Exercise 3: MARTINGALES and CONVERGENCE
Due day announced in my web page.

1. (Review of Modes of Convergence)
(a) Suppose $X_n \xrightarrow{r} X$ where $r \geq 1$. Show that $E|X_n|^r \to E|X|^r$.
(b) Suppose $X_n \xrightarrow{L_1} X$. Show that $E(X_n) \to E(X)$. Is the converse true?
(c) Suppose $X_n \xrightarrow{L_2} X$. Show that $var(X_n) \to var(X)$.
(d) Suppose $|X_n| \leq Z$ for all n, where $E(Z) < \infty$. Prove that if $X_n \xrightarrow{p} X$ then $X_n \xrightarrow{L_1} X$.

2. Let $X_1, X_2, ...$ be square integrable random variables such that the partial sums $S_n = X_1+X_2+...+X_n$ determine a martingale. Show that $E(X_iX_j) = 0$ if $i \neq j$.

3. Let $X_0, X_1, X_2...$ be a sequence of random variables with finite means and satisfying $E(X_{n+1}|X_0, X_1, ..., X_n) = aX_n + bX_{n-1}$ for $n \geq 1$ where $a \neq 1$ and $a + b = 1$. Find a value of α for which $S_n = \alpha X_n + X_{n-1}, \quad n \geq 1$, defines a martingale with respect the sequence X.

4. Let S be a martingale with respect to X, such that $E(S_n^2) < K < \infty$ for some $K \in R$. Suppose that $var(S_n) \to 0$ as $n \to \infty$, and prove that $S_\infty = \lim_{n\to\infty} S_n$ exists and is constant almost surely. (S_n is defined in question 2.)

5. A bag contains red and green balls. A ball is drawn from the bag, its colour noted, and then it is returned to the bag together with a new ball of the same colour. Initially the bag contained one ball of each colour. If R_n denotes the number of red balls in the bag after n additions, show that $S_n = R_n/(n + 2)$ is a martingale. Deduce that the ratio of red to green balls converges almost surely to some limit as $n \to \infty$.

6. Let S_n be a symmetric random walk, that is,

$$S_n = X_1 + ... + X_n,$$

where $X_1, X_2, ...$ is a sequence of independent identically distributed random variables such that

$$P(X_n = 1) = P(X_n = -1) = 1/2.$$

Show that $S_n^2 - n$ is a martingale with respect to the filtration

$$\Im_n = \sigma(X_1, ..., X_n).$$

7. Show that if $X_1, X_2, ...$ is a martingale and $E(X_n^2) < \infty$ for all n, then the martingale differences $X_1, X_2 - X_1, ..., X_n - X_{n-1}, ...$ are orthogonal.

8. [*Exercise from a Peter Phillips Take Home Exam, Fall 1998, Yale University*]. The time series X_t is generated by the model

$$X_t = \phi X_{t-1} + u_t, \qquad t = 1, ..., n \tag{1}$$

where $|\phi| < 1$ and $X_0 = 0$. The errors u_t form a strictly stationary and ergodic martingale difference sequence with respect the natural filtration $\Im_t$ and satisfy the following conditions (so that u_t is and ARCH(1) error process):

- **C1** $E(u_t|\Im_{t-1}) = 0, \quad E(u_t^2|\Im_{t-1}) = \sigma_t^2, \quad E(u_t^2) = \sigma^2.$
- **C2** $\sigma_t^2 = \alpha_0 + \alpha_1 u_{t-1}^2$**,** $\alpha_0, \alpha_1 > 0$ **and** $\alpha_1 < 1.$
- **C3** $E((u_t^2 - \sigma_t^2)^2|\Im_{t-1}) = \mu_t^c, \quad E((u_t^2 - \sigma_t^2)^2) = \mu_4.$

Let

$$\widehat{\phi} = (\sum_{t=2}^{n} X_{t-1}^2)^{-1}(\sum_{t=2}^{n} X_{t-1}X_t) \tag{2}$$

be the least squares regression estimator of ϕ in (1). Define the following alternative estimator of ϕ

$$\widehat{\phi}_g = (\sum_{t=2}^{n} X_{t-1}^2/\sigma_t^2)^{-1}(\sum_{t=2}^{n} X_{t-1}X_t/\sigma_t^2). \tag{3}$$

- 1. Show that $\widehat{\phi}$ is strongly consistent for ϕ as $n \to \infty$.
- 2. Find the limit distribution of $\widehat{\phi}$ as $n \to \infty$.

- 3. Compare this limit distribution with the case where u_t is $iid(0, \sigma^2)$.
- 4. Show that $\widehat{\phi}_g$ is strongly consistent for ϕ.
- 5. Find the limit distribution of $\widehat{\phi}_g$.
- 6. Compare the limit distribution of $\widehat{\phi}_g$ and $\widehat{\phi}$. Is $\widehat{\phi}_g$ more efficient than $\widehat{\phi}$? Is $\widehat{\phi}_g$ efficient in some sense?

9. [*Exercise from a Peter Phillips Take Home Exam, Fall 1998, Yale University*]. Let the time series X_{t1}^n be generated by the following linear process

$$X_t = \Sigma_{i=0}^{\infty} d_i u_{t-i}, \quad with \quad \Sigma_{i=0}^{\infty} j^{1/2} |d_j| < \infty, \quad d_0 = 1 \tag{4}$$

and where the innovations u_t constitute a strictly stationary and ergodic sequence of martingale differences satisfying the following conditions:

- **C1** $E(u_t|\Im_{t-1}) = 0, \quad E(u_t^2|\Im_{t-1}) = \sigma_t^2, \quad E(u_t^2) = \sigma^2.$
- **C2'** $E(u_t^2 u_{t-k}^2) = \mu_{kk}$**, for all integer** k**.**
- **C3'** $E(u_t^2 u_{t-k} u_{t-l}) = \mu_{kl}$**, for all integers** $k \neq l$**, and** $sup_l \Sigma_{k=1}^{\infty} |\mu_{kl}| < \infty$**.**

And econometrician proposes to use a simple first order autoregression to model X_t and fits the following regression equation by ordinary least squares

$$X_t = \widehat{\phi} X_{t-1} + \widehat{\nu}_t,$$

where $\widehat{\phi}$ is defined in (2) as in the previous question.

- 1. Find the almost sure limit of $\widehat{\phi}$ as $n \to \infty$.
- 2. Find the limit distribution of $\widehat{\phi}$ as $n \to \infty$.
- 3. Show that when $d_i = \phi^i$ (i.e. the econometrician's AR(1) model for the conditional means is satisfied) and u_t satisfies **C2**, limit distribution reduces to your result in the previous question.
- 4. Show that when $X_t = u_t$ and u_t satisfies **C2** (i.e. the data form a martingale difference and follow an ARCH(1) process) then

$$\sqrt{n}\widehat{\phi} \Rightarrow N(0, \mu_{11}/\sigma^4)$$

where

$$\mu_{11} = \alpha_1 E(u_t^2 - \sigma^2)^2 + \sigma^4 > \sigma^4,$$

so that the limit distribution is not standard normal in general.

- 5. Use the results obtained above to suggest a procedure for testing that X_t is a martingale difference (i.e. test the null hypothesis that $X_t = u_t$ is serially correlated).

10. If an AR(2) model is fitted, construct an LM test against and ARMA(2,Q) alternative and compare the form of this test to the Box-Pierce test applied to the residuals. (Hint: Harvey's book on Time Series)

11. In ONLY one page tell me which things have you already done on the course project.