Appendix 1

Functional Limit Theory and Unit Root Asymptotics

(A must read book: Convergence of Probability Measures by Patrick Billingsley)

Random elements on function spaces

Let's start by considering the following two function spaces:

(1) $C[0,1]$ = space of cont. functions on the interval $[0,1]$ endowed with the uniform metric

$$(*) \quad d(f,g) = \sup_t |f(t)-g(t)| \qquad f,g \in C[0,1]$$

makes it a complete metric space

(2) $D[0,1]$ = space of real valued functions with left limits and right continuous (CADLAG). This space is complete with the Skorohod metric

For us we will be enough to work with (*) because all our limit processes will be in $C[0,1]$.

A very natural random element in $D[0,1]$ is the partial sum process

$$X_n(r,\omega) = \frac{1}{\sqrt{n}} \sum_{1}^{[nr]} u_j(\omega) \qquad [nr] = \text{integer part of } nr$$

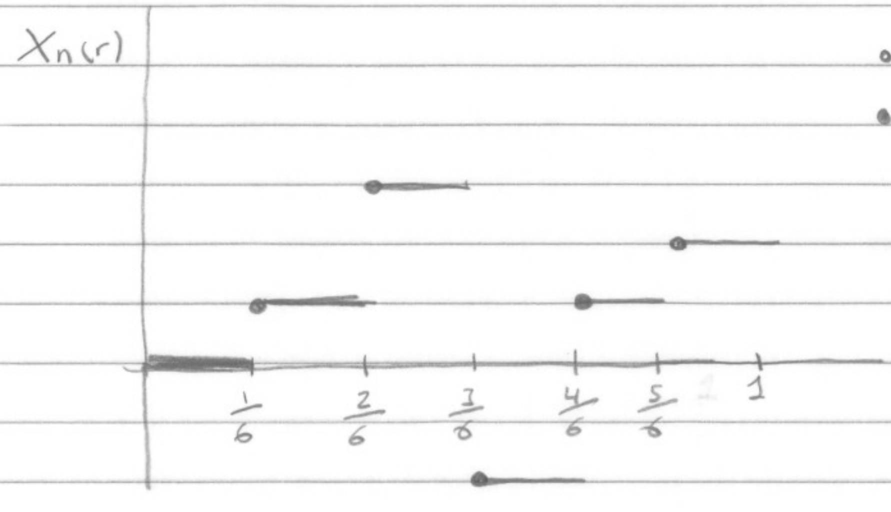
$$\{u_j\} \equiv I(0)$$

$$= \frac{1}{\sqrt{n}} \sum_{1}^{k-1} u_j(\omega) \qquad \text{for } \frac{k-1}{n} \leq r \leq \frac{k}{n}$$

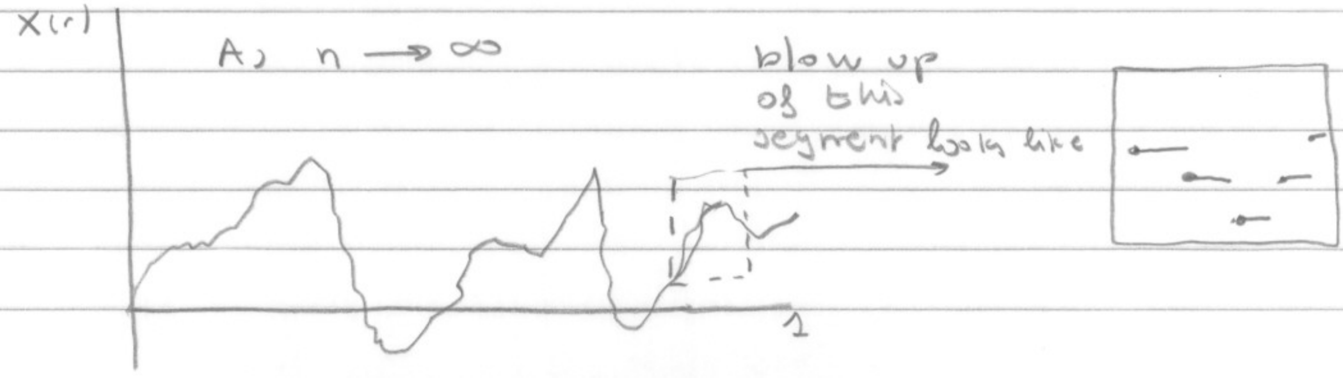
$$= \frac{1}{\sqrt{n}} S_{[nr]}^{(\omega)} \qquad (\text{with } S_0 = 0)$$

$X_n(r) \in D[0,1]$

A typical realization

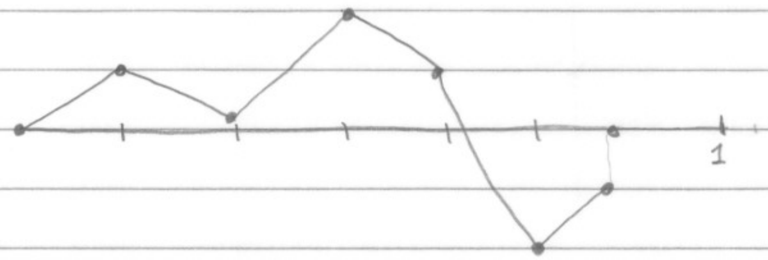


- $n = 6$
- The interval $[0,1]$ is splitted into segments of length $\frac{1}{n}$.
- $X_n(r)$ is cte in each segment



We could also define a random element of $C[0,1]$ in a similar fashion

$$\bar{X}_n(r,w) = \frac{1}{\sqrt{n}} S_{[nr]}(w) + \frac{nr-[nr]}{\sqrt{n}} u^{(w)}_{[nr]+1} \in C[0,1]$$



The jumps in $\frac{1}{\sqrt{n}} S_{[nr]}$ are now eliminated by the line segments based on

$$0 \le nr - [nr] < 1$$
$$\frac{nr-[nr]}{\sqrt{n}} = O\left(\frac{1}{\sqrt{n}}\right)$$

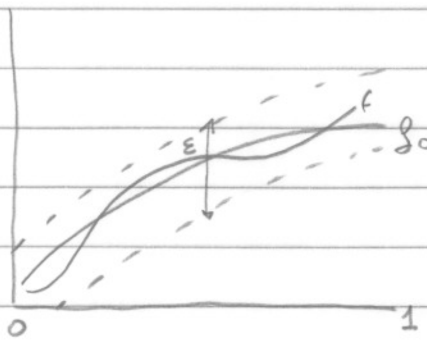
and the asymptotic behaviour of $\bar{X}_n(w,r)$ is the same as that of $X_n(w,r)$.

Probability in Metric Spaces

Let (D, d) be a metric space. Using d we can construct open sets

$$S(f_0, \varepsilon) = \{ f \mid d(f, f_0) < \varepsilon \} = \text{open sphere around } f_0$$

Example: $D = C[0,1]$
$d =$ uniform metric



$$d(f, f_0) = \sup_r |f(r) - f_0(r)| < \varepsilon$$

As we do with standard random variables we can construct

$$(D, \mathcal{D}, P) = \text{probability space}$$

$\mathcal{D}$: Borel σ-field of D

P: probability function

Weak Convergence in $(D, \mathcal{D}, P)$

When $(D, \mathcal{D}, P) = (\mathbb{R}, \mathcal{B}, P)$ we usually define weak convergence by

$$F_n(x) \longrightarrow F(x) \quad \text{at all points of continuity of } F.$$

In the general prob. space $(D, \mathcal{D}, P)$ it is more convenient to work directly with the sequence of prob. measures itself $\{P_n\}$.

Let $\{P_n\}$ be a sequence of prob. measures on $(D, \mathcal{D})$. Then

$$P_n \Rightarrow P \quad (P_n \text{ converges weakly to } P)$$

if

$$(*) \quad P_n(A) \longrightarrow P(A) \quad \text{for all events in } \mathcal{D} \text{ s.t. } P(\delta A) = 0$$

where δA = boundary of A.

The sets A in $(*)$ for which $P(\delta A) = 0$ are called (P) continuity sets.

Example 1 $\quad D = R$, $\mathcal{D} = \mathcal{B}$

$P_n \Rightarrow P$ iff $F_n(x) \longrightarrow F(x)$ at all continuity points of F
$(P(X=x) = 0)$

Here we have

$F(x) = P(X \le x)$ and $\partial A = x$ when $A = (-\infty, x]$

So $P(\partial A) = P(X = x) = 0$ is required

This example illustrates another useful point, the sets

(*) $\quad \{ y : y \le x \} = (-\infty, x] \quad \forall x$

forms a "Convergence determining class"
i.e. if $P_n \Rightarrow P$ for all sets in this class then $P_n \Rightarrow P$ in general.

Note also that these sets are a "determining class", in the sense that if P and Q are the same on (*) they are the same in $\mathcal{B}$.

<u>Example</u> 2 $(R_\infty, \mathcal{B}_\infty)$ = coordinate representation space for a sequence such as $\{X_n\}$.

$$x = (x_1, x_2, \ldots) \in R_\infty$$

Recall that $\mathcal{B}_\infty$ is generated by product cylinder sets of the form

$$(*) \quad \left(\mathop{\times}_1^r R\right) \left(\mathop{\times}_{r+1}^s B_i\right) \left(\mathop{\times}_{s+1}^\infty R\right) \qquad B_i \in \mathcal{B}.$$

We can think of these sets another way: as the preimage of a projection

$$\pi_k : R_\infty \to R^k$$

defined by

$$\pi_k(x) = (x_1, \ldots x_k) \in R^k$$

Then for $H \in \mathcal{B}_k = \mathcal{B}(R^k)$, a finite dimensional product cylinder,

$$\pi_k^{-1} H = \text{preimage of } H \text{ under } \pi_k$$

This preimage is just a cylinder set of the form $(*)$. The sets $\{\pi_k^{-1} H\}$ $\forall k$ form out to be a determining and convergence determining class of sets

i.e $\quad P_n \Rightarrow P$ on $(R_\infty, \mathcal{B}_\infty)$

iff

$P_n(A) \longrightarrow P(A)$ $\forall$ finite dimensional P-continuity set A.

Example 3 : $(C, \mathcal{D})$

$C = C[0,1]$ with uniform metric $\|f-g\| = \sup_r |f(r)-g(r)|$

$\mathcal{D}$ = Borel σ-field of $C[0,1]$ = σ field generated by subsets of $C[0,1]$ that are open wrt uniform metric.

projections We define these here by

$\pi_{t_1,\ldots t_k}(x) = (x(t_1), \ldots, x(t_k)) \in R^k$

So

$\pi_{t_1,\ldots t_k} : C[0,1] \to R^k$ is projection mapping

$\pi_{t_1,\ldots t_k}^{-1} H$ = finite dimensional set (preimage) for $H \in \mathcal{B}_k = \mathcal{B}(R^k)$

$\in \mathcal{D}$

Remark: In fact, the finite dimensional sets do generate $\mathcal{D}$ and form a determining class

But The finite dimensional sets are not convergence determining. To illustrate this look at the example given by Billingsley (1968, p. 20)

The example in Billingsley shows that fidi cgce (finite-dimensional convergence) is not enough to establish that

$$P_n \Rightarrow P \qquad \text{(weak cgce of } P_n\text{)}$$

We need something more than just fidi cge.

In fact what we need are:

(a) fidi cgce; and

(b) tightness of $\{P_n\}$

Definition: A family $\mathcal{P}$ of probability measures P on $(D, \mathcal{D})$ is tight if $\forall \varepsilon > 0$ $\exists$ compact set K s. that

$$P(K) > 1-\varepsilon \qquad \forall P \in \mathcal{P}.$$

i.e. is there a compact set in the space that contains almost all the mass for all the measures in the family $\mathcal{P}$.

Clearly, P is tight if it has compact support (i.e $\exists A \in \mathcal{D}$ s.th $P(A) = 1$ and A is compact)

Examples

$$P_n(X = n) = 1 \quad , \quad \text{cdf} \quad F_n(x) = 0 \quad x \leq n$$
$$= 1 \quad x \geq n$$

$$F_n(x) \longrightarrow F(x) = 0 \quad \forall x$$

- P_n is not tight because $\exists$ no compact K for which given $\varepsilon > 0$
$$P_n(K) > 1 - \varepsilon \quad \forall n$$

[take any compact K and choose n_0 such that $n_0 > \sup_{x \in K} |x|$. Then $P_n(K) = 0 \quad \forall n > n_0$]

- Note that although we get cgce of $F_n \to F$ at all points of continuity of F, we have lost all the mass of P_n in the limit function F, which is not a cdf.