"Beginning Functional Analysis" by Karen Saxe

Additions to Basic Concepts L. Notes

Sequence Spaces and Function Spaces

Some Examples:

(•) **Sequence spaces** are linear spaces whose elements are sequences.

The first example is the collection ℓ^∞ (pronounced "Little ell infinity") of all bounded sequences $\{x_n\}_{n=1}^{\infty}$.

The next example is the collection c_0 of all sequences that converge to 0. Notice that $c_0 \subset \ell^\infty$. Both of these collections become normed linear spaces with norm defined by

$$\|(x_n)\|_\infty = \sup\{|x_n| \ (1 \le n < \infty)\}$$

Next we define the ℓ^p-spaces for $1 \le p < \infty$ consisting of all the sequences

$\{x_n\}_{n=1}^{\infty}$ such that

$$\sum_{n=1}^{\infty} |x_n|^p < \infty.$$

With norm denoted by

$$\|\{x_n\}\| = \left(\sum_{n=1}^{\infty} |x_n|^p\right)^{\frac{1}{p}}$$

ℓ^p becomes a normed space.

Notice that $\ell^p \subset c_0$

The space ℓ^1 is somewhat especial. It consists of all absolutely convergent sequences. That is $\{x_n\}_{n=1}^{\infty}$ is in ℓ^1 iff $\sum_{n=1}^{\infty} |x_n|$ converges.

The space ℓ^2 is undoubtedly the most important of all the ℓ^p spaces.

(••) Now function spaces. These are linear spaces consisting of functions. As with sequence spaces, addition and scalar multiplication are defined pointwise.

It is not surprising that function and sequence spaces are much alike. Sequences can be considered as functions defined on $\mathbb{N}$.

$$V = \{ f : [a,b] \longrightarrow \mathbb{R} \mid \text{there exists } B \geq 0 \text{ such that } |f(x)| \leq B \text{ for all } x \in [a,b] \}$$

This is a linear space.

The collection

$$\{ f : [a,b] \longrightarrow \mathbb{R} \mid f \text{ is continuous} \}$$

is a subspace of V.

This subspace of all continuous functions on $[a,b]$ will be very important for you.

It is represented by $C[a,b]$.

With norm defined by

$$\|f\|_\infty = \sup\{|f(x)| \,/\, x \in [a,b]\}$$

both V an $C[a,b]$ become normed linear space.

It can be proved that the supremum norm on $C[a,b]$ does not come from an inner product. However we can endow this linear space with an inner product via

$$\langle f, g \rangle = \int_a^b f(x)\, g(x)\, dx.$$

The induced norm is then

$$\|f\|_2 = \sqrt{\int_a^b |f(x)|^2\, dx}$$

and not

$$\|f\|_\infty = \sup\{|f(x)| \,/\, x \in [a,b]\}$$

Is then the supremum norm, in some sense, less desirable than the norm $\|f\|_2$ on $C[a,b]$?

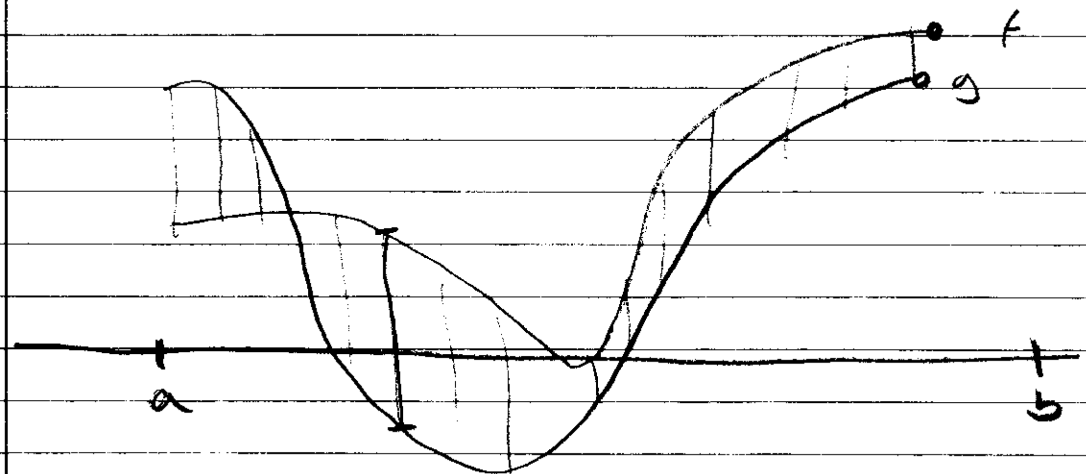
There are always advantages and disadvantages of every norm

The sup norm does have the very nice property that it is "complete" on $C[a,b]$ while the norm $\|\cdot\|_2$ on $C[a,b]$ is not complete

A subset A of a metric space is called complete if every Cauchy sequence in A converges to a point of A.

If a metric space is complete in itself, the space is called a <u>complete metric space</u>. A complete normed linear space is called BANACH space. A complete inner product space is called HILBERT space

<u>Theorem</u>: $C([a,b],\mathbb{R})$ with the sup norm is complete



The sup metric

Why do we need to introduce the Skorohod metric when we are in the metric space $D[0,\infty)$?

$D[0,\infty)$ is the space of functions

$$X : [0,\infty) \longrightarrow R$$ that are

right continuous with left limits.

Consider the following example

$$X_n(t) = \begin{cases} 0 & t < \frac{1}{n} \\ 1 & t \geq \frac{1}{n} \end{cases}$$

Then prove that

$$X_n(t) \not\longrightarrow X(t) \equiv 1$$

with the supremum metric.

Prove also that

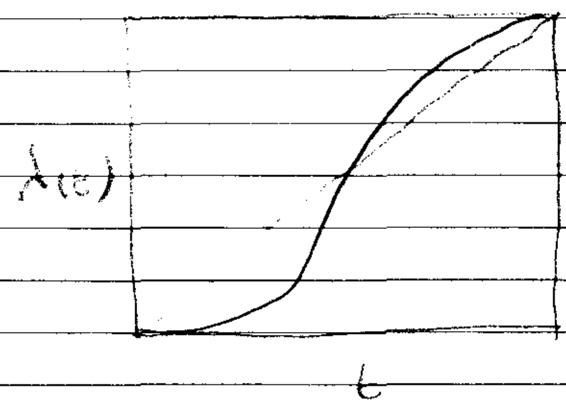
$$X_n(t) \longrightarrow X(t) \equiv 1$$

with the Skorohod metric

$$d_S(x,y) = \inf_{\lambda \in \Lambda} \left\{ \varepsilon > 0 : \sup_t |\lambda(t) - t| \leq \varepsilon, \ \sup_t |x(t) - y(\lambda(t))| \leq \varepsilon \right\}$$

Λ denote the collection of all homeomorphisms

$\lambda : [0,1] \longrightarrow [0,1]$ with $\lambda(0) = 0$
$\lambda(1) = 1$



Prove that d_S is a metric. It helps to note that

$\sup_t |\lambda(t) - t| = \sup_t |t - \lambda^{-1}(t)|$ and

$\sup_t |x(t) - y(\lambda(t))| = \sup_t |x(x(\lambda^{-1}(t)) - y(t)|$,

where $\lambda^{-1} \in \Lambda$ if $\lambda \in \Lambda$

While in the sup (or uniform) metric two functions are closed only if their vertical separation is uniformly small, the Skorohod metric also takes into account the possibility that the horizontal separation is small.

If x is uniformly close to y except that it jumps slightly befor or slightly after y, the functions would be consider close as measured by $\underline{\underline{d_S}}$ but not as by sup norm

Shift Operator (Linear operators)

Two linear spaces X and Y.

A Linear operator T, is a mapping

$$T : X \longrightarrow Y$$

such that the domain D_T is a linear subspace of X and if

$$T(\alpha x + \beta y) = \alpha T(x) + \beta T(y)$$

$$\forall x, y \in D_T$$

Notice that $T(0) = 0$

Example 1 : $T : \mathbb{R}^n \longrightarrow \mathbb{R}^m$

$$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ & & & \\ \cancel{a_{m1}} & & & a_{mn} \end{pmatrix} \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = \left(\sum_{j=1}^{n} a_{1j} x_j, \ \ldots \ \sum_{j=1}^{n} a_{mj} x_j \right)$$

Example 2 : An "infinite" matrix

$$\begin{pmatrix} 0 & 1 & 0 & 0 & \cdots \\ 0 & 0 & 1 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & & \vdots & \end{pmatrix}$$

represents the linear operator T acting on $\ell^2 = \ell^2(\mathbb{N})$ or ℓ^∞ given by

$$T(x_1, x_2, \ldots) = (x_2, x_3 \ldots)$$

The shift operator (left shift)

Another important shift is the "right shift" defined by

$$S(x_1, x_2, \dots) = (0, x_1, x_2, \dots)$$

Which is the infinite matrix representing this shift?

Thm: Let X and Y be normed linear spaces. A linear operator $T: X \to Y$ is continuous at every point if it is continuous at a single point (for instance zero).

Operator Norms

$T: X \longrightarrow Y$ is said to be bounded

if there exists an $M > 0$ s.t

$$\|Tx\|_Y \leq M\|x\|_X \qquad \forall x \in X$$

Example:

$T: \ell^2 \longrightarrow \ell^2$ defined by

$$T(x_1, x_2, \dots) = (x_2, x_3 \dots)$$

is bounded because

$$\|T(x_1, x_2, \dots)\|_{\ell^2} = \sum_{k=2}^{\infty} |x_k|^2 \leq \sum_{k=1}^{\infty} |x_k|^2 =$$
$$= \|(x_1, x_2 \dots)\|_{\ell^2}$$

We may choose $M = 1$.

An element S of $B(X)$ is called

algebra normed space

invertible if there exists $T \in B(X)$ such that

$$ST = I = TS \quad \text{where} \quad Ix = x.$$

Note that S the right shift on $l^2 \to l^2$ given by

$$S(x_1, x_2, \dots) = (0, x_1, x_2 \dots)$$

has "left inverse"

$$T(x_1, x_2 \dots) = (x_2, x_3, \dots)$$
$$TS = I$$

but no right inverse.

This doesn't occur with finite matrices.

Some theorems that help to test whether an operator is invertible or not

Thm: Let X be a Banach space. Suppose that $T \in B(X)$ is such that $\|T\| < 1$. Then the operator $I - T$ is invertible in $B(X)$, and its inverse is given by

$$(I - T)^{-1} = \sum_{k=0}^{\infty} T^k$$

Thm: Let X be a Banach space. Suppose that $S, T \in B(X)$, T is invertible, and $\|T - S\| < \|T^{-1}\|^{-1}$. Then S is invertible in $B(X)$

How to calculate the norm of an operator T?

Invariant Subspaces

Given a bounded linear operator T on a Banach space X, a subspace Y of X is called an invariant subspace for T if $T(Y) \subseteq Y$.

Invariant subspaces of the right shift on $\ell^2 = \ell^2(\mathbb{N})$:

$$S(x_1, x_2, \ldots) = (0, x_1, x_2, \ldots)$$

and

$$M_n = \{(x_1, x_2, \ldots) \in \ell^2 : x_k = 0,\ 1 \leq k \leq n\}$$

It is obvious to check that M_n is an invariant subspace for S. The difficult question is to study if this is the only invariant subspace.