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Limit Theory for Mixing Dependent Random Variables

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Chapter 1 Definitions and Basic Inequalities

In this book, we always assume that $\{X_n, n \geq 1\}$ is a sequence of random variables defined on a probability space $(\Omega, \mathcal{F}, P)$. There are many ways to describe weak dependence or asymptotic independence of $\{X_n\}$. In Section 1.1, we give some common and important definitions of this kind. In Section 1.2, some basic inequalities on covariances of $\{X_n\}$ are established, which are useful for studying limit properties of $\{X_n\}$. In these sections, we also discuss the relations between each other for different definitions.

1.1 Definitions

Let $\mathcal{A}$ and $\mathcal{B}$ be sub-σ-fields of $\mathcal{F}, L_p(\mathcal{A})$ a set of all $\mathcal{A}$-measurable random variables with *p-th* moments. Define

$$\begin{aligned}
\alpha(\mathcal{A},\mathcal{B}) &= \sup_{A\in\mathcal{A},B\in\mathcal{B}} |P(AB)-P(A)P(B)|,\\
\rho(\mathcal{A},\mathcal{B}) &= \sup_{X\in L_2(\mathcal{A}),Y\in L_2(\mathcal{B})} \frac{|EXY-EXEY|}{\sqrt{\mathrm{Var}X\,\mathrm{Var}Y}},\\
\varphi(\mathcal{A},\mathcal{B}) &= \sup_{A\in\mathcal{A},B\in\mathcal{B},P(A)>0} |P(B|A)-P(B)|,\\
\psi(\mathcal{A},\mathcal{B}) &= \sup_{A\in\mathcal{A},B\in\mathcal{B},P(A)P(B)>0} \frac{|P(AB)-P(A)P(B)|}{P(A)P(B)},\\
\beta(\mathcal{A},\mathcal{B}) &= E(\mathrm{tvar}_{B\in\mathcal{B}}|P(B|\mathcal{A})-P(B)|),\\
\lambda(\mathcal{A},\mathcal{B}) &= \sup_{X\in L_{1/\alpha}(\mathcal{A}),Y\in L_{1/\beta}(\mathcal{B})} \frac{|EXY-EXEY|}{\|X\|_{1/\alpha}\|Y\|_{1/\beta}},
\end{aligned}$$

where tvar means total variation and $\|X\|_p = (E|X|^p)^{1/p}$. Let $\mathcal{F}_a^b = \sigma(X_i,\, a \leq i \leq b)$, $\mathbb{Z}$ a set of all integers, $\mathbb{Z}^+$ a set of all non-negative integers, $\mathbb{N}$ a set of all positive integers. Some commom and important definitions of mixing sequences are as follows:

Definition 1.1.1. A sequence $\{X_n, n \geq 1\}$ is said to be *α-mixing* or *strong mixing* if

$$\alpha(n) = \sup_{k \in \mathbb{N}} \alpha(\mathcal{F}_1^k, \mathcal{F}_{k+n}^\infty) \to 0 \quad \text{as } n \to \infty.$$

Definition 1.1.2. A sequence $\{X_n, n \geq 1\}$ is said to be *ρ-mixing* if

$$\rho(n) = \sup_{k \in \mathbb{N}} \rho(\mathcal{F}_1^k, \mathcal{F}_{k+n}^\infty) \to 0 \quad \text{as } n \to \infty.$$

Definition 1.1.3. A sequence $\{X_n, n \geq 1\}$ is said to be *φ-mixing* or *uniformly strong mixing* if

$$\varphi(n) = \sup_{k \in \mathbb{N}} \varphi(\mathcal{F}_1^k, \mathcal{F}_{k+n}^\infty) \to 0 \quad \text{as } n \to \infty.$$

Definition 1.1.4. A sequence $\{X_n, n \geq 1\}$ is said to be *ψ-mixing* or **-mixing* if

$$\psi(n) = \sup_{k \in \mathbb{N}} \psi(\mathcal{F}_1^k, \mathcal{F}_{k+n}^\infty) \to 0 \quad \text{as } n \to \infty.$$

Definition 1.1.5. A sequence $\{X_n, n \geq 1\}$ is said to be *absolutely regular* if

$$\beta(n) = \sup_{k \in \mathbb{N}} \beta(\mathcal{F}_1^k, \mathcal{F}_{k+n}^\infty) \to 0 \quad \text{as } n \to \infty.$$

Definition 1.1.6. Let $0 \leq \alpha, \beta \leq 1, \alpha + \beta = 1$. A sequence $\{X_n, n \geq 1\}$ is said to be (α, β) *-mixing* if

$$\lambda(n) = \sup_{k \in \mathbb{N}} \lambda(\mathcal{F}_1^k, \mathcal{F}_{k+n}^\infty) \to 0 \quad \text{as } n \to \infty.$$

Remark 1.1.1. The versions of the above definitions for a sequence with time-parameter set R^+ or R or $\mathbb{Z}$ are trivial.

Remark 1.1.2. The concept of α-mixing was introduced by Rosenblatt (1956). The concept of ρ-mixing was introduced by Kolmogorov and Rozanov (1960). Dobrushin (1956) first introduced the definition of φ-mixing for a Markov process. This definition for a stationary process was presented by Ibragimov(1959) and Rozanov and Volconski (1959) respectively (one can also trace back to Hirschfeld 1935 and Gebelein 1941).

Absolute regularity was introduced by Kolmogorov (1959),(cf. Rosanov and Volconski 1959). Blum, Hanson and Koopmans (1963) presented the concept of ψ-mixing. (α, β)-mixing was introduced by Bradley (1985a) and Shao(1989a) independently.

Remark 1.1.3. Doob(1953) showed that a Döeblin irreducible Markov chain is φ-mixing with $\varphi(n) \le ab^n$ for some $a > 0$ and $0 \le b < 1$; Rosenblatt (1971) showed that a purely non-deterministic Markov chain is α-mixing; Davydov (1973) gave a class of Markov chains which are β-mixing.

Remark 1.1.4. For simplicity, we always assume that the mixing coefficients $\alpha(n), \rho(n), \cdots, \lambda(n)$ all are non-increasing.

It is clear from the definitions that

$$\rho(n) = \lambda_{1/2,1/2}(n), \quad \lambda_{1,0}(n) = \varphi(n) \le \psi(n),$$

and further

$$\alpha(n) \le \rho(n)$$

by taking $X = 1_A$ and $Y = 1_B$ in the definition of ρ-mixing.

Kolmogorov and Rozanov (1960) investigated the relation between α-mixing and ρ-mixing for a Gaussian sequence.

Theorem 1.1.1. *For a Gaussian sequence* $\{X_n, n \ge 1\}$, *we have*

$$\alpha(\mathcal{F}_1^k, \mathcal{F}_{k+n}^\infty) \le \rho(\mathcal{F}_1^k, \mathcal{F}_{k+n}^\infty) \le 2\pi\alpha(\mathcal{F}_1^k, \mathcal{F}_{k+n}^\infty).$$

Proof. The former inequality is obvious.

For any $\varepsilon > 0$, there exist two normal random variables $X \in L_2(\mathcal{F}_1^k), Y \in L_2(\mathcal{F}_{k+n}^\infty)$ such that $EX = EY = 0, \mathrm{Var}X = \mathrm{Var}Y = 1$ and

$$r := EXY \ge \rho(\mathcal{F}_1^k, \mathcal{F}_{k+n}^\infty) - \varepsilon.$$

Noting that $A := \{X > 0\} \in \mathcal{F}_1^k, \quad B := \{Y > 0\} \in \mathcal{F}_{k+n}^\infty$, we have

$$P(AB) = \frac{1}{4} + \frac{1}{2\pi}\arcsin r, \quad P(A)P(B) = \frac{1}{4} \tag{1.1.1}$$

by elementary calculations (see Cramér 1946, p.290). If $\alpha(\mathcal{F}_1^k, \mathcal{F}_{k+n}^\infty) > \frac{1}{4}$, it is clear that

$$2\pi\alpha(\mathcal{F}_1^k, \mathcal{F}_{k+n}^\infty) > \frac{\pi}{2} \ge \rho(\mathcal{F}_1^k, \mathcal{F}_{k+n}^\infty);$$

if $\alpha(\mathcal{F}_1^k, \mathcal{F}_{k+n}^\infty) \le \frac{1}{4}$, by (1.1.1) we obtain

$$\alpha(\mathcal{F}_1^k, \mathcal{F}_{k+n}^\infty) \ge P(AB) - P(A)P(B) = \frac{1}{2\pi}\arcsin r,$$

which implies

$$\rho(\mathcal{F}_1^k, \mathcal{F}_{k+n}^\infty) - \varepsilon \le r \le \sin 2\pi\alpha \le 2\pi\alpha.$$

The theorem is proved by arbitrariness of ε.

Kolmogorov and Rozanov (1960) also studied the relation between the spectral function of a (weakly) stationary sequence and ρ-mixing property. At first, we give some notations and concepts about a stationary sequence $\{X_n,\ n \in \mathbb{N}\}$. Let the covariance function of $\{X_n\}$

$$R(n) = EX_m X_{m+n}.$$

By the Herglotz theorem, there exists the spectral resolution for $R(n)$ as follows:

$$R(n) = \int_{-\pi}^{\pi} e^{in\lambda}\, dF(\lambda),$$

where $F(\lambda)$ is called the spectral function of the stationary sequence. When the spectral function is absolutely continuous, its derivative $f(\lambda) = F'(\lambda)$ is called the spectral density of the stationary sequence.

Theorem 1.1.2. *If the spectral function of a stationary sequence is not absolutely continuous, then $\rho(n) \equiv 1$, i.e. the sequence is not ρ-mixing. Conversely, if the spectral function is absolutely continuous, then*

$$\rho(n) = \inf_h \operatorname{ess\,sup}_\lambda \left| f(\lambda) - e^{i\lambda n} h(e^{-i\lambda}) \right| / f(\lambda),$$

where the inf is extended over all functions which is analytically continuable in unit circle; and further, if there exists an analytic function $h_0(z)$ in unit circle with the boundary value $h_0(e^{-i\lambda})$ such that $|f(\lambda)/h_0(e^{-i\lambda})| \ge \varepsilon > 0$ and $(f(\lambda)/h_0(e^{-i\lambda}))^{(k)}$ is bounded uniformly, then

$$\rho(n) \le cn^{-k}$$

for some $c > 0$. In particular, when $f(\lambda)$ is a rational function of $e^{i\lambda}$,

$$\rho(n) = e^{-cn}$$

for some $c > 0$.

The Proof of Theorem 1.1.2 is omitted(Kolmogorov, Rozanov 1960).

1.2 Basic inequalities

Let X be $\mathcal{F}_{-\infty}^{k}$ measurable and Y be $\mathcal{F}_{k+n}^{\infty}$ measurable.

In this section, we establish some bounds of the covariance $\text{Cov}(X,Y) = EXY - EXEY$ for the various mixing sequences.

At first, we consider the α-mixing case.

Lemma 1.2.1. *Let $\{X_n, n \in \mathbb{Z}\}$ be an α-mixing sequence, $X \in \mathcal{F}_{-\infty}^{k}$ and $Y \in \mathcal{F}_{k+n}^{\infty}$ with $|X| \le C_1$ and $|Y| \le C_2$. Then*

$$|EXY - EXEY| \le 4C_1C_2\alpha(n). \tag{1.2.1}$$

Proof. By the property of conditional expectation, we have

$$\begin{aligned}|EXY - EXEY| &= \left|E\{X(E(Y|\mathcal{F}_{-\infty}^{k}) - EY)\}\right| \\ &\le C_1 E|E(Y|\mathcal{F}_{-\infty}^{k}) - EY| \\ &= C_1|E\xi\{E(Y|\mathcal{F}_{-\infty}^{k}) - EY\}|,\end{aligned}$$

where $\xi = \text{sgn}(E(Y|\mathcal{F}_{-\infty}^{k}) - EY) \in \mathcal{F}_{-\infty}^{k}$, i.e.

$$|EXY - EXEY| \le C_1|E\xi Y - E\xi EY|.$$

With the same argument procedure it follows that

$$|E\xi Y - E\xi EY| \le C_2|E\xi\eta - E\xi E\eta|,$$

where $\eta = \text{sgn}\ (E(\xi|\mathcal{F}_{k+n}^{\infty}) - E\xi)$. Therefore

$$|EXY - EXEY| \le C_1C_2|E\xi\eta - E\xi E\eta|. \tag{1.2.2}$$

Put $A = \{\xi = 1\}, B = \{\eta = 1\}$. It is clear that $A \in \mathcal{F}_{-\infty}^{k}, B \in \mathcal{F}_{k+n}^{\infty}$. Using the definition of α-mixing, we obtain

$$\begin{aligned}&|E\xi\eta - E\xi E\eta| \\ &\quad = |P(AB) + P(A^cB^c) - P(A^cB) - P(AB^c) \\ &\qquad - (P(A) - P(A^c))(P(B) - P(B^c))| \\ &\quad \le 4\alpha(n).\end{aligned}$$

Inserting it into (1.2.2) yields (1.2.1).

Lemma 1.2.2. *Let $\{X_n, n \in \mathbb{Z}\}$ be an α-mixing sequence, $X \in \mathcal{F}_{-\infty}^{k}$ and $Y \in \mathcal{F}_{k+n}^{\infty}$ with $E|X|^p < \infty$ for some $p > 1$ and $|Y| \le C$. Then*

$$|EXY - EXEY| \le 6C\|X\|_p(\alpha(n))^{1/q}, \tag{1.2.3}$$

where $1/p + 1/q = 1$.

Proof. Let $X_N = XI(|X| \le N)$, $X_N' = X - X_N$. Write

$$|EXY - EXEY| \le |EX_NY - EX_NEY| + |EX_N'Y - EX_N'EY|.$$

By Lemma 1.2.1, $|EX_NY - EX_NEY| \le 4CN\alpha(n)$. For the second term of the right hand side of the above inequality, we have

$$|EX_N'Y - EX_N'EY| \le 2CE|X_N'| \le 2CN^{-p+1}E|X|^p.$$

Taking $N = \|X\|_p(\alpha(n))^{-1/p}$ yields (1.2.3).

For a random variable X and a continuous non-decreasing function $f(x)$ on R^+ with $f(0) = 0$, which doesn't identically equal to zero, define

$$\|X\|_f = \inf\{t > 0, Ef(|X|/t) \le 1\}.$$

From this definition, it is easy to know that

$$\|X\|_f = 0 \iff X = 0 \quad \text{a.s.} \tag{1.2.4}$$

and if $0 < \|X\|_f < \infty$, then $Ef(|X|/\|X\|_f) \le 1$. Moreover, if $|X_1| \le |X_2|$ a.s., then $\|X_1\|_f \le \|X_2\|_f$.

Lemma 1.2.3. *Let $\{X_n,\ n \in \mathbb{Z}\}$ be an α-mixing sequence, $X \in \mathcal{F}_{-\infty}^{k}$, $Y \in \mathcal{F}_{k+n}^{\infty}$, $f(x)$ and $g(x)$ be two continuous functions on R^+ with $f(0) = g(0) = 0$, $f(x)/x^{\frac{r+s}{r}} \nearrow \infty$ and $g(x)/x^{\frac{r+s}{s}} \nearrow \infty$ for some $r > 0, s > 0, \|X\|_f < \infty, \|Y\|_g < \infty$. Then*

$$|EXY - EXEY| \le 10\,\mathrm{inv}\,f\Big(\frac{1}{\alpha(n)}\Big)\,\mathrm{inv}\,g\Big(\frac{1}{\alpha(n)}\Big)\alpha(n)\|X\|_f\|Y\|_g. \tag{1.2.5}$$

Proof. It is easy to see that $E|X|^{1+s/r} < \infty$ and $E|Y|^{1+r/s} < \infty$ by the conditions of the lemma . If either $\|X\|_f = 0$ or $\|Y\|_g = 0$, (1.2.4) implies that (1.2.5) holds. If $\alpha(n) = 0$, (1.2.5) is trivial by independence of X and Y. Now we assume that $\|X\|_f > 0, \|Y\|_g > 0$ and $\alpha(n) > 0$. There are $M > 0$ and $N > 0$ such that

$$\alpha(n) = 1/f(M/\|X\|_f) = 1/g(N/\|Y\|_g).$$

Let

$$X_M = XI(|X| \le M), \quad X_M' = X - X_M,$$
$$Y_N = YI(|Y| \le N), \quad Y_N' = Y - Y_N.$$

We have

$$\begin{aligned}
&|EXY - EXEY| \\
&\quad \le |EX_M Y_N - EX_M EY_N| + |EX_M' Y_N - EX_M' EY_N| \\
&\qquad + |EX_M Y_N' - EX_M EY_N'| + |EX_M' Y_N' - EX_M' EY_N'| \\
&\quad =: I_1 + I_2 + I_3 + I_4. \qquad (1.2.6)
\end{aligned}$$

By Lemma 1.2.1, $I_1 \le 4MN\alpha(n)$. Noting that $f(x)/x \nearrow \infty$ and $g(x)/x \nearrow \infty$, we have

$$\begin{aligned}
E|X_M'| &= E(|X_M'|/\|X_M'\|_f)\|X_M'\|_f \\
&\le Ef(|X_M'|/\|X_M'\|_f)M/f(M/\|X_M'\|_f) \\
&\le M/f(M/\|X\|_f).
\end{aligned}$$

Therefore

$$I_2 \le 2MN/f(M/\|X\|_f) = 2\,\mathrm{inv}\, f\Big(\frac{1}{\alpha(n)}\Big)\,\mathrm{inv}\, g\Big(\frac{1}{\alpha(n)}\Big)\alpha(n)\|X\|_f\|Y\|_g.$$

Similarly, we have the same estimation for I_3.

Furthermore, noting that $f(x)/x^{\frac{r+s}{r}} \nearrow \infty$ and $g(x)/x^{\frac{r+s}{s}} \nearrow \infty$, we have

$$\begin{aligned}
EX_M'Y_N' &\le \Big(E(|X_M'|/\|X_M'\|_f)^{\frac{r+s}{r}}\Big)^{\frac{r}{r+s}} \\
&\qquad \cdot \Big(E(|Y_N'|/\|Y_N'\|_g)^{\frac{r+s}{s}}\Big)^{\frac{s}{r+s}}\|X_M'\|_f\|Y_N'\|_g \\
&\le \Big(Ef(|X_M'|/\|X_M'\|_f)\Big)^{\frac{r}{r+s}}\Big(Eg(|Y_N'|/\|Y_N'\|_g)\Big)^{\frac{s}{r+s}} \\
&\qquad \cdot MN/\Big(f(M/\|X_M'\|_f)\Big)^{\frac{r}{r+s}}\Big(g(N/\|Y_N'\|_g)\Big)^{\frac{s}{r+s}} \\
&\le MN/\Big(f(M/\|X\|_f)\Big)^{\frac{r}{r+s}}\Big(g(N/\|Y\|_g)\Big)^{\frac{s}{r+s}}.
\end{aligned}$$

Hence

$$\begin{aligned}
I_4 &\le 2MN/\Big(f(M/\|X\|_f)\Big)^{\frac{r}{r+s}}\Big(g(N/\|Y\|_g)\Big)^{\frac{s}{r+s}} \\
&\le 2\,\mathrm{inv}\, f\Big(\frac{1}{\alpha(n)}\Big)\,\mathrm{inv}\, g\Big(\frac{1}{\alpha(n)}\Big)\alpha(n)\|X\|_f\|Y\|_g.
\end{aligned}$$

Now, inserting these estimations into (1.2.6) yields (1.2.5).

As some consequences of this lemma, we have

Lemma 1.2.4. *Let* $\{X_n,\ n \in \mathbb{Z}\}$ *be an* α*-mixing sequence,* $X \in \mathcal{F}_{-\infty}^k$ *and* $Y \in \mathcal{F}_{k+n}^{\infty}$ *with* $E|X|^p < \infty$ *and* $E|Y|^q < \infty, \frac{1}{p} + \frac{1}{q} < 1$. *Then*

$$|EXY - EXEY| \le 10\|X\|_p\|Y\|_q(\alpha(n))^{1-\frac{1}{p}-\frac{1}{q}}. \tag{1.2.7}$$

Lemma 1.2.5. *Let* $\{X_n,\ n \in \mathbb{Z}\}$ *be an* α*-mixing sequence,* $X \in \mathcal{F}_{-\infty}^k$ *and* $Y \in \mathcal{F}_{k+n}^{\infty}$ *with* $E|X|^{2+\delta} \le C_1, E|Y|^{2+\delta} \le C_2$. *Then*

$$|EXY - EXEY| \le 10(C_1C_2)^{\frac{1}{2+\delta}}(\alpha(n))^{\frac{\delta}{2+\delta}}. \tag{1.2.8}$$

For an (α, β)-mixing sequence and a ρ-mixing sequence, we have the following lemmas.

Lemma 1.2.6. *Let* $\{X_n,\ n \in \mathbb{Z}\}$ *be an* (α, β)*-mixing sequence,* $X \in L_p(\mathcal{F}_{-\infty}^k)$ *and* $Y \in L_q(\mathcal{F}_{k+n}^{\infty})$ *with* $p, q \ge 1$ *and* $1/p + 1/q = 1$. *Then*

$$|EXY - EXEY| \le 4\lambda(n)^{\frac{1}{\alpha p}\wedge\frac{1}{\beta q}}\|X\|_p\|Y\|_q. \tag{1.2.9}$$

Proof. Without loss of generality, assume that $\alpha p \ge 1$, which implies that $\beta q \le 1$. Put

$$Y_1 = YI(|Y| \le C), \quad Y_2 = Y - Y_1,$$

where C is a positive constant specified later on. Write

$$|EXY - EXEY| \le |EXY_1 - EXEY_1| + |EXY_2 - EXEY_2|. \tag{1.2.10}$$

By the definition of (α, β)-mixing and the Hölder inequality

$$\begin{aligned}|EXY_1 - EXEY_1| &\le \lambda(n)\|X\|_{1/\alpha}\|Y_1\|_{1/\beta}\\ &\le \lambda(n)C^{1-\beta q}\|X\|_p\|Y\|_q^{\beta q},\end{aligned}$$

$$\begin{aligned}|EXY_2| &\le (E|Y_2|^q)^{1-\frac{1}{\alpha p}}(E|X|^{\alpha p}|Y_2|^{\beta q})^{\frac{1}{\alpha p}}\\ &\le (E|Y_2|^q)^{1-\frac{1}{\alpha p}}\left(E|X|^{\alpha p}E|Y_2|^{\beta q} + \lambda(n)(E|X|^p)^{\alpha}(|Y_2|^q)^{\beta}\right)^{\frac{1}{\alpha p}}\\ &\le (E|Y|^q)^{1-\frac{1}{\alpha p}}\left(E|X|^{\alpha p}E|Y|^qC^{-\alpha q} + \lambda(n)(E|X|^p)^{\alpha}(E|Y|^q)^{\beta}\right)^{\frac{1}{\alpha p}}\\ &\le \|X\|_p\|Y\|_q^qC^{-\frac{q}{p}} + \lambda^{\frac{1}{\alpha p}}(n)\|X\|_p\|Y\|_q\end{aligned}$$

and

$$|EXEY_2| \le \|X\|_p \|Y\|_q^q C^{-q/p}.$$

Inserting these estimations into (1.2.10) and taking $C = \|Y\|_q (\lambda(n))^{-1/\alpha q}$ we obtain (1.2.9).

Let $p = q = 2$ in (1.2.9). It is easy to see that

$$\rho(n) \le 4\lambda(n)^{\frac{1}{2\alpha} \wedge \frac{1}{2\beta}}. \qquad (1.2.11)$$

As a consequence of Lemma 1.2.6, noting that $\rho(n) = \lambda_{1/2,1/2}(n)$, we have

Lemma 1.2.7. *Let $\{X_n, n \in \mathbb{Z}\}$ be a ρ-mixing sequence, $X \in L_p(\mathcal{F}_{-\infty}^k)$ and $Y \in L_q(\mathcal{F}_{k+n}^\infty)$ with $p, q \ge 1$ and $1/p + 1/q = 1$. Then*

$$|EXY - EXEY| \le 4\rho(n)^{\frac{2}{p} \wedge \frac{2}{q}} \|X\|_p \|Y\|_q.$$

For the φ-mixing case, we have the following three results.

Lemma 1.2.8. *Let $\{X_n, n \in \mathbb{Z}\}$ be a φ-mixing sequence, $X \in L_p(\mathcal{F}_{-\infty}^k)$ and $Y \in L_q(\mathcal{F}_{k+n}^\infty)$ with $p, q \ge 1$ and $1/p + 1/q = 1$. Then*

$$|EXY - EXEY| \le 2(\varphi(n))^{\frac{1}{p}} \|X\|_p \|Y\|_q. \qquad (1.2.12)$$

Proof. At first, we assume that $X$and $Y$are simple functions, i.e.

$$X = \sum_i a_i I_{A_i}, \quad Y = \sum_j b_j I_{B_j},$$

where both $\sum_i$ and $\sum_j$ are finite sums and $A_i \cap A_k = \emptyset \quad (i \ne k)$, $B_j \cap B_l = \emptyset \quad (j \ne l)$, $A_i \in \mathcal{F}_{-\infty}^k$, $B_j \in \mathcal{F}_{k+n}^\infty$. So

$$EXY - EXEY = \sum_{i,j} a_i b_j P(A_i B_j) - \sum_{i,j} a_i b_j P(A_i) P(B_j).$$

By the Hölder inequality we have

$$
\begin{aligned}
&|EXY-EXEY| \\
&\quad = \Big|\sum_i a_i(P(A_i))^{1/p} \sum_j (P(B_j|A_i) - P(B_j))b_j(P(A_i))^{1/q}\Big| \\
&\quad \le \Big(\sum_i |a_i|^p P(A_i)\Big)^{1/p} \Big(\sum_i P(A_i)\Big|\sum_j b_j(P(B_j|A_i) - P(B_j))\Big|^q\Big)^{1/q} \\
&\quad \le \|X\|_p \Big|\sum_i P(A_i)\Big(\sum_j |b_j|^q(P(B_j|A_i) \\
&\qquad + P(B_j))\Big)\Big(\sum_j |P(B_j|A_i) - P(B_j)|\Big)^{\frac{q}{p}}\Big|^{\frac{1}{q}} \\
&\quad \le 2^{1/q}\|X\|_p\|Y\|_q \max_i \Big(\sum_j |P(B_j|A_i) - P(B_j)|\Big)^{1/p}. \qquad (1.2.13)
\end{aligned}
$$

Note that

$$
\begin{aligned}
\sum_j |P(B_j|A_i) - P(B_j)| &= (P(\cup_j^+ B_j|A_i) - P(\cup_j^+ B_j)) \\
&\qquad - (P(\cup_j^- B_j|A_i) - P(\cup_j^- B_j)) \\
&\le 2\varphi(n), \qquad (1.2.14)
\end{aligned}
$$

where the union $\cup_j^+ (\cup_j^-)$ is carried out over j such that $P(B_j|A_i) - P(B_j) > 0$ $(P(B_j|A_i) - P(B_j) < 0)$. Inserting (1.2.14) into (1.2.13) yields (1.2.12) for the simple function case.

In order to complete the proof of the lemma, let

$$
X_N = \begin{cases} 0 & \text{if } |X| > N. \\ k/N & \text{if } k/N < X \le (k+1)/N,\ |X| \le N; \end{cases}
$$

$$
Y_N = \begin{cases} 0 & \text{if } |Y| > N. \\ k/N & \text{if } k/N < Y \le (k+1)/N,\ |Y| \le N. \end{cases}
$$

We have showed that (1.2.12) is true for X_N and Y_N. Moreover, note

$$
E|X - X_N|^p \to 0,\ E|Y - Y_N|^q \to 0, \quad \text{as } N \to \infty.
$$

Letting $N \to \infty$, we obtain (1.2.12) for the general case.

Let $p = q = 2$ in (1.2.12). It is easy to see that

$$
\rho(n) \le 2\varphi^{1/2}(n). \qquad (1.2.15)
$$

From the proof of Lemma 1.2.8, we can see that

Lemma 1.2.9. *Let $\{X_n, n \in \mathbb{Z}\}$ be a φ-mixing sequence, $X \in \mathcal{F}_{-\infty}^{k}$ and $Y \in \mathcal{F}_{k+n}^{\infty}$ with $|X| \le C_1$ and $|Y| \le C_2$. Then*

$$|EXY - EXEY| \le 2C_1C_2\varphi(n). \tag{1.2.16}$$

Let $p = 1$ and $q = \infty$ in (1.2.12). From Lemma 1.2.8, we also have

Lemma 1.2.10. *Let $\{X_n, n \in \mathbb{Z}\}$ be a φ-mixing sequence, $X \in \mathcal{F}_{-\infty}^{k}$ and $Y \in \mathcal{F}_{k+n}^{\infty}$ with $E|X| < \infty$ and $|Y| \le C$. Then*

$$|EXY - EXEY| \le 2C\varphi(n)E|X|. \tag{1.2.17}$$

Finally, we consider the ψ-mixing case.

Lemma 1.2.11. *Let $\{X_n, n \in \mathbb{Z}\}$ be a ψ-mixing sequence, $X \in \mathcal{F}_{-\infty}^{k}$ and $Y \in \mathcal{F}_{k+n}^{\infty}$ with $E|X| < \infty$ and $E|Y| < \infty$. Then $E|XY| < \infty$ and*

$$|EXY - EXEY| \le \psi(n)E|X|E|Y|. \tag{1.2.18}$$

Proof. At first, we assume that X and Y are non-negative simple functions. We have

$$\begin{aligned}|EXY - EXEY| &= |\sum_{i,j} a_i b_j (P(A_iB_j) - P(A_i)P(B_j))| \\ &\le \sum_{i,j} a_i b_j \psi(n) P(A_i)P(B_j) \\ &= \psi(n)EXEY.\end{aligned}$$

From this, (1.2.18) holds for non-negative random variables X and Y.

For the general case, write $X = X^+ - X^-$, $Y = Y^+ - Y^-$. We have

$$\begin{aligned}&|EXY - EXEY| \\ &\quad \le |EX^+Y^+ - EX^+EY^+| + |EX^+Y^- - EX^+EY^-| \\ &\quad\quad + |EX^-Y^+ - EX^-EY^+| + |EX^-Y^- - EX^-EY^-| \\ &\quad \le \psi(n)(EX^+ + EX^-)(EY^+ + EY^-) \\ &\quad \le \psi(n)E|X|E|Y|.\end{aligned}$$

Finally, we summarize the relations between one and another of variaous mixing properties. It is easy to verify that

$$2\alpha(n) \le \beta(n) \le \varphi(n). \tag{1.2.19}$$

With a necessary and sufficient condition for Markov processes to be ψ-mixing, one can show that a φ-mixing (Markov) sequence is not ψ-mixing (Blum, Hanson and Koopmans 1963). Ibragimov and Solev (1969) given an example of a stationary α-mixing Gaussian process which is not β-mixing; such a process is ρ-mixing but not β-mixing. Davydov (1973) constructed a stationary α-mixing Markov process with less than geometric rate of decay of the mixing coefficients, which is not ρ-mixing. It is possible that a geometrically ergodic Markov process which is not Doeblin recurrent is β-mixing and not φ-mixing (Andrews 1984). Combining these results and recalling Remark 1.1.4, (1.2.11) and (1.2.15) we have

$$\psi-\text{ mixing}\begin{Bmatrix}\Longrightarrow\\ \not\Longleftarrow\end{Bmatrix}\varphi-\text{ mixing}\begin{Bmatrix}\Longrightarrow\\ \not\Longleftarrow\end{Bmatrix}\left\{\begin{array}{l}\beta-\text{ mixing}\begin{Bmatrix}\Longrightarrow\\ \not\Longleftarrow\end{Bmatrix}\alpha-\text{ mixing}\\ \quad\not\Uparrow\quad\not\Downarrow\\ \rho-\text{ mixing}\begin{Bmatrix}\Longrightarrow\\ \not\Longleftarrow\end{Bmatrix}\alpha-\text{ mixing}\end{array}\right.$$

$$\Uparrow$$

$$\lambda-\text{mixing}$$