y_t = 0.8y_{t-1} + 3x_t + \varepsilon_t.$$

Solution:

• Note that $C(L) = 1 - 0.8L$ and $B(L) = 3$. That is process y_t is ARDL(1,0)

• Further, we have that the roots of $C(L) = 0$ are $L = \frac{1}{0.8} = 1.25 > 1$. Hence model is stable.

• $D(L) = \frac{B(L)}{C(L)} = \frac{3}{1-0.8L} = 3(1 + 0.8L + 0.8L^2 + ...)$
$= 3 + 2.4L + 1.92L^2 + 1.54L^3 + ... = \delta_0 + \delta_1 L + \delta_2 L^2 + \delta_3 L^3 + ...$

ARDL - Mean and Medial Lag

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ARDL - Mean and Medial Lag

Example (cont.)

- The latest result in turn gives:
  - $m_T = D(1) = B(1)/C(1) = 3/(1-0.8) = 15$
  - Mean lag $= B'(1)/B(1) - C'(1)/C(1) = -\frac{-0.8}{1-0.8} = 4$
  - Define the cumulative multiplier as

$$m_0^q = \sum_{j=0}^{q} \delta_j$$

$$\begin{array}{cccccc} Lag(q) & 0 & 1 & 2 & 3 & \\ \frac{m_0^q}{m_T} & \frac{3}{15} & \frac{5.4}{15} & \frac{7.32}{15} & \frac{8.86}{15} & \Rightarrow \text{Median Lag} = 3 \end{array}$$

Estimation of Dynamic Regression Models

We have assumed that both y_t and x_t are stationary (we'll see later on what happens with non-stationary variables). However, the estimation of ARDL models using OLS face various (and serious) problems:

(1) Multicollinearity;

(2) Large number of parameters;

(3) Possible correlation between regressors and errors.

Estimation of Dynamic Regression Models

Solutions:

Problems **1** and **2** can be partly solved by choosing low values of p and r.

Problem **3** is more serious, since, in the presence of both lagged dependent variables and serially correlated errors, OLS estimators are generally inconsistent. This is due to the correlation between lagged dependent variable (which contains lagged errors) and the current error term (which also contains lagged errors). To solve this problem we can use the Instrumental Variables (IV) estimation.

Estimation of Dynamic Regression Models

A closer look at OLS drawbacks

- Consider the following model:

$$\begin{aligned} Y_t &= \alpha + \gamma Y_{t-1} + U_t \\ U_t &= \rho U_{t-1} + \epsilon_t. \end{aligned}$$

- It can be shown that:

$$plim(\hat{\gamma}) = \frac{plim \frac{1}{T} \sum y_t y_{t-1}}{plim \frac{1}{T} \sum y_{t-1}^2} = \frac{\gamma + \rho}{1 + \gamma\rho}.$$

Hence, plim$(\hat{\gamma}) = \gamma$ only if $\rho = 0$. Consequently, the OLS estimator of the coefficient of the lagged dependent variable is inconsistent if the errors are serially correlated.