Structural Breaks and Broken Trends

(Stock, Handbook of Econometrics)

Breaks in coefficients in time series regression

- TEST FOR A SINGLE BREAK DATE

$$\boxed{Y_t = \beta_t' X_t + \varepsilon_t}$$

- ε_t is a m.d.s w.r.t $\mathcal{F}_{t-1}(\varepsilon_{t-1}, X_{t-1}, \varepsilon_{t-2}, X_{t-2} \ldots)$
- $\underset{k \times 1}{X_t}$ are constant and/or $I(0)$

  with $E X_t X_t' = \Sigma_x$
- For convenience $E(\varepsilon_t^2 | \varepsilon_{t-1}, X_{t-1}, \varepsilon_{t-2}, X_{t-2} \ldots) = \underline{\underline{\sigma^2}}$
- $T^{-1} \sum_{s=1}^{[T\lambda]} X_s X_s' \xrightarrow{p} \lambda \Sigma_x$

  uniformly in λ for $\lambda \in [0,1]$

Note that X_{t-1} can include lagged dependent variables as long as they are $I(0)$ under H_0

$$\boxed{H_0: \beta_t = \beta} \quad \forall t$$

$$\left[H_a: \quad \beta_t = \beta \quad t \leq r \quad \text{and} \quad \beta_t = \beta + \gamma \quad t > r \right]$$

*The break date is "r"

* $\gamma \neq 0$

When the potential break date is known, a natural test for a change in β is the Chow (1960) test

$$F_T\left(\frac{r}{T}\right) = \frac{SSR_{1,T} - (SSR_{1,r} + SSR_{r+1,T})}{(SSR_{1,r} + SSR_{r+1,T})/(T-2k)}$$

For fixed $\left(\frac{r}{T}\right)$, $F_T\left(\frac{r}{T}\right)$ has an asymptotic χ^2_k

What if $\frac{r}{T}$ is unknown?

* Estimate the break and plug it in $F_T\left(\frac{\hat{r}}{T}\right)$

The distribution of $F_T(\cdot)$ under the null is not the same as if the break where chosen without regard the data.

* Quandt (1960, JASA) proposed to estimate r_B and $\lambda_B = \frac{r_B}{T}$ by finding the value λ_B which maximizes the Chow or Wald statistic

$$\hat{\lambda}_B = \underset{\lambda_B = \lambda_{min}, \ldots, \lambda_{max}}{\operatorname{argmax}} F_{\gamma=0}(\lambda_B)$$

i.e, sequentially calculate $F_{\gamma=0}(\lambda_B)$ for $\lambda_B = \frac{k+1}{T}, \frac{k+2}{T}, \ldots \frac{T-1}{T}$ (all possible break dates)

The value λ_B which produces the largest $F_{\gamma=0}(\ )$ statistic is denoted by $\hat{\lambda}_B$

Remark: $\hat{\lambda}_B \xrightarrow{p} \lambda_B$

but $\hat{r}_B \overset{p}{\not\rightarrow} r_B$

And what about the test?

$$QLR = \max_{r = r_0, \ldots r_T} F_T\left(\frac{r}{T}\right)$$

Quandt

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(i) QLR $\overset{d}{\not\to} \chi^2_{(\,)}$

(ii) We use the max to solve the problem of unidentification of $\underline{\underline{\gamma}}$ under H_0

(iii) We use the max because the intuition suggests that this statistic will have power against a change in β even though the break is unknown.

Think on $F_{\gamma=0}(\lambda_B)$ as a function of $\lambda_B \in [0,1]$. This is similar to the unit root partial sum process

$$X_T(r) = T^{-1} \sum_{t=1}^{[Tr]} \varepsilon_t$$

with is a function of $r \in [0,1]$

Then to find the A-D of QLR we can use the FCLT and CMP.

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Let $\tilde{F}_T(r/T) = SSR_{1,T} - (SSR_{1,r} + SSR_{r+1,T})$

and use $y_t = \beta_t' x_t + \varepsilon_t$ to write

$$\tilde{F}_T\left(\frac{r}{T}\right) = -\left(\sum_{t=2}^{T} x_{t-1}\varepsilon_t\right)'\left(\sum_{t=2}^{T} x_{t-1}x_{t-1}'\right)^{-1}\left(\sum_{t=2}^{T} x_{t-1}\varepsilon_t\right)$$

$$+\left(\sum_{t=2}^{r} x_{t-1}\varepsilon_t\right)'\left(\sum_{t=2}^{r} x_{t-1}x_{t-1}'\right)^{-1}\left(\sum_{t=2}^{r} x_{t-1}\varepsilon_t\right)$$

$$+\left(\sum_{t=r+1}^{T} x_{t-1}\varepsilon_t\right)'\left(\sum_{t=r+1}^{T} x_{t-1}x_{t-1}'\right)^{-1}\left(\sum_{t=r+1}^{T} x_{t-1}\varepsilon_t\right)$$

$$= -\nu_T(1)'\, V_T(1)^{-1}\,\nu_T(1)$$

$$+\,\nu_T\left(\frac{r}{T}\right)'\, V_T\left(\frac{r}{T}\right)^{-1}\nu_T\left(\frac{r}{T}\right)$$

$$+\left[\nu_T(1)-\nu_T\left(\frac{r}{T}\right)\right]'\left[V_T(1)-V_T\left(\frac{r}{T}\right)\right]^{-1}\left[\nu_T(1)-\nu_T\left(\frac{r}{T}\right)\right]$$

where $\nu_T(\lambda) = T^{-\frac{1}{2}}\sum_{t=2}^{[T\lambda]} x_{t-1}\varepsilon_t$

$$V_T(\lambda) = T^{-1}\sum_{t=2}^{[T\lambda]} x_{t-1}x_{t-1}'$$

Because ε_t is a m.d.s then $X_{t-1}\varepsilon_t$ is a m.d.s too. Assume additionally

- X_{t-1} has sufficiently limited dependence
- $X_{t-1}\varepsilon_t$ enough moments

Then we can apply a FCLT for m.d.s

$$\boxed{\nu_T(\cdot) \Rightarrow \sigma_\varepsilon \Sigma_x^{\frac{1}{2}} W_k(\cdot)}$$

where

$W_k(\lambda)$ = k-dimensional Std BMotion

$$= \begin{pmatrix} W_1(\lambda) \\ W_2(\lambda) \\ \vdots \\ W_k(\lambda) \end{pmatrix}$$

independent univariate standard BMotions

$$\boxed{V_T(\lambda) \xrightarrow{p} \lambda \Sigma_x}$$ uniformly in λ

Then

$$\tilde{F}_T(\cdot) \Rightarrow \sigma_\varepsilon^2 F^*(\cdot)$$

where

$$F^*(\lambda) = -W_k(1)'W_k(1) + \frac{W_k(\lambda)'W_k(\lambda)}{\lambda} + \frac{[W_k(1)-W_k(\lambda)]'[W_k(1)-W_k(\lambda)]}{1-\lambda}$$

$$= \frac{B_k^\mu(\lambda)'B_k^\mu(\lambda)}{\lambda(1-\lambda)}$$

with $B_k^\mu(\lambda) = W_k(\lambda) - \lambda W_k(1)$

a k-dimensional Brownian Bridge

Because $\tilde{F}_T \Rightarrow \sigma_\varepsilon^2 F^*$ and $SSR_{1,T}/(T-k) \xrightarrow{p} \sigma_\varepsilon^2$ under the null,

$$\frac{(SSR_{1,r} + SSR_{r+1,T})}{T-2k} \xrightarrow{p} \sigma_\varepsilon^2$$

uniformly in r.

Thus $\boxed{F_T \Rightarrow F^*}$

From the CMP

$$\boxed{QLR \Rightarrow \sup_{\lambda \in [\lambda_0, \lambda_1]} \left\{ \frac{B_k^{\mu}(\lambda)' B_k^{\mu}(\lambda)}{\lambda(1-\lambda)} \right\}}$$

Note that for a fixed $\bar{\lambda}$, $\underline{F^*(\bar{\lambda}) \Rightarrow \chi^2}$

$\underline{\underline{k=1}}$

$QLR_{5\%}$ 8.85

$\chi^2_{5\%}$ 3,84

$k=10$

$QLR_{5\%}$ 27.03

$\chi^2_{5\%}$ 18.3

The A.Ds obtained so far have to be modified if

(*) ε_t is serially correlated

(**) X_t contains $I(1)$ or trended regressors

Trended regressors

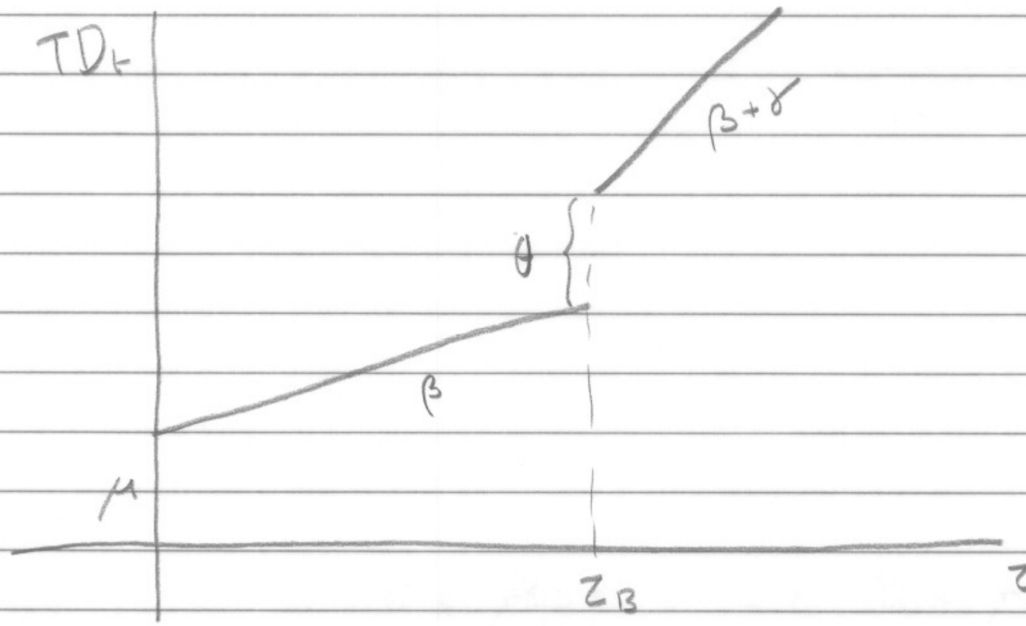
$$y_t = \mu + \theta\, DU(\tau_B)_t + \beta t + \gamma\, DT(\tau_B)_t + \sum_{j=1}^{k} c_j y_{t-j} + \varepsilon_t \qquad \varepsilon_t \sim WN(0, \sigma^2)$$

where

$$DU(\tau_B)_t = 1 \qquad t > \tau_B$$
$$= 0 \qquad \text{otherwise}$$

$$DT(\tau_B) = t - \tau_B \qquad \text{if } t > \tau_B$$
$$= 0 \qquad \text{otherwise}$$

$$TD_t = \mu + \theta\, DU(\tau_B)_t + \beta t + \gamma\, DT(\tau_B)_t$$



There are two possibilities for the no-structural change model,

(1) H_0^1: $\theta = 0$, $\gamma = 0$ and $y_t \sim I(0)$

(2) H_0^2: $\theta = 0$, $\gamma = 0$ and $y_t \sim I(1)$

In each case we compute

$$QLR = \max_{\lambda_B \in [\lambda_{min}, \lambda_{max}]} F(\lambda_B)_{\theta=\gamma=0}$$

Vogelsang (1997) shows that the AD of QLR is different in case (1) and (2).

Trend Breaks and tests for autoregressive unit roots

Shift in mean $\quad d_t = \beta_0 + \beta_1 I(t > r)$

Shift in trend $\quad d_t = \beta_0 + \beta_1 t + \beta_2 (t-r) I(t>r)$

Now we can run the DF test with these broken trends in the regression:

$$\Delta y_t = d_t + \alpha y_{t-1} + \varepsilon_t$$

* Test for $\alpha = 0$ vs $\alpha < 0$ assuming we know the break date

* We can test for $\alpha = 0$ vs $\alpha < 0$ with the break date unknown

$$t_{DF}^{min} = \min_{\delta \in [\delta_0, \delta_1]} \hat{t}^d(\delta)$$

where

$$\hat{t}^d(\delta) = \frac{T^{-1} \sum_{2}^{T} \Delta y_t^d(\delta)\, y_{t-1}^d(\delta)}{\left\{ (\hat{\sigma}^d)^2(\delta)\, T^{-2} \sum_{t=2}^{T} \left(y_{t-1}^d(\delta)\right)^2 \right\}^{\frac{1}{2}}}$$