Canonical Correlations

$$X = \begin{pmatrix} X^1 \\ X^2 \end{pmatrix} \begin{array}{l} m_1 \text{ components} \\ m_2 \quad \text{''} \\ \hline m \quad \text{TOTAL} \end{array} \; ; \; \mu = \begin{pmatrix} \mu^1 \\ \mu^2 \end{pmatrix} ; \; \Sigma = \begin{pmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{21} & \Sigma_{22} \end{pmatrix}$$

Def

$$Z = \alpha' X^1 \; ; \quad W = \beta' X^2$$

with

$$\rho = \frac{\mathrm{Cov}(\alpha' X^1, \beta' X^2)}{[\mathrm{var}(\alpha' X^1)\, \mathrm{var}(\beta' X^2)]^{\frac{1}{2}}}$$

If α and β are chosen such as to maximize ρ, then ρ is termed the (first) canonical correlation between X^1 and X^2, while Z and W are called the (first set of) canonical variables associated to X^1 and X^2.

Canonical correlations and variables are the solution of

$$\boxed{\max_{\alpha, \beta} \quad \frac{\alpha' \Sigma_{12} \beta}{[(\alpha' \Sigma_{11} \alpha)(\beta' \Sigma_{22} \beta)]^{\frac{1}{2}}}}$$

This maximand is homogenous of degree zero, then we are faced with a scale problem. To eliminate this indeterminancy we impose

$$\alpha' \Sigma_{11} \alpha = 1 \quad \text{and} \quad \beta' \Sigma_{22} \beta = 1$$

The mathematical form of the canonical correlation problem is

$$L = \alpha' \Sigma_{12} \beta + \frac{1}{2} \lambda_1 (1 - \alpha' \Sigma_{11} \alpha) + \frac{1}{2} \lambda_2 (1 - \beta' \Sigma_{22} \beta)$$

F.O.C

$$\frac{\partial L}{\partial \alpha} = \Sigma_{12} \beta - \lambda_1 \Sigma_{11} \alpha = 0 \Rightarrow$$

$$\frac{\partial L}{\partial \beta} = 0$$

$$\frac{\partial L}{\partial \lambda_1} = 0$$

$$\frac{\partial L}{\partial \lambda_2} = 0$$

The solution must satisfy

$$\lambda_1 = \lambda_2 = \lambda \quad \begin{bmatrix} -\lambda \Sigma_{11} & \Sigma_{12} \\ \Sigma_{21} & -\lambda \Sigma_{22} \end{bmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = 0$$

This will have a non-trivial solution iff

$$\begin{vmatrix} -\lambda \Sigma_{11} & \Sigma_{12} \\ \Sigma_{21} & -\lambda \Sigma_{22} \end{vmatrix} = 0.$$

How do we find the <u>second</u> set of canonical correlations?

$$L = \alpha' \Sigma_{12} \beta + \frac{1}{2} \lambda_1 (1 - \alpha' \Sigma_{11} \alpha) + \frac{1}{2} \lambda_2 (1 - \beta' \Sigma_{22} \beta) + \mu_1 \underbrace{\alpha'_{(1)} \Sigma_{11} \alpha}_{=0} + \mu_2 \underbrace{\beta'_{(1)} \Sigma_{22} \beta}_{=0}$$

F.O.C

- - - $\Rightarrow$ The solution is <u>again</u> the vectors that satisfy

- - -

$$\begin{bmatrix} -\lambda \Sigma_{11} & \Sigma_{12} \\ \Sigma_{21} & -\lambda \Sigma_{22} \end{bmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = 0$$

now linked to the second eigenvalue of $\begin{vmatrix} -\lambda \Sigma_{11} & \Sigma_{12} \\ \Sigma_{21} & -\lambda \Sigma_{22} \end{vmatrix} = 0.$

Proposition

$$X = \begin{pmatrix} X^1 \\ X^2 \end{pmatrix} ; \quad \mu = \begin{pmatrix} \mu^1 \\ \mu^2 \end{pmatrix} \quad \text{and} \quad \Sigma = \begin{bmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{21} & \Sigma_{22} \end{bmatrix}$$

The canonical correlations are obtained as the roots of

$$(*) \quad \begin{vmatrix} -\lambda \Sigma_{11} & \Sigma_{12} \\ \Sigma_{21} & -\lambda \Sigma_{22} \end{vmatrix} = 0 \quad (\lambda^{(1)} \geq \lambda^{(2)} \geq \ldots > \lambda^{(m)})$$

while the the canonical variables are obtained as the vectors that solve

$$\begin{bmatrix} -\lambda \Sigma_{11} & \Sigma_{12} \\ \Sigma_{21} & -\lambda \Sigma_{22} \end{bmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = 0$$

Some properties of the canonical correlations

(1) They ought to be less than one

(2) Their squares has an interpretation as a characteristic root of a certain positive semidefinite matrix

The determinant of a partitioned matrix

$$(*) \quad (-1)^m \lambda^{m_2 - m_1} |\Sigma_{22}| \, |\lambda^2 \Sigma_{11} - \Sigma_{12} \Sigma_{22}^{-1} \Sigma_{21}| = 0$$

So $(*)$ has at least $m_2 - m_1$ zero roots, and its nonzero roots are exactly the nonzero roots of

$$|\lambda^2 \Sigma_{11} - \Sigma_{12} \Sigma_{22}^{-1} \Sigma_{21}| = 0$$

Cointegration and Reduced Rank Regression

$$\Delta X_t = \alpha\beta' X_{t-1} + \sum_{i=1}^{k-1} \Gamma_i \Delta X_{t-i} + \varepsilon_t, \quad t=1 \dots T$$

Regress ΔX_t on $\Delta X_{t-1} \dots \Delta X_{t-k+1} \rightarrow \boxed{R_{0t}}$

Regress X_{t-1} on $\Delta X_{t-1}, \dots \Delta X_{t-k+1} \rightarrow \boxed{R_{1t}}$

$$\text{Log} L(\alpha, \beta, \Omega) = -\frac{1}{2} T \log|\Omega| - \frac{1}{2} \sum_{t=1}^{T} (R_{0t} - \alpha\beta' R_{1t})' \Omega^{-1} (R_{0t} - \alpha\beta' R_{1t})$$

Define $\boxed{S_{ij} = T^{-1} \sum_{t=1}^{T} R_{it} R_{jt}'}$ $\quad i,j = 0,1$

Note that for fixed β

$$\hat{\alpha}(\beta) = S_{01}\beta(\beta' S_{11}\beta)^{-1} \quad \text{and}$$

$$\hat{\Omega}(\beta) = S_{00} - S_{01}\beta(\beta' S_{11}\beta)^{-1}\beta' S_{10}$$

$$= S_{00} - \hat{\alpha}(\beta)(\beta' S_{11}\beta)\hat{\alpha}(\beta)'$$

$$L_{max}^{-2/T}(\hat{\alpha}(\beta), \beta, \hat{\Omega}(\beta)) = L_{max}^{-2/T}(\beta) = |\hat{\Omega}(\beta)| = |S_{00} - S_{01}\beta(\beta' S_{11}\beta)^{-1}\beta' S_{10}|$$

$$= |S_{00}| \frac{|\hat{\beta}'(S_{11} - S_{10} S_{00}^{-1} S_{01})\hat{\beta}|}{|\hat{\beta}' S_{11}\hat{\beta}|} = \boxed{|S_{00}| \prod_{i=1}^{r} (1 - \hat{\lambda}_i)}$$

since by the choice of $\hat{\beta} S_{11} \hat{\beta} = I$, as well as, $\hat{\beta}' S_{10} S_{00}^{-1} S_{01} \hat{\beta} = \text{diag}(\hat{\lambda}_1, \dots, \hat{\lambda}_r)$

The likelihood ratio test of rank r vs rank p is

$$Q\left(H(r)\,|\,H(p)\right)^{-\frac{2}{T}} = \frac{|S_{00}| \prod_{i=1}^{r}(1-\hat{\lambda}_i)}{|S_{00}| \prod_{i=1}^{p}(1-\hat{\lambda}_i)}$$

so

$$\boxed{-2\log Q\left(H(r)\,|\,H(p)\right) = -T\sum_{i=r+1}^{p}\log(1-\hat{\lambda}_i)}$$

Theorem: Under the hypothesis

$$H(r): \Pi = \alpha\beta'$$

the ML estimator of β is found by the following procedure

(1) Solve

$$|\lambda S_{11} - S_{10}S_{00}^{-1}S_{01}| = 0$$

for the eigenvalues $1 > \hat{\lambda}_1 > \dots > \hat{\lambda}_p > 0$
and eigenvector $\hat{V} = (\hat{v}_1, \dots \hat{v}_p)$ which normalize

$$\hat{V}' S_{11} \hat{V} = I$$

The cointegrating relations are estimated by

$$\hat{\beta} = (\hat{v}_1, \dots \hat{v}_r),$$

and

$$L_{max}^{-2/T}(H(r)) = |S_{00}| \prod_{i=1}^{r}(1-\hat{\lambda}_i)$$

<u>Theorem</u>: Under hypothesis $H_0: \boxed{\beta = \underset{p\times s\; s\times r}{H\varphi}}$ we find the maximum likelihood estimator of β by reduced rank regression of ΔX_t on $H'X_{t-1}$

(corrected by lagged differences), that is, first we solve

$$|\lambda H'S_{11}H - H'S_{10}S_{00}^{-1}S_{01}H| = 0$$

for $1 > \lambda_1^* > \dots > \lambda_s^* > 0$ and $V = (v_1, \dots, v_s)$ which we normalize by $V'H'S_{11}HV = I$. Then choose

$$\hat{\varphi} = (v_1, \dots, v_r) \text{ and } \hat{\beta} = H\hat{\varphi}$$

and find the estimates of the remaining parameters by OLS for $\beta = \hat{\beta}$. Then

$$L_{max}^{-2/T}(H_0) = |S_{00}| \prod_{i=1}^{r}(1-\lambda_i^*)$$

and the likelihood ratio test $Q(H_0|H(r))$ of the hypothesis H_0 in $H(r)$ is

$$-2\log Q(H_0|H(r)) = T\sum_{i=1}^{r}\log\{(1-\lambda_i^*)/(1-\hat{\lambda}_i)\}$$

which is asymptotically as χ^2 with $r(p-s)$.