Econometrics III (Time Series) Jesús Gonzalo

PROBLEM SET 1: Basics of Stochastic Processes

Due day announced in my web page.

1. Let X_n, $n = ..., -1, 0, 1, ...$ be a stationary real sequence with mean 0 and autocovariance function γ.

- (i) Show that the infinite series $\sum_{n=0}^{\infty} a_n X_n$ converges almost surely, and in mean square, whenever $\sum_{n=0}^{\infty} |a_n| < \infty$.
- (ii) Let
$$Y_n = \sum_{k=0}^{\infty} a_k X_{n-k}, \quad n = ..., -1, 0, 1, ... \tag{1}$$
where $\sum_{n=0}^{\infty} |a_n| < \infty$. Find an expression for the autocovariance function γ_Y of Y_n. Provided $\sum_{m=-\infty}^{\infty} |\gamma(m)| < \infty$, show that
$$\sum_{m=-\infty}^{\infty} |\gamma_Y(m)| < \infty \tag{2}$$

2. Let Y be uniformly distributed on $[-1, 1]$ and let $X = Y^2$.

- (a) Find the predictor of X given Y, and of Y given X.
- (b) Find the best linear predictor of X given Y, and of Y given X.

3. Let X be a (weakly) stationary sequence with zero mean and autocovariance function γ.

- (i) Find the best linear predictor $\widehat{X}_{n+1}$ of X_{n+1}, given X_n.
- (ii) Find the best linear predictor $\widetilde{X}_{n+1}$ of X_{n+1}, given X_n and X_{n-1}.

- (iii) Find an expression for

$$D = E[(X_{n+1} - \widehat{X}_{n+1})^2] - E[(X_{n+1} - \widetilde{X}_{n+1})^2] \tag{3}$$

and evaluate this expression when

  - (a) $X_n = cos(nU)$ where $U \sim U[-\pi, \pi]$
  - (b) X is an autoregressive scheme with $\gamma(k) = \alpha^{|k|}$ where $|\alpha| < 1$.

4. Show that a Gaussian process is strongly stationary if and only if it is weakly stationary.

5. Let X_t be a stationary Gaussian process with zero mean, unit variance, and autocovariance function γ. Find the autocovariance functions of the processes X_t^2 and X_t^3, where $t = ..., -1, 0, 1, ...$

6. A filter $A(L)$ is said to be stable if, for any x_t bounded, then $y_t = a(L)x_t$ is bounded. (x_t is bounded if there exists a real number $M < \infty$ such that $|x_t| \leq M$ with probability one for all t.) Show that an absolutely summable filter is stable.

7. A sequence of random variables $\{x_t\}$ is bounded in probability if, for every $\varepsilon > 0$, there exists a real number $M_\varepsilon < \infty$ such that $P[|x_t| \leq M_\varepsilon] > 1 - \varepsilon$ for all t. This is also referred to as $\{x_t\}$ being $O_p(1)$, or $x_t = O_p(1)$. Show that

- (a) If $\sup_t E|x_t| < \infty$, then $x_t = O_p(1)$.
- (b) If $\{x_t\}$ is covariance stationary, then $x_t = O_p(1)$.

8. Assume that $X_1, X_2, ...$ is a stationary sequence with autocovariance function γ. Show that

$$\text{var}\left[\frac{1}{n}\sum_{i=1}^{n} X_i\right] = \frac{2}{n^2}\sum_{j=1}^{n}\sum_{i=0}^{j-1}\gamma_i - \frac{\gamma_0}{n} \tag{4}$$

Assuming that $j^{-1}\sum_{i=0}^{j-1}\gamma_i \to \sigma^2$ as $j \to \infty$, show that

$$\text{var}\left[\frac{1}{n}\sum_{i=1}^{n} X_i\right] \to \sigma^2 \text{ as } n \to \infty \tag{5}$$

9. Deduce the Strong Law of Large Numbers from an appropriate ergodic theorem.

10. Let Q be a stationary measure on (R^T, B^T) where $T = 1, 2, ...$ Show that Q is ergodic if and only if

$$\frac{1}{n}\sum_{i=1}^{n} Y_i \to E[Y] \text{ a.s. and in mean} \tag{6}$$

for all $Y : R^T \to R$ for which $E[Y]$ exists, where $Y_i : R^T \to R$ is given by $Y_i(x) = Y(\tau^{i-1}(x))$. As usual, τ is the natural shift operator on R^T.

11. The stationary measure Q on (R^T, B^T) is called *strongly mixing* if $Q(A \cap \tau^n B) \to Q(A)Q(B)$ as $n \to \infty$, for all $A, B \in B^T$; as usual $T = 1, 2, ...$ and τ is the shift operator on R^T. Show that every strongly mixing measure is ergodic.

12. Let $U \sim U[0, 1]$ with binary expansion

$$U = \sum_{i=1}^{\infty} X_i 2^{-i} \tag{7}$$

Show that the sequence

$$V_n = \sum_{i=1}^{\infty} X_{i+n} 2^{-i}, \quad n \geq 0 \tag{8}$$

is strongly stationary, and calculate its autocovariance function.

13. Let $y_t = \eta_t + b\eta_{t-1}^2$ where η_t is iid $N(0, \sigma^2)$ and b is a constant.

- (a) Compute the autocovariance of y_t.
- (b) Give the Wold representation for y_t. What is the variance of the 1-step ahead linear forecast error?
- (c) Let $\hat{y}_t$ be the optimal forecast of y_t based on knowledge of b, σ^2 and $\eta_{t-1}, \eta_{t-2}, ...$ in the sence that $\hat{y}_t$ minimizes the 1-step ahead mean square forecast error. What is $\text{var}[y_t - \hat{y}_t]$?
- (d) Suggest consistent estimators of σ^2 and b. Are any conditional restrictions needed on (b, σ^2) for identifiability, other than $\sigma > 0$?