Exercise 2: ARMA Processes

(Due day will be announced on my web page)

.

1. Let $\{X_t,\, t = 0, \pm 1, \pm 2, ...\}$ be the stationary solution of the non-casual AR(1) equations,

$$X_t = \phi X_{t-1} + Z_t,\ Z_t \sim WN(0, \sigma^2),\ |\phi| > 1.$$

Show that $\{X_t\}$ also satisfies the causal AR(1) equations,

$$X_t = \phi^{-1} X_{t-1} + \tilde{Z}_t,\ \tilde{Z}_t \sim WN(0, \tilde{\sigma^2}),$$

for a suitable chosen white noise process $\{\tilde{Z}_t\}$. Determine $\tilde{\sigma^2}$.

2. Show that there is no stationary solution of the difference equations

$$X_t = \phi X_{t-1} + Z_t,\ \{Z_t\} \sim WN(0, \sigma^2),$$

if $|\phi| = \pm 1$.

3. Find the elements ψ_j, j=0, 1, 2 ...., in the representation

$$X_t = \sum_{j=0}^{\infty} \psi_j Z_{t-j}$$

of the ARMA(2, 1) process,

$$(1 - .5B + .04B^2)X_t = (1 + .25B)Z_t,\ \{Z_t\} \sim WN(0, \sigma^2).$$

4. Find the mean and the autocovariance function of the ARMA(2,1) process,

$$(1 - 1.3B + .4B^2)X_t = 2 + (1 + B)Z_t,\ \{Z_t\} \sim WN(0, \sigma^2).$$

Is the process causal and invertible?

5. Let the ε_t be a white noise. Verify that the processes defined as $x_t = \varepsilon_t$ and $y_t = (-1)^t \varepsilon_t$ are stationary. Show that their sum

$$z_t = x_t + y_t$$

is not stationary.

6. Let us consider the stationary process y_t defined as

$$y_t = \varepsilon_t - \theta\varepsilon_{t-1},\ -1 \leq \theta \leq 1,$$

where ϵ_t is a Gaussian white noise zero mean and variance σ^2. Let us define the process x_t as

$$x_t = \begin{cases} 1 & \text{if } y_t > 0, \\ 0 & \text{otherwise.} \end{cases}$$

1. Show that the process x_t is stationary. What is the autocorrelation function $\rho(.)$?

2. What is the range of $\rho(1)$?

7. The process $X_t = Z_t - Z_{t-1}$, $Z_t \sim WN(0, \sigma^2)$, is not invertible according to the standard definition. Show however that $Z_t \in \overline{sp}\{X_j, -\infty < j \leq t\}$ by considering the mean square limit of the sequences $\sum_{j=0}^{n}(1 - j/n)X_{n-j}$ as $n \to \infty$.

8. Suppose X_t is the two-sided moving average

$$X_t = \textstyle\sum_{j=-\infty}^{\infty} \psi_j Z_{t-j}, \qquad Z_t \sim WN(0, \sigma^2)$$

where $\sum_j |\psi_j| < \infty$. Show that $\sum_{h=-\infty}^{\infty} |\gamma(h)| < \infty$, where $\gamma(.)$ is the autocovariance function of X_t.

9. Let X_t be an ARMA process with $\phi(z) \neq 0$, $|z| = 1$, and autocovariance function $\gamma(.)$. Show that there exist constants $C > 0$ and $s \in (0, 1)$ such that $|\gamma(h)| \leq Cs^{|h|}, h = 0, \pm 1, ...$ and hence that $\sum_{j=-\infty}^{\infty} |\gamma(h)| < \infty$.

10. Simulate 100 observations from an ARMA(1,1) model.

- (a) Fit the simulated series with an AR(1) or an MA(1) model. Carry out diagnostic checking, and modify your fitted model from the result of residual analysis.

- (b) Estimate the parameters of your modified model, and compare with the true parameters values of the model

11. In Granger, and Andersen (1978), "On the invertibility of time series models", *Stoch. Proc. and Appl.*, 8, 87-92, the concept of invertibility for non-linear models is defined in the following way:

The process $x_t = g(x_{t-1}, \epsilon_{t-1}, ..., x_{t-p}, \epsilon_{t-p}) + \epsilon_t$ is invertible if

$$\lim_{t\to\infty} E(v_t^2) = 0$$

with

$$v_t = \epsilon_t - \hat{\epsilon}_t = \epsilon_t - (x_t - g(x_{t-1}, \hat{\epsilon}_{t-1}, ..., x_{t-p}, \hat{\epsilon}_{t-p})).$$

Using a linear MA(1) model show the relationship between this definition of invertibility and the standard one.

12. Chapter 3 from Wei's book: (3.5); (3.6); (3.13) and (3.15).

13. In the next homeworks you will have to do a lot of simulations. Please start learning the software *Matlab*. You can find tutorials and a lot of *Matlab* code in the web. The program is installed in several machines of our computer lab and it is a very common program in Time Series Econometrics.