Appendix 2

Functional Central Limit Theory
[Phillips - LECTURE NOTES]

Idea: Our object is to characterise the distribution of random elements like the partial sum

$$X_n(r) = \frac{1}{\sqrt{n}} \sum_{1}^{[nr]} u_j$$

as $n \to \infty$. Since $X_n(r) \in D[0,1]$, the limit theory that we obtain is described as FUNCTIONAL LIMIT THEORY - or CENTRAL LIMIT THEORY ON FUNCTION SPACES.

Assumptions: • Note that $X_n(r)$ is just a function space version of the usual (Euclidean space) random element

$$\frac{1}{\sqrt{n}} S_n.$$

In fact, when $r=1$ we have the equivalence

$$X_n(r) = \frac{1}{\sqrt{n}} S_n$$

Thus, we would expect many of the ideas and requirements of CLT theory for $\frac{1}{\sqrt{n}} S_n$ to carry over to function space theory for $X_n(r)$. In particular:

(i) We need to control for outlier occurrences, usually by moment condition

(ii) We need to control for temporal dependence

- In general, $X_n(r)$ just gives the standardised partial sum of u_t up to a certain fraction (r) of the overall sample.

Working with fractions of the sample ($X_n(r)$) turns out to be critical in unit root limit theory, where the whole trajectory of the process (rather than its end points) is important due to persistence in the shocks.

Procedure

- Find a limit law for $X_n(r)$ on a function space like $D[0,1]$ or $C[0,1]$ via a FCLT

- Then use continuous mapping theorems to map the limit law of $X_n(r)$ into the limit law for the functional like

$$\sup_r X_n(r), \quad \int_0^1 X_n(r)\,dr, \ \ldots$$

Notation Weak cgce of P_n to P is denoted by $P_n \Rightarrow P$. Similarly if $X_n(r) =_d P_n$ we write $X_n(r) \Rightarrow X(r)$ or $X_n(r) \xrightarrow{d} X(r)$ to signify cgce of $P_n \Rightarrow P$ when $P = \mathcal{L}(X(r))$

Partial sums of iid sequences

Theorem (Donsker's theorem for partial sums)

If $\{u_j\} \equiv iid(0, \sigma^2)$

Then

$$X_n(r) = \frac{1}{\sqrt{n}\,\sigma} \sum_{1}^{[nr]} u_j \Rightarrow W(r) \equiv BM(\sigma^2 = 1)$$

Wiener process, or standard BM on $C[0,1]$

Recall that $W(r)$ is completely defined by its properties

(i) $W(0) = 0$
(ii) $W(r) \equiv N(0, r)$
(iii) $W(s)$ indep of $W(r) - W(s)$
(iv) $W(r)$ has continuous sample paths

The proof can be found in Billingsley (1968) or Davidson (1994)

The proof has two parts (a) fidi cge

(b) tightness

Time Series Extensions

Theorem (Phillips and Solo)

$$X_t = C(L)\varepsilon_t \qquad C(L) = \sum_{u=0}^{\infty} C_u L^u$$

If (i) $\{\varepsilon_t\} \equiv iid(0, \sigma^2)$

(ii) $\sum_0^{\infty} |u|^{\frac{1}{2}} |C_u| < \infty$

Then

$$\frac{1}{\sqrt{n}} \sum_1^{[nr]} X_t \Rightarrow B(r) \equiv BM(\sigma^2 C^2(1))$$

Proof: Under (ii) we have the valid decomposition

$$X_t = C(1)\varepsilon_t + \tilde{\varepsilon}_{t-1} - \tilde{\varepsilon}_t \quad , \quad \tilde{\varepsilon}_t = \tilde{C}(L)\varepsilon_t$$

so that

$$(*) \quad \frac{1}{\sqrt{n}} \sum_1^{[nr]} X_t = C(1) \frac{1}{\sqrt{n}} \sum_1^{[nr]} \varepsilon_t + \frac{1}{\sqrt{n}} \left(\tilde{\varepsilon}_0 - \tilde{\varepsilon}_{[nr]} \right)$$

Note the first member of $(*)$ satisfies the FCLT for iid sequences

$$C(1) \frac{1}{\sqrt{n}} \sum_1^{[nr]} \varepsilon_t \Rightarrow BM(\omega^2) \quad , \quad \omega^2 = \sigma^2 C^2(1)$$

To prove the FCLT for $\frac{1}{\sqrt{n}} \sum_{1}^{[nr]} x_t$ we now need only show that

$$(**) \qquad \sup_r \left| \frac{1}{\sqrt{n}} \sum_{1}^{[nr]} x_t - C(1) \frac{1}{\sqrt{n}} \sum_{1}^{[nr]} \varepsilon_t \right| \xrightarrow[p]{} 0$$

To do so we write $(**)$ as

$$\sup_r \left| \frac{\tilde{\varepsilon}_0 - \tilde{\varepsilon}_{[nr]}}{\sqrt{n}} \right| \le \underbrace{\frac{\tilde{\varepsilon}_0}{\sqrt{n}}}_{\downarrow p \; 0} + \sup_r \frac{|\hat{\varepsilon}_{[nr]}|}{\sqrt{n}}$$

$$\begin{aligned}\sup_r \left| \frac{\tilde{\varepsilon}_{[nr]}}{\sqrt{n}} \right| &= \max_{0 \le k \le n} \frac{1}{\sqrt{n}} \left| \sum_{u=0}^{\infty} \tilde{c}_u \varepsilon_{k-u} \right| \\ &= \max_{0 \le k \le n} \frac{1}{\sqrt{n}} |\tilde{\varepsilon}_k|\end{aligned}$$

Note that by (i) and (ii) $\tilde{\varepsilon}_k$ is stationary and has finite variance so

$$\begin{aligned}P\left(\max_k \frac{|\tilde{\varepsilon}_k|}{\sqrt{n}} \ge \varepsilon \right) &= P\left(\max_k \frac{\tilde{\varepsilon}_k^2}{n} \ge \varepsilon^2 \right) \\ &\le \sum_{k=0}^{n} P\left(\frac{1}{n} \tilde{\varepsilon}_k^2 \ge \varepsilon^2 \right) \\ &= (n+1)\, P\left(\frac{1}{n} \tilde{\varepsilon}_0^2 \ge \varepsilon^2 \right) \\ &\le \frac{n+1}{n \varepsilon^2} E\left(\tilde{\varepsilon}_0^2 \, 1\!\!1\left(|\tilde{\varepsilon}_0| > \varepsilon \sqrt{n} \right) \right) \\ &\to 0\end{aligned}$$

The final piece of apparatus we need is the continuous mapping theorem (cmt). Again this in Billingsley (1968)

Theorem

Let h be any continuous functional on $D[0,1]$. Then

$$h(X_n(r)) \Rightarrow h(B(r))$$

if $X_n(r) \Rightarrow B(r)$ on $D[0,1]$.

Application

$$h(X_n(r)) = \int_0^1 X_n(r)\,dr \Rightarrow \int_0^1 B(r)\,dr = h(B(r)$$

Note that $h(\cdot) = \int_0^1$ is a cts function. In particular for any sequence $g_n \to g_\infty$ we have

$$|h(g_n) - h(g_\infty)| \leq \int_0^1 |g_n - g_\infty|\,dr \leq \int_0^1 \sup_r |g_n(r) - g_\infty(r)|\,dr$$

$$\to 0 \text{ as } n \to \infty$$

Distribution of $h(B(r))$

$$h(B(r)) = \int_0^1 B(r)\,dr \equiv N(0, v)$$ linear functional of Gaussian process

and

$$v = E\left(\int_0^1 B\right)^2$$

$$= 2\int_0^1 \int_0^r E(B(r)B(s))\,ds\,dr$$

$$= 2\sigma^2 \int_0^1 \int_0^r s\,ds\,dr \qquad E(B(r)B(s)) = \sigma^2 r \wedge s$$

$$= 2\sigma^2 \int_0^1 \left[\frac{s^2}{2}\right]_0^r dr$$

$$= \sigma^2 \int_0^1 r^2\,dr$$

$$= \underline{\underline{\sigma^2/3}}$$

Asymptotics for integrated processes

In general we are going to deal with processes like y_t based on

$$(1-L)\, y_t = C(L)\,\varepsilon_t = u_t \qquad \{\varepsilon_t\} \equiv iid\,(0,\sigma^2)$$

Key point:

$$y_t = \Sigma_1^t u_j + y_0 = C(1)\, \Sigma_1^t \varepsilon_j + (\tilde{\varepsilon}_0 - \tilde{\varepsilon}_t) + y_0$$

$$\frac{1}{\sqrt{t}}\, y_t \sim C(1)\, \frac{1}{\sqrt{t}} \Sigma_1^t \varepsilon_j + o_p(1)$$

behaves asymptotically like a BM

Hence, functions of y_t like sample moments behave asymptotically like functions of BM

mechanism: Transform from y_t to the corresponding random elements that live in the function space. Then sample moments (etc) of y_t become functionals of the random element ($X_n(r)$ say). But $X_n(r) \Rightarrow B(r)$ so that the required limit theory is a consequence of the CMT.

This works well for all cases except some that involve quantities of different stochastic orders like

$$\frac{1}{n} \Sigma_1^n y_t u_t \;;\; y_t \equiv I(1),\; u_t \equiv I(0)$$